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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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INTERNATIONAL CENTRE FOR SCIENCE AND HIGH TECHNOLOGY
c/o INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS 34100 TRIESTE (ITALY) VIA GRIGNANO, 9 (ADRIATICO PALACE) P.O. BOX 586 TELEPHONE 040-224572 TELEFAX 040-224575 TELEX 460449 APH I

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"Risk Analysis of Crop Allocation Strategies"

P. Racsko
Dept. of Biomathematics & Computer Science
University of Veterinary Science
1400 Budapest
Hungary

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RISK ANALYSIS OF CROP ALLOCATION STRATEGIES

Peter Racsko
Hungary

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Abstract

In most crop production systems the output depends on the actual weather conditions. That implies the randomness of the yield and the risk of the production. Several management practices are applied to reduce risk. In this paper the risk reduction is achieved by a good crop sort allocation policy. A full analysis and optimal strategy is given for two sorts with different - random - productivity.

Introduction

The problem of allocation of sowing areas for different crop sorts arises in several cases, from allocation of sowing areas for genetic varieties to the distribution of land between different crop sorts in a given year's production strategy. See Hammond (1974), Falkoner (1952), Jinks (1982), Pesek (1974), Dragovcev (1984).

Selection criteria between two allocation strategies might be sevaral. The production quantities are always random variables and to compare random variables is not straigth-forward. Beyond that one can find that greater expected production is generally associated with greater variance.

Usually the evaluation of two strategies is performed either by the comparison of the two expected values with an upper bound set for the two variances or by the comparison of the two variances, a lower bound set for the expected values. Sometimes a linear combination of the two parameters is used for the analysis (Harnos, 1987.).

In this paper we describe a multicriterial optimization procedure for the allocation problem, based on risk analysis. Time series of the Hungarian crop production are used for model identification and evaluation.

1. Productivity as a stochastic process

Let us denote the area allocated to two crop sorts by S_1 and S_2, $S_1 + S_2 = S$. In year t the per hectare production is represented by the dynamic equations

$$Y_t^1 = L_t^1 + \xi_t^1 \qquad (kg/ha)$$
$$Y_t^2 = L_t^2 + \xi_t^2 \qquad (kg/ha) \qquad (1)$$

where Y_t^1 and Y_t^2 are the per hectare production for the two crops. We assumed that the crop productivity can be represented by the sum of a deterministic factor L_t^1, and L_t^2, and a stochastic process, ξ_t^1 and ξ_t^2.

The analysis of the wheat and maize production figures in Hungary from 1951 shows a t pical logistic growth, we could easily fit logistic curves of the following form to the data by a maximum likelihood algorithm:

$$L_t = C + \frac{e^{\alpha t}}{\left(\frac{1}{Y_0}\right) + \left(\frac{1}{K}\right)\left(e^{\alpha t} - 1\right)} \quad . \qquad (2)$$

Further analysis showed that the stochastic processes ξ_t^1 and ξ_t^2 are not necessarily stationary, in most cases they have increasing variances.

Thus, the following model was identified for maize and wheat productivity:

$$\xi_t^1 = g^1(t)\eta^1$$
$$\xi_t^2 = g^2(t)\eta^2 \qquad (3)$$

where ξ_t^1 stands for maize, ξ_t^2 for wheat.

Regression analysis of the variances showed that the following models give a good fit to our data:

$$\xi_t^1 = c\eta^1$$
$$\xi_t^2 = (a+bt)^{1/2}\eta^2 = \lambda_t\eta^2 \qquad (4)$$

where a, b, and c are constants, η^1, and η^2 are normally distributed random variables with zero mean and unit variance.

2. Risk analysis

The conception of risk estimation applied in the further analysis is close to that introduced by Wald (1945) - it is done by the probability distribution of the events that the damage caused by the difference of the expected and actual values does not exceed a given quantity. We have a zero cost of the decision if the actual productivity is greater than the expected value, and a positive cost if the actual one is less than the expected. The risk will be associated to the probability distribution of the positive decision costs.

