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Green's Functions on Curves

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Appendix:

The Green's function for curves uniformized by the upper half-plane

B.H. Gross and D.B. Zagier

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Lecture Notes for
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A) Hermitian Geometry and Volume Forms

For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ define the complex differential operators

$$\partial = \sum_{\mu=1}^{m} \frac{\partial}{\partial z_\mu} dz_\mu , \qquad \bar\partial = \sum_{\mu=1}^{m} \frac{\partial}{\partial \bar z_\mu} d\bar z_\mu$$

and the real operators

$$d = \partial + \bar\partial, \qquad d^c = (4\pi i)^{-1}(\partial - \bar\partial) ,$$

so that $dd^c = (i/2\pi)\partial\bar\partial$. If $m = 1$, and $z = x+iy$, then $dx \wedge dy = (i/2) dz \wedge d\bar z$, and dd^c is related to the Laplacian by

$$dd^c = \frac{1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) dx \wedge dy .$$

In Cartesian and polar coordinates, respectively, d and d^c become

$$d = \sum_{\mu=1}^{m} \left(\frac{\partial}{\partial x_\mu} dx_\mu + \frac{\partial}{\partial y_\mu} dy_\mu\right) = \sum_{\mu=1}^{m} \left(\frac{\partial}{\partial r_\mu} dr_\mu + \frac{\partial}{\partial \theta_\mu} d\theta_\mu\right)$$

$$d^c = \frac{1}{4\pi} \sum_{\mu=1}^{m} \left(- \frac{\partial}{\partial y_\mu} dx_\mu + \frac{\partial}{\partial x_\mu} dy_\mu\right) = \frac{1}{4\pi} \sum_{\mu=1}^{m} \left(r_\mu \frac{\partial}{\partial r_\mu} d\theta_\mu - r_\mu^{-1} \frac{\partial}{\partial \theta_\mu} dr_\mu\right)$$

where $z_\mu = x_\mu + i y_\mu = r_\mu e^{i\theta_\mu}$ with $x_\mu, y_\mu, r_\mu, \theta_\mu \in \mathbb{R}$.

Let X be an m-dimensional complex manifold. Locally at every point on X, a (1,1)-form φ can be written

$$\varphi = (i/2) \sum_{\mu,\nu=1}^{m} f_{\mu\nu}(z)\, dz_\mu \wedge d\bar z_\nu .$$

It is called *smooth* if the coordinate functions $f_{\mu\nu}(z)$ are $\mathcal{C}^\infty$, *real* if at each $z \in X$ the matrix $M_z = (f_{\mu\nu}(z))$ is Hermitian (which is equivalent to φ having real coefficients, when it is expressed in terms of the real differentials $dx_\mu \wedge dx_\nu$, $dx_\mu \wedge dy_\nu$, $dy_\mu \wedge dy_\nu$), and *positive* if M_z is positive definite Hermitian.

Throughout these notes, we will assume differential forms are smooth.

The positive (1,1)-forms on X are in one-to-one correspondence with Hermitian metrics. Specifically, if $T_z'(X)$ denotes the holomorphic tangent space to X at z (the set of $\mathbb{C}$-linear combinations of the operators $\partial/\partial z_\mu$), a *Hermitian metric* on X is a family of positive definite Hermitian products

$$ds^2 : T'_z(X) \otimes \overline{T'_z(X)} \to \mathbb{C}$$

depending smoothly on z. Thus, ds^2 can locally be written

$$ds^2 = \sum_{\mu,\nu=1}^{m} h_{\mu\nu}(z)\, dz_\mu \otimes d\bar{z}_\nu$$

where the $h_{\mu\nu}(z)$ are smooth, and the matrix $(h_{\mu\nu}(z))$ is positive definite Hermitian for each z. Applying the Gram-Schmidt process, locally near any z one can find a *coframe*, an m-tuple of (1,0)-forms $(\omega_1,\dots,\omega_m)$, such that

$$ds^2 = \sum_{\mu=1}^{m} \omega_\mu \otimes \bar{\omega}_\mu \quad .$$

Writing $\omega_\mu = \alpha_\mu + i\beta_\mu$, where α_μ and β_μ are real differential forms,

$$ds^2 = \sum_{\mu=1}^{m} (\alpha_\mu \otimes \alpha_\mu + \beta_\mu \otimes \beta_\mu) + i \sum_{\mu=1}^{m} (-\alpha_\mu \otimes \beta_\mu + \beta_\mu \otimes \alpha_\mu) \quad .$$

Hence, taking the real and imaginary parts of ds^2, one gets an ordinary inner product and an alternating quadratic form, respectively, on the underlying real vector spaces. The real part of ds^2 is a Riemannian metric on X; using it, one obtains notions of length, volume, etc. on the Real manifold underlying X (the volume form is $\alpha_1 \wedge \beta_1 \wedge \dots \wedge \alpha_m \wedge \beta_m$). The imaginary part of ds^2 determines the (1,1)-form

$$\varphi = \sum_{\mu=1}^{m} \alpha_\mu \wedge \beta_\mu = (i/2) \sum_{\mu=1}^{m} \omega_\mu \wedge \bar{\omega}_\mu = -\tfrac{1}{2}\mathrm{Im}(ds^2)$$

which is seen to be positive using the middle expression. Conversely, ds^2 can be recovered from φ, since if $H(u,v)$ is any Hermitian inner product on $\mathbb{C}^m$, and $E(u,v) = \mathrm{Im}(H)(u,v)$, then

