INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.637/7

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

The Hodge decomposition theorems (I)

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These are preliminary lecture notes, intended only for distribution to participants

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The Hodge decomposition theorem I

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References

[G, H] P. Griffiths, J. Harris, Principles of Algebraic Geometry, John Wiley 1978

[Wa] F. W. Warner, Foundations of diff. manifolds and Lie groups, Springer

[We] R.O. Wells, Diff. analysis on complex manifolds, Springer GTM 65, 1980

In the above references general elliptic operators are treated. The text [Wa] is more elementary than [G, H] and [We]. In [We] the theory is based on pseudo-differential operators which are an interesting topic in itself. We will mostly follow [We].

We begin with generalities about the $*$-operator first for hermitian vector spaces and then for Riemannian or Hermitian Manifolds.

Let V be an n-dimensional real vector space with an inner product $\langle , \rangle$. It induces inner products on all exterior powers $\Lambda^p V$ as follows: If $e_1, \dots, e_n$ is an orthonormal basis ...

$e_{i_1} \wedge \dots \wedge e_{i_p}$ for $1 \le i_1 < \dots < i_p \le m$ form an ON-basis of $\Lambda^p V$.

Since $\dim_{\mathbb{R}} \Lambda^m V = 1$ there are exactly two ON-bases of $\Lambda^m V$. These are called orientations. They correspond to an ordering of $e_1, \dots, e_m$ up to an even permutation, since $e_{\sigma_1} \wedge \dots \wedge e_{\sigma_m} = \mathrm{sgn}(\sigma)\, e_1 \wedge \dots \wedge e_m$ for all $\sigma \in \mathfrak{S}_m$.

Fix an orientation $\mathrm{vol} \in \Lambda^m V$. Define the $*$-operator as follows

$$* : \Lambda^p V \longrightarrow \Lambda^{m-p} V .$$

Having fixed an ON-basis of V, $e_1, \dots, e_m$ s.t. $\mathrm{vol} = e_1 \wedge \dots \wedge e_m$ set

$$*(e_{i_1} \wedge \dots \wedge e_{i_p}) = \delta\, e_{j_1} \wedge \dots \wedge e_{j_{m-p}}$$

where $\{i_1, \dots, i_p, j_1, \dots, j_{m-p}\} = \{1, \dots, m\}$ and $\delta \in \{\pm 1\}$ is s.t.

$$(e_{i_1} \wedge \dots \wedge e_{i_p}) \wedge *(e_{i_1} \wedge \dots \wedge e_{i_p}) = \mathrm{vol} .$$

For the $\mathbb{R}$-linear extension of $*$ to $\Lambda^p V$ we then have

$$(1.1) \qquad \alpha \wedge * \beta = \langle \alpha, \beta \rangle \mathrm{vol} \quad \text{for all } \alpha, \beta \in \Lambda^p V.$$

Setting $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, $\langle , \rangle$ on V extends to a hermitian scalar product $\langle , \rangle$ on $V_{\mathbb{C}}$. If $*$ is extended $\mathbb{C}$-linearly to $V_{\mathbb{C}}$ by $* \mathrel{\hat=} * \otimes \mathrm{id}$ we have

$$(1.2) \qquad \alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle \mathrm{vol} \quad \text{for all } \alpha, \beta \in \Lambda^p V_{\mathbb{C}} = \Lambda^p V \otimes_{\mathbb{R}} \mathbb{C}.$$

If $*$ is extended $\mathbb{C}$-antilinearly to $V_{\mathbb{C}}$ by $\bar{*} \triangleq * \otimes c$, c = complex conjugation on $\mathbb{C}$ then

(1.3) $$\alpha \wedge \bar{*} \beta = \langle \alpha, \beta \rangle \, \mathrm{vol}.$$

The $*$-operator is almost an involution:

Set $\Lambda^* V = \bigoplus_{p=0}^{m} \Lambda^p V$ and denote by

$$\pi_p : \Lambda^* V \longrightarrow \Lambda^p V \quad \text{the } p\text{-th projection}.$$

If we set

$$\omega : \Lambda^* V \longrightarrow \Lambda^* V \quad , \quad \omega = \sum_{p=0}^{m} (-1)^{mp+p} \pi_p$$

then we have (exercise!)

(1.4) $$* * = \omega .$$

Note that if m is even $\omega = \sum (-1)^p \pi_p$.

