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"Dynamic and Stochastic Fracture"

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Notes on Rapid Fracture

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1. Introduction

The goal of these notes is to provide an introduction to dynamic fracture. They are not a substitute for the book by L. B. Freund, *Dynamic Fracture Mechanics*, but sometimes cover topics in different ways.

The main achievement of dynamic fracture mechanics is the complete solution of the motion of a crack in a rather general setting. The steps which lead to this solution are the following:

1. The stress field which surrounds the tip of a moving crack has a dominant, universal, singularity, which depends only upon its instantaneous velocity and upon a single constant, called the stress intensity factor, which sets the overall scale.
2. The flow of energy to the crack tip is completely determined by this singular field. and therefore by the instantaneous velocity of the crack, and the stress intensity factor.
3. For a semi-infinite crack in an infinite plate, the stress intensity factor is completely determined by integrals over the known applied forces which cause the crack to move. As a consequence, the energy flow to the crack tip is a known function of crack velocity and applied loads.
4. Given a model for how the flow of energy to the crack tip causes it to move, one can now find an equation of motion for the crack tip.

To describe this accomplishment in other words, dynamic fracture mechanics succeeds in integrating out all the complicated behavior of two-dimensional elastic fields surrounding a crack, and turning the crack tip into an effective particle which responds to external driving forces. The crack tip is in fact completely insensitive to its own history, and behaves like a particle completely without inertia, capable of changing velocity instantaneously in response to changes of load.

Although this calculation is extremely powerful, it does not answer all questions. For example, what happens when the boundaries of the plate in which the crack moves are included? The notes show that in this case the crack can acquire an effective mass, by virtue of wave reflections from the boundaries. A more fundamental problem has to do with the response of the crack tip to energy flux. Dynamic fracture mechanics is incomplete without specifying precisely how the crack tip is to move when energy comes pouring in. Experiments show that the crack tip decides to move in mysterious ways. Even in the most brittle of materials, the energy required for crack motion begins to diverge at unexpectedly low velocities, and the crack tip becomes unstable. My belief is that these phenomena have the best chance of being understood within models that do not make a fundamental distinction between the inside and outside of the crack tip, but treat them simultaneously on an equal footing. Preliminary calculations along these lines occupy the last sections of the notes. However, the problem is open.

2. Linear Elasticity

Consider a collection of point masses located at points $\mathbf{r}_i$, and interacting by a potential V, so that their energy is

$$U = V\{\mathbf{r}_i\}. \tag{2.1}$$

The theory of linear elasticity is concerned with the case in which the potential is minimized by some collection of equilibrium locations $\vec{d}_i$, and the masses execute small motions about this minimum;

$$\mathbf{r}_i = \mathbf{d}_i + \mathbf{u}_i, \tag{2.2}$$

where $\mathbf{u}_i$ will be taken to be small. Saying that the $\mathbf{u}_i$ are small means that it is legitimate to expand V in powers of the $\mathbf{u}_i$. One is expanding about an equilibrium configuration, so the first order contributions vanish, and the energy of the system is to leading order

$$U_h = \sum_{ij\alpha\gamma} \frac{\partial^2 V}{\partial u_i^\alpha \partial u_j^\gamma} u_i^\alpha u_j^\gamma. \tag{2.3}$$

The next goal is to turn this expression into a continuum theory of elasticity. From the continuum viewpoint, one writes $\mathbf{u}_i = \mathbf{u}(\mathbf{x})$, with the continuous field chosen so that

$$\mathbf{u}(\mathbf{d}_i) = \mathbf{u}_i. \tag{2.4}$$

In other words, the displacements are indexed by the equilibrium locations. (It would be equally possible to index the displacements by where they end up, rather than by where they started, which would use the definition $\mathbf{u}(\mathbf{r}_i) = \mathbf{u}_i$.) Adopting the summation convention, the energy now takes the form

$$U_h = \int d\mathbf{x}d\mathbf{x}' \frac{\delta^2 V}{\delta u_\alpha(\mathbf{x})\,\delta u_\gamma(\mathbf{x}')} u_\alpha(\mathbf{x})\, u_\gamma(\mathbf{x}') \tag{2.5a}$$

$$\equiv \int d\mathbf{x}d\mathbf{x}' W_{\alpha\gamma}(\mathbf{x},\mathbf{x}')\, u_\alpha(\mathbf{x})\, u_\gamma(\mathbf{x}'). \tag{2.5b}$$

It is helpful to notice a few symmetries of W. Since the system is translationally invariant, W depends only upon $\mathbf{x} - \mathbf{x}'$. From its definition

$$W_{\alpha\gamma}(\mathbf{x}) = W_{\gamma\alpha}(-\mathbf{x}). \tag{2.6}$$

So one can rewrite Eq. (2.5) in $\mathbf{k}$ space as

$$\int \frac{d\mathbf{k}}{(2\pi)^3} W_{\alpha\gamma}(\mathbf{k})\, u_\alpha(\mathbf{k})\, u^*_\gamma(\mathbf{k}). \tag{2.7}$$

To be consistent with the continuum picture, one should concentrate only on the long-wavelength behavior of the system, which means looking at the behavior only for small $\mathbf{k}$ through a Taylor expansion. The zero'th order term in $\mathbf{k}$ must vanish because of translational invariance–if it did not vanish, then each mass would act as if it were tied by a set of springs to some definite point in space. The linear terms vanish because of the symmetry Eq. (2.6). So the leading contribution to the energy in **k** space is

$$\int \frac{d\mathbf{k}}{(2\pi)^3} E_{\alpha\beta\gamma\delta}\, k_\beta k_\delta\, u_\alpha(\mathbf{k})\, u^*_\gamma(\mathbf{k}). \tag{2.8}$$

Inverting this expression to real space, one finally has

$$U_h = \int d\mathbf{x}\, \frac{\partial u_\alpha}{\partial x_\beta} E_{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial x_\delta}. \tag{2.9}$$

Because of the way that E has been defined, it is unchanged after interchange of α with γ or of β with δ. The theory of elasticity is not normally couched in terms of the shears $\partial u_\alpha/\partial x_\beta$, but instead in terms of the strain tensor

$$e_{\alpha\beta} \equiv \frac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right]. \tag{2.10}$$

In fact, the energy U_h depends only upon the combinations that appear in this tensor. Intuitively, since

$$\omega_{\alpha\beta} \equiv \frac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\alpha}\right] \tag{2.11}$$

changes when **u** is rotated a slight amount, the energy cannot depend upon terms of this sort. To be more explicit, one can perform the following computation. Rewrite U_h as

$$U_h = \int d\mathbf{x} \begin{array}{l} \frac{1}{32}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right]\left[E_{\alpha\beta\gamma\delta} + E_{\beta\alpha\gamma\delta} + E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\gamma\delta}\right]\left[\frac{\partial u_\gamma}{\partial x_\delta} + \frac{\partial u_\delta}{\partial x_\gamma}\right] \\ +\frac{1}{32}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right]\left[E_{\alpha\beta\gamma\delta} - E_{\beta\alpha\gamma\delta} - E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\gamma\delta}\right]\left[\frac{\partial u_\gamma}{\partial x_\delta} - \frac{\partial u_\delta}{\partial x_\gamma}\right] \end{array}. \tag{2.12}$$

The first term involves only the strain tensor, and the second can be shown to vanish because it is not rotationally invariant. One consequnce of rotational invariance is that if the whole system in equilibrium is picked up and twisted through an infinitesimal angle s about the a axis, so that

$$u_\alpha = \epsilon^{\alpha\alpha' a} x_{\alpha'}, \tag{2.13}$$

the energy must be unchanged. Placing Eq. (2.13) into Eq. (2.9) and demanding that the result vanish gives (don't sum on a!)

$$\int d\mathbf{x}\, \epsilon^{\alpha\beta a} E_{\alpha\beta\gamma\delta} \epsilon^{\gamma\delta a} = 0 \tag{2.14}$$

$$\Rightarrow E_{\alpha\beta\gamma\delta} - E_{\beta\alpha\gamma\delta} - E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\gamma\delta} = 0. \tag{2.15}$$

Comparison with Eq. (2.12) shows that one can write

$$U_h = \int d\mathbf{x}\, \frac{1}{2} e_{\alpha\beta}\, C_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}, \tag{2.16}$$

with

$$C_{\alpha\beta\gamma\delta} = \frac{1}{4}\left[E_{\alpha\beta\gamma\delta} + E_{\beta\alpha\gamma\delta} + E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\gamma\delta}\right]. \tag{2.17}$$

The tensor C is invariant under interchange of α with β, of γ with δ, and also under the interchange $\alpha\beta \longleftrightarrow \gamma\delta$. Therefore, C has at most 21 components.

To find the equation of motion for **u**, one computes

$$\rho\ddot{u}_\alpha(\mathbf{x}) = -\frac{\delta U_h}{\delta u_\alpha(\mathbf{x})} = \frac{\partial}{\partial x_\beta}\sigma_{\alpha\beta}(\mathbf{x}), \tag{2.18}$$

with the stress tensor σ given by

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\, e_{\gamma\delta} \tag{2.19}$$

Since the acceleration of small sections of mass is given by the divergence of the stress tensor, the stress tensor is physically interpreted as giving the forces that each section of the body exerts upon its neighbor. Specifically, if one imagines taking a knife and using it to sever bonds in a small two-dimensional region, let us say perpendicular to the x axis, then σ_{xx} gives the force per unit area required to pull the faces of the region together along x, and σ_{xy} and σ_{xz} give the forces per unit area required to stretch the faces in the directions perpendicular to x so that each atom is directly across from the atom that was its neighbor in equilibrium.