$$H(u,v) = E(iu,v) + iE(u,v) \quad .$$

Now specialize to the case $m = 1$. Let X be a compact, connected Riemann surface: e.g. the complex points of a complete non-singular curve. Let φ be a positive (1,1)-form on X, and let ds^2 be its associated Hermitian metric. Then φ is itself the volume form on X determined by the Riemannian metric $\mathrm{Re}(ds^2)$: specializing the formulas above, locally near every point there exists a coframe $\omega = \alpha + i\beta$ such that $ds^2 = \alpha \otimes \alpha + \beta \otimes \beta$, and the volume form associated to ds^2 is precisely $\alpha \wedge \beta = \varphi$. For this reason, a positive (1,1)-form on a curve is called a *volume form*. A volume form φ is called *normalized* if it satisfies

$$\int_X \varphi = 1 \quad .$$

B) <u>Green's functions.</u>

Arakelov introduced Green's functions in number theory in order to complete the intersection pairing on an arithmetic surface, so that it behaved well under linear equivalence. His idea was that specifying Hermitian metrics on the archimedean fibres of an arithmetic surface was like specifying a model at the finite fibres. Most naturally, Arakelov's

Green's functions arise as extensions of Nêron's local height pairing at the archimedean fibres; and, as has become clearer through the work of Harbater ([Ha]), Rumely ([R]), and Zhang ([Zh]), it was a historical accident that because of the close relation between intersection theory and Nêron's pairing, they could be forced to give an extension of the intersection pairing.

In the literature, Green's functions come in two flavors: additive and multiplicative. The additive ones, "Green's functions with logarithmic singularities", are the ones which mesh with intersection theory most readily; they are usually denoted $g_P(\cdot)$, or $g(\cdot,P)$, with a small "g". The multiplicative ones are exponentials of the additive ones, and are usually written with a large "G": more precisely

$$G(\cdot,P) = e^{-\frac{1}{2}g(\cdot,P)} .$$

As will be seen, the function $G(Q,P)$ for $P, Q \in X$ is symmetric, continuous as a function of two variables, and has a simple zero along the diagonal; it is comparable, above and below, with a metric on X.

There is a good deal of variation in the literature concerning the notation for, and normalization of, Green's functions. The variation comes from trying to avoid different normalization constants which arise in differential equations, in applying the product formula from number theory, or in comparing with absolute values. Our $g_P(\cdot)$ agree with those in Lang ([L2]) and Gillet-Soulé ([GS]), but are the negative of those in Gross ([CS],Ch.XIV), and are -2 times those in Faltings ([F]) and Arakelov ([A1]).

Fix a Hermitian metric on X, and let φ be the corresponding normalized volume form. For each $P \in X$, the Green's function $g_P(\cdot)$ associated to φ is the unique function

$$g_P : X\backslash\{P\} \to \mathbb{R}$$

satisfying the axioms

(G1) [Logarithmic Singularities] On some neighborhood U of P, there exists a smooth function $\alpha(z)$ such that for all $z \neq P$

$$g_P(z) = -\log(|z-P|^2) + \alpha(z) .$$

(G2) [Uniform Laplacian] On $X\backslash P$, g_P satisfies the equation

$$dd^c(g_P) = \varphi .$$

(G3) [Normalization] $\int_X g_P \, \varphi = 0$.

Assuming g_P exists, the first two axioms determine it up to an additive constant, as they show the difference of two such functions extends to a bounded function on X which is harmonic on $X\backslash P$, and such a function is necessarily constant. The third axiom eliminates the ambiguous constant. This normalization is important for the Adjunction Formula and Faltings' Riemann Roch, but in many contexts it can be ignored.

Granted enough machinery, the existence of g_P is easy. The basic problem is to solve a certain PDE; the fundamental tool needed is the

dd^c-Lemma. Let X be a compact Kähler Manifold, and let $p,q \geq 1$. Let ω be a (p,q)-form on X which is exact, e.g. $\omega = d\phi$ for some form ϕ. Then there exists a $(p-1,q-1)$-form β such that

$$dd^c(\beta) = \omega .$$

A *Kähler Manifold* is a complex manifold with a Hermitian metric ds^2 whose associated positive (1,1)-form φ satisfies $d\varphi = 0$. Curves are trivially Kähler, since their cohomology vanishes in dimensions ≥ 2. More general, nonsingular quasi-projective algebraic varieties are Kähler. The dd^c-Lemma is a direct consequence of the Hodge decomposition of cohomology; see ([GH],p.149) for a proof.

To prove the existence of g_p for curves, fix the volume form φ, and let $\|\cdot\|_0$ be any (smooth) Hermitian metric on the line bundle $\mathcal{O}_X(P)$. This means that in any local coordinate patch U_i on X, there is a positive function $h_i(z)$ giving the square-norm on the fibres: if s is a section of $\mathcal{O}_X(P)$ over U_i, then for each $z \in U_i$,

$$\|s(z)\|_0 = h_i(z)|s(z)|^2;$$

if U_j is another coordinate patch, and φ_{ij} is the transition function, then $h_i(z) = |\varphi_{ij}(z)|^2 h_j(z)$. The *first Chern form* of $\|\cdot\|_0$ is the smooth (1,1)-form on X defined in the coordinate patch U_i by

$$\rho = -dd^c(\ln\|s\|_0) = -dd^c(\ln(h_i(z))) .$$

It is independent of s, since $s(z)$ is holomorphic, which implies $dd^c(\log|s(z)|^2) = 0$; it is well-defined on X since the transition functions are holomorphic. By Stokes' theorem,

$$\int_X \rho = \deg(\mathcal{O}_X(P)) = 1.$$

We now seek to modify $\|\cdot\|_0$ to obtain another metric $\|\cdot\|$ whose first Chern form is the specified volume form φ. To do this, note first that $\rho-\varphi$ is exact: Since X is a curve, the deRham cohomology group $H^2(X,\mathbb{C})$ = (2-forms)/d(1-forms) is 1-dimensional, and by Stokes' theorem, for any 1-form θ, $\int_X d\theta = 0$. Therefore, since $\int_X (\rho-\varphi) = 1-1 = 0$, there must be a form ϕ such that $d\phi = \rho-\varphi$.