Now let M be a compact Riemannian manifold of dimension m. We assume that there exists a (C^∞-) section vol of $\Lambda^m T^*(M)$ s.t. for all $\xi \in M$

$$\mathrm{vol}_\xi \in \Lambda^m T^*_\xi(M) \quad \text{is an orientation (with respect}$$

to the inner product coming from the Riemannian structure).

If such a section vol exists the manifold is called orientable. The orientability of M depends in fact only on the manifold structure of M and not on the choice of a Riemannian metric.

Henceforth we assume M to be [illegible]
a choice of orientation vol. We obtain smooth bundle maps

$$* : \Lambda^p T^*(M) \to \Lambda^{m-p} T^*(M) \quad \text{and} \quad * : \Lambda^p T^*(M)_{\mathbb{C}} \to \Lambda^{m-p} T^*(M)_{\mathbb{C}}$$

by defining $*$ fibrewise. They give isomorphisms of sections

$$A^p(M) = \Gamma(M, \Lambda^p T^*(M)_{\mathbb{C}})$$

$$* : A^p(M) \xrightarrow{\sim} A^{m-p}(M) \quad \text{s.t.} \quad \alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle \, \mathrm{vol}$$

for all $\alpha, \beta \in A^p(M)$. Note that this means

$$\alpha_\xi \wedge * \bar{\beta}_\xi = \langle \alpha_\xi, \beta_\xi \rangle_\xi \, \mathrm{vol}_\xi \quad \text{for all } \xi \in M.$$

Before going on we have to recall how one integrates m-forms on an orientable m-dimensional compact manifold:

For $\varphi \in A^m(M)$ define $\int_M \varphi$ as follows:

Fix a finite atlas $\{ \mathcal{U}_\alpha, f_\alpha : V_\alpha \subset \mathbb{R}^m \xrightarrow{\sim} \mathcal{U}_\alpha \subset M \}$ and choose a subordinate partition of unity $\{\psi_\alpha\}$ i.e. $\operatorname{supp} \psi_\alpha \subset\subset \mathcal{U}_\alpha$, $\sum_\alpha \psi_\alpha = 1$. Set:

$$\int_M \varphi := \sum_\alpha \int_{V_\alpha} f_\alpha^*(\psi_\alpha \varphi) = \sum_\alpha \int_{\mathbb{R}^m} g_\alpha(x) \, dx$$

where $f_\alpha^*(\psi_\alpha \varphi) = g_\alpha(x) \, dx_1 \wedge \dots \wedge dx_m$ for some $g_\alpha \in C_0^\infty(V_\alpha)$.

The definition of $\int_M \varphi$ is independent of all choices (exercise)!

On $A^{\cdot}(M) = \bigoplus_{p=0} A^{\cdot}(M)$ we can now expand

a hermitian inner product by

$$(\alpha,\beta) := \int_M \alpha \wedge *\bar\beta = \int_M \langle\alpha,\beta\rangle \,\mathrm{vol} \quad \text{for } \alpha,\beta\in A^p(M)$$

(1.5)

$$(\alpha,\beta) := 0 \quad \text{for } \alpha\in A^p(M),\ \beta\in A^q(M) \text{ with } p\neq q.$$

This is called the <u>Hodge inner product</u> on $A^*(M)$.

If M is a Hermitian complex manifold of real dimension $2m$ define the Hodge inner product with respect to the underlying Riemannian metric and the orientation coming from the complex structure. Set

$$A^{pq}(M) = (p,q)\text{-forms in } A^{p+q}(M).$$

<u>Prop. (1.6)</u> $A^n(M) = \bigoplus_{p+q=m} A^{pq}(M)$ is orthogonal with respect to the Hodge inner product.

<u>Proof</u>: $\alpha\in A^{p,q}(M)$, $\beta\in A^{r,s}(M)$, $p+q=r+s$. Then

$\alpha\wedge *\bar\beta$ has type $(m-r+p,\ m-s+q)$. It is a $2m$-form iff $\begin{matrix} m-r+p = m \\ m-s+q = m\end{matrix}$ i.e. $p=r,\ q=s$.