Depending upon the symmetries of the underlying lattice, the stress tensor may acquire additional symmetries. The case we will be concerned with exclusively is the case in which the equations of motion are rotationally invariant, resulting in

$$C_{\alpha\beta\gamma\delta} = 2\lambda\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu\left\{\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right\} \tag{2.20}$$

$$\Rightarrow \sigma_{\alpha\beta} = \lambda\,\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma} + \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right). \tag{2.21}$$

The constants μ and λ are called the Lamé constants, have dimensions of energy per volume, and are typically of order 10^{10} ergs/cm^3. The equation of motion Eq. (2.18) becomes explicitly

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}, \tag{2.22}$$

originally found by Navier.

3. Elasticity in Two Dimensions

The tensor structure of elasticity makes it particularly difficult to solve fully three-dimensional problems, and it is difficult to carry out controlled three-dimensional experiments as well. Fortunately, there are cases in which the theory naturally reduces to two dimensions, where most of the analytical results have been obtained.

A first case is called anti-plane shear. Imagine one is tearing a telephone book, with one hand gripping on the left, the other gripping on the right, one pushing up and the other pulling down. The only non-zero displacement is u_z, and it is a function of x and y alone. The only non-vanishing stresses in this case are

$$\sigma_{xz} = \mu \frac{\partial u_z}{\partial x}, \tag{3.1a}$$

and

$$\sigma_{yz} = \mu \frac{\partial u_z}{\partial y}. \tag{3.1b}$$

The equation of motion for u_z is

$$\frac{1}{c^2}\frac{\partial^2 u_z}{\partial t^2} = \nabla^2 u_z, \tag{3.2}$$

where

$$c = \sqrt{\frac{\mu}{\rho}}. \tag{3.3}$$

Therefore, the vertical displacement obeys the ordinary wave equation.

A second case corresponds to pulling on a thin plate, and is called plane stress. Let the z direction be the direction that goes through the plate. If the scale over which stresses are varying in x and y is large compared with the thickness of the plate, then one might expect that the displacements in the z direction will come quickly into equilibrium with the local x and y stresses. When the material is being stretched, (think of pulling on a balloon), the plate will contract in the z direction, and when it is being compressed, the plate will thicken. Therefore, one guesses that

$$u_z = z f(u_x, u_y), \tag{3.4}$$

and that u_x, and u_y are independent of z. One can deduce the function f by noticing that σ_{zz} must vanish on the face of the plate. This means that

$$\lambda\left\{\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right\} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} = 0 \tag{3.5}$$

at the surface of the plate, which implies that

$$f(u_x, u_y) = \frac{\partial u_z}{\partial z} = -\frac{\lambda}{\lambda + 2\mu}\left\{\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right\}. \tag{3.6}$$

So

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{2\mu}{\lambda + 2\mu}\left\{\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right\}, \tag{3.7}$$

and one can write

$$\sigma_{\alpha\beta} = \tilde{\lambda}\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma} + \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right), \tag{3.8a}$$

where

$$\tilde{\lambda} = \frac{2\mu\lambda}{\lambda + 2\mu}, \tag{3.8b}$$

and α and β now range only over x and y. Therefore, a thin plate obeys the equations of two-dimensional elasticity, with an effective constant $\tilde{\lambda}$, so long as u_z is dependent upon u_x and u_y according to Eq. (3.6). In the following discussion, the tilde over $\tilde{\lambda}$ will usually be dropped, with the understanding that the relation to three-dimensional materials properties is given by Eq. (3.8a). The equation of motion is still Navier's equation, Eq. (2.22), but restricted to two dimensions.

A few random useful facts: materials are frequently described by the Young's modulus E and Poisson ratio ν. In terms of these constants,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\ \tilde{\lambda} = \frac{E}{2(1-\nu^2)},\ \mu = \frac{E}{2(1+\nu)}. \tag{3.9}$$

The following relation will be useful in discussing two-dimensional static problems. First note that

$$\nabla \cdot \mathbf{u} = (\lambda + 2\mu)\,\sigma_{\alpha\alpha}. \tag{3.10}$$

Second, taking the divergence of Eq. (2.22), one finds that

$$\frac{\rho}{\lambda + 2\mu}\frac{\partial^2 \sigma_{\alpha\alpha}}{\partial t^2} = \nabla^2 \sigma_{\alpha\alpha} \tag{3.11}$$

Therfore, $\nabla \cdot \mathbf{u}$ obeys the wave equation, with the longitudinal wave speed

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{3.12}$$

Similarly, $\nabla \times \mathbf{u}$ also obeys the wave equation, but with the shear wave speed

$$c_t = \sqrt{\frac{\mu}{\rho}}. \tag{3.13}$$

The fact that the local thickness of the plate is tied to stresses in the x and y directions leads to two optical methods to determine stress fields experimentally. The first method relies on the fact that when light reflects off a curved surface, the reflected intensity becomes singular at certain points that depend on the details of the geometry. In practice, when light is shined on a crack tip, a sharp dark spot surrounds the tip, and its shape and size can be used to deduce the stresses. This technique is known as the method of caustics. A second method, older and more reliable, relies upon the fact that materials under stress will typically rotate the plane of polarization of transmitted light. It is possible to determine the basic structure of this rotation without detailed calculation. The rotation must depend upon features of the stress tensor which are rotationally invariant, and therefore can depend only upon the two principal stresses, which are the the diagonal elements in a reference frame where the stress tensor is diagonal. In addition, there should be no rotation of polarization when the material is stretched uniformly in all directions, in which case the two principal stresses are equal. So the angular rotation of the plane of polarization must be of the form

$$\Delta = K(\sigma_1 - \sigma_2), \tag{3.14}$$

where σ_1 and σ_2 are the principal stresses (eigenvalues of the stress tensor), and K is a constant that will have to be determined experimentally. Whenever stresses of a two-dimensional problem are calculated analytically, the results can be placed into Eq. (3.14), and compared with experimental fringe patterns. Fast optical systems have been developed to carry out this procedure for rapidly moving cracks, although I am not sure to what extent Eq. (3.7) is obeyed when cracks move at speeds on the order of the speed of sound.

3.1. Static problems in Anti-plane Shear

For an equilibrium situation, the equation for anti-plane shear, Eq. (3.2) takes a particularly simple form:

$$\nabla^2 u_z = 0. \tag{3.15}$$

Since u_z is a solution of Laplace's equation, it can be represented by

$$u_z = \phi(\zeta) + \overline{\phi(\zeta)}, \tag{3.16}$$

where ϕ is analytic, and $\zeta = x + iy$.

Consider now a sheet under uniform tension at infinity, so that far from the center of the sheet,

$$\phi = \Gamma\zeta, \tag{3.17}$$

with Γ complex. At the center of the sheet, cut some sort of hole out and allow the boundaries to relax. The goal is now to find the stress and strain fields in the body, as a result of having cut the hole.

Because the edges of the hole are free, the stress normal to the edge must vanish. If s is a variable which parameterizes the edge of the hole, so that

$$[x(s), y(s)] \tag{3.18}$$

travels around the boundary of the hole as s moves along the real axis, then requiring normal stress to vanish means that

$$\frac{\partial u_z}{\partial y}\frac{\partial x}{\partial s} - \frac{\partial u_z}{\partial x}\frac{\partial y}{\partial s} = 0 \tag{3.19}$$

Using the representation of u_z, Eq. (3.16), one finds that

$$\frac{\partial \phi}{\partial s} = \frac{\partial \overline{\phi}}{\partial s} \tag{3.20}$$

on the boundary, or since ϕ is arbitrary up to a constant,

$$\phi(\zeta) = \overline{\phi(\zeta)} \tag{3.21}$$

when ζ lies on the boundary.

To illustrate the use of Eq. (3.21), suppose the hole in the sheet is elliptical, and described by

$$\zeta = \omega + \frac{m}{\omega}, \tag{3.22}$$

with w contained in the unit circle. When $m = 0$, the boundary is circular, and when $m = 1$, the boundary is a cut along the real axis. Considering ϕ as a function of ω, one has

$$\phi(\omega) = \bar{\phi}\left(\frac{1}{\omega}\right), \tag{3.23}$$

since $\bar{\omega} = 1/\omega$ on the unit circle. This equation can now be analytically continued off the unit circle. Notice that when ω is outside the unit circle, ϕ must be completely regular, except for the fact that it diverges as $\Gamma\omega$ for large ω. One concludes that $\bar{\phi}$ is regular within the unit circle, except for a pole at the origin that goes as Γ/ω. These being the only singularities of ϕ and $\bar{\phi}$, one concludes that

$$\phi(\omega) = \Gamma\omega + \frac{\bar{\Gamma}}{\omega} \tag{3.24}$$

and that

$$\phi(\zeta) = \Gamma\frac{\zeta}{2}\left(1 + \sqrt{1 - 4m/\zeta^2}\right) + \bar{\Gamma}\frac{\zeta}{2m}\left(1 - \sqrt{1 - 4m/\zeta^2}\right). \tag{3.25}$$

The case in which $m \to 1$ is particularly interesting. The hole becomes a straight crack along [-2,2]. Notice that ϕ has a branch cut over exactly the same region. The displacement u_z is finite approaching the tip of the crack, but the stress $\sigma_{yz} = \mu\partial u_z/\partial y$ diverges approaching the tip as $1/\sqrt{z-2}$.

