INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.637/9

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

Homotopy, Homology and Cohomology "Calculations"

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0 - Introduction: The main general idea in Topology is to study spaces which can be continously deformed into one another. This idea is given mathematical substance by introducing homeomorphisms: If we take two topological spaces T_1 and T_2 then a map $f: T_1 \to T_2$ is homeomorphism if it is both continous and has an inverse which is also continous.

A pair of spaces T_1 and T_2 belong to the same equivalence class if they are homeomorphic. The next stage in the topologists work is to introduce enough mathematical criteria to characterise any particular equivalence class. The idea behind the characterizatio is to produce enough topological invariance, i.e Things which do not change under homeomorphisms, to uniquely specify each equivalence class. These topological invariants take many forms: They can be integers such as the dimensions n of $\mathbb{R}^n$, they can be certain specific properties of topological spaces such as connectedness or compactness, they can be whole mathematical structures such as homotopy groups, homology and cohomology groups and so on. In the search for these invariants two other notions are used isotopy and homotopy.

Homeomorphism generates equivalence classes whose members are topological spaces, while homoto generates equivalence classes whose members are continous maps. Take two continous maps α_1 and α_2 from topological space X to another topological space Y; then the map α_1 is said to be homotopic to the map α_2 if there exist a continous map .

$F : X \times [0,1] \to Y$ such that $F(x,0) = \alpha_1(x)$; $F(x,1) = \alpha_2(x)$. That is as real variable t in $F(x,t)$ varies continuously from 0 to 1 in the unit interval $[0,1]$ the map α_1 is deformed to the map α_2. Homotopy is clearly an equivalence relation and equivalence classes are denoted by $C(X,Y)$, which is unchanged under homeomorphism of X or Y. Therefore topological invariant.

1. Invariance of the dimension of $\mathbb{R}^n$.

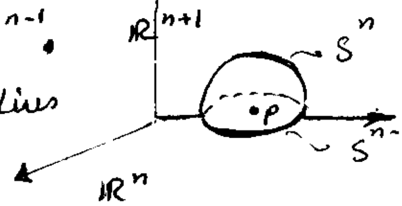
We begin by comparing $\mathbb{R}$ with $\mathbb{R}^2$, suppose that there is a homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}^2$. If such an α existed then the dimension of $\mathbb{R}$ and $\mathbb{R}^2$, could not be topological invariants. To see that α can not exist, let $\mathbb{R}^2$ be (x,y) plane and $\mathbb{R}$ be its x-axis. Now delete the point $(0,0)$ from $\mathbb{R}^2$ and hence the point $x=0$ from $\mathbb{R}$, this transforms $\mathbb{R}$ into a disconnected set, but does not make $\mathbb{R}^2$ disconnected. Thus $\mathbb{R}^2-\{(0,0)\}$ and $\mathbb{R}-\{0\}$ are not homeomorphic (connectedness is topological invariant). However if α exists then consider the restriction of α to $\mathbb{R}-\{0\}$, this will be a homeomorphism of $\mathbb{R}-\{0\}$ to $\mathbb{R}^2-\alpha(0)$, but these to sets are not homeomorphic and hence neither are $\mathbb{R}$ and $\mathbb{R}^2$.

Now consider $\mathbb{R}^2 = \{(x,y) \,;\, x,y \in \mathbb{R}\}$ and $\mathbb{R}^3 = \{(x,y,z) \,;\, x,y,z \in \mathbb{R}\}$ and delete a point p from both spaces. If we suppose the existance of a homeomorphism $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ then since $\mathbb{R}^2-\{p\}$ and $\mathbb{R}^3-\{p\}$ are both open sets in relative topologies, then the restriction α to $\mathbb{R}^2-\{p\} \to \mathbb{R}^3-\{p\}$ will be a homeomorphism too. Now we consider the equivalence classes of $C(S^1, \mathbb{R}^2-\{p\})$ and those of $C(S^1, \mathbb{R}^3-\{p\})$.

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[dr, "g"'] & \mathbb{R}^2-\{p\} \arrow[d, "\alpha"] \\ & \mathbb{R}^3-\{p\} \end{tikzcd}$$

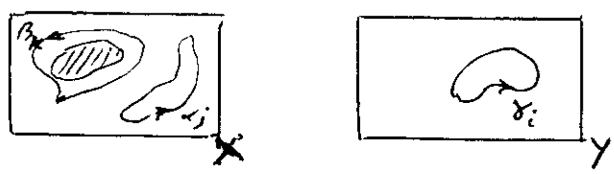
In this case two equivalence classes should coinsid, but they are not, since $C(S^1, \mathbb{R}^2-\{p\})$ is not contractible to a point (not simply connected), but $C(S^1, \mathbb{R}^3-\{p\})$ is contractible to a point (simply connected), and simply connectedness is a topological property, therefore, there is no such homeomorphism α.

In the general induction step, we compair $\mathbb{R}^n$ with $\mathbb{R}^{n+1}$. Then if we delete a point p from both spaces to obtain $\mathbb{R}^n-\{p\}$ and $\mathbb{R}^{n+1}-\{p\}$ then we surround the point p with spher S^{n-1}. and see what happence when the radius tends to zero.



Now consider the equivalence classes of $C(S^{n-1}, \mathbb{R}^n-\{p\})$, and those of $C(S^{n-1}, \mathbb{R}^{n+1}-\{p\})$, in the same way it is easy to show that they are not the same.

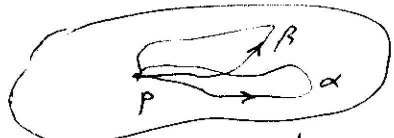
<u>2 - The fundamental group:</u> Let us start with an example, take two rectangular region X and Y of $\mathbb{R}^2$, X contains a hole shown as a shaded area, while Y does not contain any hole. It is clear that any loop in Y can be shrunk to a point, but it is not so in X.