<u>Prop. (1.7)</u> M m-dimensional compact oriented Riemannian manifold. Then exterior derivation $d: A^*(M) \to A^*(M)$ has an adjoint d^* with respect to the Hodge inner product and we have:

i) $d^* = \delta$ on $A^p(M)$ where $\delta = (-1)^{m+mp+1} * d * = (-1)^{m+mp+1} \bar{*} d \bar{*}$

ii) For the operator $\Delta = d d^* + d^* d$ on $A^*(M)$ or $A^p(M)$ we have $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$, $*\Delta = \Delta *$, $\bar{*}\Delta = \Delta\bar{*}$

<u>Remark:</u> The operator Δ is the Laplace operator on M.

<u>Ex.</u> Using the isomorphism $S^1 \cong \mathbb{R}/\mathbb{Z}$, $e^{2\pi i\theta} \leftarrow \theta$ the standard metric on $\mathbb{R}$ induces a Riemannian structure on S^1. Let $T^m = \underbrace{S^1 \times \dots \times S^1}_{m}$ with the induced Riemannian structure. Then we have for $\alpha \in A^0(T^m)$

$$\Delta\alpha = -\left(\frac{\partial^2}{\partial\theta_1^2} + \dots + \frac{\partial^2}{\partial\theta_m^2}\right)\alpha . \qquad \text{(exercise!)}$$

(The orientation chosen is the "obvious" one).

<u>Proof of (1.7):</u> i) For $\alpha \in A^{p-1}(M)$, $\beta \in A^p(M)$ we have

$$d(\alpha \wedge *\bar{\beta}) = d\alpha \wedge *\bar{\beta} + (-1)^{p-1}\alpha \wedge d(*\bar{\beta})$$
$$\overset{(1.4)}{=} d\alpha \wedge *\bar{\beta} - \alpha \wedge *\overline{\delta\beta} .$$

Hence

$$0 = \int_M d(\alpha \wedge *\bar{\beta}) = (d\alpha, \beta) - (\alpha, \delta\beta) \quad \Rightarrow \text{i)}, \text{ rest: exercise} .$$

Stokes theorem! $\partial M = \emptyset$

Notation: The kernel of Δ on $A^p(M)$

$$\mathcal{H}^p(M) := \mathrm{Ker}\,(\Delta : A^p(M) \to A^p(M))$$

is called the space of "harmonic p-forms on M".

Ex.: For $p=0$ we have $\Delta = d^* d$ on $A^0(M)$. Hence

$\Delta\alpha = 0 \Rightarrow (\Delta\alpha, \alpha) = 0 \Rightarrow \|d\alpha\|^2 = 0 \Rightarrow d\alpha = 0 \Rightarrow \alpha$ constant

(M being connected). It follows that

$$\mathcal{H}^0(M) = \mathbb{C} .$$

For the $\bar\partial$-operator there is the entirely analogous result

Prop. (1.8) M compact Hermitian complex manifold. Then

$$\bar\partial : A^{p,q}(M) \to A^{p,q+1}(M)$$

has an adjoint $\bar\partial^*$ with respect to the Hodge inner product an we have

$$\bar\partial^* = - \bar{*}\, \bar\partial\, \bar{*} .$$

If $\bar\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ denotes the "complex Laplacian" we have: $\bar\Box\, \bar{*} = \bar{*}\, \bar\Box$ and $(\bar\Box\alpha, \beta) = (\alpha, \bar\Box\beta)$.

Notation: The kernel of $\bar\Box$ on $A^{pq}(M)$

$$\mathcal{H}^{pq}(M) := \mathrm{Ker}\,(\bar\Box : A^{pq}(M) \to A^{pq}(M))$$

is called the space of "harmonic (p,q)-forms on M".

We are now in the position to state the famous Hodge decomposition theorem for forms:

<u>Th. (1.9)</u> Let M be as in (1.7). For each p the space $\mathcal{H}^p(M)$ is finite dimensional and we have the following orthogonal sum decompositions of $A^p(M)$:

$$A^p(M) \overset{(1)}{=} \mathcal{H}^p(M) \oplus \Delta A^p(M)$$

$$\overset{(2)}{=} \mathcal{H}^p(M) \oplus d d^* A^p(M) \oplus d^* d A^p(M)$$

$$\overset{(3)}{=} \mathcal{H}^p(M) \oplus d A^{p-1}(M) \oplus d^* A^{p+1}(M).$$

In particular the equation $\Delta \omega = \alpha$ has a solution $\omega \in A^p(M)$ iff $\alpha \in A^p(M)$ is orthogonal to $\mathcal{H}^p(M)$.