Let us denote the covariance matrix of η^1 and η^2 by $\underline{\underline{\tau}}$ with the elements $\tau_{ij}(i,j=1,2)$ and suppose the utility of each crop linearly depends on the quantity with the utility coefficients u_1 and u_2.

Let $\tilde{\tau}_{11} = c^2u_1^2\tau_{11}$; $\tilde{\tau}_{12} = \tilde{\tau}_{21} = c\lambda_t u_1u_2\tau_{12}$; $\tilde{\tau}_{22} = \lambda_t^2u_2^2\tau_{22}$ be in year t.

Let the total utility of the two crops in year t at the allocation strategy (S_1,S_2) be $\phi_t(S_1S_2)$. ϕ is a random variable with the expected value

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$$U(S_1,S_2) = E(\phi_t(S_1S_2)) = L_t^1 S_1 u_1 + L_t^2 S_2 u_2 \tag{5}$$

and variance

$$\sigma(S_1S_2) = \mathrm{Var}(\phi_t(S_1S_2)) = \sum_{i=1}^{2}\sum_{j=1}^{2} \tilde{\tau}_{ij} S_i S_j \tag{6}$$

The optimality criterion $\max\limits_{S_1S_2} E(\phi_t(S_1S_2))$ gives us the following trivial optimal allocation:

$$\begin{aligned} &S_1 = S,\ S_2 = 0 \quad \text{if} \quad L_t^1 u_1 > L_t^2 u_2 \\ &S_1 = 0,\ S_2 = S \quad \text{otherwise.} \end{aligned} \tag{7}$$

It is also easy to get the optimal allocation for the optimality criterion $\min\limits_{S_1S_2} \sigma(\phi_t(S_1S_2))$. Without loss of generality we set $S = 1$, thus $S_2 = 1 - S_1$. Now

$$\sigma(S_1S_2) = \sigma(S_1) = \tilde{\tau}_{11} S_1^2 + 2\tilde{\tau}_{12} S_1(1 - S_1) + \tilde{\tau}_{22}(1 - S_1)^2$$

and

$$S_1^{opt} = \frac{\tilde{\tau}_{22} - \tilde{\tau}_{12}}{\tilde{\tau}_{11} + \tilde{\tau}_{22} - 2\tilde{\tau}_{12}}; \qquad S_2^{opt} = 1 - S_1^{opt} \tag{8}$$

τ_{12}, τ_{12}, and τ_{22} are the elements of a covariance matrix, $\tau_{11}\cdot\tau_{22} \geq \tau_{12}^2$. From this relation and from $(\sqrt{\tau_{11}}\, u_1 c - \sqrt{\tau_{22}}\, u_2\lambda)^2 > 0$ it is obvious that the divisor in (8) is positive. Now, if $\tilde{\tau}_{22} < \tilde{\tau}_{12}$, $S_1^{opt} < 0$ and the minimum of $\sigma(S_1)$ is reached in $\tilde{S}_1^{opt} = 0$. If $\tilde{\tau}_{11} < \tilde{\tau}_{12}$, then $\tilde{S}_1^{opt} = 1$. If $\tilde{\tau}_{12} < \min(\tilde{\tau}_{11}, \tilde{\tau}_{12})$ then $0 < S_1^{opt} < 1$. The last case means a low positive or negative correlation between η^1 and η^2.

If $\tilde{\tau}_{11} = \tilde{\tau}_{22}$, or $\dfrac{\tau_{11}}{\tau_{22}} = \dfrac{\lambda_t^2 u_2^2}{c^2 u_1^2}$ in year t, $S_1^{opt} = S_2^{opt} = \frac{1}{2}$, and minimum variance is obtained at equal allocation. This last result is somewhat peculiar for the common sense, at equal variances we should prefer the crop with higher utility.

It has been shown, that both optimality criteria do not meet the empirical "common sense" requirements to the optimal allocation problem.

