INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.637/10

ADVANCED WORKSHOP ON ARITHMETIC ALGEBRAIC GEOMETRY

(31 August - 11 September 1992)

Riemann Surfaces

V. Pati
Statistics-Mathematics Unit
Indian Statistical Institute
8th mile, Mysore Road
Bangalore - 560 059
India

These are preliminary lecture notes, intended only for distribution to participants

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

Riemann Surfaces

V. Pati.

§0. Basics.

0.1. Def: A 2nd-countable, Hausdorff complex manifold of dimension 1 (complex dimension, that is) is called a Riemann Surface. More precisely, X is a Riemann Surface if

i) X is 2nd-countable + Hausdorff

ii) $\exists$ a covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of X by open sets U_α and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \underset{\text{open}}{\subset} \mathbb{C}$ such that the coordinate changes

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic maps (between open sets in $\mathbb{C}$) $\forall$ α and β.

Note that this makes a Riemann Surface a (real) differentiable manifold of dim 2. Hence the term "Surface". $\{\varphi_\alpha, U_\alpha\}$ are called holomorphic charts or coordinate systems on X.

Examples :-

1. $\mathbb{C}$. (clearly). Similarly, any open subset U of $\mathbb{C}$. For example,
2. $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ called the upper-half plane, or
3. $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ called the Poincare-Disc.

It is easy to see that Δ + $\mathbb{H}$ are conformally equivalent, whereas $\mathbb{C}$ and $\mathbb{H}$ (or Δ) are not. (Liouville's Theorem)

4. $\mathbb{C} \cup \{\infty\}$ – the Riemann sphere. Topologically, it is the one-point compactification of $\mathbb{C} = \mathbb{R}^2$ and so (using stereographic projection) is seen to be S^2. The complex structure is given by

$$U_1 = \mathbb{C}, \quad U_2 = (\mathbb{C} - \{0\}) \cup \{\infty\}.$$

$\varphi_1 : U_1 \to \mathbb{C}$ is the identity map $\varphi_1(z) = z$ and $\varphi_2 : U_2 \to \mathbb{C}$ is given by $\varphi_2(z) = \frac{1}{z}$ $(z \neq \infty)$; $\varphi_2(\infty) = 0$. The coordinate change on $\varphi_1(U_1 \cap U_2) = \mathbb{C}^* = \mathbb{C} - \{0\}$: $\varphi_2\varphi_1^{-1}(z) = \frac{1}{z}$, which is clearly holomorphic on $\mathbb{C}^*$.

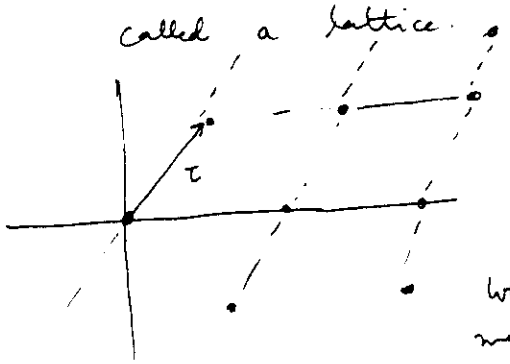
This space is called $\mathbb{P}^1$ (or $\mathbb{CP}^1$) for the following reason:–
$\mathbb{CP}(1) = \mathbb{P}^1 \overset{\text{def}}{=} \{[z_1 : z_2] : [\lambda z_1 : \lambda z_2] = [z_1 : z_2] \text{ for } \lambda \in \mathbb{C}^*, (z_1,z_2) \neq (0,0)\}$ and we define the map from $\mathbb{P}^1$ to the Riemann sphere by taking $[z_1 : z_2] \mapsto z_1/z_2 \in \mathbb{C} \cup \{\infty\}$. Which is easily checked to be a homeomorphism.

5. Smooth projective varieties of $\dim_{\mathbb{C}} = 1$ are Riemann surfaces. (Holomorphic Implicit Function Theorem). Later it will be seen that all compact Riemann Surfaces are smooth projective varieties of $\dim_{\mathbb{C}} = 1$. (§2.24 – Kodaira Embedding Theorem)

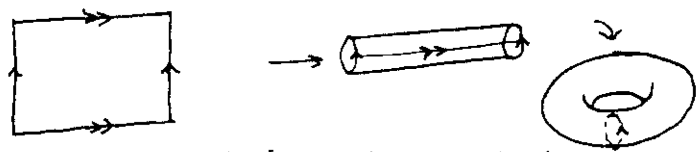
5. Toruses: Let $\tau \in \mathbb{C}$ such that $(\operatorname{Im}\tau) > 0$. (i.e. $\tau \in \mathbb{H}$)
Consider the discrete abelian subgroup of $\mathbb{C}$ defined by

$$\Gamma = \{m + n\tau : m, n \in \mathbb{Z}\}$$

called a lattice.



The quotient space $\mathbb{C}/\Gamma$ is a compact Riemann surface. If U is an open set contained in a fundamental domain, then $\pi_{|U} : U \to \pi(U)$ is a homeomorphism where $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is the natural quotient map. Thus $(\pi_{|U})^{-1} : \pi(U) \to U$ gives a chart on $X = \mathbb{C}/\Gamma$. Clearly such $\pi(U)$'s cover X, so X is a Riemann Surface. Note that $\pi : \mathbb{C} \to X = \mathbb{C}/\Gamma$ is a covering space. For example, if we take $\tau = i$, we have the familiar gluing picture of a torus



For any Γ of the above type, the topological space will be this torus. However, the complex structure depends on τ (!) In fact, there will be whole complex 1-parameter family of "distinct" complex structures possible on the (topological) torus. However the C^∞-manifold structure is unique and independent of $\tau \in \mathbb{H}$. (Exercise!)

0.2 Def:- A continuous map $f : X \to Y$ between Riemann surfaces is said to be holomorphic if for each $p \in X$, and a small nbd. U of p contained in a chart U_α of X and a small nbd V of $f(p)$ contained in a chart V_β of Y, with $f(U) \subset V$, the composite

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \underbrace{\phi_\alpha(U)}_{\text{open} \subset \mathbb{C}} \longrightarrow \underbrace{\psi_\beta(V)}_{\text{open} \subset \mathbb{C}}$$

is holomorphic. The fact that this notion is sensible follows from (ii) in the Def. (0.1) of a Riemann Surface.

A holomorphic map $f : X \to \mathbb{P}^1$ is called a meromorphic function. Note that (by open mapping, Liouville, whatever), the only holomorphic maps $f : X \to \mathbb{C}$ for compact connected Riemann surfaces X are constants.

Def:- Riemann surfaces X and Y are said to be biholomorphic, or conformally equivalent if $\exists$ $f : X \to Y$ holomorphic such that f is bijective and f^{-1} is also holomorphic. f is called a biholomorphism then.

0.3 Exercise:- Show that Δ and $\mathbb{H}$ are biholomorphic, but $\mathbb{C}$ and Δ are not. Thus $\mathbb{C}$ and $\mathbb{H}$ are also not biholomorphic.

(3.4) Uniformisation. We shall hitherto assume that all Riemann Surfaces under study are connected, as it is clearly enough to do so.

Riemann Mapping Theorem If X is a simply connected Riemann Surface, then X is biholomorphically equivalent to $\mathbb{P}^1$, if X is compact. If X is not compact, then X is biholo. to either $\mathbb{C}$ or $\mathbb{H}$ $(\cong \Delta)$.

Proof :- May be found in [2], Ch IV (§ IV.4).

Automorphism gps. of Simply connected Riemann Surfaces.

Def :- A biholo. map $f : X \to X$ is called an automorphism of X. Thus we get a group $\mathrm{Aut}(X)$ (under composition of maps being the gp. operation).

Propn :-

known as the Möbius groups

$$\begin{cases} \mathrm{Aut}\,\mathbb{C} = \{ f : f(z) = az+b,\ a \neq 0 \in \mathbb{C},\ b \in \mathbb{C} \} \\ \mathrm{Aut}\,\Delta = \{ f : f(z) = e^{i\theta}\dfrac{(z-\alpha)}{1-\bar{\alpha}z}, \text{ for some } \alpha \in \Delta \}. \\ \mathrm{Aut}\,\mathbb{H} = \{ f : f(z) = \dfrac{az+b}{cz+d}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \} \Big/ \{\pm 1\} \overset{def}{:=} PSL_2(\mathbb{R}) \\ \mathrm{Aut}\,\mathbb{P}^1 = \{ f : f(z) = \dfrac{az+b}{cz+d}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C}) \} \Big/ \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}^* \} \underset{def}{:=} PGL_2(\mathbb{C}) \end{cases}$$

Proof :- First Consider $\mathrm{Aut}(\mathbb{P}^1)$ Note that the map f_A corresponding to $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defined by $f_A(z) = \dfrac{az+b}{cz+d}$ maps the point $-b/a$ to 0 and the point $-\dfrac{d}{c}$ $(\neq -\dfrac{b}{a})$ to ∞. Clearly any two distinct points on $\mathbb{P}^1$ can be taken as $-b/a$ & $-d/c$ respectively, so we can post compose any automorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ with a (Möbius) transfn. f_A and assume that f maps 0 to 0 and $\infty \to \infty$. We claim that any auto. of $\mathbb{P}^1$ taking $0 \to 0$ and $\infty \to \infty$ is of the type $f(z) = \lambda z. = \dfrac{\lambda z + 1}{0.z+1}$ $= f_{\begin{pmatrix} \lambda & 1 \\ 0 & 1 \end{pmatrix}}$. Look at the two coord. charts U_0 & U_1 on $\mathbb{P}^1$

on U_0, $f_0(z) = z\,h(z)$ and on U_1, $f_1(w) = \dfrac{1}{f_0(\frac{1}{w})} = w.g(w)$

where h & g are holo. fns. of z and w respectively. So on $U_0 \cap U_1$ we get $w\,g(w) = \dfrac{1}{\frac{1}{w}h(\frac{1}{w})} = \dfrac{w}{h(1/w)}$ Since $w \neq 0$, $g(w)\,h(\frac{1}{w}) = 1$

Now one consequence of the fact that f is biholo is that $f_0'(0) = \dfrac{df_0}{dz}(0) = h(0) \neq 0$ Similarly $g(0) \neq 0$. Now if g is nonconst.

, there exist a sequence of points $w_n \in \mathbb{C} = U_1$ such that $w_n \to \infty$
and $|g(w_n)| \to \infty$ (since g is entire, non-constant it is unbounded, by Liouville's Thm). So $\lim_{n\to\infty} \left| h\left(\frac{1}{w_n}\right)\right| = \lim_{n\to\infty} \frac{1}{g(w_n)} = 0$.

i.e. $h(0) = 0$

which is a contradiction. So g is a const and so is h therefore,
$\Rightarrow f_0(z) = z\lambda$ where $\lambda = h(z)$. $\Rightarrow \operatorname{Aut}(\mathbb{P}^1) = GL_2(\mathbb{C}) / (\lambda \cdot Id,\ \lambda \neq 0)$

For Aut $\mathbb{C}$, we note that an automorphism f of $\mathbb{C}$ is a proper map. Since $f^{-1}(K)$ = continuous image under f^{-1} of K = compact. This means f extends to a continuous map $\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1$ as does its inverse, to say $\widehat{f^{-1}}$. Clearly $\widehat{f^{-1}} = \hat{f}^{-1}$. Also $\hat{f}$ is holomorphic since by the Riemann Removable Singularity Theorem, it is mapping a small nbd. of ∞ ($|w| < \epsilon$) to another small nbd of ∞, and it is holo. on $(|w| < \epsilon) - \{w = 0\}$. Similarly for $\hat{f}^{-1}$. So $\hat{f}$ is in Aut $\mathbb{P}^1$ taking ∞ to ∞. But then $f(z) = \frac{az+b}{cz+d}$, and the inverse image of ∞ under f is $-d/c$ ~~which~~ is ∞ so $c = 0$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\infty}$

So $f(z) = \frac{a}{d}z + \frac{b}{d} = \alpha z + \beta$ where $d \neq 0$ since $ad - bc = ad \neq 0$

For Aut Δ, one notes that a Möbius transformation $z \to \frac{z - \alpha}{1 - \bar{\alpha}z}$ is a biholo map of Δ taking α to 0. So by post-composing with such a map, we may assume $f \in \operatorname{Aut}(\Delta)$, $f(0) = 0$. If $g = f^{-1}$, $g(0) = 0$ as well. So $(f \circ g)(w) = w$ and $(f \circ g) = id$.

Thus $|w| = |f \circ g(w)| \leq |g(w)| \leq |w|$.

(Schwartz Lemma on Δ)

So Equality holds Everywhere, and so $|g(w)| = |w|$ $\forall\, w$. Again Schwartz lemma implies that g (and hence f) must be rotations.

For Aut $\mathbb{H}$, use the Conformal equivalence $\mathbb{H} \to \Delta$, $z \to \frac{z-i}{z+i}$ and use Aut Δ above

We say a group G acts *freely* on the space X if $gx = x$ for some $x \in X \Rightarrow g = id$ (i.e. Stabiliser$(x) = \{id\}$ $\forall\, x \in X$). We say G acts *properly discontinuously* on X if (1) Stabiliser(x) is finite $\forall x \in X$, and given any $x \in X$, $\exists$ a nbd. U of x with $\begin{cases} gU \cap U = \varphi & \text{for } g \notin \text{Stabiliser}(x) \\ gU = U & \text{for } g \in \text{Stabiliser}(x). \end{cases}$

The Uniformisation Theorem:-

If X is any Riemann surface, since it is semi-locally 1-connected, it has a universal cover $\tilde{X}$ which is simply connected. By the Riemann mapping theorem, $\tilde{X}$ (with the Riemann surface structure induced from X, to which it is locally homeomorphic via $\pi : \hat{X} \to X$) must be $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H}$. Further the group of covering transformations Γ must be a subgroup of $\mathrm{Aut}(\tilde{X})$, which we know from the previous Proposition to be $PGL_2(\mathbb{C})$, $\mathbb{C}$-Affine maps of $\mathbb{C}$, $PSL_2(\mathbb{R})$ respectively. Thus we have

(0.5) Uniformisation Theorem. Every Riemann surface X is the quotient space of $\tilde{X} = \mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H}$ under a discrete subgroup Γ of $PGL_2(\mathbb{C})$, $\mathbb{C}$-Affine maps of $\mathbb{C}$, or $PSL_2(\mathbb{R})$ respectively, acting freely and properly discontinuously on $\tilde{X}$. Further $\Gamma \cong \pi_1(X)$.

Remarks:- In the case of $\tilde{X} = \mathbb{P}^1$, it will turn out by the Riemann-Hurwitz formula (0.11) that Γ must be trivial and $X = \mathbb{P}^1$ as well. (genus $g = 0$)

In the case of $\tilde{X} = \mathbb{C}$, if we require X to be compact, Γ will be forced to be a lattice $\Lambda = \{m + n\tau : m, n \in \mathbb{Z}, \ \mathrm{Im}\,\tau > 0\}$ as in Example 6 of toruses, and X will then be a torus. (genus $g = 1$)

In the case of $\tilde{X} = \mathbb{H}$, if we require X to be compact, Γ will turn out to be the group

$$\frac{\{\text{Free group on } 2g\text{-generators } a_1, b_1, a_2, b_2 \cdots a_g, b_g\}}{\{\text{Normal subgp. generated by } a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2 \cdots a_g b_g a_g^{-1} b_g^{-1}\}}$$

with $g \geq 2$. (genus $g \geq 2$).

All the compact Riemann surfaces fall under the above three cases. This theorem was, again, known in the last century.

The study of Riemann surfaces via the group-theoretic/arithmetic study of Γ above (in the case when $\tilde{X} = \mathbb{H}$) is the deep and vast subject of Fuchsian Groups. For a flavour of this theory we point the reader to Ch IV of [2].

(0.6) Branched coverings, and the Riemann-Hurwitz Formula.

Let X and Y be compact connected Riemann surfaces, and let f be a holomorphic map between them $f : X \to Y$. Let us also assume that f is not the constant map.

It then follows, because holomorphic maps (which are not constant) are open maps, that $f(X)$ is open in Y. Also $f(X)$ is compact, and hence closed in Y. Thus $f(X) = Y$ and f is surjective.

(0.7) Local structure of f.