3.2. Muskhelishvili's Formalism

In the absence of body forces, the equations of two-dimensional plane elasticity, for an isotropic medium at rest are found from Eq. (2.22) and Eq. (3.11) to be

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} = 0. \tag{3.26}$$

$$\nabla^2 \sigma_{\alpha\alpha} = 0 \tag{3.27}$$

$$\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma} + \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) \tag{3.28}$$

These are the equations for plane strain. For plane stress, one must replace λ by $2\lambda\mu/(\lambda + 2\mu)$.

From Eq. (3.26) it follows that there exists a real function U, the Airy stress function, such that

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \tag{3.29}$$

Using Eq. (3.27), one has then that

$$\nabla^2\nabla^2 U = 0 \tag{3.30}$$

This equation is due to Maxwell.

Writing the biharmonic operator as

$$\nabla^2\nabla^2 = \frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial \bar{z}^2}$$

one can argue immediately that since U is real, it must be of the form

$$\frac{1}{2}U(z,\bar{z}) = \bar{z}\phi(z) + z\overline{\phi(z)} + \int^z dz'\psi(z') + \overline{\int^z dz'\psi(z')}, \tag{3.31}$$

where $z = x + iy$, and ϕ and ψ are analytic. Acting on U with

$$\frac{\partial^2}{\partial z\partial\bar{z}} \text{ and } \frac{\partial^2}{\partial z^2},$$

one finds immediately that

$$\sigma_{xx} + \sigma_{yy} = 2\left[\phi'(z) + \overline{\phi'(z)}\right]; \quad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right] \tag{3.32}$$

One can also find the displacements in terms of ϕ and ψ. The result is

$$2\mu(u_x + iu_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{3.33}$$

where

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

A final useful relation gives the force exerted on a boundary in terms of ϕ and ψ. In particular, if the normal force on a boundary vanishes, then on that boundary

$$\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = 0. \tag{3.34}$$

3.2.1. The stress field of a straight crack

Suppose we have a plate under uniform tension,

$$\sigma_{yy} = 2\Gamma; \quad \sigma_{xx} = \sigma_{xy} = 0.$$

One finds immediately that

$$2\phi = \psi = \Gamma z.$$

Next suppose we have an infinite plate with the same condition at infinity, but a crack stretching from -1 to 1 on the real axis. On the crack we have

$$\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = 0.$$

Now define $\hat{\phi}(z)$ to be the analytic function such that

$$\hat{\phi}(x) = \bar{\phi}(x)$$

on the real axis. Define $\hat{\psi}$ similarly. Then our boundary condition gives us that

$$\phi(z) + z\hat{\phi}'(z) + \hat{\psi}(z) = 0.$$

Argue that ϕ goes as z at infinity, that it can have a branch cut between -1, and 1, that as κ is arbitrary, the only power of $\sqrt{z^2-1}$ allowed is the first, if $\vec{u}$ is not to diverge. The answer then is

$$\phi(z) = \Gamma\sqrt{z^2-1} - \frac{\Gamma}{2}z$$

$$\psi(z) = \Gamma z - \Gamma\frac{1}{\sqrt{z^2-1}}.$$

4. Motion of a Crack in an Infinite Plate

4.1. Mott's Scaling Analysis

The first analysis of rapid fracture was carried out by Mott. It is a scaling analysis which clarifies the basic physical processes, despite being wrong in many details, and consists in writing down an energy balance equation for crack motion. Consider a crack of length $l(t)$ growing at rate $v(t)$ in a plate under stress σ_∞ far from the crack. When the crack extends, its faces separate, causing the plate to relax within a circular region centered on the middle of the crack and with diameter of order l. The kinetic energy involved in moving a region of this size is guessed to be of the form

$$\text{KE} = c_K l^2 v^2, \tag{4.1}$$

and the potential energy gained in releasing stress from the region is guessed to be of the form

$$\text{PE} = -c_P l^2. \tag{4.2}$$

These guesses are correct for slowly moving cracks, but fail qualitatively as the crack velocity approaches the speed of sound, in which case both kinetic and potential energies diverge. This divergence will be demonstrated later, but for the moment, let us proceed fearlessly. The final process contributing to the energy balance equation is the creation of new crack surfaces, which takes energy Γl, where Γ is a phenomenological fracture energy. So the total energy of the system containing a crack is given by

$$E = c_K l^2 v^2 + E_{qs}(l), \tag{4.3}$$

with

$$E_{qs}(l) = -c_P l^2 + \Gamma l. \tag{4.4}$$

Consider first the problem of quasi-static crack propagation. If a crack moves forward only slowly, its kinetic energy will be negligible, so only the quasi-static part of the energy, E_{qs}, will be important. It costs energy for very short cracks to elongate, and in fact such cracks would heal and travel backwards if it were not for irreversible processes, such as oxidation of the crack surface, which typically prevent this from happening. That the crack grows at all is due to additional irreversible processes, sometimes chemical attack on the crack tip, sometimes vibration or other irregular mechanical stress. It should be emphasized that the system energy E increases as a result of these processes. Eventually, at length l_0, the energy gained by relieving elastic stresses in the body exceeds the cost of creating new surface, and the crack becomes able to extend spontaneously. One sees that at l_0, the energy functional $E_{qs}(l)$ has a quadratic maximum, so that Eq. (4.4) can be rewritten

$$E_{qs}(l) = E_{qs}(l_0) - c_P(l - l_0)^2\,;\ l_0 = \Gamma/2c_P. \tag{4.5}$$

The whole study of engineering fracture mechanics boils down to calculating l_0, given things such as external stresses, which in the present case have all been condensed into the constant c_P. Dynamic fracture starts in the next instant, and because it is so rapid, the energy of the system is conserved, remaining at $E_{qs}(l_0)$. Using Eq. (4.3) and Eq. (4.5), with $E = E_{qs}(l_0)$ gives

$$v(t) = \sqrt{\frac{c_P}{c_K}}\left(1 - \frac{l_0}{l}\right) = v_{\max}\left(1 - \frac{l_0}{l}\right). \tag{4.6}$$

This equation predicts that the crack will accelerate until it approaches the speed $v_{\max}$. The maximum speed cannot be deduced from these arguments, but Stroh correctly argued that $v_{\max}$ should be the Rayleigh wave speed, the speed at which sound travels over a free surface. One needs only to know the length at which a crack begins to propagate in order to predict all the following dynamics.

4.2. Steady States in Antiplane Shear

The goal of this section is to compute properties of a crack moving at steady velocity, in the case of antiplane shear. The starting point is the equation of motion Eq. (3.2), which is just the wave equation for the displacement u_z. If a crack moves at constant velocity v, then in a co-moving frame it will obey the equation

$$\alpha^2\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = 0 \tag{4.7}$$

where

$$\alpha = \sqrt{1 - v^2/c^2}. \tag{4.8}$$

Defining the complex variable

$$\zeta = x + i\alpha y, \tag{4.9}$$

the steady state equation can be rewritten as

$$\frac{\partial^2 u_z}{\partial\zeta\partial\bar\zeta} = 0. \tag{4.10}$$

In other words, $u_z(\zeta)$ obeys Laplace's equation, and can therefore be written as

$$u_z = \phi(\zeta) + \overline{\phi(\zeta)}, \tag{4.11}$$

where ϕ is an analytic function.

To proceed further, of course, one has to specify boundary conditions, and details about the problem. However, Eq. (4.11) is sufficient to extract important information about the singularity at the tip of a crack, assuming only the the crack is loaded symmetrically, so that

$$u_z(y) = u_z(-y)\,. \tag{4.12}$$

Letting the crack tip be located at the origin, one can assert the following:

1 The displacement u_z vanishes in front of the crack.
2 Behind the crack, u_z must flip sign as y goes from negative to positive.
3 The stress $\sigma_{zy} = \mu\partial u_z/\partial y$ just vanish behind the crack, since the crack face is a free surface.

Just as in the static case, these conditions are met by a square root singularity in ϕ. Freund shows that ϕ must have this form by asymptotic expansion, and later in these notes an explicit calculation will show that ϕ has this form near the tip in a fairly general case. So let us take

$$\phi(\zeta) = -i\frac{K}{\mu\alpha\sqrt{2\pi}}\sqrt{\zeta}. \tag{4.13}$$

This equation is meant to describe the tip of a crack moving from left to right, which in the moving frame has its tip at the origin. The branch cut of the square root extends backwards towards negative x. The constant K is arbitrary, and has been normalized in a peculiar way to accord with convention, but must be real so that σ_{yz} vanishes along the crack surface, for negative x and $y \to 0$. One has then the following general structure for the displacement and stresses near the tip of the crack.

$$u_z = -i\frac{K}{\mu\alpha\sqrt{2\pi}}\left(\sqrt{\zeta} - \sqrt{\bar\zeta}\right) \tag{4.14a}$$

$$\sigma_{yz} = \frac{1}{2}\frac{K}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{\zeta}} + \frac{1}{\sqrt{\bar\zeta}}\right] \tag{4.14b}$$

$$\sigma_{xz} = \frac{-i}{2}\frac{K}{\sqrt{2\pi}\alpha}\left[\frac{1}{\sqrt{\zeta}} - \frac{1}{\sqrt{\bar\zeta}}\right]. \tag{4.14c}$$

Note that only the constant K is unknown; it is given by

$$K = \lim_{x\to 0^+}\sqrt{2\pi x}\,\sigma_{yz}, \tag{4.15}$$

and is called the stress intensity factor.