Now we shall introduce a new, risk-type criterion. Let us denote the loss of utility in year t by $F(S_1 S_2)$:

$$F(S_1 S_2) = (Y_t^1 - L_t^1)u_1 S_1 + (Y_t^2 - L_t^1)u_2 S_2 = \xi_t^1 u_1 + \xi_t^2 u_2$$

$$= \eta_t^1 c u_1 + \eta_t^2 \lambda_t u_2 \qquad (8)$$

The expected value $E(F(S_1, S_2)) = 0$ and the variance is $\sigma(S_1 S_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} \tilde{\tau}_{ij} S_i S_j$ for any pair S_1, S_2. The random variables η_t^1 and η_t^2 have standard normal distribution. $F(S_1 S_2)$ is also distributed normally with zero mean and variance $\sigma(S_1, S_2)$. For a fixed pair S_1 and S_2 let us denote $F = F(S_1, S_2)$; $\sigma(S_1, S_2) = \sigma$.

The probability of the event, that the loss of utility will not be greater then F^*:

$$P(F \leq F^*) = \phi(F^* / \sqrt{\sigma}) \qquad (10)$$

where $\Phi(x)$ is the probability integral:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

Having derived the distribution of the loss of utility we can define a significance level (e.g. $\Phi^* = 0.95$ or 95% significance) and find the associated F^* loss of utility. It is easy to find in the table of the Gauss integrals that $\Phi(1.65) \approx 0.95$, thus the event $F \leq 1.65\sqrt{\sigma}$ will have the probability 0.95.

Let us fix now a significance level Φ^*. The associated argument will be denoted by x^*, and $F^* = x^* \sqrt{\sigma}$.

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Now it is possible to formulate a more natural criterion of selecting an allocation strategy:

$$\max_{S_1} \left\{ U(S_1,S_2) - x^* \sqrt{\sigma(S_1 S_2)} \right\}$$

$$S_1 + S_2 = 1 \;;\; S_1 \geq 0 \qquad (11)$$

The allocation strategy (11) provides the maximum utility with an a priori probability. This definition can be used as a new risk concept. Any other allocation strategy will provide less utility at the same significance level, or the same utility with a lower significance level.

Let us find the optimum for (11) in year t (we shall omit t again).

Let:

$$U(S_1) = U(S_1,S_2) = (L^1 u_1 - L^2 u_2)S_1 + L^2 u_2 \;;\; \text{and}$$

$$\sigma(S_1) = \sigma(S_1,S_2) = AS_1^2 + 2BS_1 + C \qquad (12)$$

where

$$A = \tilde{\tau}_{11} + \tilde{\tau}_{22} - 2\tilde{\tau}_{12} > 0; \quad B = \tilde{\tau}_{12} - \tilde{\tau}_{22}\;; \quad C = \tilde{\tau}_{22}\,.$$

From the necessary optimality condition

$$\frac{\partial U(S_1)}{\partial S_1} - x^* \frac{\partial \sqrt{\sigma(S_1)}}{\partial S_1} = 0$$

we obtain

$$(L^1 u_1 - L^2 u_2)\sqrt{\sigma} = x^*(A\hat{S}_1 + B) \qquad (13)$$

where $\hat{S}_1$ is the optimal solution.

Let us denote $\hat{S}_1$ by p and $\varepsilon = [(L^1 u_1 - L^2 u_2)/x^*]^2$. Then the solution of the quadratic equation (12):

$$p = \hat{S}_1 = -\frac{B}{A} \pm \frac{1}{A}\sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}} \qquad (14)$$

The sign of the second term in (14) is defined by the sign of $(L^1_{u_1} - L^2_{u_2})$ (see (13)), because from (14)

$$A\hat{\hat{S}}_1 + B = \pm\sqrt{\frac{\varepsilon(AC - B^2)}{A = \varepsilon}} .$$

Suppose $L^1u_1 > L^2u_2$, then $A\hat{S}_1 + B > 0$, and

$$p = -\frac{B}{A} + \frac{1}{A}\sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}} \qquad (15)$$