Let $p \in X$ and $f(p) \in Y$, and let z be a local coordinate around p such that $z = 0$ at p. (for convenience). Using a local coordinate system w around $f(p)$, with $w = 0$ at $f(p)$, we may write f (in terms of these coordinate systems) as

$$f(z) = z^n h(z) \quad , \quad n \geq 1 \quad \text{———} \quad (*)$$

where $h(0) \neq 0$. Now since $h(0) \neq 0$, $h(z) \neq 0$ in a neighborhood U of $p : z = 0$, so that we may take a holomorphic n-th root of h on U, call it $h^{1/n}$. Then clearly by taking $z' = z h^{1/n}$ on U, we see that

$$f(z') = (z')^n \quad \text{on } U, \text{ with } n \geq 1. \quad \text{—} \quad (**)$$

And z' can replace the coordinate z on the neighborhood U. So in terms of suitable coordinates around a point p and its image $f(p)$, f "looks" like $z \mapsto z^n$ for $n \geq 1$. Also note that this number n in $(*)$ is well-defined, and does not depend on the local coordinates. When $n = 1$, we say p is a simple point of f and if $n \geq 2$, we say p is a branch point or ramification point of f. The number n is called the branching order or ramification index of f at p and denoted by $b_f(p)$.

(0.8) Exercise: Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $f(z) = z^2 + z^3$ on $\mathbb{P}^1 - \{\infty\} = \mathbb{C}$ and extended to ∞ in the unique manner. Show that $z = 0$, $z = -2/3$ and $z = \infty$ are the only branch points of f with branching orders 2, 2 and 3 respectively.

(0.9) Defn. - The ramification locus of f is defined to be the divisor

$$B_f = \sum_{p\in X} (b_f(p)-1)\cdot p \quad \text{on } X$$

or its "image"

$$\sum_{p\in X} (b_f(p)-1) f(p) \quad \text{on } Y.$$

We note here, from the local description of f in $(**)$ that branch points are isolated points in X, so form a discrete subset of compact X, so are finite in number. The images of the branch points are also finite in number, therefore.

Now we define another number associated with f, called its degree $\deg f$. For a point $q\in Y$, define the multiplicity of f at q to be

$$\mathrm{mult}_q(f) = \sum_{p\in f^{-1}(q)} b_f(p).$$

(0.10) Claim: If $f: X\to Y$ is a non-constant holomorphic map between compact connected R.S's the number $\mathrm{mult}_q(f)$ is independent of $q\in Y$, and denoted by $(\deg f)$ (the degree of f).

Proof:- First we remark that the fibre of a point $q := f^{-1}(q)$ consists of isolated points and hence always finite since X is compact. (from the local description $(**)$ again). Consider the map now

$$M : Y \longrightarrow \mathbb{Z}_+$$
$$q \longrightarrow \mathrm{mult}_q(f)$$

By the local description of f in $(**)$, it is clear that M is lower semi-continuous. viz. $M^{-1}[k,\infty)$ is open $\forall\ k = 1,2,\ldots$

We claim that $M^{-1}[k,\infty)$ is also closed $\forall\ k=1,2,\ldots$ Let $q\in Y$ be a limit point of a sequence $\{q_i\}_{i=1}^{\infty}\in M^{-1}[k,\infty)$. Since $\{f(p) : b_f(p)\geq 2\}$ is a finite set, we can assume that no q_i is the image of a branch point, viz., $b_f(p)=1\ \forall\ p\in f^{-1}(q_i)$, and $\forall\ i$. Since $q_i\in M^{-1}[k,\infty)$, $\exists$ at least k points $p_{i1},\ldots,p_{ik}\in f^{-1}(q_i)$. Fix $j\in\{1,\ldots,k\}$. Then $\{p_{ij}\}_{i=1}^{\infty}$ contains a convergent subsequence which we denote again by $\{p_{ij}\}_{i=1}^{\infty}$; $p_{ij}\in f^{-1}(q_i)$ with $p_{ij}\to p_j$ say. Clearly, $f(p_j) = \lim_i f(p_{ij}) = \lim_i q_i = q$. Finally, Since $1 \le j \le k$ is arbitrary, we have found p_j, $j = 1, \dots, k$ which lie in $f^{-1}(q) \Rightarrow q \in M^{-1}[k, \infty)$. Which implies $M^{-1}[k, \infty)$ is closed. Since Y is connected, $M^{-1}[k, \infty) = Y$ or ϕ $\forall$ $k \in \mathbb{Z}_+$

Now clearly $M^{-1}[1, \infty) = Y$, and $M^{-1}[r, \infty) = \phi$ for r large enough, so $\exists$ n such that $M^{-1}[n, \infty) = Y$ and $M^{-1}[n+1, \infty) = \phi$. We call this $n = \text{mult}_q(f)$ $\forall$ q, the degree of f.

—. This claim also shows f is a genuine covering map on the complement of the branch locus, which is why called a branched covering. #

(0.11) The Riemann-Hurwitz Formula:- Let $f : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. Let $n = (\deg f)$ as defined above by the Claim (0.10). Then we have the following formulas:

$$\chi(X) = n\,\chi(Y) - \sum_{p \in X} (b_f(p) - 1).$$

Equivalently $$g(X) = n\big(g(Y) - 1\big) + 1 + \frac{1}{2} \sum_{p \in X} (b_f(p) - 1)$$

Proof:- The second formula clearly follows from the first formula by using the relation $\chi(X) = 2 - 2g(X)$. To show the first formula, take a triangulation of Y so that all the images of the branch points are vertices, and a triangulation of X so that all the branch points (of X) are vertices. (If we start with arbitrary triangulation of X & Y, we can refine these triangulations so that this happens.). Also choose these triangulations so that f is simplicial.

Then,
$$\#(2\text{-simplices of } X) = n[\#(2 \text{ simplices of } Y)]$$
$$\#(1\text{-simplices of } X) = n[\#(1 \text{ simplices of } Y)]$$
$$\#(0\text{-simplices of } X) = n[\#(0\text{-simplices of } Y)] - \underbrace{\sum_{p \in X} (b_f(p) - 1)}_{\text{Correction term}}$$

(a branch point p has been counted $b_f(p) - 1$ extra times)

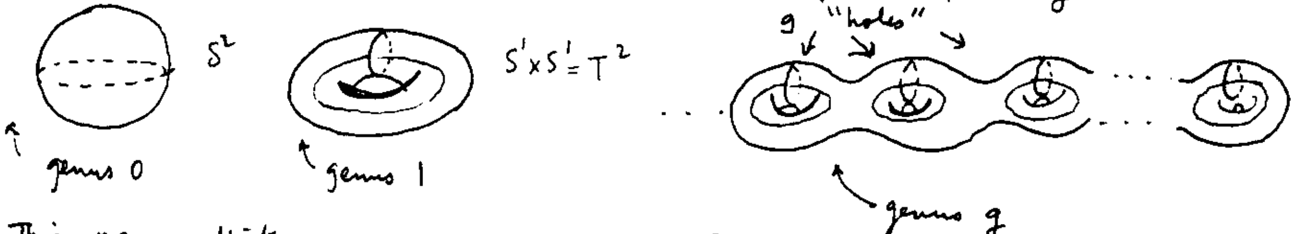
$$\Rightarrow \quad \chi(X) = \sum_{i=0}^{2} (-1)^i (\#\ i\text{-simplices of } X) = n\,\chi(Y) - \sum_{p \in X} (b_f(p) - 1)$$

which proves the formula. #

For our example/Exercise (0.8); $\chi(X) = \chi(Y) = 2$; $n = 3$; $\sum_{p \in X} (b_f(p) - 1) = 1 + 1 + 2 = 4$; clearly $2 = 3 \cdot 2 - 4$

§ 1. Topological Theory of Riemann Surfaces.

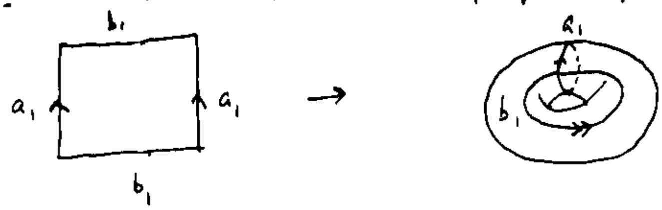
(1.1) Since the coordinate change functions ($\varphi_\alpha \circ \varphi_\beta^{-1}$) of a Riemann surface are holomorphic, their real jacobian determinant is the absolute square of their complex derivative (Cauchy-Riemann equations!), and so these jacobian determinants are all positive. Therefore they are 2nd countable, T_2 orientable, real differentiable manifolds of dimension 2. It was already known in the last century that if we further assume them to be connected and compact, these are diffeomorphic to one of the following:-



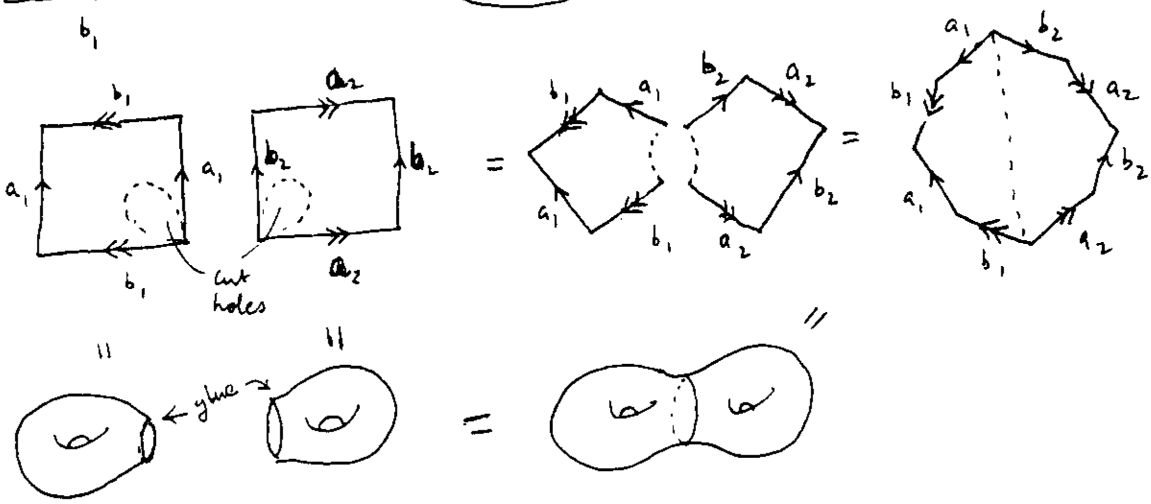
This means that every compact connected Riemann surface is C^∞ diffeomorphic to one of the above list. The proof of this fact is to be found in [5] or [6] (actually only "homeomorphic" is shown there, but with some extra effort "diffeomorphic" is also true). The proof is purely combinatorial, using the triangulability of the surface, which follows, for example by the fact that differentiable manifolds are triangulable. We'll see a proof of triangulability for Riemann surfaces later, using the existence of a non-constant meromorphic function. (which in fact requires the Riemann-Roch Theorem!)

(1.2). Description by gluing the edges of a polygon. X compact, connected Riemann surface. Since S^2 is quite well understood, assume $g \geq 1$. Then topologically

genus $g = 1$:- The torus $S^1 \times S^1$ as a topological space is realised as:

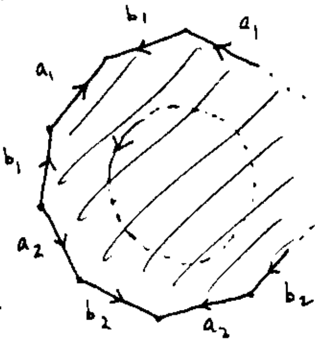


genus $g = 2$



More generally, for genus g :-

genus g.- An easy induction then shows that a genus-g Riemann Surface is topologically nothing but a 4g-sided polygon glued according to the pattern below:- (An a_i edge is glued to the other a_i edge in the direction of the arrow, and so also the b_i, for $i=1,\dots,g$)



Note that all the vertices get identified to a single point.

(1.3) Fundamental group. The fundamental group of S^2 (genus 0) is trivial. From the Van-Kampen theorem, and drilling a disc from the middle of the polygon above, and observing that the remainder has the homotopy type of the (glued) boundary of the polygon – which is a bouquet of 2g-circles with fundamental group equal to the free-product $(\mathbb{Z}_{a_1} * \mathbb{Z}_{b_1} * \mathbb{Z}_{a_2} * \mathbb{Z}_{b_2} * \cdots)$ of 2g copies of $\mathbb{Z} = \pi_1(S^1)$, and the fact that the common circle boundary of the (punched-out) disc and punctured polygon is (see figure above) is equivalent to $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

we see

$$\boxed{\pi_1(\text{Surface of genus } g) = \begin{cases} \text{the group with } 2g \text{ generators } a_1, b_1, \dots a_g b_g \\ \text{modulo the single relation } a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \end{cases}}$$

NB:- From now on we concentrate on $g \geq 1$. $g=0$, the sphere is a "trivial" case, topologically

(1.4) Homology groups: The cell-complex structure from the polygon above gives exactly 1 0-cell (all the vertices get identified to a single point), say p. 2g 1-cells $\{a_i\}\{b_i\}_{i=1\dots g}$ corresponding to the 2g-circles of the bouquet that results from the boundary of the polygon, and 1. 2-cell – i.e. the polygon itself. The boundary maps are all 0, clearly. ($\partial(2\text{cell}) = a_1 + b_1 - a_1 - b_1 + a_2 + b_2 - \cdots = 0$
$\partial(a_i) = \partial(b_i) = p - p = 0$)

Thus the integer homology groups of a surface S_g of genus g are

$$\boxed{H_0(S_g) = H_2(S_g) = \mathbb{Z}, \quad H_1(S_g) = (\oplus_i \mathbb{Z}a_i) \oplus (\oplus_i \mathbb{Z}b_i)}$$

As a corollary the Euler characteristic :- $\chi(S_g) = \sum_i (-1)^i \dim H_i(S_g, \mathbb{Q})$
$= 1 - 2g + 1 = 2 - 2g$.

(The genus g, or equivalently the Euler-characteristic χ is thus a complete topological invariant for connected compact Riemann Surfaces).

(1.5) Cohomology Ring :- By the universal coefficient theorem (or from the cellular cochain complex got from taking the dual of the above cellular chain complex) it is clear that the cohomology groups (with $\mathbb{Z}$-coefficients) are the same as the homology groups. Note that as a complex manifold, there is a preferred orientation, and with respect to that orientation the (polygonal) picture above shows that the intersection numbers of the cycles $\{a_i\}$, $\{b_i\}$ $i=1,\dots,g$ are :-

$$a_i \# a_j = b_i \# b_j = 0 \quad \forall \ i, j$$
$$a_i \# b_j = -b_j \# a_i = \delta_{ij} \quad \forall \ i, j$$

If one denotes the dual cohomology classes to $\{a_i\}$, $\{b_i\}$ as $\{a_i^*\}$, $\{b_i^*\}$ respectively, then the cup-products are

$$a_i^* \cup a_j^* = b_i^* \cup b_j^* = 0 \quad \forall \ i, j$$
$$a_i^* \cup b_j^* = -b_j^* \cup a_i^* = \delta_{ij} \ \forall \ i, j$$

(These facts are consequences of Poincare duality).

(1.6) De-Rham Cohomology, Harmonic Theory, Hodge Theorem.

From the de-Rham Theorem, and the computations of (1.4) above, one sees that for a compact-connected Riemann surface of genus g, the de-Rham cohomology vector spaces are :-

$$H^0(S_g, \mathbb{C}) \cong H^2(S_g, \mathbb{C}) \cong \mathbb{C}. \qquad H^1(S_g, \mathbb{C}) \cong \mathbb{C}^{2g}$$

(we shall always be using complex coefficients for de-Rham cohomology)

As a generator for H^0, one may take the constant 0-form $\equiv 1$ and for a generator of H^2 one may take the normalised volume form $dVol$ (w.r.t. a metric) satisfying $\int_{[S]} dVol = 1$. What about H^1? It turns out that H^1 has a lot of fascinating structure which holds the clue to the precise complex structure on S! Indeed, a Riemann surface of genus $g \geq 1$ has a whole family of distinct complex structures, and a certain algebraic variety called the moduli space $\mathcal{M}_g$ parametrises these complex structures. (More on this later). However, there is a purely algebraic construction (arising out of essentially $H^1(S, \mathbb{Z})$, the cup-product pairing and the "Hodge decomposition" of $H^1(S, \mathbb{C})$ which completely determines the underlying complex structure of S. This is called Torelli's Theorem.. we will discuss it in detail later).