That is the space Y has only one kind of loops

γ_i, which can be shrunk to a point. However there are two kinds of loops in X, loops like α_j which can be shrunk to a point, and loop like β_k which contain the hole and cannot be shrunk to a point. Two loops are equivalent, or homotopic, if one can be obtained from the other by a process of continous deformations. Now it is just one homotopy class of loops in Y denoted by $[\gamma]$, but two classes in X; $[\alpha]$ and $[\beta]$. This example suggests that the study of equivalence classes of loops or closed paths in any topological space M might be a way of determining the "holes" in the space M. Let $\alpha : [0,1] \to M$ be a closed path (a loop) i.e α is continous and $\alpha(0)=\alpha(1)=$ ($p \in M$ is base point). The inverse loop is $\alpha^{-1}(t) = \alpha(1-t)$ $0 \le t \le 1$ and the constant loop c is $c : [0,1] \to M$, with $c(t) = p$ $0 \le t \le 1$. If α, β are two loops based at $p \in M$, then $\gamma = \alpha * \beta$ (product loop) is defined as

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1 . \end{cases}$$



Two loops α and β based at $p \in M$ are homotopic $\alpha \simeq \beta$ if there exists a continous map $H : [0,1] \times [0,1] \to M$ such that

$$H(t,0) = \alpha(t) \qquad 0 \le t \le 1$$
$$H(t,1) = \beta(t) \qquad 0 \le t \le 1$$
$$H(0,s) = H(1,s) = p \qquad 0 \le s \le 1$$

H is called homotopy between α and β

Now let $\pi_1(M,p)$ be the set of all homotopy classes of loops on M based at $p \in M$. If $[\alpha]$ and $[\beta]$ belong to $\pi_1(M,p)$ then the product $[\alpha] \circ [\beta]$ defines as

$$[\alpha] \circ [\beta] = [\alpha * \beta].$$

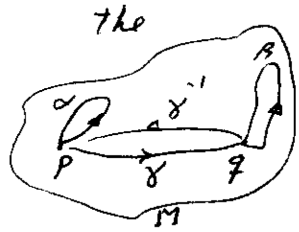
One should not that the set of loops are not form a group but the set of equivalent classes of loops ; since it is easy to see that for every loop α ; $\alpha * \alpha^{-1}$ is not the constant loop (identity element of the group) but it is homotopy to the constant loop. It is easy to check that if α, β, γ belong to $\Pi_1(M,p)$ then $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ which implies $([\alpha] \circ [\beta]) \circ [\gamma] = [\alpha] \circ ([\beta] \circ [\gamma])$. Therefore $\Pi_1(M,p)$ will have structure of a group called the first fundamental group for M. It appears that $\Pi_1(M,p)$ depends on the base point $p \in M$ choosen for the loops. This would be something of a disadvantage if it were generally true. It is not true if the space M is path connected as we have assumed it to be. Then $\Pi_1(M,p)$ is isomorphic to $\Pi_1(M,q)$ for different points $p, q \in M$.

It suffices to take a path γ from p to q (not loop) using γ and γ^{-1} we have the homomaphism

$$\sigma_\gamma : \Pi_1(M,p) \longrightarrow \Pi_1(M,q)$$

given by

$$\sigma_\gamma([\alpha],p) = ([\gamma^{-1} * \alpha * \gamma], q)$$



and

$$\sigma_{\gamma^{-1}}([\beta], q) = ([\gamma * \beta * \gamma^{-1}], p).$$

Then $\Pi_1(M,p)$ for a path connected topological space depends, up to isomorphism, on the space M.

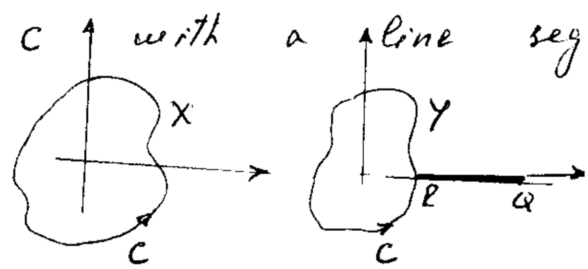
Definition 2-1 Two spaces X and Y are of the same homotopy type if we have continous maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \simeq 1_Y$, $g \circ f \simeq 1_X$.

Thus, if X and Y are homeomorphic path connected topological spaces then $\Pi_1(X,p)$ is isomorphic to $\Pi_1(Y,q)$ for all $p \in X$, $q \in Y$.

Therefore, fundamental groups is a topological invariant.

<u>corollary:</u> A pair of spaces X and Y which are of the same homotopy type, they may not be homeomorphic. This means that if two spaces have isomorphic fundamental groups they need not be homeomorphic.

For an example take X to be the closed curve C in $\mathbb{R}^2$ and Y to be C with a line segment PQ added to C.



Let $f : X \to Y$ be the map $f(x) = x$ and $g : Y \to X$ be the map $g(y) = \begin{cases} y & \text{if } y \in C \\ p & \text{if } y \in PQ \end{cases}$ where p is the point where the line segment joins the curve C.

Now X and Y are not homeomorphic, because removal of a point from X does not make X disconnected, but removal of a point from the line segment PQ of Y does change Y to a disconnected space.

X and Y are of the same homotopy type i.e f and g satisfy:

$g \circ f \simeq I_X$ the identity on X

$f \circ g \simeq I_Y$ " " Y.

To see that $g \circ f \simeq I_X$ is obvious, to show that $f \circ g \simeq I_Y$ just think of a continuous deformation which contracts the line segment down to the point P.

This example leads directly to the notion of retract and deformation retracts. Intuitively if

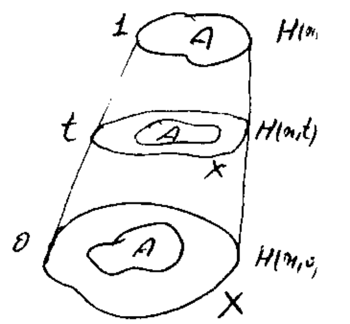
$A \subset X$ is a deformation retract of X, it means that X can be continously deform to A without moving points of A at any stage.