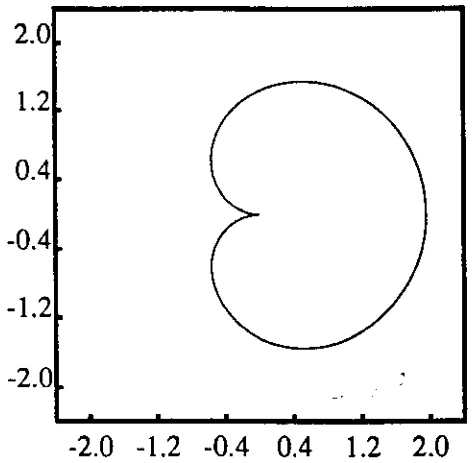
An interesting calculation, first done in a more difficult context by Yoffe, is to imagine approaching the crack tip and cutting an incision that approaches that crack tip radially at an angle θ. The normal stress required to glue the cuts back together is

$$\sigma_{\theta\theta} = \cos\theta\frac{\partial u_z}{\partial y} - \sin\theta\frac{\partial u_z}{\partial x} \tag{4.16a}$$

This stress is to be evaluated at the point

$$\zeta = r(\cos\theta + i\alpha\sin\theta). \tag{4.16b}$$

Upon evaluating Eq. (4.16) for velocities $v < .573\ldots$, one finds that the maximum value of the stress is for $\theta = 0$, so it seems sensible that the crack will continue to propagate forwards. However, for larger values of v, the maximum stress is off at an angle, as shown in the figures. For $v > .573$, it seems likely that the crack would become unstable, either to crack branching, or to some complicated unsteady motion.



The stress normal to a cut along θ is indicated by drawing a radius vector proportional to the stress for each angle θ. This picture is for $v = .2$

4.3. Steady States in Plane Stress

These notes show how to find the structure of the singularity around a crack tip in plane stress.

Begin with the dynamical equation for the strain field $\vec{u}$ of a steady state in the moving frame,

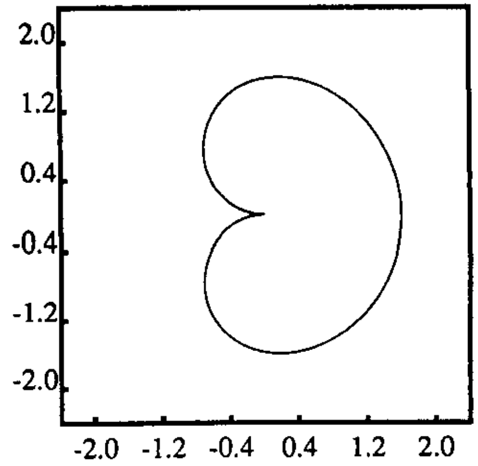
$$(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u} = \rho v^2 \frac{\partial^2 \vec{u}}{\partial x^2} \tag{4.17}$$

Divide $\vec{u}$ into transverse and longitudinal parts so that

$$\vec{u} = \vec{u}_t + \vec{u}_l,$$

with

$$\vec{u}_l = \vec{\nabla} v_l \quad \text{and} \quad \vec{u}_t = \left(\frac{\partial v_t}{\partial y}, -\frac{\partial v_t}{\partial x}\right).$$



The stress normal to a cut along θ is indicated by drawing a radius vector proportional to the stress for each angle θ. This picture is for $v = .6$

It follows immediately that

$$\left[(\lambda + 2\mu)\nabla^2 - \rho v^2 \frac{\partial^2}{\partial x^2}\right]\vec{u}_l = \vec{f} = -\left[\mu\nabla^2 - \rho v^2 \frac{\partial^2}{\partial x^2}\right]\vec{u}_t, \tag{4.18}$$

for some function $\vec{f}$ which must be harmonic ($f_x - if_y$ is a function of $x + iy$). We have then that

$$\left[\alpha^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\nabla^2 v_l = 0,$$

$$\left[\beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\nabla^2 v_t = 0,$$

where

$$\alpha^2 = 1 - \frac{\rho v^2}{\lambda + 2\mu} \tag{4.19a}$$

$$\beta^2 = 1 - \frac{\rho v^2}{\mu} \tag{4.19b}$$

Therefore the general form of the potentials is

$$v_l = v_l^0(z) + \overline{v_l^0(z)} + v_l^1(x + i\alpha y) + \overline{v_l^1(x + i\alpha y)} \tag{4.20a}$$

$$v_t = v_t^0(z) + \overline{v_t^0(z)} + v_t^1(x + i\beta y) + \overline{v_t^1(x + i\beta y)}, \tag{4.20b}$$

subject to the constraint of Eq. (4.18), which gives a relation between v_l^0 and v_t^0.

In fact, the purely harmonic pieces v_l^0 and v_t^0 disappear entirely from the expressions for $\vec{u}$. They result from the freedom one has to add a harmonic function to v_l and v_t

simultaneously, and can be neglected. Defining $\phi(z) = \partial v_l^1(z)/\partial z$ and $\psi(z) = \partial v_t^1(z)/\partial z$ we have for $\vec{u}$

$$u_x = \phi(z_\alpha) + \overline{\phi(z_\alpha)} + i\beta\left[\psi(z_\beta) - \overline{\psi(z_\beta)}\right] \tag{4.21a}$$

$$u_y = i\alpha\left[\phi(z_\alpha) - \overline{\phi(z_\alpha)}\right] - \left[\psi(z_\beta) + \overline{\psi(z_\beta)}\right], \tag{4.21b}$$

where

$$z_\alpha = x + i\alpha y, \qquad z_\beta = x + i\beta y.$$

Equation Eq. (4.21) gives a general solution for steady state cracks. Define also $\Phi = \partial\phi(z)/\partial z$ and $\Psi = \partial\psi(z)/\partial z$. Then the stresses are given by

$$\sigma_{xx} + \sigma_{yy} = 2(\lambda + \mu)\left[\Phi(z_\alpha) + \overline{\Phi(z_\alpha)}\right](1 - \alpha^2) \tag{4.22a}$$

$$\sigma_{xx} - \sigma_{yy} = 2\mu\left\{(1 + \alpha^2)\left[\Phi(z_\alpha) + \overline{\Phi(z_\alpha)}\right] + 2i\beta\left[\Psi(z_\beta) - \overline{\Psi(z_\beta)}\right]\right\} \tag{4.22b}$$

$$2\sigma_{xy} = 2\mu\left\{2i\alpha\left[\Phi(z_\alpha) - \overline{\Phi(z_\alpha)}\right] - (\beta^2 + 1)\left[\Psi(z_\beta) + \overline{\Psi(z_\beta)}\right]\right\} \tag{4.22c}$$

we will also need the rotation, which means

$$\nabla = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = -(1 - \beta^2)\left[\Psi(z_\beta) + \overline{\Psi(z_\beta)}\right]. \tag{4.22d}$$

It is worth writing down the stresses directly as well:

$$\begin{aligned} \sigma_{yy} &= -\mu(1 + \beta^2)\left[\Phi(z_\alpha) + \overline{\Phi(z_\alpha)}\right] - 2i\beta\mu\left[\Psi(z_\beta) - \overline{\Psi(z_\beta)}\right] \\ \sigma_{xx} &= \mu(1 + 2\alpha^2 - \beta^2)\left[\Phi(z_\alpha) + \overline{\Phi(z_\alpha)}\right] + 2i\beta\mu\left[\Psi(z_\beta) - \overline{\Psi(z_\beta)}\right] \end{aligned}. \tag{4.22e}$$

The definitions of α and β in Eq. (4.19) have been used to simplify the expressions.

To solve a general problem, one has to find the functions ϕ and ψ which match boundary conditions. It is interesting to notice that when $v \to 0$, the right hand side of Eq. (4.22a) goes to zero as well. Since one will be finding the potentials from given stresses at the boundaries, Φ must diverge as $1/v$, and the right hand side of *Eq.* (4.22) will turn into a derivative of Φ with respect to α. That is why the static theory has a different structure than the dynamic theory. In fact, the dynamic theory is more straigtforward.

As a first application, we will show that a moving crack under symmetric loading becomes unstable at a certain speed. We assume the crack to lie along the negative x axis, terminating at $x = 0$, and moving forward. The problem is assumed symmetric under reflection about the x axis, but no other assumption is needed. This instability was first found in a particular case by Yoffe.