1.7 : The (complex) de-Rham complex

If U is a coordinate chart on the Riemann surface X, with the coordinate chart given by the complex coordinate $z = x+iy : U \to \mathbb{C}$ one makes the following dictionary between the real and complex coordinate systems for the coordinate 1-forms and coordinate vector fields

$$x = \tfrac{1}{2}(z+\bar z) \qquad y = \tfrac{1}{2i}(z-\bar z)$$

$$dz = dx + i\,dy$$

$$d\bar z = dx - i\,dy$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$dz \wedge d\bar z = -2i\, dx\wedge dy$$

Then $\Lambda^0(X,\mathbb{C})$ = vector space $\{ f : X \to \mathbb{C} ;\ f \text{ a } C^\infty\text{-function}\}$

$\Lambda^1(X,\mathbb{C})$ = vector space $\{\omega : \omega \text{ a } C^\infty \text{ 1-form}\}$

A C^∞ 1-form may be represented on the coordinate patch U by

$$\omega_{|U} = p\,dz + q\,d\bar z \quad , \quad p \,\&\, q \ C^\infty \text{ complex valued functions on } U.$$

If w is another coordinate system, the coordinate change $w \to z(w)$ being holomorphic implies $dz = \frac{dz}{dw}dw$, $d\bar z = \overline{\left(\frac{dz}{dw}\right)}d\bar w$ so that ω in the w-coordinate system has the representation $p'\,dw + q'\,d\bar w$ with $p' = p\cdot\frac{dz}{dw}$, $q' = q\,\frac{d\bar z}{d\bar w}$. Thus we may define the complex subspaces of

viz. $\Lambda^{1,0}(X) = \{\omega \in \Lambda^1(X,\mathbb{C}) : \text{(all) local expressions of } \omega \text{ involve only } dz\}$

$\Lambda^{0,1}(X) = \{\omega \in \Lambda^1(X,\mathbb{C}) : \text{(all) local expressions of } \omega \text{ involve only } d\bar z\}$

and clearly $\Lambda^1(X,\mathbb{C}) = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$

Similarly, $\Lambda^2(X,\mathbb{C}) = \Lambda^{1,1}(X) = \{\omega \text{ a } C^\infty \text{ 2-form} : \omega \text{ locally} = f\,dz\wedge d\bar z \text{ with } f \text{ a } C^\infty \text{ fn}\}$

These decompositions of Λ^1 & Λ^2 are called the decompositions into $(p\text{-}q)$-type

By plugging in the identities at the top of the page one sees

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \underbrace{\frac{\partial f}{\partial z}dz}_{\substack{\text{call this } \partial f \\ \text{(of type 1-0)}}} + \underbrace{\frac{\partial f}{\partial \bar z}d\bar z}_{\substack{\text{call this } \bar\partial f \\ \text{(of type 0-1)}}} = \partial f + \bar\partial f$$

Similarly for $\omega = p\,dz + q\,d\bar z$, one immediately checks from the identities on the top that the exterior derivative is precisely

$$d\omega = \left(\frac{\partial q}{\partial z} - \frac{\partial p}{\partial \bar z}\right) dz\wedge d\bar z \ .$$

Again we denote by ∂ the operator taking $p\,dz$ to 0 and $q\,d\bar z$ to $\frac{\partial q}{\partial z}dz\wedge d\bar z$

and by $\bar\partial$ the operator taking $p\,dz$ to $\left(-\frac{\partial p}{\partial \bar z}\,dz\wedge d\bar z\right)$ and $q\,d\bar z$ to 0, we have the following picture of the de-Rham complex

$$\begin{tikzcd}
 & \Lambda^{1,0}(X) \arrow[dr, "\bar\partial"] & \\
\Lambda^0(X,\mathbb{C}) = \Lambda^{0,0}(X) \arrow[ur, "\partial"] \arrow[r, "d"'] \arrow[dr, "\bar\partial"'] & \Lambda^1(X,\mathbb{C}) = \Lambda^{1,0}(X)\oplus\Lambda^{0,1}(X) \arrow[u, "\pi_{1,0}"'] \arrow[d, "\pi_{0,1}"] \arrow[r, "d"'] & \Lambda^2(X,\mathbb{C}) = \Lambda^{1,1}(X) \\
 & \Lambda^{0,1}(X) \arrow[ur, "\partial"'] &
\end{tikzcd}$$

where $\pi_{1,0}$ and $\pi_{0,1}$ are the projections to the summands $\Lambda^{1,0}(X)$ & $\Lambda^{0,1}(X)$ of $\Lambda^1(X,\mathbb{C})$ respectively. From the fact that $d^2=0$, $\partial^2=0$, $\bar\partial^2=0$, it follows that $(\partial\bar\partial+\bar\partial\partial)\equiv 0$ (on $\Lambda^0(X,\mathbb{C})$). The 2-term complex

$$\Lambda^0(X,\mathbb{C}) = \Lambda^{0,0}(X) \xrightarrow{\bar\partial} \Lambda^{0,1}(X)$$

is called the <u>Dolbeault</u> <u>complex</u> of <u>X</u>. It is clear that the 0-th cohomology of this Dolbeault-complex is precisely the vector space of <u>global</u> <u>holomorphic</u> <u>functions</u> on <u>X</u>, i.e. the <u>0-th</u> <u>sheaf</u> <u>cohomology</u> $H^0(X,\mathcal{O})$ where $\mathcal{O}$ is the sheaf of holomorphic fns. on X. If X is compact and connected (Liouville's theorem says that) these are precisely the <u>constants</u> $=\mathbb{C}$. We will see later that $H^{0,1}_{\bar\partial}(X) := \Lambda^{0,1}(X)/\mathrm{Im}\,\bar\partial$ can be interpreted as $H^1(X,\mathcal{O})$, the 1st-sheaf cohomology with coefficients in $\mathcal{O}$. This follows from the <u>Dolbeault</u> <u>Grothendieck</u> (or $\bar\partial$-Poincaré) <u>Lemma</u> (see [3] p. 25), namely that the <u>sequence</u> of <u>sheaves</u> (on the complex mfd.)

$$0 \to \mathcal{O} \to \underline{\underline{\Lambda}}^{0,0} \xrightarrow{\bar\partial} \underline{\underline{\Lambda}}^{0,1} \to 0$$

is <u>exact</u>. From the long-exact sheaf-cohomology sequence we have

$$\begin{tikzcd}
H^0(X,\underline{\underline{\Lambda}}^{0,0}) \arrow[r, "\bar\partial"] \arrow[d, equal, "\text{global sections}"'] & H^0(X,\underline{\underline{\Lambda}}^{0,1}) \arrow[r, "\delta"] \arrow[d, equal] & H^1(X,\mathcal{O}) \arrow[r] & H^1(X,\underline{\underline{\Lambda}}^{0,0}) \\
\Lambda^{0,0}(X) \arrow[r, "\bar\partial"'] & \Lambda^{0,1}(X) & &
\end{tikzcd}$$

The last term is 0 because $\underline{\underline{\Lambda}}^{0,0}$ is a <u>fine</u> <u>sheaf</u> (admits partitions of unity) so the <u>connecting homomorphism</u> δ is surjective implying that

$$\boxed{H^1(X,\mathcal{O}) = \Lambda^{0,1}(X)/\mathrm{Im}\,\bar\partial := H^{0,1}_{\bar\partial}(X)}$$

. By using the same argument applied to the <u>sheaf</u> <u>sequence</u> $0\to\Omega^1\to\underline{\underline{\Lambda}}^{1,0}\xrightarrow{\bar\partial}\underline{\underline{\Lambda}}^{1,1}\to 0$ one shows

$$\boxed{\text{vector space of global holo. 1-forms} = H^0(X,\underline{\Omega}^1) = H^{1,0}_{\bar\partial}(X)}$$

.8) <u>Hermitian metrics</u>: The tangent bundle of a Riemann surface X is a holomorphic complex vector bundle of rank-1 (i.e. each fibre is a complex vector space of dim 1), or a <u>holomorphic line bundle</u>. A <u>hermitian metric</u> on <u>any</u> complex line bundle is a smooth assignment of a positive-definite hermitian inner product on each fibre. More precisely, on a coordinate chart U with coordinate system z, it has the locall expression

$$h(z)\, dz \otimes d\bar{z} \qquad h(z)\ (C^\infty\text{- on } U)$$

which means that for tangent vectors $X = a\frac{\partial}{\partial z}$, $Y = b\frac{\partial}{\partial z}$ at the point $p \in U$, $(X, Y)_p = h(z(p))\, a\bar{b}$.

The positive definiteness forces $h(z) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)$ to be real-valued and everywhere > 0 on U. If we use another coordinate system w, Then $h\, dz \otimes d\bar{z} = h'\, dw \otimes d\bar{w}$ where $h' = \left|\frac{dz}{dw}\right|^2 h$. Such global hermitian metrics can be constructed by first defining them locally on coordinate patches and then gluing them using a <u>C^∞ partition of unity</u>. See [3]. (for example), p. 27.

(1.9) <u>Volume form</u>: The transformation law for h above shows that the 2-form given in local coordinates by $h\, dx \wedge dy = \frac{i}{2} h\, dz \wedge d\bar{z}$ is globally well-defined, and the volume form associated to the (Riemannian metric $h(dx \otimes dx + dy \otimes dy)$ associated to) hermitian metric above. There also results the dual-hermitian metric $\frac{1}{h}\left(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}}\right)$ on the <u>holomorphic cotangent bundle $T^*(X)$</u>.

Note that complex valued 0-forms, 1-forms, 2-forms are C^∞-sections of the vector bundles $\Lambda^0(T^*_{\mathbb{R}}(X) \otimes \mathbb{C})$, $\Lambda^1(T^*_{\mathbb{R}}(X) \otimes \mathbb{C})$, $\Lambda^2(T^*_{\mathbb{R}}(X) \otimes \mathbb{C})$ where $T^*_{\mathbb{R}}(X)$ denotes the <u>real</u> <u>cotangent</u> bundle of the underlying <u>real</u> 2-dim. differentiable manifold. Of course our identities of (1.7) <u>imply</u> that

$$T^*_{\mathbb{R}}(X) \otimes \mathbb{C} \approx T^*(X) \oplus \overline{T^*(X)}$$

as complex vector bundles, where $\overline{T^*(X)}$ is the conjugate of the complex bundle $T^*(X)$. The hermitian metric on $T^*(X)$ naturally defines the hermitian metric on $\overline{T^*(X)}$ (viz. $\frac{1}{h}\left(\frac{\partial}{\partial \bar{z}} \otimes \frac{\partial}{\partial z}\right)$), since $\overline{T^*(X)}$ has the local basis $d\bar{z}$ instead of dz). Thus we have a <u>hermitian metric on</u> $\underline{T^*_{\mathbb{R}}(X) \otimes \mathbb{C}}$ and hence on $\Lambda^i(T^*_{\mathbb{R}}(X) \otimes \mathbb{C})$ for $i = 0, 1, 2$.

Namely, in local coordinates

$(f, f) = |f|^2$

$(\omega, \omega) = \frac{1}{h}(|p|^2 + |q|^2)$ where $\omega = p\, dz + q\, d\bar{z}$ is a 1-form

$(\omega, \omega) = \frac{1}{h^2}|f|^2$ where $\omega = f\, dz \wedge d\bar{z}$ is a 2-form (i.e. 1-1 form)

(1.10). The Hodge-Star operator :-

We define an operator $* : \Lambda^{p,q}(X) \to \Lambda^{1-p,1-q}(X)$ as follows ::

$$\left.\begin{aligned} * : \Lambda^{0,0}(X) &\longrightarrow \Lambda^{1,1}(X) \\ f &\mapsto i\bar{f}.\, h\, dz \wedge d\bar{z} \end{aligned}\right\} \text{ so } * : \Lambda^0(X, \mathbb{C}) \to \Lambda^2(X, \mathbb{C})$$

$$\left.\begin{aligned} * : \Lambda^{0,1}(X) &\longrightarrow \Lambda^{1,0}(X) \\ q\, d\bar{z} &\mapsto -i\bar{q}\, dz \\ * : \Lambda^{1,0}(X) &\longrightarrow \Lambda^{0,1}(X) \\ p\, dz &\mapsto i\bar{p}\, d\bar{z} \end{aligned}\right\} \text{ so } * : \Lambda^1(X, \mathbb{C}) \to \Lambda^1(X, \mathbb{C})$$

$$\begin{aligned} * : \Lambda^{1,1}(X) &\longrightarrow \Lambda^{0,0}(X) \quad \text{so } * : \Lambda^2(X, \mathbb{C}) \to \Lambda^0(X, \mathbb{C}) \\ f\, dz \wedge d\bar{z} &\mapsto \frac{i}{h}\bar{f} \end{aligned}$$

Clearly $(*)$ is conjugate linear, and it is easy to check that these local expressions make global sense. It is also checked directly that for $\omega \in \Lambda^i(X, \mathbb{C})$ $i = 0, 1, 2$ we have

$$\omega \wedge *\omega = 2(\omega, \omega)\, dVol.$$ Indeed $\omega \wedge (*\tau) = 2(\omega, \tau)\, dVol$

Where (ω, ω) is defined in (1.9) above. From the above formulas it also follows that $** = (-1)^j$ on $\Lambda^j(X, \mathbb{C})$. for $j = 0, 1, 2$. Note that the $*$ operator depends on the hermitian metric h.

(1.11) : The Laplacians :- For $i = 0, 1, 2$, $* : \Lambda^i(X, \mathbb{C}) \to \Lambda^{2-i}(X, \mathbb{C})$.

Define the operator

$$\begin{tikzcd} \delta : \Lambda^i(X, \mathbb{C}) \arrow[d, "-*"'] \arrow[r] & \Lambda^{i-1}(X, \mathbb{C}) \\ \Lambda^{2-i}(X, \mathbb{C}) \arrow[r, "d"] & \Lambda^{2-i+1}(X, \mathbb{C}) \arrow[u, "*"'] \end{tikzcd}$$

by $\delta = (-*d*)$

This is the global adjoint of the exterior derivative in the following sense:- If ω and τ are i and $(i+1)$-forms respectively (for $i = 0, 1, 2$). we have

$$d(\omega \wedge *\tau) = d\omega \wedge (*\tau) + (-1)^i \omega \wedge \underbrace{d(*\tau)}_{(2-i)\text{ form}} = d\omega \wedge (*\tau) + \omega \wedge (**d(*\tau)$$

$$= d\omega \wedge (*\tau) - \omega \wedge (*\delta\tau)$$

Since $\partial X = \phi$, $\int_X d(\omega \wedge *\tau) = 0$ so we have the so called Stokes formula

$$\int_X (d\omega, \tau)\, dVol = \int_X d\omega \wedge (*\tau) = \int_X (\omega, \delta\tau)\, dVol.$$

Thus if we define the global ~~inner~~ hermitian inner product on $\Lambda^i(X, \mathbb{C})$ by

$(\omega, \tau)_X = \int_X (\omega, \tau)\, dVol$ we have

$$\boxed{(d\omega, \tau)_X = (\omega, \delta\tau)_X}$$

(Notation: because of this relation, the operator δ is also sometimes written as d^*)

The completion of $\Lambda^i(X, \mathbb{C})$ with respect to $(\ ,\ )_X$ is denoted by $L_2^i(X, \mathbb{C})$, which has the structure of a Hilbert space over $\mathbb{C}$.

The Laplace-Beltrami operator on $\Lambda^i(X, \mathbb{C})$ is defined as

$$\Delta = d\delta + \delta d : \Lambda^i(X, \mathbb{C}) \to \Lambda^i(X, \mathbb{C})$$

Note that $d\delta$ (resp δd) is 0 on $\Lambda^0(X, \mathbb{C})$ (resp. $\Lambda^2(X, \mathbb{C})$). Also

$$(\Delta\omega, \tau)_X = ((d\delta + \delta d)\omega, \tau)_X = (\delta\omega, \delta\tau)_X + (d\omega, d\tau)_X = (\omega, (d\delta + \delta d)\tau)_X = (\omega, \Delta\tau)_X$$

(from the adjointness of d and δ above)

So that Δ is self-adjoint (or "symmetric") with respect to $(\ \ )_X$.