<u>Definition 2-2</u>: A subset A of a topological space X is called a retract of X if there exists a continuous map called a retraction $r: X \to A$ such that $r(a) = a \quad \forall a \in A$.

<u>Definition 2-3</u>: A subset A of a topological space X is a deformation retract of X if there is a retraction $r: X \to A$ and a homotopy $H: X \times [0,1] \to X$ such that

$H(x,0) = x \quad , \quad H(x,1) = r(x) \quad , \quad H(a,t) = a \quad a \in A \quad , \quad t \in [0,1]$

It is easy to see that deformation retractedness property of A implies that A and X are of the same homotopy type therefore $\pi_1(X, a) \overset{isom}{\simeq} \pi_1(A, a) \cdot \quad a \in A$.



<u>Examples</u> (1) Let $\mathbb{R}^n = \{x : x = (x_1, \ldots, x_n)\}$ and $Y = \{0\}$ then Y is a deformation retract of $\mathbb{R}^n$. Define $H: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by formula $H(x,t) = tx \quad 0 \le t \le 1 \quad x \in \mathbb{R}^n$. We have

$\pi_1(\mathbb{R}^n, 0) \simeq \pi_1(\{0\}, 0) =$ identity element.

(2) The unit $(n-1)$-sphere S^{n-1} is a deformation retract of $D^n - \{0\}$. Since we define $H: (D^n - \{0\}) \times [0,1] \longrightarrow (D^n - \{0\})$ by

$H(x,t) = (1-t)x + t \dfrac{x}{|x|} \quad 0 \le t \le 1 \qquad |x| = (\Sigma x_i^2)^{1/2}$

for $n = 2$.

$D^2 - \{0\}$

therefore $\pi_1(D^2 - \{0\}, p) \simeq \pi_1(S^1, p) \simeq \mathbb{Z}$.

For calculation of fundamental groups, we first introduce the notion of m-simplex denoted by σ^m in a topological space X.

Definition 2-4. Let $x_1, x_2, \ldots x_{m+1}$ be distinct points in $\mathbb{R}^n$, the set of points $x_1, x_2, \ldots, x_{m+1}$ are independent if the m vectors $x_2 - x_1, x_3 - x_1, \ldots, x_{m+1} - x_1$ are linearly independent vectors.

Definition 2-5: An m-simplex σ^m is the set of points x in $\mathbb{R}^n$ given by $\sigma^m = \{ x = \sum_{i=1}^{m+1} \lambda_i x_i \ ; \ \lambda_i \geq 0 \ , \sum_{i=1}^{m+1} \lambda_i = 1 \}$ where $x_1, x_2, \ldots, x_{m+1}$ are independent.

We write $\sigma^m = [x_1, x_2, \ldots x_{m+1}]$ and call $x_1 \ldots x_{m+1}$ the vertices of m-dimensional simplex, λ_i are called the barycentric coordinates of the simplex.

The set $\{ \lambda_1 x_1 + \cdots + \lambda_{m+1} x_{m+1} \ ; \ \lambda_j = 0 \}$ is called the j-th face of the simplex σ^m. It lies opposite the j-th vertex x_j.

Definition 2-6: A simplecial complex K is a finite collection of simplexes in some $\mathbb{R}^n$ satisfying

(1) If $\sigma^p \in K$, then all faces of σ^p belong to K.

(2) If $\sigma^p, \sigma^q \in K$, then either $\sigma^p \cap \sigma^q = \phi$ or $\sigma^p \cap \sigma^q$ is a common face of σ^p and σ^q.

Dimention of K is defined to be the maximum of the dimensions of the simplexes of K.

A simplicial complex K is path connected if for every pair of vertices u and v of K there is a sequence $v_0, v_1, \ldots, v_n$ of vertices in K such that $v_0 = u$ and $v_n = v$ and $v_i v_{i+1}$ is a 1-simplex of K for all $i = 0, 1, \ldots, n-1$.

Definition 2-7 : The union of the members of K with the Eucledian subspace topology is called the polyhedron associated with K. A polyhedron is path connected if the simplicial complex with which it is associated is path connected.

3 - Calculations:

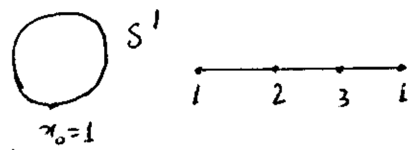
Let K be a path connected polyhedron with a_0 a vertex of K. Let L be a one dimensional sub-polyhedron of K which is contractible (has the same homotopy type as a point), and contains all the vertices of K. Let G be the group generated by the symbols g_{ij}, one for each ordered 1-simplex $\{a_i, a_j\}$ of K, subjected to the relation $g_{ij}\, g_{jk}\, g_{ik}^{-1} = 1$. one for each ordered 2-simplex $\{a_i, a_j, a_k\}$ of K-L. If $\{a_i, a_j\}$ belong to L then $g_{ij} = 1$. Then G is isomorphic to the fundamental group $\pi_1(K, a_0)$. In this case to determine the fundamental group of a given topological space X we have to find a polyhedron which is homeomorphic to it. It is convenient to introduce triangulations:

3-1 triangulation of a space

A topological space X which is homeomorphic to a polyhodron K is said to be triangulable and the polyhedron K (which is not unique) is called a triangulation of X.

Examples (1) S^1 is a one dimensional space, this means we should find a collection of 1-simplexes which is homeomorphic to S^1

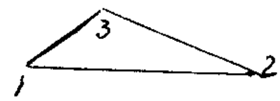
The polyhedra K_0 associated with S^1 can be written as
$K_0 = \{1\} \cup \{2\} \cup \{3\} \cup \{1,2\} \cup \{1,3\} \cup \{2,3\}$



that is K_0 is the union of 3 . 0-simplexes and 3 1-simplexes hence L_0, the contractible subpolyhedra contained in K_0 is $L_0 = \{1,3\} \cup \{2,3\}$.