<u>Th. (1.10)</u> Let M be as in (1.8). For all p, q the spaces $\mathcal{H}^{pq}(M)$ are finite dimensional and we have orth. sum decompositions:

$$A^{pq}(M) = \mathcal{H}^{pq}(M) \oplus \bar{\square} A^{pq}(M)$$

$$= \mathcal{H}^{pq}(M) \oplus \bar{\partial} \bar{\partial}^* A^{pq}(M) \oplus \bar{\partial}^* \bar{\partial} A^{pq}(M)$$

$$= \mathcal{H}^{pq}(M) \oplus \bar{\partial} A^{p,q-1}(M) \oplus \bar{\partial}^* A^{p,q+1}(M).$$

In particular the equation $\bar{\square} \omega = \alpha$ has a solution $\omega \in A^{pq}(M)$ iff $\alpha \in A^{pq}(M)$ is orthogonal to $\mathcal{H}^{pq}(M)$.

Remark: There is in fact a general Hodge decomposition theorem for any "elliptic operator" on M: Δ and $\bar{\Box}$ are examples of elliptic operators. The proof which we will sketch roughly below for Δ carries over in fact to elliptic operators.

Elementary remarks on (1.9): We have the important fact:

(1.11) Fact: $\Delta\alpha = 0 \iff d\alpha = 0$ and $d^*\alpha = 0$

Proof: "$\Leftarrow$" clear, "$\Rightarrow$" $0 = (\Delta\alpha, \alpha) = (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha)$

$$= (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2$$

$\Rightarrow d\alpha = 0$, $d^*\alpha = 0$.

Now it is clear that the decompositions in (1.9) are orthogonal (also use $d^2 = 0$) and hence (2) and (3) follow from (1).

(1.12) Def.: Let $\mathcal{H} : A^p(M) \to \mathcal{H}^p(M)$ denote the (orthogonal) projection with respect to the Hodge decomposition. The Green's operator

$$G : A^p(M) \to \mathcal{H}^p(M)^\perp$$

is defined by $G(\alpha) = \omega$ where ω is the unique (!) solution of $\Delta\omega = \alpha - \mathcal{H}(\alpha)$ in $\mathcal{H}^p(M)^\perp$.

Similarly for the $\bar{\Box}$-Hodge decomposition (1.10)

Theorems (1.9) and (1.10) have important consequences for the study of de Rham and Dolbeault cohomology.

Recall that

$$H^p_{dR}(M,\mathbb{C}) = \frac{\operatorname{Ker}(d : A^p(M) \to A^{p+1}(M))}{\operatorname{Im}(d : A^{p-1}(M) \to A^p(M))} = \frac{\{d\text{-closed } p\text{-forms}\}}{\{d\text{-exact } p\text{-forms}\}} .$$

If a form $\beta \in d^* A^{p+1}(M)$ is d-closed $d\beta = 0$ then $\beta = 0$ (since $\beta = d^*\alpha \Rightarrow \|\beta\|^2 = (d^*\alpha, d^*\alpha) = (dd^*\alpha, \alpha) = (d\beta,\alpha) = 0$

Hence the Hodge decomposition (3) of Th. (1.9) implies using (1.11) that

$$\operatorname{Ker}(d : A^p(M) \to A^{p+1}(M)) = \mathcal{H}^p(M) \oplus d A^{p-1}(M) .$$

Hence the map

(1.13) $\quad \mathcal{H}^p(M) \xrightarrow{\sim} H^p_{dR}(M,\mathbb{C}) \quad , \quad \omega \mapsto [\omega]$

is an <u>isomorphism</u> with inverse $[\alpha] \mapsto \mathcal{H}(\alpha)$.

Every dR-cohomology class thus contains a <u>unique</u> harmonic representative. In particular (1.13) implies

$$\dim_{\mathbb{C}} H^p_{dR}(M,\mathbb{C}) < \infty \; !$$

Entirely analogous remarks apply to Dolbeault cohomology:

Setting

$$H^{pq}_{\bar\partial}(M) := \frac{\operatorname{Ker}(\bar\partial : A^{pq}(M) \to A^{p,q+1}(M))}{\operatorname{Im}(\bar\partial : A^{p,q-1}(M) \to A^{pq}(M))}$$

we have canonically

$$\mathcal{H}^{pq}(M) \xrightarrow{\sim} H^{pq}_{\bar\partial}(M)$$

and in particular the $H^{pq}_{\bar\partial}(M)$ are finite dimensional. Note that by Dolbeault's theorem we have

$$H^q(M, \Omega^p) \cong H^{pq}_{\bar\partial}(M).$$

In our second section we give some indications about the proof of (1.9). We begin with Sobolev spaces and L^2-spaces.