Further since for a C^∞ i-form ω:

$$(\Delta\omega, \omega)_X = (\delta\omega, \delta\omega)_X + (d\omega, d\omega)_X$$

we have $\Delta\omega = 0$ iff $\delta\omega = d\omega = 0$ for $\omega \in \Lambda^i(X, \mathbb{C})$. A form ω with $\delta\omega = 0$ is called a co-closed form and a form $\omega = \delta\tau$ is called coexact). ~~Thus~~ A -form ω is called harmonic if $\Delta\omega = 0$. Thus, i-form is harmonic iff it is closed and coclosed.

Local expressions for the Laplace-Beltrami operator :-

For Example let $f \in \Lambda^0(X, \mathbb{C})$, a C^∞- function. Then $\Delta f = \delta d f$

$$= \delta\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right) = -*d*\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right)$$

$$= -*d\left(i\frac{\partial \bar f}{\partial \bar z}d\bar z - i\frac{\partial \bar f}{\partial z}dz\right) \quad \text{(by defn of $*$ in (1.10))}$$

$$= -*i\left(\frac{\partial^2 \bar f}{\partial z \partial \bar z}dz\wedge d\bar z - \frac{\partial^2 \bar f}{\partial \bar z \partial z}d\bar z\wedge dz\right) = -*\left(2i\frac{\partial^2 \bar f}{\partial z\partial \bar z}dz\wedge d\bar z\right)$$

$$= -\frac{i}{h}\left(-2i\frac{\partial^2 f}{\partial \bar z \partial z}\right) = -\frac{2}{h}\frac{\partial^2 f}{\partial \bar z \partial z} = -\frac{2}{h}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \text{ — (}$$

Thus, $\Delta f = 0$ iff $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ in ~~every~~ local coordinate system $z = x + iy$, and f being harmonic does not depend on the hermitian metric h.

(9)

Similarly for $\omega = f\,dz\wedge d\bar z \in \Lambda^2(X,\mathbb{C})$, one may show that

$$\Delta\omega = -2\frac{\partial^2}{\partial z\,\partial\bar z}\left(f/_h\right) dz\wedge d\bar z \qquad \text{——— (ii)}$$

and the expression for $\omega\in\Lambda^1(X,\mathbb{C})$ is left as an exercise. The formula is, for $\omega = p\,dz + q\,d\bar z$ (locally) in $\Lambda^1(X,\mathbb{C})$

$$\Delta\omega = -2\left[\frac{\partial}{\partial z}\left(\frac{1}{h}\frac{\partial p}{\partial\bar z}\right)dz + \frac{\partial}{\partial\bar z}\left(\frac{1}{h}\frac{\partial q}{\partial z}\right)d\bar z\right] \qquad \text{—— (iii)}$$

In (1.7) we have already defined the operators

$$\bar\partial : \Lambda^0 = \Lambda^{0,0}(X) \longrightarrow \Lambda^{0,1}(X)$$

$$\bar\partial : \Lambda^{1,0}(X) \longrightarrow \Lambda^{1,1}(X)$$

and $\bar\partial = 0$ elsewhere. In the same way as we defined δ as $-*d*$, we can define $\bar\partial^*$ by $(-*\bar\partial*)$. This again, for the same reason as before, is the adjoint to $\bar\partial$ in the sense that

$$(\omega,\ \bar\partial\tau)_X = (\bar\partial^*\omega,\ \tau)_X$$

for ω a $(p+q)$-form of type (p,q) and τ a $(p+q-1)$-form of type $(p,q-1)$. Thus we may form another self-adjoint operator, defined on $\Lambda^i(X,\mathbb{C})$ for $i=0,1,2$ by

$$\Box_{\bar\partial} = \bar\partial\,\bar\partial^* + \bar\partial^*\bar\partial$$

called the <u>Dolbeault Laplacian</u> of X. It is immediate from the definition that $\Box_{\bar\partial} : \Lambda^{p,q}(X)\to\Lambda^{p,q}(X)\quad \forall\ p,q.$

<u>Relation between Δ and $\Box_{\bar\partial}$</u> :-

First note that

$$\bar\partial^*(p\,dz + q\,d\bar z) = -*\bar\partial(i\bar p\,d\bar z - i\bar q\,dz) = i*\bar\partial(\bar p\,d\bar z - \bar q\,dz)$$

$$= i*\left(-\frac{\partial\bar q}{\partial\bar z}d\bar z\wedge dz\right) = i*\left(\frac{\partial\bar q}{\partial\bar z}dz\wedge d\bar z\right) = i\cdot\frac{i}{h}\frac{\partial q}{\partial z} = -\frac{1}{h}\frac{\partial q}{\partial z} \qquad \text{— (iv)}$$

and

$$\bar\partial^*(f\,dz\wedge d\bar z) = -*\bar\partial\left(\frac{i}{h}\bar f\right) = -*\left(\frac{\partial}{\partial\bar z}\left(\frac{i}{h}\bar f\right)d\bar z\right)$$

$$= -\left[(-i)\frac{\partial}{\partial z}\left(-\frac{i}{h}f\right)dz\right] = \frac{\partial}{\partial z}\left(f/_h\right)dz \qquad \text{—— (v)}$$

From (iv) $\Box_{\bar\partial} f = \bar\partial^* \bar\partial f = \bar\partial^*\left(\frac{\partial f}{\partial \bar z} d\bar z\right) = -\frac{1}{h}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar z}\right) = -\frac{1}{h}\frac{\partial^2 f}{\partial \bar z\, \partial z} = \frac{1}{2}\Delta f$

(by (i) above) for $f \in \Lambda^0(X, \mathbb{C})$.

Similarly, from (iv), ~~…~~, for $\omega = p\,dz + q\,d\bar z \in \Lambda^1(X, \mathbb{C})$ we have

$$\Box_{\bar\partial}\omega = \left(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial\right)\omega = \bar\partial\left(-\frac{1}{h}\frac{\partial q}{\partial z}\right) + \bar\partial^*\left(-\frac{\partial p}{\partial \bar z}\, dz \wedge d\bar z\right)$$

(from (v)) $$= -\frac{\partial}{\partial \bar z}\left(\frac{1}{h}\frac{\partial q}{\partial z}\right) d\bar z - \frac{\partial}{\partial z}\left(\frac{1}{h}\frac{\partial p}{\partial \bar z}\right) dz$$

$$= \frac{1}{2}\Delta\omega \quad \text{from (iii) above}$$

Finally, for $\omega = f\, dz \wedge d\bar z$, and (v) above

$$\Box_{\bar\partial}\omega = \bar\partial\bar\partial^*\omega = \bar\partial\left(\frac{\partial}{\partial z}(f/h)\,dz\right) = \frac{\partial^2}{\partial\bar z\,\partial z}(f/h)\, d\bar z \wedge dz$$

$$= -\frac{\partial^2}{\partial z\,\partial \bar z}(f/h)\, dz\wedge d\bar z = \frac{1}{2}\Delta\omega \quad \text{from (ii) above.}$$

Thus we see the basic relation between Dolbeault Laplacian and Laplace-Beltrami:-

$$\boxed{\Box_{\bar\partial} = \frac{1}{2}\Delta}$$

Remark:- This relation is not true for a complex manifold of complex dimension ≥ 2. The condition which ensures it is that the complex Manifold be Kähler, viz if $\sum_{ij} h_{i\bar j}\, dz_i \otimes d\bar z_j$ is the hermitian metric, the associated Kähler form $\omega = \sum h_{i\bar j}\, dz_i \wedge d\bar z_j$ should be a closed (1-1)-form i.e. $d\omega = 0$. On a complex manifold of $\dim_{\mathbb{C}}$ 1 (our present situation) all 2-forms are closed, so every hermitian metric is a Kähler metric (i.e. has closed Kähler form). The above identity for Kähler metrics in general may be found in [3], Ch. 1. A consequence of the above relation is that Δ maps forms of type (p,q) to forms of the same type (p,q) $(p,q = 0 \text{ or } 1)$.

(1.12). The Hodge-Decomposition Theorem:

This section depends on the theory of elliptic operators on compact manifolds, which we shall not have time to explore in detail. A good refrence is [3] or [8]. However we will deduce Hodge-decomposition using the well known Elliptic regularity theorem and Rellich's lemma.

The local expressions derived for $\Box_{\bar{\partial}}$ (and Δ) in the last section (1.11) show that these are elliptic operators on X.

On $\Lambda^{p,q}(X)$ we define another inner product by

$$\langle \omega, \tau \rangle := (\omega, \tau)_X + (\Box\omega, \Box\tau)_X \geq (\omega, \tau)_X \quad \text{——} \quad (i)$$

and complete $\Lambda^{p,q}(X)$ with respect to this (+ve-definite inner product, called the Sobolev-inner product) to get the Sobolev space $W^{p,q}(X)$. The inequality on the right above implies that convergence in $\langle\ \rangle \Rightarrow$ convergence in $(\ )_X$ so that

$$i : W^{p,q}(X) \hookrightarrow L_2^{p,q}(X) \quad \text{——} \quad (ii)$$

[where $L_2^{p,q}(X)$ is the completion of $\Lambda^{p,q}(X)$ with respect to $(\ )_X$] is a bounded (continuous) operator between Hilbert spaces.

Now by (i), it is clear that the operator between Hilbert space

$$\Box : W^{p,q}(X) \to L_2^{p,q}(X) \quad \text{——} \quad (iii)$$

is a bounded operator. (in fact of norm ≤ 1). (The definition of the operator in (iii) follows by first defining it on $\Lambda^{p,q}(X)$ as the Dolbeault-Laplacian, and using the inequality $(\Box\psi, \Box\psi)_X \leq \langle \psi, \psi \rangle$ to pass to the completions)

Let us say that $\psi \in W^{p,q}(X)$ (resp. $L_2^{p,q}(X)$) is weakly harmonic if $\langle \psi, \Box\eta \rangle = 0 \quad \forall \ \eta \in \Lambda^{p,q}(X)$ (resp. $(\psi, \Box\eta)_X = 0 \quad \forall\ \eta \in \Lambda^{p,q}(X)$). By self-adjointness of $\Box$, harmonic forms are weakly harm.

The Elliptic Regularity Theorem for $\Box$:- If $\psi \in W^{p,q}(X)$ (resp $L_2^{p,q}(X)$) is weakly harmonic, then ψ is in fact C^∞ and hence belongs to $\Lambda^{p,q}(X)$. Further ψ is then actually harmonic: $\Box\psi = 0$

Since both $W^{p,q}(X)$, $L_2^{p,q}(X)$ contain $\Lambda^{p,q}(X)$, we may define $\mathcal{H}^{p,q} = N(\Box)$ unambiguously as the space of harmonic ($\Leftrightarrow$ weakly harmonic) (p,q) forms in $W^{p,q}(X)$ or $L_2^{p,q}(X)$.

Let $(\mathcal{H}^{p,q})^\perp$ denote the orthogonal complement of $\mathcal{H}^{p,q}(X)$ in $W^{p,q}(X)$ and $(\mathcal{H}^{p,q})^{perp}$ denote the orthogonal compl. of $\mathcal{H}^{p,q}(X)$ in $L_2^{p,q}(X)$.

Clearly, by definition $\square : (\mathcal{H}^{p,q})^\perp \longrightarrow (\operatorname{Im}\square) \hookrightarrow L_2^{p,q}(X)$ bounded operator, an and $\square : (\mathcal{H}^{p,q})^\perp \to (\operatorname{Im}\square)$ is (1-1)-onto so has a bounded inverse

$$S : (\operatorname{Im}\square) \to (\mathcal{H}^{p,q})^\perp$$

by the open mapping theorem, so that $\boxed{\square S\psi = \psi \ \forall\, \psi \in (\operatorname{Im}\square)}$ —— (iv)

<u>Claim</u>:- $(\operatorname{Im}\square) = (\mathcal{H}^{p,q})^{perp}$ in $L_2^{p,q}(X)$.

Enough to show $(\operatorname{Im}\square)^{perp} = \mathcal{H}^{p,q}$ in $L_2^{p,q}(X)$. But if $\psi \in (\operatorname{Im}\square)^{perp}$, $(\psi, \square\eta)_X = 0 \ \forall\, \eta \in \Lambda^{p,q}(X) \Rightarrow \psi$ is weakly harmonic in $L_2^{p,q}(X) \Rightarrow \psi$ is harmonic $\Rightarrow \psi \in \mathcal{H}^{p,q}$

<u>Proves the claim</u>:

Thus $L_2^{p,q}(X) = \mathcal{H}^{p,q} \oplus (\operatorname{Im}\square)$. Now let's define the so called Green operator

$$G : L_2^{p,q}(X) \longrightarrow L_2^{p,q}(X)$$

by

$$G|\mathcal{H}^{p,q} \equiv 0$$

$$G|(\operatorname{Im}\square) = \text{the composite} \quad (\operatorname{Im}\square) \xrightarrow{S} (\mathcal{H}^{p,q})^\perp \xhookrightarrow{\text{incl.}} W^{p,q} \xhookrightarrow{i} L_2^{p,q}$$

—— (v)

<u>Facts about G</u>:

(a) For $\psi \in L_2^{p,q}(X)$, we have the so-called <u>Kodaira decomposition</u> ($\pi_{\mathcal{H}^{p,q}}$ = projection onto $\mathcal{H}^{p,q}$ of $L_2^{p,q}$)

$$\psi = \pi_{\mathcal{H}^{p,q}}(\psi) + \square G\psi.$$

This follows because $\psi \in L_2^{p,q}(X)$, $\psi - \pi_{\mathcal{H}^{p,q}}(\psi) \in (\mathcal{H}^{p,q})^{perp}$

i.e. $\psi - \pi_{\mathcal{H}^{p,q}}(\psi) \in (\mathrm{Im}\,\square)$ by the Claim above, which implies

$$\square G\psi = \square G\left(\psi - \pi_{\mathcal{H}^{p,q}}(\psi)\right) = {}'\square S\left(\psi - \pi_{\mathcal{H}^{p,q}}(\psi)\right) = \psi - \pi_{\mathcal{H}^{p,q}}(\psi)$$

by relation (iv) above.

Thus $\psi = \pi_{\mathcal{H}^{p,q}}(\psi) + \square G\psi$. proving (a).

(b). G is a compact operator.

This follows because it is enough to show $G|_{(\mathrm{Im}\,\square)}$ is compact, but $G|_{(\mathrm{Im}\,\square)}$ is defined in equation (v) above as the composite of two bounded operators followed by the inclusion $i : W^{p,q} \hookrightarrow L_2^{p,q}$. But a famous theorem called Rellich's Lemma states that this inclusion i is compact; and of course compact operators composed with bounded operators (pre or post-composed in fact) continue to be compact.

(c) G commutes with $*$, $\bar\partial$, $\bar\partial^*$ and $\square$.

It is easy to check directly that $*$ commutes with $\square$, as does $\bar\partial$ and $\bar\partial^*$. Since $G|_{(\mathrm{Im}\,\square)}$ is basically $S = \square^{-1}$, so S also commutes with all these operators. And $G|_{(\mathrm{Im}\,\square)^{\perp}}$ is 0 which certainly commutes with everything. So G commutes with $*$, $\bar\partial$, $\bar\partial^*$.

Now we are ready to prove the Hodge-Decomposition Theorem namely:- Let X be a compact Riemann surface

(i) $\mathcal{H}^{p,q}(X) \cong H^{p,q}_{\bar\partial}(X)$ (the Dolbeault Cohomology) $\forall\ p,q \in \{0,1\}$

(ii) $H^i(X,\mathbb{C}) \cong \mathcal{H}^i(X) \cong \bigoplus_{p+q=i} \mathcal{H}^{p,q}(X) \cong \bigoplus_{p+q=i} H^{p,q}(X)$, $i = 0,1,2$

(iii) $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$ $\forall\ p,q \in \{0,1\}$ such that $p+q=1$

(iv) $\dim_{\mathbb{C}} H^{0,1}(X) = \dim_{\mathbb{C}} H^{1,0}(X) = g$ (Thus the space of holo. 1-forms has $\dim_{\mathbb{C}} = g$).