Applying the dd^c-Lemma, we find a function (= (0,0)-form) β such that $dd^c\beta = \rho-\varphi$. Let $\|\cdot\| = e^\beta\|\cdot\|_0$. For any section s,

$$\begin{aligned} -dd^c(\ln\|s\|) &= -dd^c(\ln(e^\beta\|s\|_0) \\ &= -dd^c(\beta) - dd^c(\ln\|s\|_0) \\ &= \varphi-\rho + \rho = \varphi . \end{aligned}$$

If "1" denotes the canonical section of $\mathcal{O}_X(P)$, then the function $-\ln\|"1"\|$ satisfies axiom G1), since "1" vanishes only at P, and has a simple zero there. It satisfies axiom G2) by the computation above. To satisfy the normalization G3) it is only necessary to add on an appropriate constant. Thus, for an appropriate C, we may take

$$g_p(\cdot) = -\ln \|"1"\| + C .$$

There are many other proofs of the existence of Green's functions.

The chief defect of the proof above is that it fails to establish one of the main properties of $g(Q,P)$, which is that off the diagonal it is *continuous as a function of two variables.* To prove this, it seems important to build in continuity from the start. Generally, one first constructs the Green's function for a specific volume form and shows it is continuous, then obtains the continuity of the Green's functions for other volume forms by a change of metric formula.

There are many proofs of the existence of Green's functions. Arakelov's original proof, found in ([A1]), ([Sz],§III), ([L2],Ch.II.2), constructed $g(Q,P)$ as the section $-\ln\|``1"\|$ for an appropriate metric on the line bundle $\mathcal{O}_{X\times X}(\Delta)$, where Δ is the diagonal embedding. He also noted its role as the kernel function for Green's operator inverting the φ-Laplace operator on $\mathcal{E}^{\infty}(X)$ (see [SS]). Rumely ([R],§2.1,§2.3) gave a construction using the Prime Function of a marked Riemann Surface. Both of these proofs establish the joint continuity. Hriljac ([Hr]) gave a construction using theta-functions and the embedding of a curve in its Jacobian; this proof is given in ([L1],Ch.13.5). Coleman gave a construction using differentials of the third kind, which may be found in ([L2],§II.4).

Gillet and Soulé's proof of the existence of "Green's forms with logarithmic singularities" for higher-dimensional varieties ([GS]) is a generalization of the proof above, based on the dd^c-Lemma. It uses many other tools, in particular Hironaka's resolution of singularities.

Recall that the Green's function depends on the choice of Hermitian metric. For curves of positive genus, two metrics seem favored arithmetically: the "canonical metric", gotten by pulling back the flat metric on the curve's Jacobian using the Abel map; and the "constant curvature metric", gotten by using the uniformization theorem, and projecting the flat metric on $\mathbb{C}$ or the Poincaré metric on the unit disc. The canonical metric is required for the Adjunction Formula and Faltings' Riemann-Roch, while the constant curvature metric behaves well for deformation-theoretic arguments.

The Green's functions for the constant curvature metric also have the advantage that one can construct them more-or-less explicitly, thus providing examples for the theory, and giving arithmetic information about them. For curves of genus 0 the formulas are classical; for genus 1 they were known to Néron, and for genus $g \geq 2$ they are due to Hejhal, Gross and Zagier. In the remainder of the paper we will give these formulas.

Case 1. Curves of Genus 0.

Over $\mathbb{C}$, any curve of genus 0 becomes isomorphic to $\mathbb{P}^1(\mathbb{C})$, the Riemann sphere. We identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C}\cup\{\infty\}$, letting z be the coordinate function on $\mathbb{C}$. By stereographic projection, $\mathbb{P}^1(\mathbb{C})$ becomes isomorphic to a sphere with diameter 1 and south pole placed on the origin. Up to scaling, there is a unique Hermitian metric on the sphere invariant under the rotation group $U(2,\mathbb{C})$, the Fubini-Study metric. In the affine patch $\mathbb{C}$, the Fubini-Study metric and its associated normalized volume form are

$$ds^2 = \frac{dz\otimes d\bar{z}}{(1+|z|^2)^2}, \qquad \varphi = \frac{i}{2\pi}\,\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$$

For $P,\ z \in \mathbb{C}$,

$$g(z,P) = -\ln\left(\frac{|z-P|^2}{(1+|z|^2)(1+|P|^2)}\right) - 1, \qquad g(z,\infty) = -\ln\left(\frac{1}{1+|z|^2}\right) - 1 \quad .$$

Direct computations show that $g(z,P)$ satisfies $dd^c(g_P) = \varphi$ and is invariant under the action of $U(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$. Hence, to verify its normalization, it suffices to take $P = 0$ and compute $\int_X g_0\,\varphi$, which can readily be done using polar coordinates.

Correspondingly, $G(z,P) = e^{\frac{1}{2}}\|z,P\|$ where

$$\|z,P\| = \frac{|z-P|}{(1+|z|^2)^{\frac{1}{2}}(1+|P|^2)^{\frac{1}{2}}}$$

is the *chordal metric*. It has the geometric interpretation of being the length of the chord between the points corresponding to P and Q, when $\mathbb{P}^1(\mathbb{C})$ is identified with the unit sphere via stereographic projection.