Let M be a compact oriented Riemannian manifold of dimension m and $E \to M$ a hermitian vector bundle [in applications to Δ, $E = \Lambda^p T^*(M)_{\mathbb{C}}$]

Set $\quad (\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_E \operatorname{vol} \quad$ for sections α, β of E

[in appl. to Δ this is the Hodge inner product]

Let $L^2(M, E)$ be the completion of the space of (C^∞-) sections $\Gamma(M, E)$ with respect to $(\,,\,)$. This is a Hilbert space.

Define Sobolev norms $\| \ \|_{E,s}$ for any $s \in \mathbb{R}$ on $\Gamma(M, E)$ as follows:

Let $\{ \mathcal{U}_i, f_i, f_{E,i} \}$ be a finite trivializing cover for M, E:

$$\begin{tikzcd} V_i \times \mathbb{C}^r \arrow[r, "\sim", "f_{E,i}"'] \arrow[d] & E \mid \mathcal{U}_i \arrow[d] \\ \mathbb{R}^m \supset V_i \arrow[r, "\sim", "f_i"'] & \mathcal{U}_i \subset M \end{tikzcd} \qquad ; \quad r = \operatorname{rank} E$$

and let $f_i^* : \Gamma(\mathcal{U}_i, E) \xrightarrow{\simeq} C^\infty(V_i)^r$, $\sigma \mapsto f_{E,i}^{-1} \circ \sigma \circ f_i$ be the induced map. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{\mathcal{U}_i\}$ and set for $\alpha \in \Gamma(M, E)$:

$$\| \alpha \|_{E,s} := \sum_i \| \underbrace{f_i^*(\psi_i \alpha)}_{\text{has compact support in } \mathbb{R}^m} \|_{\mathbb{R}^m, s} .$$

Here for $f : \mathbb{R}^m \to \mathbb{C}^r$ a C^∞-function with compact support

$$\| f \|^2_{\mathbb{R}^m, s} = \sum_{j=1}^{m} \| f_j \|^2_{\mathbb{R}^m, s}$$

where finally for $f \in C_0^\infty(\mathbb{R}^m)$ one sets:

(2.1) $\|f\|^2_{\mathbb{R}^m,s} := \int_{\mathbb{R}^m} |\hat{f}(y)|^2 (1+|y|^2)^s \, dy$; $|y|^2 = \sum_{j=1}^m y_j^2$

where $\hat{f}(y) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x)\, dx$

is the Fourier transform.

Remark: On $C_0^\infty(\mathbb{R}^m)$ the $\| \, \|_{\mathbb{R}^m,s}$ - norm for $s = 0, 1, \ldots$ is equivalent to $\| \, \|'_{\mathbb{R}^m,s}$ defined by:

(2.1.1) $\|f\|'_{\mathbb{R}^m,s} := \left(\sum_{|\gamma| \le s} \int_{\mathbb{R}^m} |D^\gamma f|^2 dx \right)^{1/2}$.

This follows from : $\widehat{D^\gamma f}(y) = y^\gamma \hat{f}(y)$, $\|f\|_2 = \|\hat{f}\|_2$ where $y^\gamma = y_1^{\gamma_1} \ldots y_m^{\gamma_m}$, $D^\gamma = (-i)^{|\gamma|} D_1^{\gamma_1} \ldots D_m^{\gamma_m}$, $D_j = \frac{\partial}{\partial x_j}$, $|\gamma| = \sum_{j=1}^m \gamma_j$.

Proof is an exercise! Start with proving the inequality:

$$(1+|y|^2)^s \le \sum_{|\gamma| \le s} y^{2\gamma} \le c\,(1+|y|^2)^s \quad \text{for suitable } c > 0.$$

The norms $\| \, \|_{E,s}$ actually depend on our choices. However due to the compactness of M different choices of trivializations $(\mathcal{U}_i, t_{E,i}, t_i)$ and partitions of unity give rise to equivalent norms. We fix such choices in the following. Every norm comes from a scalar product (which?) Completing $\Gamma(M,E)$ with respect to $\| \, \|_{E,s}$ therefore defines a Hilbert space $W^s(M,E)$ for every $s \in \mathbb{R}$. These is a natural isomorphism of topological vector spaces

$$(2.2) \qquad L^2(M,E) = W^0(M,E)$$

such that the norms $\| \|_2 = \sqrt{(,)}$ and $\| \|_{E,0}$ are equivalent.