We know that in the static case, the stress fields have a square root singularity at the crack tip. We will assume the same to be true in this case (the assumption is verified in all cases that can be worked explicitly.) Near the crack tip, we assume that

$$\begin{aligned} \phi(z) &\sim (B_r + iB_i)\, z^{1/2} \\ \psi(z) &\sim (D_r + iD_i)\, z^{1/2} \end{aligned}. \tag{4.23}$$

We first appeal to symmetry. Observe that

$$u_x(-y) = u_x(y), \qquad u_y(-y) = -u(y). \tag{4.24}$$

Placing Eq. (4.23) into Eq. (4.21) and using Eq. (4.24), we find immediately that $B_i = D_r = 0$. Thus

$$\Phi(z) \sim \frac{B_r}{z^{1/2}}, \quad \Psi(z) \sim \frac{iD_i}{z^{1/2}}. \tag{4.25}$$

We also observe that the square roots in Eq. (4.23) must be interpreted as having their cuts along the negative x axis, corresponding to the crack. On the crack surface, we have two boundary conditions, which require that σ_{xy} and σ_{yy} vanish. Upon substituting Eq. (4.25) into Eq. (4.22e) we find that the condition upon σ_{yy} is satisfied identically for $x < 0, y = 0$. However, substituting into Eq. (4.22d) with $y = 0$ we find that

$$\sigma_{xy} = \mu i\left\{2\alpha B_r - (\beta^2 + 1)D_i\right\}\left\{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{\bar{x}}}\right\}. \tag{4.26}$$

Thus

$$\frac{D_i}{B_r} = \frac{2\alpha}{\beta^2 + 1}. \tag{4.27}$$

This relation is enough to find the maximum velocity at which a crack can proceed stably along the x axis.

Using Eq. (4.27) we find that

$$\begin{aligned} \sigma_{xx} &= \frac{K}{\sqrt{2\pi}D}\left[(\beta^2 + 1)(1 + 2\alpha^2 - \beta^2)\left\{\frac{1}{\sqrt{z_\alpha}} + \frac{1}{\sqrt{\bar{z}_\alpha}}\right\} - 4\alpha\beta\left\{\frac{1}{\sqrt{z_\beta}} + \frac{1}{\sqrt{\bar{z}_\beta}}\right\}\right] \\ \sigma_{yy} &= \frac{K}{\sqrt{2\pi}D}\left[-(1 + \beta^2)^2\left\{\frac{1}{\sqrt{z_\alpha}} + \frac{1}{\sqrt{\bar{z}_\alpha}}\right\} + 4\alpha\beta\left\{\frac{1}{\sqrt{z_\beta}} + \frac{1}{\sqrt{\bar{z}_\beta}}\right\}\right] \\ \sigma_{xy} &= \frac{K}{\sqrt{2\pi}D}2i\alpha(\beta^2 + 1)\left\{\frac{1}{\sqrt{z_\alpha}} - \frac{1}{\sqrt{\bar{z}_\alpha}} - \frac{1}{\sqrt{z_\beta}} + \frac{1}{\sqrt{\bar{z}_\beta}}\right\} \end{aligned}, \tag{4.28a}$$

with

$$D = 4\alpha\beta - (1 + \beta^2)^2. \tag{4.28b}$$

The constant K is again called the stress intensity factor, and is given by

$$K = \lim x \to 0^+ \sqrt{2\pi x}\sigma_{yy}. \tag{4.29}$$

In order to find the direction of maximum stress, one must approach the tip of the crack along a line at angle θ to the x axis, and compute the stress perpendicular to that line. So one wants to choose

$$z_\alpha = r\cos\theta + ri\alpha\sin\theta, \quad z_\beta = r\cos\theta + ri\beta\sin\theta, \tag{4.30}$$

and to evaluate the stress

$$\sigma_{\theta\theta} = \cos^2\theta\sigma_{yy} + \sin^2\theta\sigma_{xx} - \sin(2\theta)\sigma_{xy}. \tag{4.31}$$

When this is carried out, one finds, as in the case of antiplane shear, that above a certain velocity (for Poisson ratio $\nu = 1/3$, the critical velocity is about .61) the direction of maximum tearing stress points away from the axis, and the crack would presumably become unstable.

4.4. Energy Flux

4.4.1. Antiplane Shear

Freund has emphaisized the importance of computing the energy current in fracture problems. It may be found from the time derivative of the total energy. The energy of the field u_z is

$$\mathcal{E} = \frac{\mu}{2}\int dxdy \frac{1}{c^2}(\dot{u}_z)^2 + (\nabla u_z)^2 . \tag{4.32}$$

So

$$\frac{d\mathcal{E}}{dt} = \mu \int \frac{1}{c^2}\dot{u}_z\ddot{u}_z + \nabla u_z \cdot \nabla \dot{u}_z \tag{4.33}$$

$$= \mu \int \dot{u}_z \nabla^2 u_z + \nabla \cdot (\dot{u}_z \nabla u_z) - \dot{u}_z \nabla^2 u_z \tag{4.34}$$

$$= \mu \int_{\partial S} d\hat{n} \cdot \dot{u}_z \nabla u_z . \tag{4.35}$$

So the energy current is

$$\vec{J} = \mu \dot{u}_z \nabla u_z . \tag{4.36}$$

In steady state, the current out of the upper half of the system thorough the x axis is

$$J_y^{\text{top}} = -v \int dx \frac{\partial u_z}{\partial x}\sigma_{yz}(x) , \tag{4.37}$$

where the integral is taken for y just a little bit above the axis.

When the crack is not loaded on its face, it appears that the current vanishes, since u_z is zero for $x > 0$ and σ_{yz} is zero for $x < 0$. However, right at the tip both of these quantities are sufficiently singular that they create a delta function which produces a finite energy flux into the tip. Only the fields right in the vicinity of the tip are important, so one can use the asymptotic forms Eq. (4.14) for the computation. One has

$$J^y = -v\mu \left[\frac{-i\alpha}{4}\right]\left[\frac{1}{x+i\alpha y} - \frac{1}{x - i\alpha y}\right]\frac{K^2}{2\pi\alpha^2\mu^2} \tag{4.38}$$

$$= \frac{vK^2}{4\pi\alpha\mu}\left[\frac{\alpha y}{x^2+\alpha^2 y^2}\right] . \tag{4.39}$$

One does in fact get a delta function as $y \to 0$. Integrating the current over x gives

$$J_y^{\text{top}} = \frac{vK^2}{4\alpha\mu} . \tag{4.40}$$

This is the energy flux from the top half of the plate into the crack tip. There will be an equal flux from the bottom half. Therefore, the total energy traveling into the crack tip per unit time is

$$J_y^{\text{tot}} = \frac{vK^2}{2\alpha\mu} . \tag{4.41}$$

4.4.2. Plane Stress

Energy flux may again be found from the time derivative of the total energy. We have

$$\frac{d}{dt}[K+P] = \frac{d}{dt}\int dxdy \left[\frac{\rho}{2}\dot{u}_\alpha\dot{u}_\alpha + \frac{1}{2}\frac{\partial u_\alpha}{\partial x_\beta}\sigma_{\alpha\beta}\right] . \tag{4.42}$$

The spatial integral is taken over an region which is static in the laboratory frame. So

$$\frac{d}{dt}[K+P] = \int dxdy \left[\rho\ddot{u}_\alpha\dot{u}_\alpha + \frac{\partial \dot{u}_\alpha}{\partial x_\beta}\sigma_{\alpha\beta}\right] , \tag{4.43}$$

where the symmetry Eq. (2.19) of the stress tensor is used for the last term. Using the equation of motion Eq. (2.18) we have

$$\int dxdy \left[\frac{\partial}{\partial x_\beta}\sigma_{\alpha\beta}\dot{u}_\alpha + \frac{\partial \dot{u}_\alpha}{\partial x_\beta}\sigma_{\alpha\beta}\right] ,$$

$$= \int dxdy \frac{\partial}{\partial x_\beta}\left[\sigma_{\alpha\beta}\dot{u}_\alpha\right] \tag{4.44}$$

$$= \int_{\partial S} \dot{u}_\alpha \sigma_{\alpha\beta} n_\beta , \tag{4.45}$$

where the integral is now over the boundary of the system, and $\hat{n}$ is an outward unit normal.

By using the asymptotic forms Eq. (4.28) for σ_{yy} and the corresponding expressions for u_y from Eq. (4.21a), one finds that the total energy flowing into the crack tip per unit time is

$$J^{\text{tot}} = v\left(1-\beta^2\right)\frac{\alpha}{2\mu}\frac{1}{4\alpha\beta - (1+\beta^2)^2}K^2 , \tag{4.46}$$

where K is

$$K = \lim_{x\to 0^+}\sqrt{2\pi x}\,\sigma_{yy}(x,0) \tag{4.47}$$

One can also derive

$$J^{\text{tot}} = \lim_{k\to\infty} 2\pi v k^2 u_y(k)\,\sigma_{yy}(-k) , \tag{4.48}$$

which is the corresponding equation in k space.