<u>Case 2.</u> Elliptic curves: $g = 1$.

An elliptic curve is isomorphic to a complex torus $\mathbb{C}/[\tau,1]$, where τ is a point in the upper half-plane, and $[\tau,1]$ is the lattice $\mathbb{Z}\tau\oplus\mathbb{Z}$; $\mathrm{Im}(\tau)$ is the volume of a fundamental parallelogram. The flat Hermitian metric ds^2 on $\mathbb{C}$ is $dz\otimes d\bar{z}$; projecting to the torus, the corresponding normalized volume form is

$$\varphi = \frac{i}{2\mathrm{Im}(\tau)}\,dz\wedge d\bar{z}$$

The Weierstrass σ-function is

$$\sigma(z;\tau) = z \prod_{\substack{\omega\neq 0\\ \omega\in[\tau,1]}} (1 - z/\omega)\; e^{(z/\omega)+\frac{1}{2}(z/\omega)^2} \quad .$$

Let $\eta_1,\ \eta_2$ be the periods of $\zeta(z) = \partial/\partial z\,(\ln\sigma(z;\tau))$ under $\tau,\ 1$ respectively. By the Legendre relation, $\eta_2\tau-\eta_1 = 2\pi i$. Given $z \in \mathbb{C}$, we can uniquely write $z = t_1\tau+t_2$ with $t_1,\ t_2 \in \mathbb{R}$. Define

$$\eta(z) = t_1\eta_1 + t_2\eta_2$$

and let

$$\mathfrak{k}(z) = e^{-\frac{1}{2}z\eta(z)}\,\sigma(z;\tau) \quad .$$

Then by the functional equation $\sigma(z+\omega;\tau) = \pm\, e^{\eta(\omega)(z+\frac{1}{2}\omega)}\,\sigma(z;\tau)$ (where the plus sign applies if $\frac{1}{2}\omega \in L$, and the minus sign if not)

$$\mathfrak{k}(z+\tau) = -e^{-\pi i t_2}\cdot\mathfrak{k}(z), \qquad \mathfrak{k}(z+1) = -e^{-\pi i t_1}\cdot\mathfrak{k}(z) \quad .$$

Consequently $|\mathfrak{k}(z)|$ is periodic for L. Since $\sigma(z,L)$ is holomorphic,

$$dd^c(-\ln|\mathfrak{k}(z)|) = dd^c(\tfrac{1}{2}z\eta(z)) = \tfrac{1}{2}\varphi$$

Further, $\sigma(z;L)$ has a simple zero at each lattice point, so it follows that up to an additive constant, the Green's function $g(P,z)$ is given by $-2\cdot\ln|\mathfrak{k}(z-P)|$ (we identify points of the elliptic curve with their pre-images in $\mathbb{C}$).

The constant can be determined by using the q-expansions for $\sigma(z;\tau)$ and the modular form $\Delta(\tau)$. Specifically, let $q_\tau = e^{2\pi i\tau}$, $q_z = e^{2\pi i z}$. We have

$$\Delta(\tau) = (2\pi i)^{-12}\cdot q_\tau\cdot\prod_{n=1}^{\infty}(1-q_\tau^n)^{24} ,$$

$$\sigma(z;\tau) = (2\pi i)^{-1}e^{\frac{1}{2}\eta_2 z^2}(q_z^{\frac{1}{2}}-q_z^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})(1-q_\tau^n)^{-2}$$

and

$$(*)\qquad \Delta(\tau)^{1/12}\mathfrak{k}(z) = q_\tau^{\frac{B_2(\mathrm{Im}(z)/\mathrm{Im}(\tau))}{2}}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})$$

where $B_2(t) = t^2-t+\frac{1}{6}$ is the second Bernoulli polynomial. Then

$$g(z,P) = -2\cdot\ln|\Delta(\tau)^{1/12}\mathfrak{k}(z-P)| .$$

This formula apparently goes back to Néron; it is given by Gross in ([CS], Ch.XIV) and Lang in ([L2],Ch.II.5). The proof that $\int_X g(z,P)\ \varphi = 0$ comes down to showing that each of the terms arising from (*) vanish. For the Bernoulli term this follows from the fact that $\int_0^1 B_2(t)\ dt = 0$; for the others, it follows because the Taylor expansion of $\ln(1-w)$ has constant term 0. The details may be found in ([L2]).

Case 3. Curves of genus $g \geq 2$.

Curves of genus $g \geq 2$ can be uniformized by the unit disc, or equivalently, by the upper half-plane $\mathfrak{H} = \{ x+iy : y > 0 \}$. For any such curve X, there is a Fuchsian group $\Gamma \subseteq PSL(2,\mathbb{R})$ of the first kind, a discrete subgroup with no elliptic or parabolic elements, such that $\Gamma\backslash\mathfrak{H} \cong X$. The Poincaré metric $y^{-2}(dx\otimes dx+dy\otimes dy)$ is invariant under $PSL(2,\mathbb{R})$ and has constant negative curvature; it is the Riemannian metric associated to a Hermitian metric

$$ds^2 = \mathrm{Im}(z)^{-2}\ dz\otimes d\bar{z}$$

with volume form $y^{-2}dx\wedge dy$. The volume of $\Gamma\backslash\mathfrak{H}$ is $2\pi(2g-2)$. Thus, the constant curvature volume form on X is

$$\varphi = \frac{1}{2\pi(2g-2)}\ y^{-2}\ dx\wedge dy .$$

Gross has given a formula for Green's function of $\Gamma\backslash\mathfrak{H}$ relative to φ, using a Poincaré series formed from Legendre functions of the second kind. This formula, which is also valid with slight modifications when Γ has cusps or parabolic elements, was important in the Gross-Zagier Theorem. We present it here, following Gross's exposition in ([CS],Ch.XIV) and the more complete exposition given in ([GZ]). The formula was known earlier to Hejhal ([He],vol.II,Ch.6) in the context of the Selberg Trace Formula.