Note that since $\| \|_{E,s} \geq \| \|_{E,t}$ on $\Gamma(M,E)$ for $s \geq t$ there are natural inclusions

$$(2.3) \qquad W^s(M,E) \hookrightarrow W^t(M,E) \quad \text{for } s \geq t$$

(2.4) <u>Proposition</u>: For every $s \in \mathbb{R}$ there is a natural continuous pairing $\mathbb{C}$-linear in the first $\mathbb{C}$-antilinear in the second argument

$$(,): W^s(M,E) \times W^{-s}(M,E) \longrightarrow \mathbb{C}$$

which identifies $W^{-s}(M,E)$ with the space $W^s(M,E)^*$ of $\mathbb{C}$-antilinear continuous linear forms on $W^s(M,E)$.

<u>Proof</u>: is reduced to pairing $f, g \in C_0^\infty(\mathbb{R}^m)$ as follows:

$$(f,g) = \int_{\mathbb{R}^m} f(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}^m} \hat{f}(y)\,\hat{\bar{g}}(y)\,dy .$$

Hence

$$|(f,g)| \leq \int_{\mathbb{R}^m} |\hat{f}(y)|\,(1+|y|^2)^{s/2}\,|\hat{\bar{g}}(y)|\,(1+|y|^2)^{-s/2}\,dy$$

$$\leq \|f\|_{\mathbb{R}^m,s}\,\|g\|_{\mathbb{R}^m,-s} .$$

Thus the nat. $\mathbb{C}$-linear, $\mathbb{C}$-antilinear pairing

$$(,): \Gamma(M,E) \times \Gamma(M,E) \longrightarrow \mathbb{C}$$

induces a non-degenerate continuous pairing

$(\, , \,) \; : W^s(M,E) \times W^{-s}(M,E) \longrightarrow \mathbb{C}$.

Since the $W^s(M,E)$ are Hilberspaces we conclude that $W^{-s}(M,E) = W^s(M,E)^*$.

The two basic results about Sobolev spaces are the following:

(2.5) Theorem (Sobolev) If $s > [\frac{m}{2}] + k + 1$ then

(2.6) $W^s(M,E) \subset \Gamma^k(M,E) :=$ space of k-times cont. diff. sections of E

and in particular

(2.7) $\bigcap_s W^s(M,E) = \Gamma(M,E)$ the C^∞-sections of E.

These assertions need some explanation: For any $s,t \in \mathbb{R}$ with $s \geq t$ there is the natural embedding (2.3)

$$W^s(M,E) \hookrightarrow W^t(M,E).$$

In particular for every $s \geq 0$

(2.8) $W^s(M,E) \subset W^0(M,E) = L^2(M,E)$.

Hence every element of $W^s(M,E)$ can be viewed as an equivalence class of measurable sections up to almost everywhere equality (with respect to the measure on Borel sets of M defined by $\mu(B) = \int_M \chi_B(x)\, \mathrm{vol}$. Assertion (2.6) says that in every such equivalence class there is a unique representative in $\Gamma^k(M,E)$. Using (2.6) the meaning of (2.7) is clear as well.

For the proof see e.g. [We] $\underline{\text{IV}}$ (1,1)'.

The second important result is this:

<u>(2.9) Theorem (Rellich):</u> The natural inclusion

$$j : W^s(M,E) \hookrightarrow W^t(M,E) \quad \text{for } s > t$$

is a compact operator i.e. j maps bounded sets to relatively compact sets.

For the proof compare e.g. [We] $\underline{\text{IV}}$ (1.2)'.

The spaces $\Gamma(M,E)$ are by definition dense in $W^s(M,E)$. Let

$$OP_k(E) = \text{"operators of order } k \text{ on } E\text{"}$$

be the space of $\mathbb{C}$-linear mappings

$$L : \Gamma(M,E) \to \Gamma(M,E)$$

which have a continuous extension

$$L_s : W^s(M,E) \longrightarrow W^{s-k}(M,E)$$

for all s.