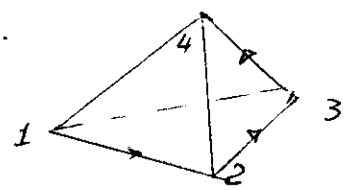
The group G generated by the symbols g_{ij}, one for each ordered 1-simplex of K_0 is generated by one element $g_{12} = g$ (say) since $g_{13} = g_{23} = 1$ being elements of L_0. Thus $\pi_1(K_0, x_0)$ is isomorphic to the group generated by one element g which is isomorphic to $\mathbb{Z}$. Therefore $\pi_1(K_0, x_0) \simeq \mathbb{Z}$.

(2) Let $D = \{(x_1, x_2) ; x_1^2 + x_2^2 \leq 1\}$, then the triangulation associated with D is



and the polyhedra associated with D is
$K = K_0 \cup \{1,2,3\}$ where K_0 is triangulation for S^1.
L is the same as L_0 for S^1, that is $L = \{1,3\} \cup \{2,3\}$
K contains only one element $g_{12} \neq 1$, but this time there is a 2-simplex $\{1,2,3\}$, thus there is the relation $g_{12}\, g_{23}\, g_{13}^{-1} = \{1\}$ (the unit element of G).
Since g_{23}, g_{13} are the elements of L they can be regarded as unit elements, thus $g_{12} = \{1\}$, hence $\pi_1(K, x_0) \simeq \{1\}$ or $\pi_1(D, x_0) \simeq \{1\}$.

(3) For S^2 we have



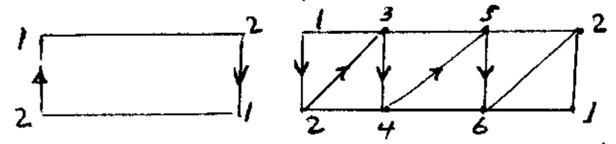
$K = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{1,2\} \cup \{2,3\}$
$\cup \{1,3\} \cup \{1,4\} \cup \{3,4\} \cup \{2,4\} \cup \{1,2,4\} \cup \{2,3,4\} \cup \{1,3,4\} \cup \{1,2,$

while $L = \{1,2\} \cup \{2,3\} \cup \{3,4\}$.

The generators g_{12}, g_{23}, g_{34} are equal to $\{1\}$, The remaining generators must satisfy 4 relations, one for

each 2-simplexes of K. Thus $g_{12}\, g_{24}\, g_{14}^{-1} = \{1\}$, ... etc.
These imply $g_{14} = g_{24} = g_{13} = \{1\}$ hence $\pi_1(S^2, x_0) \simeq \{1\}$.

(4) The Möbius band M,



then the polyhedra K corresponding to Möbius band is given by $K = \{1\}\cup\{2\}\cup\{3\}\cup\{4\}\cup\{5\}\cup\{6\}\cup\{1,2\}\cup\{1,3\}$ $\cup\{1,6\}\cup\{2,3\}\cup\{2,4\}\cup\{2,5\}\cup\{2,6\}\cup\{3,4\}\cup\{3,5\}\cup\{4,5\}\cup\{4,6\}\cup\{5,6\}$ $\cup\{1,2,3\}\cup\{1,2,6\}\cup\{2,3,4\}\cup\{2,5,6\}\cup\{3,4,5\}\cup\{4,5,6\}$.

The one dimensional contractible polyhedron L which contains all the vertices of K can be selected to be $L = \{1,2\}\cup\{2,3\}\cup\{3,4\}\cup\{4,5\}\cup\{5,6\}$.

Then the generators $g_{ij} = \{i,j\} = 1$ when the simplex $\{i,j\} \in L$. The remaining generators are g_{13}, g_{23}, $g_{35}, g_{25}, g_{46}, g_{16}$ and g_{26} which must satisfy 6 relations one for each 2-simplexes in K, namely

$g_{12}\, g_{23}\, g_{13}^{-1} = 1$ or $g_{12}\, g_{23} = g_{13}$; $g_{24}\, g_{43} = g_{23}$,

$g_{35}\, g_{54} = g_{34}$, $g_{46}\, g_{65} = g_{45}$, $g_{25}\, g_{56} = g_{26}$, $g_{12}\, g_{26} = g_{16}$,

and since $g_{12} = g_{13} = g_{34} = g_{45} = g_{56} = 1$ we get $g_{13} = 1$, $g_{24} = 1$, $g_{35} = 1$, $g_{46} = 1$ and $g_{25} = g_{26} = g_{16} = g$ (say)

Thus $\pi_1(M, x_0) \simeq \mathbb{Z}$.

In the same way for Torus T,

$g_{12} = g_{23} = g_{34} = g_{45} = g_{56} = g_{67} = 1$

$g_{13} = g_{14} = g_{47} = g_{75} = g$ (say)

$g_{17} = g_{27} = g_{37} = g_{36} = g_{35} = h$ (say)

$g_{16} = g_{26} = g_{25} = g_{24} = 1$,

Also $hg = gh$ that is the two generators commute.
Then $\pi_1(T, x_0) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Recall that if G_1 and G_2 are two groups then their direct product denoted by $G_1 \oplus G_2$ is the set of all ordered pairs (g_1, g_2), $g_1 \in G_1$, $g_2 \in G_2$ with multiplication $(g_1, g_2)\cdot(g_1', g_2') = (g_1 g_1', g_2 g_2')$.