One's first thought for constructing the Green's function might be to use the function $g_1(z,P) = -\log|(z-P)/(\bar{z}-P)|^2$, which is harmonic on $\mathfrak{H}\times\mathfrak{H}$ minus the diagonal, and invariant under $PSL(2,\mathbb{R})$. However,

$$\sum_{\gamma\in\Gamma} g_1(z,\gamma P)$$

diverges, and anyway if it converged it would have Laplacian 0. Gross's idea was to introduce a complex parameter s, and look for a family of functions $g_s(z,P)$, such that $g_s(z,P) \to g_1(z,P)$ as $s \to 1$. One could then hope to subtract off the polar part of the series formed from $g_s(z,P)$ as above, and be left with the Green's function.

More precisely, one wants a real-analytic function $g_s(z,P)$ on $\mathfrak{H}\times\mathfrak{H}$ minus the diagonal which satisfies

1) $g_s(\gamma z,\gamma P) = g_s(z,P)$ for all $\gamma \in PSL(2,\mathbb{R})$;
2) $g_s(z,P) \sim -\log|z-P|^2$ as $z \to P$, for each s;
3) $\Delta_z g_s(z,P) = s(s-1)g_s(z,P)$ where $\Delta_z = y^2(\partial/\partial x^2+\partial/\partial y^2)$ is the hyperbolic Laplacian in z, for each fixed w.

The desired $PSL(2,\mathbb{R})$-invariance of $g_s(z,P)$ means that it must be a function of the hyperbolic distance

$$r = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)} \right)$$

Putting $g_s(z,P) = f(\cosh(r))$, one finds that f satisfies the Legendre differential equation

$$(1-u^2)\frac{d^2f}{du^2} - 2u\frac{df}{du} + s(s-1)f = 0 .$$

The solution with the correct pole on the diagonal and slow growth at infinity is the Legendre function of the second kind,

$$f(u) = 2\,Q_{s-1}(u) = 2\int_0^\infty \left(u + \sqrt{u^2-1}\cosh(t) \right)^{-s} dt,$$

for $u > 1$. (This differs by a sign from Gross's formula, because our Green's functions are the negatives of his). Put

$$g_s(z,P) = 2\,Q_{s-1}\left(1 + \frac{|z-P|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(P)}\right) .$$

Now suppose $\mathrm{Re}(s) > 1$. As shown in Hejhal ([He],vol.II,Ch.6.2), the series

$$\mathfrak{G}_s(z,P) = \sum_{\gamma\in\Gamma} g_s(z,\gamma P)$$

converges absolutely. Thus, it satisfies properties 1),2) and 3) above. It can be meromorphically continued for $\mathrm{Re}(s) > \frac{1}{2}$, and has a simple pole at $s = 1$ with residue $2/(2g-2)$ independent of z, P (see [He]). Again for $\mathrm{Re}(s) > 1$, it can be shown (see ([GZ])) that for each fixed P,

$$\int\int_{\mathfrak{H}} g_s(z,P)\, y^{-2}dx\wedge dy = \frac{4\pi}{s(s-1)} .$$

Taking account of the volume of $\Gamma\backslash\mathfrak{H}$, it follows that

$$(**) \qquad \int\int_{\Gamma\backslash\mathfrak{H}} \mathfrak{G}_s(z,P)\, y^{-2} dx\wedge dy = \frac{2/(2g-2)}{s(s-1)} ,$$

We can now define the Green's function. Put

$$g(z,P) = \lim_{s\to 1} \left(\mathfrak{G}_s(z,P) - \frac{2/(2g-2)}{s(s-1)} \right)$$

Then $g(z,P)$ is symmetric and invariant under Γ in both variables. It satisfies $g(z,P) \sim -\log|z-P|^2$ as $z \to P$. From the differential equation 3) follows

$$\Delta_z\left(\mathfrak{G}_s(z,P) - \frac{2/(2g-2)}{s(s-1)} \right) = s(s-1)\cdot\left(\mathfrak{G}_s(z,P) - \frac{2/(2g-2)}{s(s-1)}\right) + 2/(2g-2)$$

Passing to the limit as $s \to 1$ shows that $\Delta_z g(z,P) = 2/(2g-2)$, which is equivalent to

$$dd^c g(z,P) = \varphi .$$

Finally, the passing to the limit as $s \to 1$ in (**) shows that

$$\int_X g(z,P)\, \varphi = 0 .$$

References

[A1] S. Ju. Arakelov, "Intersection theory of divisors on an arithmetic surface", Izv. Akad Nauk SSR Ser. Mat. 35, No. 6 (1974) (in Russian) = AMS Translations, Math USSR Izvestija Vol. 8 (1974), No. 6, 1167-1180.

[A2] S. Ju. Arakelov, "Theory of Intersections on the Arithmetic Surface", Proceedings of the ICM, Vancouver, 1974, 405-408.

[CS] G. Cornell and J. Silverman, Arithmetic Geometry, Springer-Verlag, New York, 1986

[F] G. Faltings, "Calculus on arithmetic surfaces", Ann. of Math. 119 (1984), 387-424.

[FW] G. Faltings, G. Wüstholz et al., Rational Points (Sem. Bonn/ Wuppertal 1983/1984), Aspects of Mathematics vol. E6, Vieweg, Braunschweig, 1984.