(Note that here $\mathcal{H}^i(X)$ is defined as $\{\omega \in \Lambda^i(X, \mathbb{C}) : \Delta\omega = 0\}$ where Δ is the Laplace-Beltrami operator.)

<u>Proof</u> of (i) There is clearly a map

$$j : \mathcal{H}^{p,q}(X) \longrightarrow H^{p,q}_{\bar\partial}(X)$$

Since $\Box\omega = 0 \Rightarrow (\bar\partial\omega, \bar\partial\omega)_X + (\bar\partial^*\omega, \bar\partial^*\omega)_X = 0$
$\Rightarrow \bar\partial\omega = 0$ and $\bar\partial^*\omega = 0 \Rightarrow \omega$ gives a class $j(\omega)$ in $H^{p,q}_{\bar\partial}(X)$.

j is <u>injective</u>, for if $j(\omega) = 0$ in $H^{p,q}_{\bar\partial}(X)$, $\omega = \bar\partial\eta$ for a form $\eta \in \Lambda^{p,q-1}(X)$. Also $\omega \in \mathcal{H}^{p,q}$ already says that $\bar\partial^*\omega = 0$ as well so $\bar\partial^*\bar\partial\eta = 0 \Rightarrow (\bar\partial^*\bar\partial\eta, \eta)_X = 0$
$\Rightarrow (\bar\partial\eta, \bar\partial\eta)_X = 0 \Rightarrow \bar\partial\eta = 0 \Rightarrow \omega = \bar\partial\eta = 0$.

j is <u>Surjective</u>
Let $\theta \in H^{p,q}_{\bar\partial}(X)$ represent a Dolbeault cocycle, so that <u>$\bar\partial\theta = 0$</u> is given. Now by the Kodaira decomposition (a) above and also the commutation of G with $\Box, \bar\partial$ via (c) above

$$\theta = \pi_{\mathcal{H}^{p,q}}(\theta) + G(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\theta$$
$$= \pi_{\mathcal{H}^{p,q}}(\theta) + G\bar\partial\bar\partial^*\theta$$
$$= \pi_{\mathcal{H}^{p,q}}(\theta) + \bar\partial G\bar\partial^*\theta$$

$\Rightarrow [\theta] = [\pi_{\mathcal{H}^{p,q}}(\theta)]$ in $H^{p,q}_{\bar\partial}(X)$. On the other hand $\pi_{\mathcal{H}^{p,q}}(\theta)$ by definition $\in \mathcal{H}^{p,q}(X)$, so that $[\theta] = j(\pi_{\mathcal{H}^{p,q}}(\theta))$

Proof of (ii): Since $\Box = \frac{1}{2}\Delta$ and thus Δ maps $\Lambda^{p,q}(X)$ to $\Lambda^{p,q}(X)$, it clearly follows that

$$\mathcal{H}^i(X) = \bigoplus_{p+q=i} \mathcal{H}^{p,q}(X) = \bigoplus_{p+q=i} H^{p,q}(X) \text{ by (i) above.}$$

On the other hand, the Kodaira-decompositions of $\psi \in L_2^{p,q}(X)$ when direct summed over $p+q=i$ give the Kodaira decomposition for $\psi \in L_2^i(X)$ as

$$\psi = \pi_{\mathcal{H}^i}(\psi) + \Box G\psi = \pi_{\mathcal{H}^i}(\psi) + \tfrac{1}{2}\Delta G\psi.$$

Now apply the same proof as in (i), using Δ, d, δ

in place of $\square$ $\bar\partial$, $\bar\partial^*$ to conclude that $H^i(X,\mathbb{C}) \cong \mathcal{H}^i(X)$
$= \{\omega \in \Lambda^i(X,\mathbb{C}) : \Delta\omega = 0\}$.

(i) (resp. (ii)) says that every Dolbeault (resp. de-Rham) cohomology class has a unique harmonic representative.

Proof of (iii) :- Let $p+q=1$, $p,q \in \{0,1\}$

$*$ is a complex antilinear map $\underbrace{\Lambda^{p,q} \to \Lambda^{q,p}}$ (as one sees from its definition in (1.10) and $\square * = * \square$ implies that $*$ gives a complex antilinear map from $\mathcal{H}^{p,q} \to \mathcal{H}^{q,p}$. Since $** = (\pm 1)$, it is an isomorphism which is complex antilinear.

Thus $\bar{*} : \underset{H^{p,q}_{\bar\partial}(X)}{\mathcal{H}^{p,q}} \to \underset{\overline{H^{q,p}_{\bar\partial}}(X)}{\overline{\mathcal{H}^{q,p}}}$ is a (complex) linear isomorphism..

Proof of (iv) :- follows from (ii) and (iii) and the fact that $\dim H^1(X,\mathbb{C}) = 2g$.

(1.13) The Picard Variety

X compact connected Riemann surface. Let $\mathcal{O}^*$ be the sheaf of germs of non-vanishing holomorphic functions on X, and $\mathcal{O}$ the sheaf of germs of holomorphic functions on X, and $\underline{\mathbb{Z}}$ the constant sheaf with stalk $\mathbb{Z}$ at each point.

There is the famous sheaf exact sequence on X, called Exponential sheaf sequence:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow[\exp = e^{2\pi i}]{} \mathcal{O}^* \to 0 .$$

(exp is the map taking the germ f to $e^{2\pi i f}$)

which leads to the long exact sequence of sheaf cohomology

$$\begin{array}{ccccccccccccc}
0 \to & H^0(X,\underline{\mathbb{Z}}) & \to & H^0(X,\mathcal{O}) & \to & H^0(X,\mathcal{O}^*) & \xrightarrow{\delta} & H^1(X,\underline{\mathbb{Z}}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{\delta} H^2(X,\underline{\mathbb{Z}}) \\
& \wr\wr & & \wr\wr & & \| & & \| \\
0 \to & \mathbb{Z} & \hookrightarrow & \mathbb{C} & \xrightarrow{e^{2\pi i}} & \mathbb{C}^* & \xrightarrow{0} & H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{\delta} H^2(X,\mathbb{Z})
\end{array}$$

(The second and third terms are $\mathbb{C}$ and $\mathbb{C}^*$ respectively since H^0 means global holomorphic functions $= \mathbb{C}$, respectively global non-vanishing holo. fns $= \mathbb{C}^*$ since X is compact) we then get the exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow[\delta]{} H^2(X,\mathbb{Z}).$$

$H^1(X,\mathcal{O}^*)$ represents the group (under $\otimes$ product) of isomorphism classes of holomorphic line bundles on X; the identification being $E \mapsto \{g_{ij}\}$ where $g_{ij}: U_i \cap U_j \to \mathbb{C}^*$ are the holomorphic transition functions of E. It is also known that δ is the map which takes a holomorphic line bundle (class) to its 1st Chern class.

Thus $\operatorname{Ker}\delta = \{$isomorphism classes of holomorphic line bundles on X with 1st chern class $0\}$

$$:\stackrel{\text{def}}{=} \operatorname{Pic}^0(X)$$ (the Picard Variety of X

From the above exact sequence

$$\operatorname{Pic}^0(X) \cong H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) = H^{0,1}_{\bar\partial}(X)/H^1(X,\mathbb{Z}).$$

Since by (iv) of Hodge-decomp. in § (1.12) above, $H^{0,1}_{\bar\partial}(X) \cong \mathbb{C}^g$ and $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $\operatorname{Pic}^0(X)$ is $\mathbb{C}^g/(\text{lattice } \Gamma)$, which is (topologically) a g-dimensional torus $S^1 \times S^1 \cdots \times S^1$. It has much further structure, indeed it is what is called an abelian variety. More on this subsequently, in § 3.

(1.14). The Jacobian Variety $\mathcal{J}(X)$ and Torelli's Theorem.

X compact Riemann surface, connected, of genus g throughout.

We already have a basis $\{a_i\}_{i=1}^g, \{b_i\}_{i=1}^g$ of $H_1(X,\mathbb{Z})$ satisfying the intersection relations

$$a_i \# a_j = 0 \quad \forall\, i,j \in \{1,\dots,g\}$$
$$a_i \# b_j = -b_j \# a_i = \delta_{ij} \quad \forall\, i,j \in \{1,\dots,g\}$$

as in § (1.5) above.

§ 2. Divisors, Line Bundles, Riemann-Roch Theorem.

2.1 Divisors + line bundles on compact connected Riemann surface X

Def: A formal integral linear combination

$$D = \sum_{i=1}^{k} n_i p_i$$

where $\{p_i\}_{i=1}^{k}$ are some points in X is called a divisor.

If all the $n_i \geq 0$, then D is called an effective divisor and we denote it by writing $D \geq 0$.

The sum $\sum_{i=1}^{k} n_i$ is defined as the degree $\deg D$ of the divisor.

Def: Let f be a meromorphic function on X. We define the divisor of zeros of f as

$$(f)_0 := \sum_{p \in \{f^{-1}(0)\}} (\mathrm{ord}_p f)\, p.$$

(Note $(\mathrm{ord}_p f) = k$ if f has a zero of order k at p i.e. f is locally of the form $f(z) = z^k h$ for holo. coordinate system z centred at p : $(z(p) = 0)$ and h is indivisible by z.)
Since zeros of meromorphic functions are isolated points and X is compact, the sum above is finite.

Def: Similarly the divisor of poles of f is defined as

$$(f)_\infty = \sum_{p \in \{f^{-1}(\infty)\}} (\text{order of pole of } f \text{ at } p) \cdot p.$$

(Again the sum is finite, and the order of pole p is defined as largest power of $\frac{1}{z}$ in the Laurent Expansion of f around $p := (z = 0)$). We define the divisor of f

Def: as $$(f) := (f)_0 - (f)_\infty.$$

If some divisor D is the divisor of a meromorphic function f, i.e. $D = (f)$, then we say D is a principal divisor

2.2 Propn. The degree of a principal divisor is 0.

Proof: Let $D = \sum_{i=1}^{k} n_i p_i = (f)$. Remove a small open disc D_i around the point p_i (choose D_i so that $D_i \cap D_j = \phi \ \forall i \neq j$, $i, j = 1, \dots, k$)

Then $M - (\cup_{i=1}^{k} D_i)$ is a compact 2-manifold with boundary. $\partial M = \cup_{i=1}^{k} C_i$ where $C_i = \partial D_i$ (oriented compatible with the ~~[illegible]~~ orientation on M induced by the canonical orientation on X). Now consider the 1-form on M defined by

$$\omega = \frac{df}{f}$$

clearly $d\omega = 0$, so by Stokes Theorem

$$\int_{\partial M} \omega = \sum_{i=1}^{k} \int_{C_i} \frac{df}{f} = \int_M d\omega = 0. \qquad \text{— } (*)$$

Now, around the point p_i, if we choose a coordinate z centred at p_i, by definition f has the expansion (on a neighborhood of the the small enough disc D_i)

$$f_{|D_i} = z^{n_i} h_i$$

where $h_i(0) \neq 0$, h_i holomorphic on D_i

$$df = n_i z^{n_i - 1} h_i \, dz + z^{n_i} dh_i$$

$$\frac{df}{f} = n_i \frac{dz}{z} + \frac{dh_i}{h_i}$$

so $$\int_{C_i} \frac{df}{f} = n_i \int_{C_i} \frac{dz}{z} + \int_{C_i} \frac{dh_i}{h_i}$$

The last term is 0 ~~and the~~ (why?) and therefore.

$$\int_{C_i} \frac{df}{f} = (2\pi\sqrt{-1})\, n_i$$

So $0 = \sum_{i=1}^{k} \int_{C_i} \frac{df}{f} = 2\pi i (\deg D) . \Rightarrow (\deg D) = 0$

i.e. The sum of the zeros and poles of f meromorphic counted with correct sign (+ve for zero, −ve for pole) and multiplicity is zero. //

(2.3) Remark :- The converse is not true. Let us elaborate on this briefly. Let us call $\mathrm{Div}(X)$ the abelian gp. of all divisors on X (we just add and subtract formally : if $D = \sum_{i=1}^{k} n_i p_i$ we say $n_i = \mathrm{ord}_{p_i}(D)$, so define $(D+D')$ by $\mathrm{ord}_p(D+D') = \mathrm{ord}_p(D) + \mathrm{ord}_p(D')$)

Since for meromorphic fns f and g on X, $(fg) = (f) + (g)$ the principal divisors form an abelian subgp. of $\mathrm{Div}(X)$. The quotient gp., called the divisor class group of X is defined as. $\mathrm{Div}(X) / (\text{principal divisors})$

(By this, we say that two divisors D_1 + D_2 are equivalent iff D_1 + D_2 represent the same elt. in the divisor class group, i.e. if $D_1 - D_2 = (f)$ for some meromorphic fn. f on X.). All principal divisors correspond to the 0 elt. of divisor class gp.

By what we said in Prop (2.2), the degree map factors as :-

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \\ (\text{Divisor Class gp of } X) \arrow[ur, "(\deg)"'] & \end{tikzcd}$$

Thus $\deg^{-1}(0)$ in the divisor class gp. represents the gp. of equivalence classes of divisors ~~classes~~ with ~~the~~ degree 0. We shall see later that this is in fact isomorphic to $\mathrm{Pic}^0(X)$, which of course by definition, can be nontrivial. (see § 1.13).

Next we investigate the connection between divisors + line bundles, which, in particular will illustrate why the divisor class gp. is $\mathrm{Pic}^0(X)$.

(24) The line bundle associated to a divisor.

Let D be a divisor on X, defined by $D = \sum_{i=1}^{k} n_i p_i$, $n_i \in \mathbb{Z}$. Choose a small open disc D_i around p_i, and define the function $f_i = z^{n_i}$, where z is a local coordinate on D_i centred at p_i. Let $U_0 = X - \{p_1, \dots, p_k\}$, $U_i = D_i$ so that $\{U_i\}_{i=0}^{k}$ is an open cover of X. Define f_0 on U_0 to be $\equiv 1$. Finally, define for $i, j \in \{0, \dots, k\}$

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^*$$

by $g_{ij} = f_i / f_j$

which are clearly holomorphic on $U_i \cap U_j$. (Indeed, if we take $D_1, \dots, D_k$ mutually disjoint, $U_i \cap U_j = \phi$ unless i or $j = 0$, in which case $U_i \cap U_j = U_i \cap U_0 = D_i - \{p_i\}$ and g_{ij} is just $g_{i0} = z^{n_i}$ on the punctured disc D_i). What is important is that $p_i \notin U_j$ for $j \neq i$ to ensure that g_{ij} lands in $\mathbb{C}^*$.
These g_{ij}'s clearly obey the cocycle condition

$$g_{ij}\, g_{jk}\, g_{ki} = 1 \quad \text{on } U_i \cap U_j \cap U_k$$

and this gives a line bundle $[D]$ on X which is holomorphic.
In fact since by definition

$$g_{ij} f_j = f_i \quad \forall\, i, j$$

the $\{f_i's\}$ define a global meromorphic section s of the line bundle $[D]$. Note that if all the n_i's are positive, ~~this~~ section is actually holomorphic.

So we have a correspondence

$$\text{Divisor } D \longrightarrow \left(\begin{array}{c}\text{holo. line bundle } [D] \text{ together} \\ \text{with a meromorphic section}\end{array}\right)$$

It is easy to check that if $D_1 \sim D_2$, i.e. $D_1 = D_2 + (f)$ then $[D_1]$, $[D_2]$ ~~are~~ isomorphic holo. line bundles

If we call this meromorphic section s, then defining the divisor corresponding to s (s) (as we did for a meromorphic function) as $(s)_0 - (s)_\infty$ (which can be easily seen to make sense) one recovers

$$D = (s).$$

In fact, given any holomorphic line bundle, and a meromorphic section s of that bundle, we get a divisor, namely (s)).

(2.5) Remark: If we choose two meromorphic sections s, s' of a holo. line bundle E on X, then using the trivialising charts $\{U_i\}$ of E, we see that if s_j is the meromorphic fn. on U_i defining s

$$s_i = g_{ij} s_j$$

$$s_i' = g_{ij} s_j'$$

by definition of section. Thus $\dfrac{s_i}{s_i'} = \dfrac{s_j}{s_j'}$ $\forall\ i, j$

Thus the family of meromorphic functions $\{s_i/s_i'\}_i$ patch up to give a global meromorphic function f. In other words

$$s = f s'$$

and $(s) = (f) + (s')$ therefore.