Also since a loop in $(X \times Y, x_0 \times y_0)$ is exactly a pair of loops (X, x_0) and (Y, y_0), then, it is easy to see that $\Pi_1(X \times Y, x_0 \times y_0) \simeq \Pi_1(X, x_0) \oplus \Pi_1(Y, y_0)$ $\quad x_0 \in X$, $y_0 \in Y$

Hence for the Torus T,

$$\Pi_1(T, t_0) \simeq \Pi_1(S^1, x_0) \oplus \Pi_1(S^1, y_0) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad \text{where } t_0 = (x_0, y_0)$$

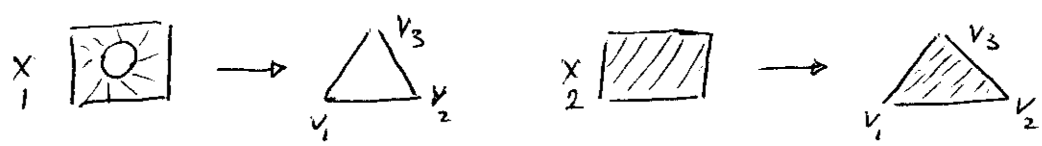
Also, a closed cylinder C is the product of a circle S^1 and a closed interval $[0,1]$, thus

$$\Pi_1(C, c_0) \simeq \Pi_1(S^1, x_0) \oplus \Pi_1([0,1], y_0) \qquad c_0 = (x_0 \times y_0)$$
$$\simeq \mathbb{Z} \oplus \{0\} \qquad \text{since } [0,1] \text{ is contractible.}$$

4- The Homology groups:

If X and Y are two topological spaces of the same homotopy type, then $H_p(X) \equiv H_p(Y)$ $\quad p = 0, 1, 2, \ldots$ hence homology groups are topological invariants. Also if K_1 and K_2 are two triangulations of the same topological space, then $H_p(K_1) = H_p(K_2)$ $\quad \forall p$.

Let X_1 and X_2 be rectangular regions of $\mathbb{R}^2$, X_1 contains a hole while X_2 does not. Thus X_2 is homeomorphic to a 2-simplex and X_1 can be replaced by its deformation retract, the edges of σ^2.



For X_2 the boundary is like the boundary of a connected region, while for X_1, the boundary consists only of the edges of the triangular region v_1, v_2, v_3 and is not the boundary of any region. This suggests a method of spotting holes in a space.

" A closed two dimensional region has a hole if its boudary (some closed curve) is not the boundary of a connected region". The idea can be generalized to higher dimension.

<u>4-1 Oriented simplexes</u>: An oriented p-simplex $p>1$ is obtained from a p-simplex $\sigma^p = [v_0, \dots, v_p]$ by choosing an ordering for its vertices. The equivalence class of even permutations of the chosen ordering determines the positive oriented simplex $+\sigma^2$.

Now we associat with each p-simplex σ_i^p $p = 0, 1, \dots, n$ of a simplicial complex K, an abelian group $C_p(K)$, called the chain group.

<u>Definition 4-2</u>: Let K be a n-dimensional simplicial complex containing l_p, p-simplexes. The p chain of K, $C_p(K)$ is the free abelian group generated by the oriented p-simplexes of K, that is every element $c_p \in C_p(K)$ can be written as: $c_p = \sum_{i=1}^{l_p} m_i \sigma_i^p$ $m_i \in \mathbb{Z}$, where

$$\sigma_i^p + (-\sigma_i^p) = 0 \quad \forall i, p. \quad \text{and}$$

$$\sum_{i=1}^{l_p} m_i \sigma_i^p + \sum_{i=1}^{l_p} n_i \sigma_i^p = \sum_{i=1}^{l_p} (m_i + n_i)\, \sigma_i^p \qquad m_i, n_i \in \mathbb{Z}.$$

The statment That K is n-dimensional simp means That $p = 0, 1, \dots, n$. It is often convenient to define to define $C_p(K) = \{0\}$ for $p > n$.

Definition 4.3 The boundary operator ∂_p is the map $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ with the following properties:

(i) It is linear. $\partial_p(\Sigma m_i \sigma_i^p) = \sum_i m_i \partial_p(\sigma_i^p)$

(ii) For an oriented p-simplex $\sigma^p = [v_0, \dots, v_p]$, $\partial[v_0, \dots, v_p] = \sum_{j=0}^{p} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_p]$. where $[v_0, \dots, \hat{v}_j, \dots, v_p]$ represent the $(p-1)$-simplex σ^{p-1} obtained from the p-simplex σ^p by omitting the vertex v_j

(iii) The boundary of every zero chain is defined to be zero.

It is easy to check that ∂_p is a homomorphism from $C_p(K)$ to $C_{p-1}(K)$.

It is easy to check that $\partial_{p-1} \circ \partial_p = 0$ (or simply $\partial^2 = 0$)

Let X_1 and X_2 be rectangular regions in $\mathbb{R}^2$, where X_1 contains a hole while X_2 does not. X_1 is a polyhedron made up of the following simplexes

$X_1 = \{[v_1, v_2]; [v_2, v_3]; [v_1, v_3]\}$ and their faces; whil $X_2 = \{[v_1, v_2, v_3]\}$ and all faces.

We observe That, in general, if K is a closed region Then ∂K, the boundary of K, is expected to be a closed surface. ∂K should not itself hav any boundary i.e $\partial(\partial K) = 0$. Thus once we have defined a boundary operator ∂, it is straight forward to spot a boundary surface B. We just verify that $\partial B = 0$. It is also possible to tell if The boundary B is the boundary of a hole or of a connected region.

This is because if B were the boundary of some connected region, then we expect to find a K such that $B = \partial K$. Let us now apply these ideas to the spaces X_1 and X_2. We note that for X_1, $B = [V_2, V_3] + [V_3, V_1] + [V_1, V_2]$, and it is a boundary, since $\partial B = 0$.

On the other hand there are no higher dimensional simplexes in X_1, that is, $B \neq \partial K$. Thus X_1 contains a hole. For X_2, B is also a boundary but it is the boundary of a 2-simplex $\sigma^2 = [V_1, V_2, V_3]$. Thus X_2 does not contain a hole.

Using the boundary B of X_1 an abelian group can be generated simply by noting that if B is a boundary then $\pm nB$, where n is an integer, is also a boundary, Therefore we can construct an abelian group (of boundaries), This is the homology group $H_1(X_1)$; which in this case is isomorphic to $\mathbb{Z}$.

Now in general case we identify all the p-dimensional boundaries (cycles) and then identify those, that are boundaries of connected region in terms of the chain group.