[GH] P. Griffeths and J. Harris, Principles of Algebraic Geometry, Wiley-Interscience, New York, 1978.

[GS] H. Gillet and C. Soulé, Arithmetic Intersection Theory, IHES Publ Math. No. 72 (1990), 93-174.

[GZ] B. Gross and D. Zagier, "The Green's function for curves uniformized by the upper half-plane", manuscript.

[Ha] D. Harbater, "Potential Theory over Local and Global Fields; I, II", Journal of Algebra 148 (1992), I:337-383, II:384-432.

[He] D. Hejhal, The Selberg Trace Formula for PSL(2,R), vols I, II: LNM 548, 1001: Springer-Verlag, New York, 1978, 1983.

[Hr] P. Hriljac, "The Néron-Tate height and intersection theory on arithmetic surfaces", PhD Thesis, MIT, 1982

[L1] S. Lang, Fundamentals of Diophantine Geometry, Springer-Verlag, New York, 1983

[L2] S. Lang, Introduction to Arakelov Theory, Springer-Verlag, New York, 1988

[R] R. Rumely, Capacity Theory on Algebraic Curves, LNM 1378: Springer-Verlag, New York, 1989

[SH] B. V. Shabat, Distribution of Values of Holomorphic Mappings, AMS Translations of Mathematical Monographs, vol. 61, Providence, 1980

[SS] M. Schiffer and D. Spencer, Functionals on Finite Riemann Surfaces, Princeton University Press, Princeton, 1954.

[Sz] L. Szpiro, ed., Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell, Soc. Math. de France, Astérisque 127 (1985)

[Zh] S. Shang, "Admissible pairing on a curve", to appear.

The Green's function for curves uniformized by the upper half-plane

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The purpose of this note is to show how the Green's function for curves uniformized by the upper half-plane can be calculated as a limit of Legendre functions of the second kind. This is not a new result (a nice exposition may be found in [Hejhal, chapter 6]), but the connections with Néron's theory of local heights [Gross, §9] and the holomorphic projections of modular forms [Gross and Zagier, chapter V] make it worth reconsidering.

§1. Let M be a compact Riemann surface of genus g , together with a Riemannian metric ds^2 which is compatible with the complex structure on M . Then ds^2 is the real part of a unique Kähler metric on M . Let $d\mu$ be the volume form of this metric, scaled so that $\int_M d\mu(p) = 1$.

There is a unique function $G(p,q)$ on $M \times M$ with values in $\mathbb{R} \cup \{\infty\}$, called the Green's function, which satisfies [Gross, §6]:

(1.1) $$G(p,q) = G(q,p)$$

(1.2) $G(p,q)$ is real analytic for $p \neq q$ and satisfies the partial differential equation

$$\frac{\partial^2 G(p,q)}{\partial p \partial \bar{p}} = 2\pi i(d\mu(p)-\delta_q)$$ where δ_q is the Dirac current at q

(1.3) $G(p,q) \sim \log\|f(q)\|$ as $q \to p$, where f is a uniformizing parameter at p and for $z \in \mathbb{C}^\times$, $\|z\| = z\bar{z} = |z|^2$.

(1.4) $$\int_M G(p,q)d\mu(p) = 0$$

The function G depends on the choice of metric ds^2 in the conformal class of the complex structure. However, if $a = \Sigma m_p(p)$ is a divisor of degree zero on M the function $g_a(q) = \Sigma m_p G(p,q)$ depends up to a constant only on the complex structure of M. This is a Green's function associated to the divisor a; if $a = \mathrm{div}(f)$ then $g_a(q) = \log\|f(q)\| - \int_M \log\|f(q)\|d\mu(q)$. If $b = \Sigma m_q(q)$ is a divisor of degree zero which is relatively prime to a, then

(1.5) $$\langle a,b\rangle_M = g_a(b) = \Sigma m_q g_a(q) = \Sigma m_p m_q G(p,q)$$

is a formula for Néron's local height pairing [Gross, §6].

§2. We now suppose that the Riemann surface M is uniformized by the upper half-plane $\mathfrak{h} = \{x+iy \in \mathbb{C} : y > 0\}$ and that the metric ds^2 arises from the invariant Poincaré metric $\frac{dx^2+dy^2}{y^2}$ of constant negative curvature on $\mathfrak{h}$. Hence there is a Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$ of the first kind, with no elliptic or parabolic elements, such that $\Gamma\backslash\mathfrak{h} = M$. In particular, the genus satisfies $g \geq 2$, so $\chi(M) = 2-2g < 0$. (We will consider general Fuchsian groups of the first kind in §5.)

The volume of M with respect to Poincaré measure $\frac{dxdy}{y^2}$ is equal to $-2\pi\chi(M)$, so $d\mu$ on M is equal to $\frac{dxdy}{-2\pi\chi(M)y^2}$ on $\mathfrak{h}$. Let $\Delta = y^2\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)$ be the standard hyperbolic Laplacian. Then the differential operator $\partial\bar{\partial}$ on $\mathfrak{h}$ takes a function $F(z)$ to the 2-form $\Delta F(z)\,\frac{dxdy}{2iy^2} = \pi i\chi(M)\Delta F(z)d\mu(z)$.

To find the Green's function of M, we represent p and q by points z and w in $\mathfrak{h}$, and look for the unique function $G(z,w)$ on $\mathfrak{h}\times\mathfrak{h}$ which satisfies:

(2.1) $$G(z,w) = G(w,z)$$

2.2) $$G(\gamma z,\gamma' w) = G(z,w) \quad \text{all } \gamma,\gamma' \in \Gamma$$

(2.3) $G(z,w)$ is real analytic for $w \notin \Gamma z$ and satisfies the differential equation $\Delta G(z,w) = 2/\chi(M)$ in that region.