So $(s) \sim (s')$. Thus we have a 1-1 correspondence

$$(\text{Divisor class gp of } X) \longleftrightarrow \left\{\begin{array}{l}\text{isomorphism classes of} \\ \text{holo line bdles on } X \\ \text{which have a meromorphic} \\ \text{section}\end{array}\right\}$$

Fact: Every holomorphic line bundle E on a compact Riemann Surface admits a meromorphic section. (See §(2.5) later)
(See Appendix for Proof:-)

Thus $(\text{Divisor class gp. of } X) \longleftrightarrow (\text{isomorphism classes of holomorphic line bundles on } X)$
addition $\equiv$ tensor product

(Remark: If E_1 & E_2 are holo. bundles which are holomorphically isomorphic by bundle map λ, then if s_i are meromorphic sections of E_i, we note that since λ is non zero everywhere, λs_1 & s_2 are both mero. sections of E_2, so $(s_1) - (s_2) = (\lambda s_1) - (s_2) = (f)$ by (2.5) above

$$\Rightarrow (s_1) \sim (s_2)\;)$$

(2.6) Degree and 1st. chern class

Recall the long-exact sheaf cohomology sequence

$$H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow[\delta = c_1]{} H^2(X, \mathbb{Z}) \to \underset{\overset{\shortparallel}{0}}{H^2(X, \mathcal{O})}$$

The connecting map takes the bundle E (or rather the isomorphism class of E) to $c_1(E)$, the 1st <u>chern class</u> of E. If E is defined by the 1-cocycle of transition functions $(g_{ij})_{i,j}$ then $c_1(E) = \delta(\{g_{ij}\})$ is defined (by the Čech Coboundary formula) as

$$n_{ijk} : U_i \cap U_j \cap U_k \to \frac{1}{2\pi i}\left(\log g_{ij} + \log g_{jk} + \log g_{ki}\right)\Big|_{U_i \cap U_j \cap U_k} \quad \text{— } (*)$$

which is a Čech <u>2-cocycle</u> $\{n_{ijk}\}$

We would like to find a <u>de-Rham representative</u> for this class in $H^2(X, \mathbb{C})$. (Actually, the representative we will find will be in $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})$). We recall the coordinate trivialisations of $[D]$ we constructed in § (2.4), and the <u>meromorphic</u> section s satisfying (with respect to that trivialisation)

$$f_i = s|_{U_i} = z^{n_i} \quad \text{on } U_i = D_i \text{ for } i = 1, \ldots, k.$$

$$f_0 = s|_{U_0} \equiv 1 \quad \text{on } U_0 = X - \{p_1, \ldots, p_k\}.$$

Let us choose a hermitian metric h on $[D]$. This means that we a collection of positive C^∞- functions $\{h_i\}_{i=0,\ldots,k}$ with a transformation law

$$h_i = |g_{ij}|^2 h_j \quad \text{on } U_i \cap U_j \quad \text{— } (**)$$

Let us also choose it so that $h_0 \equiv 1$ on a <u>neighborhood</u> of $X - \bigcup_{i=1}^k D_i$, (e.g. $X - \bigcup_{i=1}^k (\frac{1}{2} D_i)$).

<u>Claim</u>:- The 1-1 form defined by

$$\omega = \frac{1}{2\pi i} \bar{\partial}\partial \log h_i \quad \text{on } \begin{array}{l} U_i = D_i \text{ for } i = 1, \ldots, k \\ = X - \bigcup_1^k \{p_i\} \text{ for } i = 0 \end{array}$$

is a globally defined C^∞-form.

Proof: It is C^∞ by definition. On $U_i \cap U_j$,

$$\log h_i = \log |g_{ij}|^2 + \log h_j \quad \text{by } (**)$$

But g_{ij} being holomorphic on $U_i \cap U_j$,

$$\bar\partial \partial \log |g_{ij}|^2 = -\partial \bar\partial \log |g_{ij}|^2 = \tfrac{1}{4}\Delta (\log |g_{ij}|^2) = 0$$

So

$$\frac{1}{2\pi i}\, \bar\partial \partial \log h_i = \frac{1}{2\pi i}\, \bar\partial \partial \log h_j \ \text{ on } U_i \cap U_j.$$

Claim 2 :- ω represents the (Cech)-cohomology class $\{\eta_{ijk}\}$ defined in $(*)$ above in de-Rham cohomology.

Proof :- First consider the sheaf-exact sequence

$$0 \to \underline{\Omega}^1 \to \underline{\Lambda}^{1,0} \xrightarrow{\bar\partial} \underline{\Lambda}^{1,1} \to 0$$

leading to

$$\cdots \to H^0(X, \Lambda^{1,1}) \xrightarrow[\delta]{} H^1(X, \underline{\Omega}^1) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(X, \underline{\Lambda}^{1,0})}$$

(The last term is zero since $\underline{\Lambda}^{1,0}$ is a fine sheaf)

By chasing the definition of the Cech-connecting map,

$$(\delta\omega)_{ij} = \frac{1}{2\pi i}\left(\partial \log h_i - \partial \log h_j\right) = \frac{1}{2\pi i}\, \partial(\log g_{ij}) \quad \text{by } **$$

(By the computation of Claim 1, $\bar\partial\, \delta\omega_{ij} = 0$ so $\delta\omega_{ij}$ is a Cech 1-cocycle in $H^1(X, \underline{\Omega}^1)$.)

Now look at the short-exact sheaf sequence

$$0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{\partial} \underline{\Omega}^1 \to 0.$$

and the bit of the long-exact sequence

$$H^1(X, \mathcal{O}) \to H^1(X, \underline{\Omega}^1) \xrightarrow{\delta} H^2(X, \mathbb{C})$$

again,

$$\delta\left(\{(\delta\omega)_{ij}\}\right)_{ijk} = \frac{1}{2\pi i}\left(\log g_{ij} + \log g_{jk} + \log g_{ki}\right)\Big|_{U_i \cap U_j \cap U_k}$$

which is precisely the Cech-cocycle $[\eta_{ijk}]$, proving Claim 2.

(2.7) Prop:-
$$\int_{[X]} \omega = \frac{1}{2\pi i}\int_{[X]} \bar{\partial}\partial(\log h_i) = (\deg D).$$

Proof:- Since $\log h_0 \equiv 0$ on a neighborhood of $X - \bigcup_{i=1}^{k} D_i$,

$$\int_{[X]} \omega = \frac{1}{2\pi i}\sum_{i=1}^{k}\int_{D_i} \bar{\partial}\partial(\log h_i)$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{k}\int_{D_i} d(\partial(\log h_i)).$$

$$= \frac{1}{2\pi i}\sum_{i=1}^{k}\int_{\partial D_i} \partial(\log h_i).$$

But on ∂D_i, $h_0 \equiv 1$, $\forall i$, so $\log h_i = \log\left(\frac{h_i}{h_0}\right)$ on ∂D_i so $\log h_i = \log|g_{i0}|^2 = \log g_{i0} + \log \bar{g}_{i0}$ by (**)

so $\partial(\log h_i) = \partial(\log g_{i0})$ since g_{i0} is holo.

But $g_{i0} = \frac{f_i}{f_0} = z^{n_i}$ on ∂D_i.

So $$\int_{[X]} \omega = \frac{1}{2\pi i}\sum_{i=1}^{k}\int_{\partial D_i} \frac{n_i\, dz}{z} = \frac{1}{2\pi i}\sum_{i=1}^{k} 2\pi i\, n_i$$

$$= \sum_{i=1}^{k} n_i = \deg D.$$

Proving the proposition.

(2.8). Corollary:- The abelian gp of divisor classes whose degree is zero is isomorphic to $Pic^0(X)$.

Proof:- By (2.4), Remark (2.5) and Prop (2.7) we have the following commutative diagram:-

$$\begin{tikzcd}
{[D]} & \left\{\begin{array}{l}\text{Isomorphism classes of holo. line} \\ \text{bundles on } X\end{array}\right\} \arrow[dd, phantom, "\cong" rotate=90] \arrow[dr, "1^{st}\text{ chern class}"] & \\
 & & H^2(X, \mathbb{Z}) \\
D \arrow[uu] & \{\text{Divisor Class gp of } X\} \arrow[ur, "\text{degree}"'] &
\end{tikzcd}$$

which shows $Pic^0(X) \cong \{$divisors classes with degree $0\}$. #.

All this may seem like so much jargon! Let's deduce a nice, classical corollary!

(2.9) Corollary:- Let $\{p_i\}_{i=1,..k}$ be any collection of points on the Riemann sphere $\mathbb{P}^1$, and n_i be integers such that $\sum n_i = 0$. Then there exists a meromorphic function on $\mathbb{P}^1$, say f, such that

$$(f) = \sum_{i=1}^{k} n_i p_i$$

(i.e. f has zeros/poles exactly at p_i of order $n_i/-n_i$ depending on $n_i > 0$ / $n_i < 0$.).

Proof:- $H^{1,0}(\mathbb{P}^1) = H^{0,1}(\mathbb{P}^1) = 0$ since $H^1(\mathbb{P}^1) = 0$ so $Pic^0(\mathbb{P}^1) = 0$, so every degree zero divisor, in particular $D = \sum_{i=1}^{k} n_i p_i$, is a principal divisor #.

Actually, for holomorphic line bundles (resp. divisors) on $\mathbb{P}^1$, the 1^{st} chern class (resp. degree) completely determines the isomorphism class (resp. equivalence class). The reason is that $H^1(X, \mathcal{O}) \cong H^{0,1}(X) = 0 = H^2(X, \mathcal{O})$ for $X = \mathbb{P}^1$. Thus from the sheaf exact sequence

$$\underset{\overset{\parallel}{0}}{H^1(\mathbb{P}^1, \mathcal{O})} \to H^1(\mathbb{P}^1, \mathcal{O}^*) \xrightarrow[c_1]{} \underset{\overset{\cong}{\mathbb{Z}}}{H^2(\mathbb{P}^1, \mathbb{Z})} \to \underset{\overset{\parallel}{0}}{H^2(\mathbb{P}^1, \mathcal{O})}$$

one gets {iso. classes of holo line bundles on $\mathbb{P}^1$} $= H^1(\mathbb{P}^1, \mathcal{O}^*) \simeq \mathbb{Z}$, via the 1st chern class. Similarly for divisors using the commutative diagram of Cor (2.8) above. #

(2.10) <u>Exercise</u>:- Show that the <u>canonical line bundle</u> is the generator of $H^1(\mathbb{P}^1, \mathcal{O}^*)$ by computing its 1st chern class. (Use the obvious hermitian metric induced from the trivial 2-plane bundle on $\mathbb{P}^1$ and compute $\frac{1}{2\pi i}\partial\bar\partial \log h$: and show it integrates to 1 over $\mathbb{P}^1$).

(2.11) <u>The sheaf $\mathcal{O}(E)$</u>

Let E be a holomorphic line bundle on a compact Riemann Surface X. We define

$$\mathcal{O}(E) = \text{the sheaf (of germs) of holomorphic sections of } E.$$

So $\mathcal{O}(E)$ is actually just a generalisation of the sheaf $\mathcal{O}$, and is infact a sheaf of modules over the sheaf of rings $\mathcal{O}$. It is also a subsheaf of $\underline{C^\infty}(E) = \underline{\Lambda}^0(E)$ = the sheaf of C^∞ sections of E. One can also form various other sheaves of forms "with E coefficients"; viz. $\underline{\Lambda}^{p,q}(E) = \underline{\Lambda}^{p,q} \otimes_{\underline{\Lambda}^0} \underline{\Lambda}^0(E)$

$$\underline{\Omega}^p(E) = \underline{\Omega}^p \otimes_{\mathcal{O}} \mathcal{O}(E)$$

$$\underline{\Lambda}^i(E) = \underline{\Lambda}^i \otimes_{\underline{\Lambda}^0} \underline{\Lambda}^0(E)$$

etc. Since E is a <u>holomorphic</u> line bundle, the map

$$\bar\partial : \underline{\Lambda}^{p,q}(E) \to \underline{\Lambda}^{p,q+1}(E)$$

continues to make sense. (the operator d however, in general, <u>doesn't</u> make sense, unless E has a "flat connection"). By giving a hermitian metric on E, one can again define the $*$-operator (wherever we had complex conjugate, we need the <u>hermitian dual</u> in the definition in § (1.10)). So $*: \Lambda^{p,q}(X, E) \longrightarrow \Lambda^{1-p,1-q}(X, E^*)$

(2.12) Th<u>e Hodge Theorem</u> for <u>Twisted Dolbeault Cohomology</u> :-

Since by (2.11) we have $*$ and $\bar\partial$ for the "twisted Dolbeault Complex" $\Lambda^{p,q}(X,E)$, one can form the twisted Dolbeault Laplacian $\Box_E = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$; and ask whether the twisted Dolbeault-chomology $H^{p,q}_{\bar\partial}(X,E)$ is represented by harmonic forms etc. In analogy with the Hodge Theorem of § (1.12), we have the following theorems

(i) $\mathcal{H}^{p,q}(X,E) \cong H^{p,q}_{\bar\partial}(X,E)$ where $\mathcal{H}^{p,q}_{\bar\partial}(X,E) = \operatorname{Ker}\Box_E : \Lambda^{p,q}(X,E) \to \Lambda^{p,q}(X,E)$

(ii) $H^{p,q}_{\bar\partial}(X,E) \cong H^q(X, \underline{\Omega}^p(E))$ $\qquad \forall\ p, q \in \{0,1\}$.

Since the $*$ operator again commutes with $\Box_{\bar\partial}$ and sets up an isomorphism

(iii) $$*: \mathcal{H}^{p,q}(X,E) \simeq \mathcal{H}^{1-p,1-q}(X,E) \qquad p,q \in \{0,1\} — ($$

as before (§ 1.12,). The interesting case is when $p+q=1$.

In this case $*: \mathcal{H}^{p,q}(X,E) \cong \mathcal{H}^{1-p,1-q}(X,E^*)$

Now since

$$(\quad): \mathcal{H}^{p,q}(X,E) \otimes \mathcal{H}^{p,q}(X,E^*) \to \mathbb{C}.$$

$$\omega \otimes \tau \mapsto (\omega,\tau)_X = \int \omega \wedge *\tau$$

is a nondegenerate pairing, one sees from the above isomorphism (iii) the

(2.13) <u>Kodaira-Serre Duality Theorem</u> : The pairing

$$\begin{aligned} (\quad): \mathcal{H}^{p,q}(X,E) &\otimes \mathcal{H}^{1-p,1-q}(X,E^*) \to \mathbb{C}. \\ \| \qquad\qquad & \qquad\qquad \| \\ H^q(X,\underline{\Omega}^p(E)) &\otimes H^{1-q}(X,\underline{\Omega}^{1-p}(E^*)) \to \mathbb{C}. \\ \alpha &\otimes \beta \mapsto \int_{[X]} \alpha\wedge\beta. \end{aligned}$$

is nondegenerate, for a compact Riemann surface X.

Thus

$$H^1(X, \Omega^0(E)) = H^1(X, \mathcal{O}(E)) \cong (H^0(X, \Omega^1(E^*)))^*$$

$$\cong (H^0(X, \Omega^1 \otimes \mathcal{O}(E^*))$$

The sheaf Ω^1 = sheaf of sections of the holomorphic cotangent bundle $(T'(X))^*$ is often called the <u>canonical sheaf</u> of X. , and denoted k_X. The corresponding divisor is called the <u>canonical divisor</u>. $\underline{K}_X$ of X

(2.14) <u>Exercise</u> :- Using the fact that $c_1(T'(X))$ = Euler class of $T'(X)$ considered as a 2-plane real bundle.

show that the degree of the canonical divisor of X is $2g-2$ where g = genus X.