<u>Definition 4.4</u> $z_p \in C_p(X)$ is called a p-dimensional cycle or p-cycle if $\partial z_p = 0$. The family

of p-cycles is thus the kernel of the homomorphism $\partial : C_p \to C_{p-1}$ and is a subgroup of $C_p(K)$. This subgroup is called the p-dimensional cycle group of K and is denoted by $Z_p(K)$

<u>Definition 4.5</u> $b_p \in C_p(K)$ is called a p-dimensional boundary or p-boundary if there is a (p+1)-chain $C_{p+1}(K)$ or simply C_{p+1}, such that

$\partial C_{p+1} = b_p$.

The family of p-boundaries is thus the homomorphic image ∂C_{p+1} and is a subgroup of $C_p(K)$, and is denoted by $B_p(K)$. Any element b_p of $B_p(K)$ has the property that $\partial b_p = 0$. Thus $B_p(K)$ is a subgroup of $Z_p(K)$.

In order to spot the (p+1)-dimensional holes we have thus to weed out the elements belonging to $B_p(K)$ contained in $Z_p(K)$. This is achived by introducing the homology group.

<u>Definition 4.6</u>:

The p-dimensional homology group of K denoted by $H_p(K)$ is the quotient group

$$H_p(K) = Z_p(K)/B_p(K).$$

An element h_p of $H_p(K)$ is thus an equivalence class $[z_p]$ defined by the relation $z_p^1 \sim z_p^2 \Leftrightarrow$ $(z_p^1 - z_p^2) \in B_p(K)$.

$H_p(K)$ depends on the number number of $(p+1)$ - dimensional holes present in the space.

<u>Theorem 4-7</u>: If K is a contractible space i.e has the homotopy type of a single point Then

$$H_p(K) = \begin{cases} \{0\} & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}.$$

Proof: $\quad H_p(K) = H_p([v_0]) \quad \forall p \Rightarrow C_p(K) = \{0\} \Rightarrow$

$$H_p(K) = \{0\}$$

$$Z_0(K) = C_0(K), \quad B_0(K) = \{0\} \Rightarrow H_0(K) = Z_0(K) =$$

$$\{z_0 ; n[v_0] = z_0 \quad n \in \mathbb{Z}\} = \mathbb{Z}.$$

<u>Examples:</u> (1) Let $K = \sigma^2 = [v_0, v_1, v_2]$. We will calculate $H_k(K)$ for $k = 0, 1, 2, \ldots$ We first note that $\dim K = 2$. Thus by definition $C_p(K) = \{0\}$ for $p > 2$.

<u>Calculation of $H_0(K)$:</u> Since $H_0(K) = Z_0(K)/B_0(K)$, we have to determin $Z_0(K)$ and $B_0(K)$. We recall that $z_0 \in Z_0(K)$ if $z_0 \in C_0(K)$ and $\partial z_0 = 0$, since the boundary of every zero chain is defined to be zer it follows that $C_0(K) = Z_0(K)$. Also every element of $C_0(K)$ can be written as: $a_0[v_0] + b_0[v_1] + c_0[v_2]$ where $a_0, b_0, c_0 \in \mathbb{Z}$; $Z_0(K)$ thus has three independent generators so That $Z_0(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ $= \mathbb{Z}^3$.

Next we study $B_0(K)$. We recall that, $b_0 \in B_0(K)$ if $b_0 \in C_0(K)$ and $b_0 = \partial c_1$ where $c_1 \in C_1(K)$. Any element of $C_1(K)$ can be written as:

$$c_1 = a_1[v_0, v_1] + b_1[v_0, v_2] + d_1[v_1, v_2]$$

Thus $\partial c_1 = a_1([v_1] - [v_0]) + b_1([v_2] - [v_0]) + d_1([v_2] - [v_1])$

$$= (a_1 - d_1)[v_1] - (a_1 + b_1)[v_0] + (b_1 + d_1)[v_2].$$

Hence any element b_0 of $B_0(K)$ can be written as:

$$b_0 = a_0[v_0] + c_0[v_2] + d_0[v_1]$$

with $a_0 + b_0 + d_0 = 0$. This means that $B_0(K)$ has two independent generators. Hence $B_0(K) = \mathbb{Z} \oplus \mathbb{Z}$.

Finally if $h_0 \in H_0(K)$ we can write h_0 as the coset

$$h_0 = z_0 + B_0(K)$$

$$= a_0[v_0] + b_0[v_1] + c_0[v_2] + \{-a_0[v_0] - b_0[v_1] + (a_0 + b_0)[v_2]\}$$

$$= (a_0 + b_0 + c_0)[v_2] = d_0[v_2] \qquad d_0 \in \mathbb{Z}.$$

Thus $H_0(K)$ has only one independent generator and $H_0(K) = \mathbb{Z}$.

<u>Calculation of $H_1(K)$:</u>

Now $z_1 \in Z_1(K)$ if $z_1 \in C_1(K)$ and $\partial z_1 = 0$. From our previous calculation we know that if $z_1 \in C_1(K)$, then $z_1 = a_1[v_0, v_1] + b_1[v_0, v_2] + c_1[v_1, v_2]$ and

$\partial z_1 = (a_1 - c_1)[v_1] - (a_1 + b_1)[v_0] + (b_1 + c_1)[v_2]$. Hence the requirement $\partial z_1 = 0$ means that

$z_1 = a_1[v_0, v_1] - a_1[v_0, v_2] + a_1[v_1, v_2]$. i.e $Z_1(K) = \mathbb{Z}$

Next we have to study $B_1(K)$. We note that $b_1 \in B_1(K)$ if $b_1 \in C_1(K)$ and $b_1 = \partial c_2$ where $c_2 \in C_2(K)$. Any element of $C_2(K)$ can be written

as $C_2 = a_2[v_0, v_1, v_2]$, then $\partial c_2 = a_2\{[v_1,v_2] - [v_0,v_2] + [v_0,v_1]\}$
Hence any element b_1 of $B_1(K)$ can be written as:

$b_1 = a_2\{[v_1, v_2] - [v_0, v_2] + [v_0, v_1]\}$.