(2.10) Prop.: Any $L \in OP_k(E)$ has an adjoint $L^* \in OP_k(E)$ and $(L^*)_s = L^*_{k-s}$.

Proof: $L_s : W^s(M,E) \to W^{s-k}(M,E)$ induces

$$\begin{array}{ccc} L_s^* : W^{s-k}(M,E)^* & \longrightarrow & W^s(M,E)^* \\ \wr\wr \ (2.4) & & \wr\wr \\ W^{k-s}(M,E) & \longrightarrow & W^{-s}(M,E) \end{array}.$$

Thus $Q_s := L^*_{k-s} : W^s(M,E) \to W^{s-k}(M,E)$.

Uniqueness of adjoints implies that

$$Q_s|_{W^t(M,E)} = Q_t \quad \text{for } t \geq s .$$

Hence $Q|_{\Gamma(M,E)}$ takes values in $\bigcap_s W^{s-k}(M,E) \overset{(2.5)}{=} \Gamma(M,E)$.

Clearly $Q = L^*$ on $\Gamma(M,E)$ and $L^* \in OP_k(E)$.

In the following we fix a $0 \leq p \leq m$ and set $E = \Lambda^p T^*(M)_{\mathbb{C}}$ and $W^s(M) := W^s(M, \Lambda^p T^*(M)_{\mathbb{C}})$. It is easy to see that $\Delta : A^p(M) \to A^p(M)$ induces operators $\Delta_s : W^s(M) \to W^{s-2}(M)$ which are continuous. (For $s = \ \ 2,3,\ldots$ this is obvious by using (2.1.1); the general case is also not difficult)

Hence $\Delta \in OP_2 := OP_2(\Lambda^p T^* M_{\mathbb{C}})$.

According to (2.10) and the symmetry (1.7) ii) of Δ we get

(2.11) $$\Delta^*_{2-s} = \Delta_s .$$

Using e.g. pseudodifferential operators as in [We] IV one can show:

(2.12) Theorem: There exists a parametrix (or pseudo-inverse) for Δ i.e. an operator $P \in OP_{-2}$ s.t.

$$P \circ \Delta - id \in OP_{-1}$$
$$\Delta \circ P - id \in OP_{-1} .$$

In fact these techniques establish an analogue of (2.12) for general elliptic operators on M (including $\Delta, \bar{\Box}$).

We need

(2.13) Prop.: Let $S \in OP_{-1}$. Then S is a compact operator, i.e. for all extensions $\tilde{S}_s : W^s(M) \to W^s(M)$ we have: $\tilde{S}_s$ is compact.

Proof: This follows from the commutative diagram

$$\begin{tikzcd} W^s(M) \arrow[rr, "\tilde{S}_s"] \arrow[dr, "S_s"'] & & W^s(M) \\ & W^{s+1}(M) \arrow[ur, hook, "j"'] & \end{tikzcd}$$

using the compactness of j (Rellich's lemma (2.9)).

—

Recall the notation

$$\mathcal{H}^p(M) = \operatorname{Ker}(\Delta : A^p(M) \to A^p(M))$$

and let $\mathcal{H}^p(M)^\perp$ be the orthogonal complement in $L^2(M, E)$, $E = \Lambda^p T^* M_{\mathbb{C}}$.

Before we can prove that $\mathcal{H}^p(M)$ is finite dimensional we need the following useful result from functional analysis.

(2.14) Prop.: Let $\mathcal{B}$ be a Banach space, $S : \mathcal{B} \to \mathcal{B}$ a compact linear operator. Then setting $T = id - S$ one has

i) $\operatorname{Ker} T$ is finite dimensional

ii) $T(\mathcal{B})$ is closed in $\mathcal{B}$ and $\operatorname{Coker} T = \mathcal{B}/T(\mathcal{B})$ is finite dimensional.

[An operator satisfying i), ii) is called a Fredholm operator]

For the proof of (2.14) in the case of Hilbert spaces we refer to [We] IV (4.6).

We have the following corollary of (2.12), (2.13) and (2.14):

(2.15) Thm.: For any parametrix P as in (2.12) the operators $\Delta \circ P$ and $P \circ \Delta$ have continuous extensions as Fredholm operators $W^s(M) \to W^s(M)$ for each $s \in \mathbb{R}$.