4.5. Energy flux for crack with arbitrary motion

Consider a crack with tip located at $l(t)$. To the left of the crack tip, one applies known stresses σ^- to the crack face, and the body is otherwise unloaded. Consider σ and u on the x axis. Decompose σ into

$$\sigma = \sigma^+ + \sigma^-, \tag{4.49}$$

and notice that

$$u = u^-, \tag{4.50}$$

where σ^+ vanishes for $x < l(t)$, σ^- vanishes for $x > l(t)$, and u^- vanishes for $x > l(t)$. Suppose that one can obtain a relation of the following form:

$$G^+ * \sigma = G^- * u, \tag{4.51}$$

where $G^+(x,t) = 0$ for $x/t < v_{\max}$, and $G^-(x,t) = 0$ for $x/t > v_{\min}$, where $v_{\max}$ and $v_{\min}$ are the maximum and minimum velocities of the crack. Because of the causal structure of the Green's function, one always turns out to have in fact $G^+ \sim \delta(x - v_{\max}t)$, and $G^- \sim \delta(x - v_{\min}t)$. From Eq. (4.51) one can solve formally for the stress and strain fields as follows. Consider $x > l(t)$. Because u^- is zero ahead of the crack

$$G^- * u^- = \int dx'dt'G^-(x - x', t - t')\, u^-(x', t') \tag{4.52}$$

is certainly zero whenever $x' > l(t')$. The only chance for the integrand to be nonzero is for $x' < l(t')$. In this case

$$x - x' > l(t) - l(t') = \dot{l}(t^*)(t - t'), \tag{4.53}$$

where t^* is some time between t and t'. However, this means that

$$x - x' > (t - t')\, v_{\min} \tag{4.54}$$

so that by hypothesis $G^-(x - x', t - t')$ vanishes. The conclusion is that

$$\int dx'dt'G^-(x - x', t - t')\, u^-(x', t') = 0 \quad \text{for } x > l(t). \tag{4.55}$$

An identical argument shows that

$$\int dx'dt'G^+(x - x', t - t')\, \sigma^+(x', t') = 0 \quad \text{for } x < l(t). \tag{4.56}$$

Therefore, one can deduce from Eq. (4.51) that

$$G^+ * \sigma^+ = -\left[G^+ * \sigma^-\right] \theta(x - l(t)), \tag{4.57}$$

which has now been shown to be true both for $x > l(t)$, and for $x < l(t)$. But this relation can be inverted to give

$$\sigma^+ = -G^{+^{-1}} * \left\{\left[G^+ * \sigma^-\right] \theta(x - l(t))\right\}. \tag{4.58}$$

Since σ^- is a known stress to the rear of the crack tip, one has a formal solution in terms of the decomposed Green function.

Let us now try to find the stress intensity factor. This is the limit

$$K = \lim_{\epsilon \to 0_+} \sqrt{2\pi\epsilon}\, \sigma(\epsilon + l, t). \tag{4.59}$$

Examining Eq. (4.58) one can first assert that

$$G^+ * \sigma^- = \int dt'dx'G^+(x - x', t - t')\, \sigma^-(x', t') \tag{4.60}$$

goes to some finite value at the crack tip, since the applied stress σ^- is integrable, and it is $(G^+)^{-1}$ not G^+ which diverges badlye near the crack tip (this property must be verified by solving for G, but it is always true, both for strips and for plates.) So

$$K = -\lim_{\epsilon \to 0_+} \left\{\sqrt{\pi\epsilon}\, G^{+^{-1}} * \theta(x_1 - l(t_1))\right\}_{(l(t)+\epsilon, t)} \left[\sqrt{2}G^+ * \sigma^-\right]_{(l(t),t)}. \tag{4.61}$$

This result is the most general expression of the Kostrov/Eshelby/Freund calculation. It gives the stress intensity factor for a semi-infinite crack in an infinite plate with arbitrary time-dependent loading on the crack faces. Suppose one had a stationary crack of length l. Then the first term on the right hand side would give the effect of time-dependent loading, while the second term would be constant. The second term contains the information on how the stress field near a crack tip changes because of the velocity of the crack. The form of the stress field is a universal function of velocity, and loading only affects the stress intensity factor through a multiplicative term, as shown in Eq. (4.61).

4.5.1. General Equation Applied to Antiplane Shear

The goal is now to apply the general result Eq. (4.61) to the particular case of antiplane shear, and recover the result first found by Kostrov and Eshelby. The starting point is Eq. (3.2), the wave equation for u_z. Fourier transforming in both space and time by

$$\int dx\, dt\, e^{ikx + i\omega t} \tag{4.62}$$

one has that

$$\frac{\partial^2 u_z}{\partial y^2} = \left[k^2 - \omega^2/c^2 - 2ib\omega\right] u_z, \tag{4.63}$$

where an infinitesimal amount of damping has been added to take care of some convergence problems that will arise later. The result only works for a semi-infinite crack in infinite plates, so that will be the setting to try here. Only the solutions that decay as a function of y are allowed in an infinite plate, so Eq. (4.63) is solved by

$$u_z(k, y, \omega) = e^{-y\sqrt{k^2 - \omega^2/c^2 - 2ib\omega}}\, u(k, \omega). \tag{4.64}$$

Right on the x axis, taking $u = u_z(y = 0)$ and $\sigma = \sigma_{yz}(y = 0)$, one has that

$$G(k, \omega) = \frac{\sigma}{u} = -\mu\sqrt{k^2 - \omega^2/c^2 - 2ib\omega}. \tag{4.65}$$

Decompose G as

$$G = \frac{G^-}{G^+} \tag{4.66}$$

with

$$G^- = -\mu\sqrt{ik - i\omega/c + b} \tag{4.67a}$$

and

$$G^+ = \frac{1}{\sqrt{-ik - i\omega/c + b}}. \tag{4.67b}$$

To verify that this decomposition satisfies the conditions of the preceding section, find

$$G^+(x,t) = \int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\frac{e^{-ikx-i\omega t}}{\sqrt{-ik - i\omega/c + b}} \tag{4.68}$$

$$= \int \frac{dp}{2\pi}\frac{d\omega}{2\pi}\frac{e^{-ipx-i\omega(t-x/c)}}{\sqrt{-ip+b}}, \tag{4.69}$$

with $p = k + \omega/c$

$$= \delta(t - x/c)\int \frac{dp}{2\pi}\frac{e^{-ipx}}{\sqrt{-ip+b}}. \tag{4.70}$$

When $x < 0$, one must close the contour in the upper half plane, and as the branch cut is in the lower half plane one gets zero. When $x > 0$, deform the contour to surround the branch cut, and get

$$\int_0^\infty \frac{dp}{2\pi}\frac{2e^{-px}}{\sqrt{p+b}} = \frac{1}{\sqrt{\pi x}} \tag{4.71}$$

Therefore

$$G^+(x,t) = \delta(t - x/c)\frac{\theta(x)}{\sqrt{\pi x}}. \tag{4.72}$$

To find $G^{+^{-1}}$ one must do

$$G^{+^{-1}}(x,t) = \int \frac{dk}{2\pi}\frac{d\omega}{2\pi}e^{-ikx-i\omega t}\sqrt{-ik - i\omega/c + b} \tag{4.73}$$

$$= \delta(t - x/c)\int \frac{dp}{2\pi}e^{-ipx}\sqrt{-ip+b}. \tag{4.74}$$

One cannot legitimately deform the contour to perform this integral, but can instead write that

$$\int \frac{dp}{2\pi}e^{-ipx}\sqrt{-ip+b} = \frac{\partial}{\partial x}\int \frac{dp}{2\pi}\frac{e^{-ipx}}{\sqrt{-ip+b}}, \tag{4.75}$$

obtaining in this way

$$G^{+^{-1}}(x,t) = \delta(t - x/c)\frac{\partial}{\partial x}\left[\frac{\theta(x)}{\sqrt{\pi x}}\right]. \tag{4.76}$$

Returning to Eq. (4.61), there are two integrals to carry out. The singular one is

$$\sqrt{\pi\epsilon}\int dx_1 dt_1\, \delta(t_1 - x/c)\left[\frac{\partial}{\partial x_1}\frac{\theta(x_1)}{\sqrt{\pi x_1}}\right]\theta(l(t) + \epsilon - x_1 - l(t - t_1)) \tag{4.77}$$

$$= \sqrt{\pi\epsilon}\int \frac{dx_1}{\sqrt{\pi}}\left[\frac{\partial}{\partial x_1}\frac{\theta(x_1)}{\sqrt{x_1}}\right]\theta(\epsilon - x_1[1 - v(t)/c]), \tag{4.78}$$

since only very small x_1 are important

$$= \sqrt{\pi\epsilon}\int \frac{dx_1}{\sqrt{\pi}}\frac{\theta(x_1)}{\sqrt{x_1}}(1 - v(t)/c)\delta(l(t) + \epsilon - x_1 - l(t) + v(t)/cx_1), \tag{4.79}$$

$$= \sqrt{1 - v(t)/c}. \tag{4.80}$$

The next piece which is needed is

$$\tilde{K} \equiv -\sqrt{2}\int dx_1 dt_1 \delta(t_1 - x_1/c)\frac{\theta(x_1)}{\sqrt{\pi x_1}}\sigma^-(l(t) - x_1, t - t_1) \tag{4.81}$$

$$= -\sqrt{2}\int dx_1 \frac{\theta(x_1)}{\sqrt{\pi x_1}}\sigma^-(l(t) - x_1, t - x_1/c). \tag{4.82}$$

This is as far as one can take matters without an explicit expression for σ^-. However, in the case of time independent loading, one simply has

$$\tilde{K} = -\sqrt{2}\int dx_1 \frac{\theta(x_1)}{\sqrt{\pi x_1}}\sigma^-(l(t) - x_1). \tag{4.83}$$