(2.4) $$G(z,w) \sim \log\|z-w\| \quad \text{as } w \to z$$

(2.5) $$\int_{\Gamma\backslash\mathfrak{h}} G(z,w)d\mu(z) = 0\,.$$

In the next two sections, we will construct $G(z,w)$ as a limit of resolvant kernel functions $G_s(z,w)$ for $\mathrm{Re}(s) > 1$.

§3. Fix a complex number s with $\mathrm{Re}(s) \geq 1$. We begin by looking for a point-pair invariant $g_s(z,w)$ on $\mathfrak{h} \times \mathfrak{h}$ which satisfies:

(3.1) $$g_s(z,w) = g_s(w,z)$$

(3.2) $$g_s(\gamma z,\gamma w) = g_s(z,w) \quad \text{for} \quad \gamma \in PSL_2(\mathbb{R})$$

(3.3) $g_s(z,w)$ is real analytic for $z \neq w$ and satisfies the differential equation: $\Delta g_s(z,w) = s(s-1)g_s(z,w)$ in that region

(3.4) $$g_s(z,w) \sim \log\|z-w\| \quad \text{as} \quad w \to z .$$

By (3.1) and (3.2) $g_s(z,w)$ is a function of the hyperbolic distance $d(z,w) = r$. We will write $g_s(z,w) = f_s(\cosh r) = f_s\left(\frac{\|z-w\|}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)} + 1\right)$. The reparametrization via the hyperbolic cosine is suggested by moving w to the point i; then the points z of hyperbolic distance r from i in $\mathfrak{h}$ lie on a Euclidean circle with center $= i\cdot(\cosh r)$ and radius $= (\sinh r)$. We shall also require that:

(3.5) g_s has "slow growth" as $d(z,w) \to \infty$.

We rewrite the Laplacian in polar coordinates

$$\Delta = (\sinh r)^{-1}\frac{\partial}{\partial r}\sinh r\frac{\partial}{\partial r} + (\sinh r)^{-2}\frac{\partial}{\partial\theta^2} \tag{3.6}$$

and look for eigenfunctions $g_s(z,w) = f_s(\cosh r)$ which depend only on r. Writing $t = \cosh r$ so $t^2-1 = \sinh^2 r$, we see that $f_s(t)$ must satisfy the ordinary differential equation:

$$\frac{d}{dt}\left[(t^2-1)\ \frac{df_s}{dt}\right] = s(s-1)f_s \tag{3.7}$$

Writing this out gives Legendre's second order equation of index $s-1$ [Lebedev, (7.3.1)]

$$(1-t^2)f_s''(t) - 2tf_s'(t) + s(s-1)f_s(t) = 0 . \tag{3.8}$$

The general solution of (3.8) has the form $f_s(t) = A P_{s-1}(t) + B Q_{s-1}(t)$, where $P_{s-1}(t)$ is the Legendre function of the first kind, $Q_{s-1}(t)$ is the Legendre function of the second kind, and A and B are constants. (We remark that for $s \geq 1$ an integer, $P_{s-1}(t)$ is a polynomial of degree $(s-1)$ and $Q_{s-1}(t) - \frac{1}{2}\log\left(\frac{t+1}{t-1}\right) P_{s-1}(t)$ is a polynomial of degree $s-2$). The equation (3.8) has 3 regular singular points $(t = 1, -1, \infty)$, and the solutions with slowest growth as $t \to \infty$ have the form $f_s(t) = B Q_{s-1}(t)$. The correct constant $(B = -2)$ can be determined by (3.4) and the exact logarithmic singularity of $Q_{s-1}(t)$ at the point $t = 1$.

In summary, the unique point pair invariant $g_s(z,w)$ satisfying (3.1)-(3.5) is given by the formula:

$$g_s(z,w) = -2\,Q_{s-1}\left(\frac{\|z-w\|}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)} + 1\right) \tag{3.9}$$

If $s = 1$, then $Q_0(t) = \frac{1}{2}\log\left(\frac{t+1}{t-1}\right)$, so

$$g_1(z,w) = \log\left\|\frac{z-w}{\bar{z}-w}\right\| \tag{3.10}$$

is the standard $PSL_2(\mathbb{R})$-invariant harmonic function on $\mathfrak{H}\times\mathfrak{H}$ with a logarithmic singularity on the diagonal.

Now assume $\mathrm{Re}(s) > 1$; we shall prove the integral formula:

$$\iint_{\mathfrak{H}} g_s(z,w)\,\frac{dxdy}{y^2} = \frac{-4\pi}{s(s-1)} \tag{3.11}$$

for fixed w. By (3.2) we may assume $w = i$, so

$$I = \iint_{\mathfrak{H}} g_s(z,w)\,\frac{dxdy}{y^2} = -2\iint_{\mathfrak{H}} Q_{s-1}\left(1+\frac{|z-i|^2}{2y}\right)\frac{dxdy}{y^2}$$

If we make the substitution $t = 1+\frac{|z-i|^2}{2y}$ then $x^2+(y-t)^2 = t^2-1$. Hence:

$$x = \sqrt{t^2-1}\cos\theta$$

$$y = t + \sqrt{t^2-1}\sin\theta$$

$$\frac{dxdy}{y^2} = \frac{dtd\theta}{t+\sqrt{t^2-1}\sin\theta}$$

$$I = -2\int_1^\infty Q_{s-1}(t)\left(\int_0^{2\pi}\frac{d\theta}{t+\sqrt{t^2-1}\sin\theta}\right)dt \ .$$