The second derivative of $U(S_1) - x^*\sqrt{\sigma(S_1)}$ in $S_1 = p$ is negative, we have a maximum. From (14) it is clear that, when $A < \varepsilon$, there are no real p values. $AC - B^2$ is always greater then zero, because

$$AC - B^2 = \tilde{\tau}_{11}\tilde{\tau}_{22} - \tilde{\tau}^2_{12} = \lambda^2 c^2 u_1^2 u_2^2 (\tau_{11}\tau_{22} - \tau^2_{12}) > 0.$$

Consequently, if $A < \varepsilon$, the function $f(S_1) = U(S_1) - x^*\sqrt{\sigma(S_1)}$ monotonouos and reaches its maximum on the boundary of the unit interval. From the derivative of f in $S_1 = 0$ we obtain, that if $A < \varepsilon$ and $B < \sqrt{\varepsilon C}$, the maximum is in $S_1^1 = 1$, if $B > \sqrt{\varepsilon C}$, then in $S_1^1 = 0$. If $A < \varepsilon$ we have the following possibilities:

$$\text{if } B > \sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}}, \text{ then } p = 0$$

$$\text{if } B < \sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}} - A, \text{ then } p = 1$$

$$\text{if } \sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}} > B > \sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}}, \text{ then}$$

$$p = \frac{1}{A}\left(\sqrt{\frac{\varepsilon(AC - B^2)}{A - \varepsilon}} - B\right)$$

In the last case maximum is reached inside the unit interval.

Further we show a graphic representation of the results.

Let $\beta = B/A$; $\gamma = C/A$; $\bar{\varepsilon} = \varepsilon/A$; $\alpha = \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}}$. With the new notation the condition $AC - B^2 > 0$ is transformed to $\beta^2 < \gamma$. $A > \varepsilon$ which is equivalent to $\bar{\varepsilon} > 1$, the condition $B > \sqrt{\varepsilon C}$ (or $\beta > \sqrt{\bar{\varepsilon}\gamma}$) does not hold, thus, if $\bar{\varepsilon} > 1$ the only optimal solution is $p = 1$. Let us list the final summary of our results. The conditions $\gamma \geq 0$ and $\beta < \sqrt{\gamma}$ always hold.

I. If $\bar{\varepsilon} > 1$ $(\alpha < 0)$ then $p = 1$.

II. If $\bar{\varepsilon} < 1$ $(\alpha < 0)$

if $\beta > \sqrt{\bar{\varepsilon}\gamma}$ then $p = 0$

if $\beta^2 + 2(1 - \bar{\varepsilon})\beta + (1 - \bar{\varepsilon}) < \bar{\varepsilon}\gamma$, then $p = 1$

if $\beta < \sqrt{\bar{\varepsilon}\gamma}$ and $\beta^2 + 2(1 - \bar{\varepsilon})\beta + (1 - \bar{\varepsilon}) > \bar{\varepsilon}\gamma$, then

$$0 < p = \sqrt{\alpha(\gamma - \beta^2)} - \beta < 1$$

Fig. 1. illustrates the solution on the $\beta \times \gamma$ plane at $\bar{\varepsilon} < 1$.