This means that $B_1(K) = \mathbb{Z}$. Finally if $h_1 \in H_1(K)$ we write $h_1 = z_1 + B_1(K)$, and note that $z_1 \in B_1(K)$ so that $H_1(K) = \{0\}$

Calculation of $H_2(K)$:

As before $z_2 \in Z_2(K)$ if $z_2 \in C_2(K)$ and $\partial z_2 = 0$. From our previous calculations it follows that $Z_2(K) = \{0\}$ since $\partial z_2 = 0$ implies $a_2 = 0$. Again $b_2 \in B_2(K)$ if $b_2 \in C_2(K)$ and $b_2 = \partial c_3$. Since there are no 3-simplexes in K therefore $b_2 = 0$. Thus $H_2(K) = \{0\}$
Hence when $K = \sigma^2 = [v_0, v_1, v_2]$

$$H_0(K) = \mathbb{Z}, \qquad H_k(K) = \{0\} \quad k \neq 0.$$

(2) Let $K = S^1 = \partial\sigma^2$.

Now $\dim K = 1$, so that $C_k(S^1) = \{0\}$ for $k > 1$. Hence $H_k(S^1) = \{0\}$ for $k > 1$. Thus we only have to calculate $H_0(S^1)$ and $H_1(S^1)$.

Calculation $H_0(S^1)$:

Again $z_0 \in Z_0(S^1)$ if $z_0 \in C_0(S^1)$ and $\partial z_0 = 0$. From example (1) it follows that $H_0(S^1) = \mathbb{Z}$, since the 0-simplex and 1-simplex structure of σ^2 and $\partial\sigma^2$ are the same.

Calculation of $H_1(S^1)$.

From example 1, it follows that $Z_1(S^1) = \mathbb{Z}$ since $Z_1(S^1)$ only involves the 1-simplexes. However $B_1(S^1) = \{0\}$, since there are no 2-simplexes in S^1 so that $H_1(S^1) = \mathbb{Z}$. Thus

$$H_0(S^1) = \mathbb{Z} = H_1(S^1), \quad H_k(S^1) = \{0\} \quad k > 1.$$

5 - Cohomology :

Let M be compact differentiable manifold and let ω be an $(n-1)$-form and $\dim M = n$. by Stokes theorem we have $\int_M d\omega = \int_{\partial M} \omega$. Insted of integrating over M we can also integrate over some lower dimensional subset M.

Homology theory provides a class of such subspaces, that is the class of all singular p-chains C with $p = 0, 1, \dots n$. Let Δ_p be the standard n-simplex in $\mathbb{R}^p$. Every p-chain can be written as

$$C = a_1 \lambda_1 + a_2 \lambda_2 + \cdots$$

where a_i are integers and λ_i are singular p-simplex that is maps $\lambda_i : \Delta_p \longrightarrow M$, Δ_p being standard n-simplex in R^p. Now we require a_i to be real rather than integers, also $\lambda_i : \Delta_p \to M$ be C^∞.

Now take p-chain C and a p-form ω with $0 \leq p \leq n$ the integral of ω over C is define using the map λ_i^*

$$\int_C \omega = \sum_{i=1}^{k} a_i \int_{\Delta_p} \lambda_i^* \omega \qquad C = \sum_{i=1}^{k} a_i \lambda_i$$

recal that $\lambda_i^* \omega$ is a p-form on Δ_p. The integral is a real number and thus we ω as producing, from C, a real number. So if we denot by C_p the set of all C^∞ p then a p-form ω is a map from C_p to $\mathbb{R}$, that is an element of the dual of C_p (co-chain) i.e

$$\omega : C_p \to \mathbb{R} \qquad c \longrightarrow \langle \omega, c \rangle \quad \text{where}$$

$$\langle \omega, c \rangle = \int_c \omega \qquad c \in C_p \qquad \omega \in \Omega^p(M).$$

To define the cohomology group $H^p(M, \mathbb{R})$, by Stoks theorem we have $\langle dw, c \rangle = \langle w, \partial c \rangle$. That is the boundary operator ∂ and the exterior derivative operator d are formal adjoint of one another.

The dual of the space $H_p = Z_p/B_p$, we write as $H^p = Z^p/B_p$ $\left(H^p(M) = Z_p(M)/B_p(M) \right)$.

where $Z^p(M)$ are all cochains of p-forms ω for which $d\omega = 0$ and $B^p(M)$ are all p-forms ω for which $\omega = d\eta$ for some $(p-1)$ form η.

To show precisely that the space $Z^p(M)/B^p(M)$ is the dual of the space $Z_p(M)/B_p(M)$ we must define how $[\omega]$ acts on $[C]$ to give a real number

$$([\omega], [C]) = \int_C \omega .$$

Finally the space $Z_p(M)/B_p(M)$ is the p-th homology group of M with real coefficient written as $H_p(M; \mathbb{R})$ and $Z^p(M)/B^p(M)$ is the p-th cohomology group of M with real coefficients and is written as $H^p(M, \mathbb{R})$.

or:

$$\cdots \xleftarrow{\partial_{p-1}} C_{p-1} \xleftarrow{\partial_p} C_p \xleftarrow{\partial_{p+1}} C_{p+1} \xleftarrow{\partial_{p+2}} \cdots$$

$$H_p(M, \mathbb{R}) = Z_p(M)/B_p(M) = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}$$

$$\cdots \xrightarrow{d_{p-1}} \Omega^{p-1} \xrightarrow{d_p} \Omega^p \xrightarrow{d_{p+1}} \Omega^{p+1} \xrightarrow{d_{p+2}} \cdots$$

$$H^p(M, \mathbb{R}) = Z^p(M)/B^p(M) = \operatorname{Ker} d_{p+1} / \operatorname{Im} d_p .$$

this construction of cohomology groups is due to de Rham, therefore $H^p(M, \mathbb{R})$ often called de Rham cohomology groups.