Using the formula: $\displaystyle\int_0^{2\pi}\frac{d\theta}{A+B\sin\theta} = \frac{2\pi}{\sqrt{A^2-B^2}}$ we find:

$$I = -4\pi\int_1^\infty Q_{s-1}(t)dt \ ,$$

so we are reduced to proving the formula $\displaystyle\int_1^\infty Q_{s-1}(t)dt = \frac{1}{s(s-1)}$. To do this, we recall the integral formula for $Q_{s-1}(t)$: [Lebedev, (7.4.9)]

$$Q_{s-1}(t) = \int_0^\infty\left(t+\sqrt{t^2-1}\cosh u\right)^{-s}du$$

$$= \int_{t+\sqrt{t^2-1}}^\infty x^{-s}\frac{dx}{\sqrt{x^2-2xt+1}} \qquad \text{where } x = t+\sqrt{t^2-1}\cosh u$$

$$dx = \sqrt{x^2-2xt+1}\ du$$

Substituting in, we find:

$$\int_1^\infty Q_{s-1}(t)dt = \int_1^\infty \left(\int_{t+\sqrt{t^2-1}}^\infty x^{-s} \frac{dx}{\sqrt{x^2-2xt+1}} \right) dt$$

$$= \int_1^\infty x^{-s} \left(\int_1^{\frac{x^2+1}{2x}} \frac{dt}{\sqrt{x^2-2xt+1}} \right) dx$$

$$= \int_1^\infty x^{-s} \left(\frac{x-1}{x}\right) dx = \frac{1}{s(s-1)}$$

§4. We continue with the notation of the last section, and with the assumption that $\operatorname{Re}(s) > 1$. The resolvant kernel

(4.1) $$G_s(z,w) = \sum_{\Gamma} g_s(z,\gamma w)$$

is then absolutely convergent [Hejhal, 6.2]. By (3.1)-(3.4) and (3.11), this function on $\mathfrak{H} \times \mathfrak{H}$ satisfies:

(4.2) $$G_s(z,w) = G_s(w,z)$$

(4.3) $$G_s(\gamma z,\gamma' w) = G_s(z,w) \quad \text{all} \quad \gamma,\gamma' \in \Gamma$$

(4.4) $G_s(z,w)$ is real analytic for $w \notin \Gamma z$ and satisfies the differential equation $\Delta G_s(z,w) = (s)(s-1)\, G_s(z,w)$ in that region

$$G_s(z,w) \sim \log\|z-w\| \quad \text{as} \quad w \to z \tag{4.5}$$

$$\int_{\Gamma\backslash\mathfrak{H}} G_s(z,w)\,d\mu'(z) = \frac{2/\chi(M)}{s(s-1)}\ . \tag{4.6}$$

As $s \to 1$ the function $G_s(z,w)$ has a simple pole with residue equal to $2/\chi(M)$ [Hejhal, chapter 6]. This suggests defining:

$$G(z,w) = \lim_{s\to 1}\left\{G_s(z,w) - \frac{2/\chi(M)}{s(s-1)}\right\}\ . \tag{4.7}$$

By (4.2)-(4.6) this function satisfies (2.1)-(2.5), so is equal to the Green's function for M with the Poincaré metric.

§5. We may generalize the construction of the previous section to Riemann surfaces $M = \Gamma|\mathfrak{H}^*$, where Γ is an arbitrary Fuchsian group of the first kind. At each elliptic point or cusp p of M, we let $e_p = 2,3,\ldots,\infty$ denote the order of its cyclic stabilizer subgroup $\Gamma_p \subset \Gamma$. If we define $\chi(M) = 2-2g-\Sigma\left(1-\frac{1}{e_p}\right)$, then the volume of M with respect to the Poincaré measure $\frac{dxdy}{y^2}$ is equal to $-2\pi\,\chi(M)$. We will take the (singular) metric ds^2 on M with volume from $d\mu(z) = \frac{dxdy}{-2\pi\chi(M)y^2}$.

If we define $G_s(z,w)$ by (4.1) and $G(z,w)$ by (4.7), properties (2.1)-(2.3) and (2.5) continue to hold. Properly (2.4) must be replaced by

$$G(z,w) \sim e_p \log\|z-w\| \quad \text{as} \quad w \to z$$

if z is in the orbit of an elliptic point p. We remark that $G(z,w)$ has a singularity at each cusp of M. If ∞ is a cusp and

$$E_s(w) = \sum_{\Gamma_\infty \backslash \Gamma}{}' (\operatorname{Im}\gamma w)^s \tag{5.1}$$

is the associated Eisenstein series of weight zero, then [Hejhal, ?]

$$G_s(z,w) = \frac{4\pi E_s(w)}{1-2s} y^{1-s} + O(e^{-y}) \quad \text{as} \quad z \to \infty . \tag{5.2}$$

Since $E_s(w)$ has a simple pole at $s = 1$ with residue equal to $-1/2\pi\chi(M) = 1/\mathrm{Vol}(M)$, we have

$$G(z,w) \sim 2/\chi(M) \log y \quad \text{as} \quad z \to \infty . \tag{5.3}$$

Similar formulas hold at the other cusps. We remark that formula (1.5) for Néron's local pairing continues to hold in the limit, when a and b have cuspidal support.

§6. Bibliography

1. Gross, B. H. Local heights on curves.

2. Gross, B. H. and Zagier, D. B. Heegner points and derivatives of L-series.

3. Hejhal, D. The Selberg trace formula for $PSL(2,\mathbb{R})$, Vol II. Springer Lecture Notes, 1001(1983).

4. Lebedev, N. N. Special functions and their applications. Dover, New York(1972).