(2.15) <u>The Arithmetic Genus</u> :: For X a Riemann Surface, the arithmetic genus

$$\chi(X, E) := \sum_{0 \le i \le \dim_{\mathbb{C}} X} (-1)^i H^i(X, \mathcal{O}(E)) = \dim H^0(X, \mathcal{O}(E)) - \dim H^1(X, \mathcal{O}(E))$$

$$= \dim H^0(X, \mathcal{O}(E)) - \dim H^0(X, k_X \otimes_{\mathcal{O}} \mathcal{O}(E^*))$$

by the Kodaira-Serre Duality Theorem (2.13) above. For Example, if E is the trivial bundle, $= X \times \mathbb{C}$, we have

$$\chi(X, X \times \mathbb{C}) = \dim H^0(X, \mathcal{O}) - \dim H^1(X, \mathcal{O})$$
$$= \dim H^0(X) - \dim H^{0,1}_{\bar\partial}(X) = 1 - g.$$

This is a particular case of the Riemann-Roch formula (for the trivial bundle, or equivalently for a <u>principal</u> divisor).

Before we generalise to an arbitrary line bundle (or resp. divisor) let us interpret $H^0(X, \mathcal{O}(E))$ and $H^0(X, k_X \otimes \mathcal{O}(E^*))$

$$= H^1(X, \mathcal{O}(E))$$
$$= H^0(X, \Omega^1(E^*))$$

Suppose $E = [D]$ where $D = \sum_{i=1}^{k} n_i p_i$ is a divisor on X. Recall the meromorphic section $\underline{s}$ we constructed for $[D]$ in § (2.4). Consider the space $\mathcal{L}(D)$ of global meromorphic functions on X having a pole (resp. zero) of at most order n_i (resp. zero of order at least $(-n_i)$) for $n_i > 0$ (resp $n_i < 0$). at p_i. This is clearly a vector space. Now define the maps, which are clearly inverses of each other, viz

$$\mathcal{L}(D) \longrightarrow H^0(X, \mathcal{O}([D])) = \text{space of global holo. sections of } [D].$$

$$h \mapsto h.s.$$

$$t s^{-1} \longleftarrow t$$

Note that $h\,s$ on U_i is represented by $hf_i = hz^{n_i}$ for $i = 1, \ldots, k$. If $n_i > 0$, since h has a pole of order $\leq n_i$ at p_i, hz^{n_i} is holomorphic on U_i. Similarly, if $n_i < 0$, since h has a zero of order at least $(-n_i)$ at p_i, hz^{n_i} is again holomorphic. Also since

$$hf_i = g_{ij}(hf_j)$$

the (hf_i)'s do patch up to give a holomorphic section of $[D]$. The shorthand way of writing it is that $D + (f) \geq 0$.

So for a divisor D, we have identified $H^0(X, \mathcal{O}([D]))$ with a certain space of global meromorphic functions on X with zeros / poles at p_i of orders $\geq (-n_i) / \leq n_i$ for $n_i < 0 / > 0$

Now one interprets $H^0(X, \Omega^1 \otimes \mathcal{O}(E^*)) = H^0(X, \Omega^1(E^*))$.

$$= H^0(X, k_X \otimes \mathcal{O}(E^*))$$

clearly $k_X \otimes_{\mathcal{O}} \mathcal{O}(E^*) \cong \mathcal{O}([K_X]) \otimes_{\mathcal{O}} \mathcal{O}(E^*) = \mathcal{O}([K_X - E])$ where $[K_X - E]$ is the bundle representing the divisor $K_X - E$.

So $H^0(X, \mathcal{O}([K_X - E]))$ has an interpretation similar to the one we gave above for $H^0(X, \mathcal{O}(E))$, as a space of global meromorphic functions with certain order of poles and zeros. This is not very illuminating, however, since we dont know what K_X really looks like. If we regard it as $H^0(X, \Omega^1(E^*))$, we note that a global section of $\underline{\Omega}^1(E^*)$ is nothing but a collection of holomorphic 1-forms on U_i, say ω_i satisfying

$$\omega_i = g_{ij}^{-1} \omega_j \quad \text{on} \quad U_i \cap U_j$$

(g_{ij}^{-1} are the transition functions of E^*) since $g_{ij} = \frac{f_i}{f_j} = \frac{z^{n_i}}{z^{n_j}}$ when $E = [D]$, $E^* = [-D]$, and we get $z^{n_i} \omega_i = z^{n_j} \omega_j$ on $U_i \cap U_j$

So $z^{n_i}\omega_i$ patch up to give a Global meromorphic 1-form ω on X. Also since ω_i are holomorphic, ω has a zero of order at least n_i (for $n_i > 0$), and a pole of order at most $(-n_i)$ (for $n_i < 0$) at p_i. Conversely, given such a global meromorphic 1-form, one can cook up a global section of $\underline{\Omega}^1([-D])$. (Notice the reversal of roles of $n_i > 0$, $n_i < 0$ in comparison with $H^0(X, \mathcal{O}([D]))$. To sum up, the space of holomorphic sections of $[D]$, resp. $\Omega^1 \otimes [-D]$ correspond to spaces of meromorphic functions, resp meromorphic 1-forms with poles or zeros at p_i of orders prescribed by the coefficients n_i's as bounds. i.e. $D + (f) \geq 0$ respectively $(-D) + (\omega) \geq 0$

The arithmetic genus thus captures information about global meromorphic functions and meromorphic 1-forms having "prescribed zeros and poles" governed by the divisor in question. If one can find a convenient formula by which one can calculate the arithmetic genus, a lot of information about the Existence of global meromorphic functions or 1-forms would immediately follow. This is precisely the

(2.16) – Riemann-Roch Theorem Let $D = \sum_{i=1}^{k} n_i p_i$ be a divisor (actually, divisor class is all that matters). Then

$$\chi(X, [D]) = \dim H^0(X, \mathcal{O}([D]) - \dim H^0(X, \mathcal{O}([K_X - D]))$$
$$= \deg D - g + 1.$$

Proof:- By induction! we already know it for the divisor $D = 0$. (or any principal divisor for that matter, since the corresponding line bundle is trivial.)

Now suppose one knows the theorem for D. It is enough to prove it inductively for $D+p$ or $D-p$ where p is an arbitrary point $p \in X$. First let us do it for $D+p$. Note, by (2.15)

$$\mathcal{O}([D]) = \{\text{sheaf of germs of meromorphic functions } f \text{ on } X \text{ such that } D + (f) \geq 0.\}$$

$$\mathcal{O}([D+p]) = \{\text{sheaf of germs of meromorphic functions such that } D + p + (f) \geq 0\}.$$

Clearly we have a sheaf exact sequence

$$0 \to \mathcal{O}([D]) \longrightarrow \mathcal{O}([D+p]) \to \underline{\mathbb{C}}_p \to 0.$$

where $\underline{\mathbb{C}}_p$ is the skyscraper sheaf concentrated at p. Suppose $p \notin D$. Since f a germ at $p \in \mathcal{O}([D+p])_p$ means f has at most a simple pole at p, the residue at p is a complex number, and if it is 0, f is a holomorphic germ at p and comes from $\mathcal{O}([D])_p$. Similarly, if $p \in D$, then $\mathrm{Res}(z^{\mathrm{ord}_p(D)} f)$ is a complex number, and if it is zero, f comes from $\mathcal{O}([D])_p$. This leads to the long exact cohomology sequence

$$0 \to H^0(X, \mathcal{O}([D])) \to H^0(X, \mathcal{O}[D+p]) \to H^0(X, \underline{\mathbb{C}}_p) \to H^1(X, \mathcal{O}[D]) \to H^1(X, \mathcal{O}[D]+p) \to H^1(X, \underline{\mathbb{C}}_p) \to 0$$

Claim: $H^0(X, \underline{\mathbb{C}}_p) \cong \mathbb{C}$. $\quad H^1(X, \underline{\mathbb{C}}_p) = 0$.

Since $\mathbb{C}_p(U) = \{0\}$ if $p \notin U$ $\qquad \underline{\mathbb{C}}_p(X) = H^0(X, \underline{\mathbb{C}}_p) = \mathbb{C}$.
$\qquad = \mathbb{C}$ if $p \in U$

To show that $H^1(X, \underline{\mathbb{C}}_p) = 0$, let $\mathfrak{U} = \{U_i\}_{i\in\Lambda}$ be an arbitrary open covering of X. Choose some U_0 say such that $p \in U_0$.

Define $V_i = U_i - \{p\}$ for $i \neq 0$
$\qquad = U_0$ for $i = 0$.

Clearly $\{V_i\}_{i\in\Lambda}$ is a refinement of the covering $\{U_i\}_{i\in\Lambda}$. On the other hand, $V_i \cap V_j$ __always__ excludes p for $i \neq j$. Thus $g_{ij} \in \underline{\mathbb{C}}_p(V_i \cap V_j) \equiv 0$ for __any__ 1-cocycle $\{g_{ij}\}$. Thus

$\varinjlim_{\substack{\text{open Coverings} \\ \mathfrak{U} \text{ of } X}} H^1(\mathfrak{U}, \underline{\mathbb{C}}_p) = 0 \Rightarrow H^1(X, \underline{\mathbb{C}}_p) = 0$. __Proves__ __Claim__.

Now in any exact sequence of vector space the alternating sum of the dimensions are 0.

so $\dim H^0(X, \mathcal{O}([D])) - \dim H^0(X, \mathcal{O}([D+p])) + 1$
$\qquad - \dim H^1(X, \mathcal{O}([D])) + \dim H^1(X, \mathcal{O}([D]+p)) = 0$

$$\Rightarrow \chi(X, [D]) - \chi(X, [D+p]) + 1 = 0.$$

$$\Rightarrow \chi(X, [D+p]) = \chi(X, [D]) + 1 = (\deg D) - g + 1 + 1.$$
$$= \deg(D+p) - g + 1.$$

$\Rightarrow$ inductive step.

Similarly, for $[D-p]$ we have the same argument to show $\chi(X, [D-p+p]) = \chi(X, [D-p]) + 1$.

$$\text{So } \chi(X, [D-p]) = \chi(X, [D]) - 1$$
$$= (\deg D) - g + 1 - 1$$
$$= \deg(D-p) - g + 1.$$

which proves the Riemann-Roch theorem. #.

(2.17) Line Bundles Version of Riemann-Roch : By the §'s (2.5) and (2.6) setting up the correspondence between isomorphism classes of holomorphic line bundles E and equivalence classes of divisors, and the identification between the 1st chern class of E and the degree of the corresponding divisor, we see that for a holomorphic line bundle E

$$\dim H^0(X, \mathcal{O}(E)) - \dim H^1(X, \mathcal{O}(E)) = \int_{[X]} c_1(E) - g + 1.$$

Note that the right hand side is a topological invariant of E and X, so the arithmetic genus $\chi(X, E)$ defined above is a topological invariant. This is a remarkable fact, because $H^0(X, \mathcal{O}(E))$ and $H^1(X, \mathcal{O}(E))$ are not topological invariants.

(2.18). Corollary :. Every Riemann surface admits non-constant meromorphic functions ($\equiv$ holomorphic maps to $\mathbb{P}^1$ which are non-constant).

Proof:. choose some divisor D with $\deg D - g + 1 > 1$. Then $\dim H^0(X, \mathcal{O}([D]))$, by the Riemann-Roch formula is strictly greater than 1. Since constants are a space of dim 1, we certainly have non-constant meromorphic fns f (satisfying $D + (f) \geq 0$).

Note that this makes every Riemann surface a branched covering of $\mathbb{P}^1$. Near a branch point $p \in \mathbb{P}^1$ if we choose holomorphic coordinates w around $f(p)$ and z around p (centred at those points) suitably, one can make the local expression for f look like (see § (0.7) on branched coverings)

$$w = z^n$$

where n is the branching order $b_f(p)$ of f at p. Then clearly one can triangulate a neighborhoods of p, $f(p)$ in such a way that f is simplicial on these neighborhoods. On the complement of these neighborhoods of branch points, f is an honest covering map and so a triangulation of $\mathbb{P}^1 - \bigcup\{\text{neighborhoods of all branch points}\} = A$, say, can be lifted to $f^{-1}(A)$. Thus <u>every</u> <u>Riemann surface can be triangulated</u>; a claim we made in $\S(1\cdot1)$.

(2.19). <u>Every Riemann surface of genus $g=0$ is biholomorphic to $\mathbb{P}^1$.</u>

Proof :- Take $D = p$, a simple point on X. Then $\deg D = 1$, so since $g = 0$,

$$\dim H^0(X, \mathcal{O}([p])) \geq 2.$$

So there exists a meromorphic function on X with a pole of order ≤ 1 on X. However, a pole of order 0 means a global holomorphic function, i.e. a constant. Since these constants constitute a subspace of dimension 1 of $H^0(X, \mathcal{O}([p]))$, there <u>do</u> exist meromorphic functions on X with a <u>simple</u> <u>pole</u> at p. Regarding such a function f as a map (holomorphic) to $\mathbb{P}^1$, the multiplicity $\mathrm{mult}_\infty f$ over $\infty \in \mathbb{P}^1$ is <u>one</u> (<u>simple pole</u>).* see footnote So we have a <u>1-1 mapping</u> $f: X \to \mathbb{P}^1$. Since non-constant holomorphic maps are open maps, the image is open + closed, so all of $\mathbb{P}^1$. So f is a bijective holo. map of X to $\mathbb{P}^1$. Since the sheet number is 1 everywhere, one checks (using local holo. coordinates) that f' is everywhere non-vanishing, so that f is a biholomorphism $X \longrightarrow \mathbb{P}^1$.

<u>Note</u>: From genus $g \geq 1$ onwards, we don't have such a result. In fact from $g \geq 1$, a Riemann surface of genus g will admit <u>whole</u> <u>families</u> of distinct complex structures, no two of which are biholomorphic. (phenomenon of "moduli")

* <u>Footnote</u> :- By claim (0.10) in § 0, $\mathrm{mult}_q(f)$ is <u>constant</u>, thus $\mathrm{mult}_\infty(f) = 1 \Rightarrow \mathrm{mult}_q(f) = 1 \quad \forall q \in \mathbb{P}^1$.

(2.20) Projective embeddings of a Reimann surface X.

Defn:- If D is a divisor on X, we define the complete linear system associated to D, denoted by $|D|$ as the projective space

$$|D| := \mathbb{P}(H^0(X, \mathcal{O}([D])))$$

Since $H^0(X, \mathcal{O}([D]))$ consists of the space of holomorphic sections of $[D]$, and by the Remarks (2.4), (2.5), if we let s be the meromorphic section of $[D]$ constructed there (with $D = (s)_0 - (s)_\infty = (s)$)

then any other holomorphic section t of $[D]$ satisfies

$$\begin{aligned}(t) &= (s) + (f)\\ &= D + (f) \iff (t) \sim D\end{aligned}$$

where f is a global meromorphic function. Also since t is holo, (t) is just the divisor of zeros of t, and hence an effective divisor. Thus $|D|$ can also be thought of as the space of effective divisors equivalent to D (we take the projective space because scaling a section t by a non-zero scalar makes no difference to the divisor of zeros $= (t)$ of that section).

For well-chosen line bundles L, this projective space will admit a projective embedding of X into it.

(2.21). A vanishing theorem :- if $\deg D < 0$, $\dim H^0(X, \mathcal{O}[D]) = 0$.

Proof: If an effective divisor is equivalent to D, its degree has to be ≥ 0 and also $=$ the degree of D < 0. #.

(2.22). Another vanishing theorem :- If $\deg D > \deg K_X = 2g-2$, then

$$H^1(X, \mathcal{O}[D]) = 0.$$

Proof :- Kodaira Serre duality $\Rightarrow$ (see §(2.13))

$$H^1(X, \mathcal{O}[D]) \cong H^0(X, \mathcal{O}[K_X - D]) = 0$$

when $\deg(K_X - D) = \deg K_X - \deg D < 0$ by (2.21) above.

(2.23). If $(\deg D) > 2g-2$, $\dim H^0(X, \mathcal{O}[D]) > g-1$ and $\dim |D| \geq g-1$.

Proof: By (2.22) $\dim H^0(X, \mathcal{O}[K_X - D]) = \dim H^1(X, \mathcal{O}[D]) = 0$ and the Riemann-Roch formula (2.16) $\Rightarrow$

$$\dim H^0(X, \mathcal{O}[D]) - 0 = \deg D - g + 1 > 2g-2-g+1 = g-1.$$

Conclusion: If E is a line bundle on X with $\int_X c_1(E) > 2g-2$, the space of holomorphic sections $H^0(X, \mathcal{O}(E))$ of E is of dimension at least g. In fact, using the same argument as above, if $\int_X c_1(E) > 2g-2+k$ $(k>0)$, then $\dim H^0(X, \mathcal{O}(E)) \geq g+k$. So a sufficiently positive line bundle admits sufficiently many holomorphic sections.