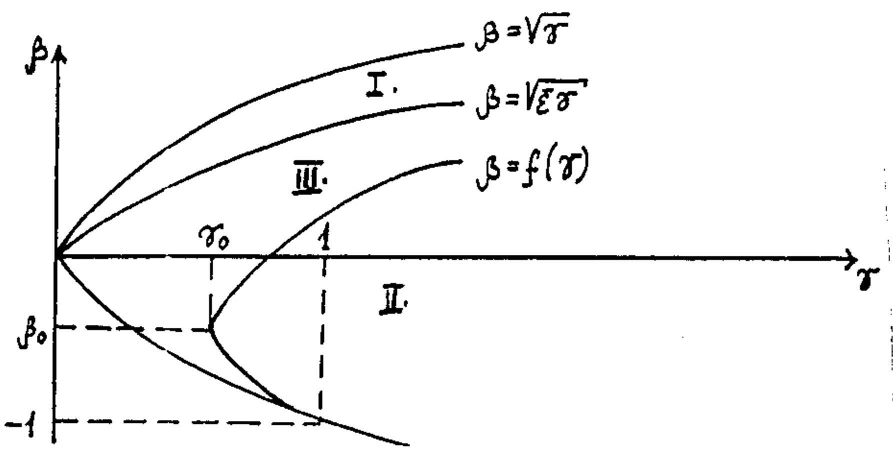

Fig. 1. The domains of solution of the allocation problem at $\bar{\varepsilon} < 1$. $-\beta_0 = \gamma_0 = 1 - \bar{\varepsilon}$; $\beta = h(\gamma)$ is given by the equation $\beta^2 + 2(1 - \bar{\varepsilon})\beta + (1 - \bar{\varepsilon}) = \bar{\varepsilon}\gamma$. The curves $\beta = \sqrt{\bar{\varepsilon}\gamma}$ and $\beta = f(\gamma)$ do not intersect.

In the domain I - the solution is $p = 0$, in II - $p = 1$; in III $p = \sqrt{\alpha(\gamma - \beta^2)}$.

3. Interpretation

The first crop is always preferred on the total area if $\overline{\varepsilon} > 1$, or if $\overline{\varepsilon} < 1$ and $(\beta, \gamma) \in II$.

Now, let us analyze the case, when $\overline{\varepsilon} > 1$, or equivalently:

$$\left(\frac{L^1 u_1 - L^2 u_2}{x^*} \right)^2 \geq \tilde{\tau}_{11} + \tilde{\tau}_{22} - 2\tilde{\tau}_{12} .$$

At a 0.95 significance level $x^* = 1.65$. The sufficient condition to prefer the first crop

$$L^1 u_1 - L^2 u_2 > 1.65\sqrt{\tilde{\tau}_{11} + \tilde{\tau}_{22} - 2\tilde{\tau}_{12}} \qquad (16)$$

It is quite trivial, that if the two crops have the same variances ($\tilde{\tau}_{11} = \tilde{\tau}_{22} = \tilde{\tau}_{12}$) of utilities, then the condition (16) will hold.

If $\overline{\varepsilon} < 1$, the first crop is preferred on the total area, if and only if

$$\beta^2 + 2(1 - \overline{\varepsilon})\beta + (1 - \overline{\varepsilon}) < \overline{\varepsilon}\gamma,$$

or:

$$\left(\frac{L^1 u_1 - L^2 u_2}{x^*} \right)^2 > \frac{(\tilde{\tau}_{11} - \tilde{\tau}_{12})^2}{(\tilde{\tau}_{11} + \tilde{\tau}_{22} - 2\tilde{\tau}_{12})(2\tilde{\tau}_{12} - \tilde{\tau}_{22} + 1)} \cdot \qquad (17)$$

Let us suppose that both crop utilities have equal variances, i.e. $\tilde{\tau}_{11} = \tilde{\tau}_{22} = \delta^2$ and the random variables η_1, and η_2 are uncorrelated ($\tau_{12} = 0$) In this case

$$L^1 u_1 - L^2 u_2 < 1.65\sqrt{2}\delta \sim 2.3\delta$$

and we have to check the condition (17). It is obvious, that the condition (17) holds if

$$L^1 u_1 - L^2 u_2 > 2.04\delta . \qquad (18)$$

Even in this trivial case the condition

$$L^1u_1 - L^2u_2 > 0$$

is not sufficient for the absolute priority of the first crop. (This is the consequence of the statistical independence of the two crops). If

$$L^1u_1 - L^2u_2 > 2.04\delta \qquad (19)$$

the optimal strategy is a "mixed" one,

$$p = \frac{1}{2}\left\{1 + \sqrt{\frac{3(L^1u_1 - L^2u_2)^2}{5.45\delta^2 - (L^1u_1 - L^2u_2)^2}}\right\} \qquad (20)$$

If the per hectare utilities are equal, $L^1u_1 = L^2u_2$, then $P = \frac{1}{2}$. If we increase L^1u_1, the value of p also increases, the first crop requires more sowing area.