5-1 Calculation of $H^p(M, \mathbb{R})$;

1) $H^p(\mathbb{R}^n, \mathbb{R})$: Since $\mathbb{R}^n$ is contractible, all closed p-forms on $\mathbb{R}^n$ are exact and we have

$$H^p(\mathbb{R}^n, \mathbb{R}) = 0 \qquad p = 1, 2, \cdots, n$$

Also for $p > \dim M$, $H^p(M, \mathbb{R}) = 0$. therefore

$$H^p(\mathbb{R}^n, \mathbb{R}) = 0 \qquad p = 1, 2, \cdots$$

We also have $H^0(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}$, because 0-forms on M are functions on M, and that a closed 0-form is a function f satisfying $df = 0$, there are no 1-form, so f can never be exact. so $f = c$ (constant). $c \in \mathbb{R}$.

So to each real number c there correspond an element of the zeroth cohomology class, i.e an element $H^0(\mathbb{R}^n, \mathbb{R})$. therefore

$$H^0(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}.$$

(2) $H^p(S^n, \mathbb{R})$; $p = 0, \cdots, n$

First we note that for any two manifolds M, N and C^∞ map $\alpha : M \to N$ we have $\alpha^* : \Omega^p(N) \to \Omega^p(M)$ hence we get $\alpha^* : H^p(N, \mathbb{R}) \to H^p(M, \mathbb{R})$ by $[\omega] \to [\alpha^* \omega]$.

Also for every closed curve on a simply connected mfd M is smoothly homotopic to the constant curve.

Let Γ be any closed curve on M and Γ_0 be the constant curve. We have two maps

$$\Gamma : S^1 \longrightarrow M$$

$$\Gamma_0 : S^1 \longrightarrow M$$

under Γ_0 all elements $\theta \in S^1$ are mapped onto the same point $x_0 \in M$. Next choose any closed 1-form ω on M then we write the integral of ω round a curve Γ as $\int_\Gamma \omega$, and define

$$\int_\Gamma \omega = \int_{S^1} \Gamma^* \omega ,$$

But since Γ is homotopic to Γ_0 and since ω is closed then we have just prove That

$$\Gamma^* \omega = \Gamma_0^{\omega} \omega + d\eta \qquad \text{for some } \eta .$$

Since Γ_0 is the constant curve it follows that $\Gamma_0^* = 0$ thus $\int_\Gamma \omega = \int_{S^1} \Gamma^* \omega = \int_{S^1} d\eta = 0$.

since, S^1 has no boundary. Now we have proved that if M is simply connected then $\int_\Gamma \omega = 0$. for all closed 1-forms.

It remains to show that ω is also exact. This is so because $\omega = df$, where f is the 0-form given by $f(x) = \int_{x_0}^{x} \omega$. Thus

$$H^1(M; R) = 0 .$$

In particular, if $M = S^n$ then S^n is simply connected for $n > 1$, so we have our first result for sphere.

$$H^1(S^n, R) = 0 \qquad n > 1$$

On can see that homotopy divides closed curves on an arbitrary manifold into equivalence classes, one for each hole in the manifold. Eeach closed curve round a particular hole is homotopic to another round the same hole. This provides us with

$$\dim H^1(M; R) = \text{the number of holes in } M.$$

Now we shall find that $H^1(S^1, \mathbb{R}) = \mathbb{R}$. Since S^1 has dimension one, all 1-forms ω on S^1 are closed. The key is to show that

$$\int_{S^1} \omega = 0 \Rightarrow \omega \text{ exact,}$$

for 1-form ω. The converse is true by Stokes theorem.

Now we are looking for all closed non-exact 1-forms. Hence all such forms have non-zero integral over S^1. Suppose then ω and ω' are two closed non-exact 1-forms on S^1. The statement that $H^1(S^1, R) = R$ means that there is some $c \in R$ for which $\omega - c\omega'$ is exact. We can prove this immediately, for consider the form $\omega - c\omega'$ where $c = \{\int_{S^1} \omega\} / \{\int_{S^1} \omega'\}$

integrating $\omega - c\omega'$ we obtain $\int_{S^1} \omega - c\omega' =$

$\int_{S^1} \omega - c\int_{S^1} \omega' = 0$. Thus $\omega - c\omega'$ is exact. c being real number shows that $H^1(S^1, \mathbb{R}) = \mathbb{R}$.

When $p = 0$ we have $\omega = f$ where f is constant function on S^n. So we obtain $H^0(S^n, \mathbb{R}) = \mathbb{R}$.

When $p > 1$ we need to prove the inductive formula

$$H^p(S^n, \mathbb{R}) = H^{p-1}(S^{n-1}, R) \qquad p > 1$$

Then we get

$$H^0(S^n, \mathbb{R}) = \mathbb{R} \quad , \quad H^1(S^1, \mathbb{R}) = \mathbb{R}$$

$$H^1(S^n, \mathbb{R}) = 0 \qquad n > 1.$$

then we see that we know $H^p(S^n, \mathbb{R})$ such that

$$H^p(S^n, \mathbb{R}) = 0 \qquad p > n$$

$$H^p(S^n, \mathbb{R}) = \mathbb{R} \qquad p = n$$

$$H^p(S^n, \mathbb{R}) = 0 \qquad 1 \leq p < n$$

$$H^0(S^n, \mathbb{R}) = \mathbb{R} \,.$$

By using wedge product we shall get,

for torus $\qquad H^p(T^n, \mathbb{R}) = \mathbb{R}^\alpha$

where. $\qquad \alpha = \binom{n}{p}$.

<u>General remarks:</u>

(1) If M is a compact, connected orientable manifold, and $\dim M = n$ then $H^n(M, \mathbb{R}) = \mathbb{R}$.

(2) If M is a non-compact, connected manifold, $\dim M = n$, then $H^n(M, \mathbb{R}) = 0$

(3) If M is a compact, connected non-orientable manifold $\dim M = n$ then $H^n(M, \mathbb{R}) = 0$