For the particular case where

$$\sigma(x) = \sigma_0\theta(x), \tag{4.84}$$

one gets

$$\tilde{K} = -\sigma_0\frac{4}{\sqrt{2\pi}}\sqrt{l}. \tag{4.85}$$

Notice that Eq. (4.80) reduces to unity when $v \to 0$. This means that in the case of time independent loading, $\tilde{K}$ is simply the stress intensity factor one would have had if the crack had been located unmoving at $l(t)$ for all time. For the moving crack, we have

$$K = \sqrt{1 - v(t)/c}\,\tilde{K}(l(t)). \tag{4.86}$$

One computes the stress singularity that would have developed if one had a static crack of the present length, $l(t)$, and multiplies by a function of the instantaneous velocity. It should be stressed that all details of the history of the crack motion are irrelevant, and only the velocity and loading configuration are needed to find the stress fields sufficiently close to the tip. As a consequence, one can use Eq. (4.41) to determine the energy flow to the tip of the crack. It is

$$J_y^{\text{tot}} = \frac{v(1 - v/c)\tilde{K}^2}{2\alpha\mu}. \tag{4.87}$$

From this expression, one can deduce an equation of motion for the crack. The rate at which energy enters the tip of the crack must be equal to $v\Gamma(v)$, which is the (phenomenological) energy needed to create new fracture surfaces. There is nothing to prevent the fracture energy from being a function of velocity, but the notions of local equilibrium which have prevailed until now strongly suggest that it should not depend upon anything else. So one must have

$$\Gamma(v) = \frac{(1 - v/c)\tilde{K}^2(l)}{2\alpha\mu}, \tag{4.88}$$

which may be rewritten as

$$\frac{\mu\Gamma\pi}{4l\sigma_0^2} = \sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{4.89}$$

or

$$\frac{l_0}{l} = \sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{4.90}$$

4.5.2. General Equation Applied to Plane Stress

The same analysis may be carried out for thin plates under tension. Everything proceeds as before, except that it is not possible to display simple analytic expressions, although there are excellent approximations that can be put in simple form. I will just record the final result, due to Freund, that gives the energy flux to the tip of the crack. The flux per unit length extension of the crack is to good approximation

$$\Gamma(v) = \frac{(1 - v/c_R)\tilde{K}^2(l)}{2\tilde{\lambda}}, \tag{4.91a}$$

$$\Rightarrow \frac{E\Gamma(v)}{(1-\nu^2)\tilde{K}^2(l)} = 1 - \frac{v}{c_R}, \tag{4.91b}$$

where c_R is the Rayleigh wave speed (the speed at which the function D given in Eq. (4.28b) vanishes), $\tilde{K}$ is still given by Eq. (4.82), using σ_{yy} on the x axis for σ. In the case of time-independent loading described by Eq. (4.86) one gets

$$\frac{l_0}{l} = 1 - v/c, \tag{4.92}$$

with

$$l_0 = \frac{\pi\Gamma\tilde{\lambda}}{4\sigma_0^2}, \tag{4.93}$$

or if Γ is velocity independent,

$$v = c_R\left(1 - \frac{l_0}{l}\right). \tag{4.94}$$

Amazingly enough, the exact analysis reproduces the result of the simplest scaling arguments, Eq. (4.6).

5. Motion of a Crack in a Strip

The power of the methods which find a general equation of motion for a semi-infinite crack in an infinite plate obscures the fact that the problem of crack motion is still not completely solved. For example, when a crack moves in a system with boundaries, wave reflections from the boundaries allow the crack to interact with its past history, and it no longer acts as a particle without inertia. Because of these reflections, it is no longer possible to write a Green function for the system in the form Eq. (4.51) The equation of motion for a crack in a strip can only be obtained in the limit where the acceleration is very small, and the net result is that the crack acquires and effective mass. The techniques needed to develop this equation provide an example of how to carry out a perturbative calculation for dynamic cracks, and also show how to work with energy conservation laws for cracks in Fourier space.

5.1. Virial Theorem

In order to calculate the total energy of a moving crack in a strip, it is useful to begin with a virial theorem. We will find a relation between the kinetic and potential energies in general. The kinetic energy of a moving crack is

$$\mathrm{KE} = \frac{1}{2}\int dxdy \sum_\alpha \dot{u}_\alpha^2, \tag{5.1}$$

while the potential energy is

$$\mathrm{PE} = \frac{1}{2}\int dxdy \sum_{\alpha\beta} \sigma_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \tag{5.2}$$

$$= \frac{1}{2}\int dxdy \sum_{\alpha\beta}\frac{\partial}{\partial x_\beta}\left[u_\alpha\sigma_{\alpha\beta}\right] - \frac{1}{2}\int dxdy \sum_\alpha \ddot{u}_\alpha u_\alpha, \tag{5.3}$$

where we have used the equation of motion

$$\ddot{u}_\alpha = \frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} \tag{5.4}$$

It follows then that the total energy is given by

$$E = \mathrm{KE} + \mathrm{PE} = \frac{1}{2}\int dxdy \sum_\alpha\left[(\dot{u}_\alpha)^2 - u_\alpha\ddot{u}_\alpha\right] + \frac{1}{2}\int_S d\hat{n}\cdot\vec{w}, \tag{5.5}$$

with the last integral over the boundaries of the system, and

$$w_\beta = u_\alpha\sigma_{\alpha\beta}. \tag{5.6}$$

We can now use this result to simplify the problem of a crack moving in the steady state. The time derivatives can all be converted to spatial derivatives, and integrating by parts we have

$$E = \int dxdyc^2 \sum_\alpha \frac{\partial u_\alpha}{\partial x} + \frac{1}{2}\int_S d\hat{n}\cdot\vec{w}. \tag{5.7}$$

5.2. Crack in a Strip: Antiplane Shear

I will now consider a semi-infinite crack moving at constant speed v in an infinite strip of unit width. The strip is loaded by a stress σ_∞ applied to the crack faces so as to push them apart, while the upper and lower surfaces of the strip are held fixed. The solution of a problem in which the upper and lower edges of the strip are clamped at a fixed height

$$\delta = \frac{\sigma_\infty}{\mu} \tag{5.8}$$

while the faces of the crack are not loaded is virtually identical. The equation of motion is Eq. (3.2), and in a frame co-moving with the crack one has boundary conditions

$$u_z(x, y = 1) = 0 \tag{5.9a}$$
$$u_z(x > 0, y = 0) = 0 \tag{5.9b}$$
$$\sigma_{yz}(x < 0, y = 0) = -\sigma_\infty. \tag{5.9c}$$

Here $\mu \partial u_z/\partial y = \sigma_{yz}$. The equation of motion in k space is

$$\frac{\partial^2 u_z}{\partial y^2} = \alpha^2 k^2 u_z, \tag{5.10}$$

where

$$\alpha^2 = 1 - \frac{v^2}{c^2}. \tag{5.11}$$

Therefore

$$u(k, y) = \frac{\sigma(k)}{\alpha k \mu} \sinh \alpha k y + u(k) \cosh \alpha k y. \tag{5.12}$$

When stresses and displacements are written without a subscript, it means they are evaluated at $y = 0$. As a consequence of boundary condition Eq. (5.9a), one sees that

$$u_z(k, 1) = 0 \tag{5.13}$$

$$\Rightarrow \frac{\sigma(k)}{u(k)} = -\mu \alpha k \coth \alpha k. \tag{5.14}$$

From the remaining two boundary conditions, one has that

$$\sigma^-(k) = \frac{-\sigma_\infty}{ik} \Rightarrow \sigma(k) = \sigma^+ + \sigma^- = \sigma^+(k) - \frac{\sigma_\infty}{ik}$$
$$u(k) = u^-(k), \tag{5.15}$$

where

$$\sigma^\pm \equiv \int dx e^{ikx} \theta(\pm x) \sigma_{yz}(x, 0), \tag{5.16}$$

and u^- is defined similarly. Define

$$-\frac{\sigma(k)}{u(k)} \equiv F(k) = \mu \alpha k \coth \alpha k, \tag{5.17}$$

using Eq. (5.14). It follows that

$$-u(k) F(k) = \left[\sigma^+(k) - \frac{\sigma_\infty}{ik}\right] \tag{5.18}$$

$$\Rightarrow -ku^-(k) F(k) = k\sigma^+(k) + i\sigma_\infty \tag{5.19}$$

$$\Rightarrow u(k) = u^-(k) = \frac{\text{const.}}{kF^-(k)}$$

$$\Rightarrow u(k) = \frac{\sigma_\infty F^-(0)}{i\mu k F^-(k)}. \tag{5.20}$$

The proper expression when the need to make the Fourier transforms converge properly is considered is

$$u(k) = \frac{E_\infty F^-(0)}{(ik + \epsilon) F^-(k)} \tag{5.21}$$

and ϵ is infinitesimal. It is in fact possible to carry out the decomposition. One has that

$$F^-(k) = \sqrt{\pi\mu} \frac{\Gamma(1 + i\alpha k/\pi)}{\Gamma(1/2 + ik\alpha/\pi)}. \tag{5.22}$$

However, this relation is not needed to compute the energy of the moving crack. The full expression for u_z is

$$u_z(k, y) = \frac{\sigma_\infty}{i\mu k} \frac{F^-(0)}{F^-(ik)} \left[\coth \alpha k \sinh \alpha k y + \cosh \alpha k y\right]. \tag{5.23}$$