Let us analyze the situation, when the variances of the utilities are fix and equal ($\tilde{\tau}_{11} = \tilde{\tau}_{22} = \delta^2$), but the correlation between the two variables η_1 and η_2 is not zero and changes ($\rho = \frac{\tilde{\tau}_{11}}{\delta^2}$ - will denote the correlation coefficient.)

Now $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2(1-\rho)}$; $\bar{\varepsilon} = \left\{\frac{L^1u_1 - L^2u_2}{\delta x^*}\right\}\frac{1}{2(1-\rho)}$

The critical value of the correlation coefficient, i.e. the value, which provides $\beta = \sqrt{\bar{\varepsilon}\gamma}$ is obtained from the equation

$$\left\{\frac{L^1u_1 - L^2u_2}{\delta x^*}\right\}^2 \frac{1}{4(1-\rho)^2} = -\frac{1}{2} \qquad (21)$$

at $x^* = 1.65$

$$L^1u_1 - L^2u_2 = 1.65\delta(1-\rho) \qquad (22)$$

Fig.2. illustrates the domains with "pure" ($p = 1$) and "mixed" ($0 < p < 1$) strategies of allocation.

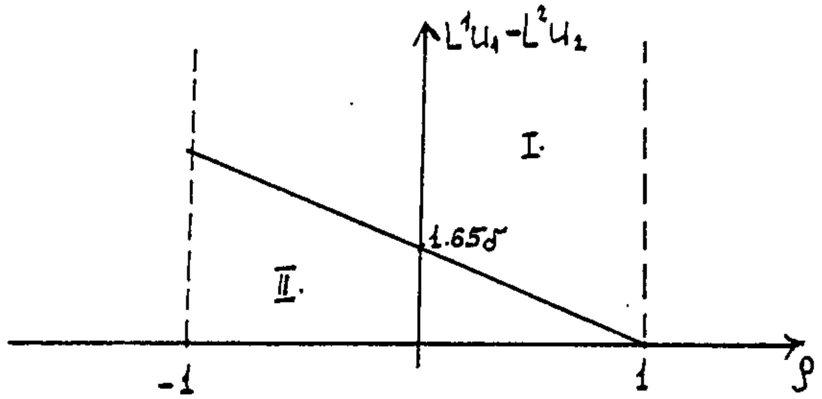

Fig.2. I-optimal "pure" strategies
II-optimal "mixed" strategies

As one can see from Fig.2. if the correlation of the two crops decreases, the difference in the utilities, required for the absolute priority of the first crop increases. Negative correlation requires even greater difference in utilities for this priority. E.g. when $\rho = -1$ the difference must be not less, then 3.3δ for the "pure" strategy.

Finally, we find the condition of the absolute priority for the second crop, which has less expected utility value.

The necessary and sufficient condition in this case is the inequality $\beta > \sqrt{\varepsilon\gamma}$. If $x^* = 1.65$, we obtain

$$L^1u_1 - L^2u_2 < 1.65 \left(\frac{\tilde{\tau}_{12}}{\sqrt{\tilde{\tau}_{12}}} - \sqrt{\tilde{\tau}_{12}} \right) \qquad (23)$$

Now, let us denote $\tilde{\tau}_{11} = \delta_1^2$; $\tilde{\tau}_{22} = \delta_2^2$; $\tilde{\tau}_{12} = \rho\sigma_1\sigma_2$. Then, from (23) we obtain the condition:

$$L^1u_1 - L^2u_2 < 1.65(\rho\delta_1 - \delta_2) \qquad (24)$$

obviously, $\rho\delta_1 > \delta_2$, or $\rho > \frac{\delta_2}{\delta_1}$, i.e. in this case the two crops must have positive correlation, and the variance of utility of the second crop must be less, then that of the first one - the second crop is more stable to the perturbation of the environment. The optimal allocation now is $p = 0$.

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