We can now prove the following basic finiteness theorem:

<u>(2.16) Thm.:</u> Setting

$$\mathcal{H}^p(M)_s = \operatorname{Ker}(\Delta_s : W^s(M) \to W^{s-2}(M))$$

we have:

i) $\mathcal{H}^p(M)_s = \mathcal{H}^p(M)$ for all $s \in \mathbb{R}$

ii) Δ_s is a Fredholm operator, in particular $\mathcal{H}^p(M)_s = \mathcal{H}^p(M)$ is finite dimensional.

<u>Proof:</u> First we show that $\dim \mathcal{H}^p(M)_s < \infty$ for all s. Let P be a parametrix for Δ as in (2.12). By (2.15) we know that the extension

$$(P \circ \Delta)_s : W^s(M) \to W^s(M)$$

has finite dimensional kernel. On the other hand because of the commutative diagram

$$\begin{tikzcd} W^s(M) \arrow[rr, "(P\circ\Delta)_s"] \arrow[dr, "\Delta_s"'] & & W^s(M) \\ & W^{s-2}(M) \arrow[ur, "P_{s-2}"'] & \end{tikzcd}$$

we have $\operatorname{Ker} \Delta_s \subset \operatorname{Ker}(P \circ \Delta)_s$ hence $\dim \mathcal{H}^p(M)_s = \dim \operatorname{Ker} \Delta_s < \infty$. Similarly one sees that Δ_s has finite dimensional cokernel. This already implies that Δ_s is Fredholm (the closedness of $\operatorname{Im} \Delta_s$ being automatic). [Hirzebruch-Scharlau, Einführg. in die Fkt.-analysis BI Bd. 296, 1971] [...]

The proof of 2) will follow from the more general statement about regularity of the Δ-operator:

(2.17) Thm.: Suppose that for $\omega \in W^s(M)$ we have

$$\Delta_s \omega \in A^p(M)$$

then $\omega \in A^p(M)$. In particular $\mathcal{H}^p(M)_s = \operatorname{Ker} \Delta_s \subset A^p(M)$.

Proof: For a parametrix P for Δ we have

$$P \circ \Delta - id = S \in OP_{-1} .$$

Now $\Delta\omega \in A^p(M)$ implies that $(P\circ\Delta)(\omega) \in A^p(M)$ (since $P(A^p(M)) \subset \bigcap_{s'} W^{s'+2}(M) \overset{(2.5)}{=} A^p(M)$). Hence from

$$\omega = (P\circ\Delta)(\omega) - S(\omega)$$

and $S(\omega) \in W^{s+1}(M)$ (recall that $S \in OP_{-1}$) we find: $\omega \in W^{s+1}(M)$. Arguing inductively we find:

$$\omega \in \bigcap_{t \geq s} W^t(M) = A^p(M) \text{ by } (2.5).$$

We can now prove the Hodge decomposition theorem (1.9). We have to show that

$$A^p(M) = \mathcal{H}^p(M) \oplus \Delta A^p(M) .$$

As $\Delta_2 : W^2(M) \to W^0(M) = L^2(M,E)$ is Fredholm by (2.16) ii) we know that $\operatorname{Im} \Delta_2$

is closed. Hence

$$L^2(M,E) = (\mathrm{Im}\,\Delta_2)^\perp \oplus \mathrm{Im}\,\Delta_2 .$$

Now clearly $(\mathrm{Im}\,\Delta_2)^\perp = \mathrm{Ker}\,\Delta_2^*$

$$\overset{(2.11)}{=} \mathrm{Ker}\,\Delta_0 = \mathcal{H}^p(M)_0$$

$$\overset{(2.16)\,i)}{=} \mathcal{H}^p(M) .$$

Hence

$$L^2(M,E) = \mathcal{H}^p(M) \oplus \mathrm{Im}\,\Delta_2 ,$$

and since $\mathcal{H}^p(M) \subset A^p(M)$ we get:

$$A^p(M) = \mathcal{H}^p(M) \oplus (\mathrm{Im}\,\Delta_2 \cap A^p(M)) .$$

The regularity theorem (2.17) shows that

$$\mathrm{Im}\,\Delta_2 \cap A^p(M) = \mathrm{Im}\,\Delta$$

and hence we get

$$A^p(M) = \mathcal{H}^p(M) \oplus \Delta A^p(M)$$

i.e. the assertion.