Let E, in the rest of this section, be a line bundle with $c_1(E)$ satisfying $\int_X c_1(E) > (2g-2)+2$; and X be a Riemann Surface of genus $g \geq 1$. By the above, $\dim H^0(X, \mathcal{O}(E)) \geq g+k$

Let $N \stackrel{\text{def.}}{=} \dim\left(\mathbb{P}\left(H^0(X, \mathcal{O}(E)\right)\right) \geq g+k-1 \geq 2$.

Let $s_0, \ldots, s_N$ denote a basis for the space of holo. sections $H^0(X, \mathcal{O}(E))$. Consider the map

$$i_E : X \to \mathbb{P}^N$$
$$p \to [s_0(p) : s_1(p) : \cdots : s_N(p)].$$

This will make sense provided we ensure that there does not exist a pt $p \in X$ such that $s_0(p), \ldots, s_N(p)$ are all 0. But if this happens, all sections in $H^0(X, \mathcal{O}(E))$ vanish at p. (In such a case we say that the complete linear system corresponding to E has base point p.) Let us exclude this possibility by using the above conditions on E. Consider the line bundle $E \otimes [-p]$ where $[-p]$ is the line bundle corresponding to the divisor $[-p]$ of degree (-1).

If $E = [D]$, then $E \otimes [-p] = [D-p]$. Using the sheaf cohomology sequence of § (2·16) for the sheaf exact sequence:

$$0 \to \mathcal{O}(E\otimes[-p]) \to \mathcal{O}(E) \xrightarrow{\text{Eval}_p} \mathbb{C}_p \to 0$$

namely

$$\cdots \to H^0(X, \mathcal{O}(E)) \xrightarrow{\text{Eval}_p} H^0(X, \mathbb{C}_p) \to H^1(X, \mathcal{O}(E\otimes[-p])) \to \cdots$$
$$H^0(X,\mathbb{C}_p) \cong \mathbb{C}_p .$$

If we ensure that $\int_X c_1(E\otimes(-p)) = \int_X c_1(E) - 1 > 2g-2$

then by the cohomology vanishing theorem (2.22), $H^1(X, \mathcal{O}(E\otimes[-p])) = 0$ so that the map Eval_p is surjective, and so $s(p)$ cannot $= 0$ $\forall$ $s \in H^0(X, \mathcal{O}(E))$.

So we need $\int_X c_1(E) > (2g-2)+1$; but we have already chosen E to satisfy $\int_X c_1(E) > (2g-2)+2$, so the map i_E makes sense. It is clearly a holomorphic map.

To make it 1-1, we need to check that for points $p \neq q$, $[s_0(p): \ldots : s_N(p)] \neq [s_0(q): \ \ : s_N(q)]$. Suppose $[s_0(p): \ldots s_N(p)] = [s_0(q): \ \ s_N(q)]$. This means that $\exists$ a non-zero scaler λ such that $\text{Eval}_p(s) = \lambda(\text{Eval}_q(s))$ for every $s \in H^0(X, \mathcal{O}(E))$. Again consider

$$0 \to \mathcal{O}(E\otimes[-p]\otimes[-q]) \to \mathcal{O}(E) \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0 .$$

leading to
$$H^0(X, \mathcal{O}(E)) \xrightarrow{\text{Eval}_p + \text{Eval}_q} H^0(\mathbb{C}_p \oplus \mathbb{C}_q) \to H^1(X, \mathcal{O}(E\otimes[-p]\otimes[-q])) \to \cdots$$

If we make $\int_X c_1(E) - 2 > 2g-2$

i.e. $\int_X c_1(E) > (2g-2)+2$

Then $(\mathrm{Eval}_p + \mathrm{Eval}_q)$ is surjective on $\mathbb{C}_p \oplus \mathbb{C}_q$ so that $\mathrm{Eval}_p(s) \neq \lambda\, \mathrm{Eval}_q(s)$ for some $\lambda \neq 0$ and some $s \in H^0(X, \mathcal{O}(E))$

But we have already assumed
$$\int_X c_1(E) > (2g-2) + 2$$

Thus the map i_E is injective.

Now one looks at the derivative of i_E. Again, for any p in X one has the exact sequence:

$$0 \to \mathcal{O}(E \otimes [-2p]) \to \mathcal{O}(E) \xrightarrow[(\mathrm{Eval}_p \oplus d_p)]{} \mathbb{C}_p \oplus \mathbb{C}_p \to 0.$$

where d_p denotes the derivative at p. Again if we ensure that $H^1(\mathcal{O}, E \otimes [-2p]) = 0$ by taking $\int_X c_1(E) > (2g-2) + 2$ we will be guaranteed that

$$H^0(X, \mathcal{O}(E)) \xrightarrow[\mathrm{Eval}_p \oplus d_p]{} \mathbb{C}_p \oplus \mathbb{C}_p \to 0.$$

and so there will exist sections with derivative at $p \neq 0$ $\forall\, p \in X$. Thus the derivatives of all the basis sections cannot simultaneously vanish at any p.

The conclusion is therefore :-

(2.24) <u>Kodaira Embedding Theorem</u> : Let E be a line bundle (holomorphic) on X, such that genus $X = g \geq 1$ and

$$\deg(E) \stackrel{\text{def}}{=} \int_X c_1(E) > (2g-2) + 2 = \deg K_X + 2$$

Then $$i_E : X \to \mathbb{P}(H^0(X, \mathcal{O}(E)))$$

is a projective embedding of X into projective space of dim $= (g-1) + \deg(E) - (\deg K_X)$

$$= (\deg E) - g + 1.$$

(2.25) : Corollary :- Every Riemann surface is a smooth projective variety. (Chow's Theorem $\Rightarrow$ analytic subvariety of $\mathbb{P}^N$ is an algebraic subvariety)

§ 3. The Jacobian Variety and the Abel-Jacobi Theorem

X: compact connected Riemann surface of genus g

Recall the basis for $H_1(X,\mathbb{Z})$ from § 1.4. $\{a_i\}_{i=1}^{g}$, $\{b_i\}_{i=1}^{g}$ satisfying the intersection relations

$$a_i \# a_j = b_i \# b_j = 0 \quad \forall\ i,j \in \{1,\dots,g\}$$
$$a_i \# b_j = -b_j \# a_i = \delta_{ij} \quad \forall\ i,j \in \{1,\dots,g\}$$

For convenience, let us denote this basis by the letters $\{\delta_i\}_{i=1}^{2g}$

where, $\delta_i = a_i$ for $1 \le i \le g$ (called the "A-cycles")

$\delta_{i+g} = b_i$ for $1 \le i \le g$ (called the "B-cycles")

and fix this basis $\{\delta_i\}_{i=1}^{2g}$ in the subsequent discussion

Now fix a basis $\{\omega_i\}_{i=1}^{g}$ (of holomorphic 1-forms) of $H^0(X,\underline{\Omega}^1)$ $= H^{1,0}(X)$.

3.1 The Period Matrix is defined as the $g \times 2g$ matrix.

$$\Omega = \begin{bmatrix} \int_{\delta_1}\omega_1 & \cdots & \int_{\delta_{2g}}\omega_1 \\ \vdots & & \vdots \\ \int_{\delta_1}\omega_g & \cdots & \int_{\delta_{2g}}\omega_g \end{bmatrix}$$

Each column is denoted by the symbol

$$\Pi_i = \begin{pmatrix} \int_{\delta_i}\omega_1 \\ \vdots \\ \int_{\delta_i}\omega_g \end{pmatrix}$$ and is a vector in $\mathbb{C}^g$

it is called a period. For $1 \le i \le g$, called an A-period. For $i+1 \le g \le 2g$, B-period

3.2 Proposition :- The periods Π_i, $i=1,\dots,2g$ are $\mathbb{R}$-linearly independent

Proof :- If $\sum_{i=1}^{2g} k_i \Pi_i = 0$ for some $k_i \in \mathbb{R}$, then for every $j \in \{1,\dots,g\}$

$$\sum_{i=1}^{2g} k_i \int_{\delta_i}\omega_j = 0 \quad \Rightarrow \quad \int_{\sum_{i=1}^{2g} k_i\delta_i} \omega_j = 0 \qquad \forall\ j \in \{1,\dots,g\}$$

But $\sum_{i=1}^{2g} k_i \int_{\delta_i} \omega_j = 0 \Rightarrow$ (by conjugating) $\sum_{i=1}^{2g} k_i \int_{\delta_i} \overline{\omega}_j = 0$

as well so
$$\int_{\sum_{i=1}^{2g} k_i\delta_i} \overline{\omega}_j = 0 \quad \forall \ j \in \{1, \ldots, g\}$$

but $\{\omega_j, \overline{\omega}_j\}_{j=1}^{g}$ form a basis for $H^1(X, \mathbb{C}) = H^{0,1}(X) \oplus H^{1,0}(X)$

So
$$\int_{\sum_{i=1}^{2g} k_i\delta_i} \alpha = 0 \quad \forall \quad \alpha \in H^1(X, \mathbb{C}) \Rightarrow \sum_{i=1}^{2g} k_i\delta_i = 0$$

$\Rightarrow k_i = 0$ since δ_i also form a basis of $H_1(X, \mathbb{C})$, and $H^1(X, \mathbb{C})$ and $H_1(X, \mathbb{C})$ are dually paired by $\int_\sigma \alpha$. #

Consider the integral lattice generated by the $2g$ periods $\{\Pi_i\}_{i=1}^{2g}$ in $\mathbb{C}^g$ called the period lattice

$$\Lambda = \{ m_1\Pi_1 + \ldots m_{2g}\Pi_{2g} \ : m_i \in \mathbb{Z} \} \subset \mathbb{C}^g$$

3.3 The Jacobian Variety (of X) $\mathcal{J}(X)$ is defined as $\mathbb{C}^g/\Lambda$

Pick a base point $p_0 \in X$. we define the map

$$\mu : X \longrightarrow \mathcal{J}(X)$$
$$p \longmapsto \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g \right)$$

where by $\int_{p_0}^{p}$ we mean $\int_c \omega_1$ where c is any path joining p_0 to p_1 i.e. $c(0) = p_0$, $c(1) = p_1$. If we choose a different path c' then $\int_c \omega_i - \int_{c'} \omega_i = \int_{c-c'} \omega_i = \int_\rho \omega_i$ where $\rho \in H_1(X, \mathbb{Z})$

$$\Rightarrow \rho = \sum_{j=1}^{2g} n_j \delta_j \Rightarrow \int_\rho \omega_i = \sum_{i=1}^{2g} n_i \int_{\delta_j} \omega_i \quad \text{so that}$$

$$\left(\int_c \omega_1, \ldots, \int_c \omega_g\right) - \left(\int_{c'} \omega_1, \ldots, \int_{c'} \omega_g\right) = \sum_{i=1}^{2g} n_i \Pi_i \in \Lambda \quad \text{so the map}$$

μ going into $\mathbb{C}^g/\Lambda$ is well-defined independent of paths.

This map μ actually extends to the so called Abel-Jacobi map as follows:

Recall the group of divisors $\mathrm{Div}(X)$ defined in (2.3). Let us denote the subgroup of <u>divisors</u> of <u>degree 0</u> as <u>$\mathrm{Div}^0(X)$</u> (which is just the inverse image of $\mathrm{Pic}^0(X)$ defined in (1.13) under the natural quotient map taking divisors to divisor classes)

Thus if $D \in \mathrm{Div}^0(X)$, we may write $D = \sum_\lambda p_\lambda - \sum_\lambda q_\lambda$ where $p_\lambda, q_\lambda \in X$ and are not necessarily distinct for distinct λ.

Define now:

<u>3.4</u> <u>The Abel-Jacobi Mapping</u>

$$\mu : \mathrm{Div}^0(X) \longrightarrow \mathcal{J}(X)$$

$$D = \textstyle\sum_\lambda p_\lambda - \sum_\lambda q_\lambda \longmapsto \left(\sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_1, \ldots, \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_g \right)$$

Again, the paths chosen from q_λ to p_λ don't matter as in the discussion of 3.3, because a different choice of path will alter $\mu(D)$ by an element of Λ, so $\mu(D)$ is well defined in $\mathcal{J}(X)$

Now we can state the

<u>3.5</u> : <u>Abel-Jacobi Theorem</u> :-

(i) $\mu(D) = 0$ in $\mathcal{J}(X)$ iff D is a principal divisor, in other words, there is a factorisation

$$\begin{tikzcd} \mathrm{Div}^0(X) \arrow[r, "\mu"] \arrow[d, "\text{quotient map}"'] & \mathcal{J}(X) \\ \mathrm{Pic}^0(X) \arrow[ur, "\tilde{\mu}"'] & \end{tikzcd}$$

<u>with $\tilde{\mu}$ injective</u>

(ii) (Jacobi-inversion theorem) μ (and hence $\tilde{\mu}$) is surjective.

References

[1] CLEMENS, C. H., Scrapbook of Complex Curve Theory, Plenum Publ.

[2] FARKAS, H., and KRA, I., Riemann Surfaces, Springer G.T.M.

[3] GRIFFITHS, P., and HARRIS, J., Principles of Algebraic Geometry, Wiley.

[4] HARTSHORNE, R., Algebraic Geometry. Springer G.T.M.

[5] MASSEY, W., Introduction to Algebraic Topology. Springer GTM

[6] MOISE, E. E., Geometric Topology in Dimensions 2 and 3.

[7] SPRINGER, G., Riemann Surfaces, Chelsea.

[8] GILKEY, P., Heat Equation, Invariance Theory and the Index Theorem, Publish or Perish.

APPENDIX:

Propn: Let E be a holo. line bundle on a cpt. Riemann Surface X. Then E has a meromorphic section.

Proof: If $p \in X$, denote by $E(p)$ the holo. line bundle $E \otimes \mathcal{L}(p)$ where $\mathcal{L}(p)$ is the line bundle associated to the divisor p. Which has a holo. section s_p with a zero of order 1 at p.

The Exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}(p) \to \mathbb{C}_p \to 0$, on tensoring with E leads to $0 \to \mathcal{O}(E) \to \mathcal{O}(E(p)) \to \mathbb{C}_p \to 0$. In general if we denote by $E(np)$ the tensor product $E \otimes \underbrace{\mathcal{L}(p) \otimes \mathcal{L}(p) \otimes \cdots}_{n \text{ times}}$ $= E \otimes (\mathcal{L}(p))^{\otimes n}$, we have

$$0 \to \mathcal{O}(E(np)) \to \mathcal{O}(E((n+1)p)) \to \mathbb{C}_p \to 0.$$

Take the long exact Coh. sequence of this sequence of sheaves to obtain that $H^1(X, \mathcal{O}(E(np))) \twoheadrightarrow H^1(X, \mathcal{O}(E((n+1)p))$ $\forall n$. (Surjects.)

So that we have surjections $H^1(X, \mathcal{O}(E)) \twoheadrightarrow H^1(X, \mathcal{O}(E(p))) \twoheadrightarrow \cdots \twoheadrightarrow H^1(X, \mathcal{O}(n+1)p$

Since by Hodge Thm: $H^1(X, \mathcal{O}(E)) = H^{0,1}_{\bar\partial}(X, E)$ is f. dim, when n gets large enough, the surjective map above becomes an isomorphism.
$\Rightarrow H^1(X, \mathcal{O}(E(np)) \cong H^1(X, \mathcal{O}(E((n+1)p))$ for some n. $\Rightarrow$ by same l. Exact sequence that $\dim H^0(X, \mathcal{O}(E((n+1)p)) = \dim H^0(X, \mathcal{O}(E(np)) + 1 > 0$.

$\Rightarrow$ $\mathcal{O}(E((n+1)p)$ has a global section t, Then $(s_p^{-n-1} t)$ is a meromorphic section of E Since $E((n+1)p) = E \otimes (\mathcal{L}(p))^{\otimes(n+1)}$

$$\Rightarrow \quad E = E((n+1)p) \otimes (\mathcal{L}(p)^{-1})^{\otimes(n+1)}$$

#