5.2.1. Energy of Antiplane Shear Crack in Strip

In steady state, the kinetic energy of the crack is

$$\text{KE} = \frac{1}{2} \int dx dy \frac{v^2}{c^2} \left(\frac{\partial u_z}{\partial x}\right)^2 \tag{5.24}$$

$$= \frac{1}{2} \int \frac{dk}{2\pi} dy \frac{v^2}{c^2} k^2 u_z(k, y) u_z(-k, y). \tag{5.25}$$

Note that

$$F(k) = F(-k) \Rightarrow \frac{F^-(k)}{F^+(k)} = \frac{F^-(-k)}{F^+(-k)} \tag{5.26}$$

$$\Rightarrow F^-(k) F^+(-k) = F^+(k) F^-(-k) = \text{const.}. \tag{5.27}$$

In the expression for $u_z(k, y) u_z(-k, y)$, the only complicated term is

$$\frac{F^-(0) F^-(0)}{F^-(k) F^-(-k)} = \frac{F(0)}{F(k)}. \tag{5.28}$$

The kinetic energy then reduces to the following integral,

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi\alpha^2 k^2} \frac{v^2\sigma_\infty^2}{\mu c^2} \left[\frac{\sinh 4\alpha k - \sinh 2\alpha k - 4\alpha k}{\sinh 2\alpha k}\right]. \tag{5.29}$$

According to Eq. (5.7), the only other contribution to the energy is a surface integral, which is

$$\int dx u_z \sigma_\infty \tag{5.30}$$

$$= l(t)\,\delta\sigma_\infty + \int dx\,(u_z - \delta\theta(-x))\,\sigma_\infty, \tag{5.31}$$

measuring l with respect to some far distant end of the strip,

$$= l(t)\,\delta\sigma_\infty + \delta\sigma H(v) \tag{5.32}$$

where the careful use of Eq. (5.21) shows that

$$H(v) = i\frac{\partial}{\partial k} \ln F^-(k)\Big|_{k=0}. \tag{5.33}$$

To calculate $H(v)$, consider

$$\int \frac{dk}{2\pi} \left[\frac{\partial}{\partial v} \frac{F^-(0)}{(ik+\epsilon)F^-(k)}\right] \frac{F^+(0)}{(-ik+\epsilon)F^+(-k)} \tag{5.34}$$

$$= \frac{-2\pi i}{2\pi} \left[\frac{\partial}{\partial v} \frac{F^-(0)}{2\epsilon F^-(-i\epsilon)}\right] \frac{F^+(0)}{(-i)F^+(i\epsilon)} = \frac{1}{2}\frac{\partial}{\partial v} H(v). \tag{5.35}$$

On the other hand one can also write this integral as

$$\int \frac{dk}{2\pi} \left[\frac{\partial}{\partial v} \frac{F^-(0)}{(ik+\epsilon)F^-(k)}\right] \frac{F(k)F^+(0)}{(-ik+\epsilon)F^-(-k)} \tag{5.36}$$

$$= \int \frac{dk}{2\pi} \frac{1}{k^2+\epsilon^2} \frac{1}{2} F(k) \frac{\partial}{\partial v} \left[\frac{F^-(0)F^+(0)}{F^-(k)F^-(-k)}\right] \tag{5.37}$$

$$= \int \frac{dk}{2\pi} \frac{1}{k^2} \frac{1}{2} F(k) \frac{\partial}{\partial v} \left[\frac{1}{F(k)}\right] \tag{5.38}$$

$$= \frac{1}{2}\frac{\partial}{\partial v} \int \frac{dk}{2\pi} \frac{1}{k^2} \ln \frac{1}{F(k)} \tag{5.39}$$

so that

$$H(v) = \int \frac{dk}{2\pi} \frac{1}{k^2} \ln \left[\frac{F(k,0)}{F(k,v)}\right]. \tag{5.40}$$

The full energy of the strip, taking both top and bottom halves into account, turns out to be

$$l(t)\,\delta\sigma_\infty + \delta\sigma_\infty \left[v\frac{\partial}{\partial v} + 1\right] H(v), \tag{5.41}$$

with

$$H(v) = \int \frac{dk}{2\pi} \frac{1}{k^2} \ln \left[\frac{F(k,0)}{F(k,v)}\right]. \tag{5.42}$$

This expression was obtained by techniques that will be demonstrated for plane stress in a later section. It must be true that if one takes the indicated derivatives of $H(v)$ one will obtain the expression for the kinetic energy in Eq. (5.29), but in fact I have never checked for antiplane shear to see that it really works!

In the case where the top edge of the strip is clamped, the energy of the system is instead

$$-l(t)\,\delta\sigma_\infty + \delta\sigma_\infty \left[v\frac{\partial}{\partial v} - 1\right] H(v). \tag{5.43}$$

One has therefore the equation of motion for the crack

$$\Gamma(v)\,v - v\delta\sigma_\infty + \dot v \delta\sigma_\infty \frac{\partial}{\partial v} \left[v\frac{\partial}{\partial v} - 1\right] H(v) = 0. \tag{5.44}$$

This expression shows explicitly how acceleration robs the crack tip of energy. It should only be valid in the limit of adiabatic acceleration, and these expressions can be recovered as the result of perturbing about steady state motion, with acceleration as the small parameter.

5.3. Crack in a Strip: Plane Stress

Consider now the problem in which a semi-infinite crack under plane stress moves at constant velocity v through a strip of half-width 1. The calculation will differ from the preceding one because it will be carried out for plane stress rather than antiplane strain. We are interested in two related problems. In both cases, the symmetry of the problem allows one to restrict attention to the upper half plane alone. The first problem, which we shall refer to as problem (A) since all the stresses are after the crack, features a crack moving in steady state at velocity v in a strip obeying boundary conditions

$$u_y(x,1) = \delta \tag{5.45a}$$

$$u_x(x,1) = 0 \tag{5.45b}$$

$$\sigma_{xy}(x,0) = 0 \tag{5.45c}$$

$$\sigma_{yy}(x,0) = 0 \text{ for } x < 0 \tag{5.45d}$$

$$u_y(x,0) = 0 \text{ for } x > 0. \tag{5.45e}$$

The second problem (B) differs from the first only because one subtracts from stress and displacement fields the solution for a uniformly stressed plate without a crack; now all stresses are behind the crack. Stresses in a uniform plate may be found from the constitutive equations Eq. (2.21) and are

$$u_x = 0; \quad u_y = \delta y;$$

$$\sigma_{yy} = \rho\delta c_l^2 \equiv \sigma_\infty; \quad \sigma_{xx} = \rho\delta\left(c_l^2 - 2c_t^2\right); \quad \sigma_{xy} = 0. \tag{5.46}$$

Subtracting these fields is equivalent to imposing boundary conditions

$$u_x(x,1)=0 \tag{5.47a}$$
$$u_y(x,1)=0 \tag{5.47b}$$
$$\sigma_{xy}(x,0)=0 \tag{5.47c}$$
$$u_y(x,0)=0, \text{for } x>0 \tag{5.47d}$$
$$\sigma_{yy}(x,0)=-\sigma_\infty, \text{for } x<0. \tag{5.47e}$$

It is most convenient to employ the Wiener-Hopf technique on problem (B); all of our equations for stress and strain will be for problem (B), and if we wish information of problem (A), we will add the required fields explicitly.

The equations of motion for an elastic medium may be expressed in terms of two scalar potentials which obey the wave equation. One of them is a potential for longitudinal waves, the other is the potential for transverse waves, and they have been derived in Eq. (4.20). From them one may derive the displacements, by

$$\vec{u}=\vec{\nabla}v_l+\vec{\nabla}\times v_t. \tag{5.48}$$

In the steady state one has

$$\begin{aligned} \alpha^2\frac{\partial^2 v_l}{\partial x^2}+\frac{\partial^2 v_l}{\partial y^2}&=0,\\ \beta^2\frac{\partial^2 v_t}{\partial x^2}+\frac{\partial^2 v_t}{\partial y^2}&=0, \end{aligned} \tag{5.49}$$

where as in Eq. (4.19) $\alpha^2=1-v^2/c_l^2$, and $\beta^2=1-v^2/c_t^2$. Then one can write

$$\begin{aligned} v_l&=A_{sl}(k)\sinh\alpha ky+A_{cl}(k)\cosh\alpha ky\\ v_t&=A_{st}(k)\sinh\beta ky+A_{ct}(k)\cosh\beta ky \end{aligned} \tag{5.50}$$

In terms of these constants one has

$$\begin{aligned} u_x=&A_{cl}(k)\beta k\sinh\beta ky+A_{st}(k)\beta k\cosh\beta ky\\ &-ik\left(A_{sl}(k)\sinh\alpha ky+A_{cl}(k)\cosh\alpha ky\right) \end{aligned} \tag{5.51a}$$

$$\begin{aligned} u_y=&ik\left(A_{st}(k)\sinh\beta ky+A_{ct}(k)\cosh\beta ky\right)\\ &+\alpha A_{cl}(k)k\sinh\alpha ky+\alpha A_{sl}(k)k\cosh\alpha ky \end{aligned} \tag{5.51b}$$

One can work out the stresses using Eq. (5.50) and Eq. (5.51).

Three of the coefficients $A_{cl}(k)\ldots A_{st}(k)$ can be found from the three boundary conditions which apply to all x, and hence all to all k, (Eq. (5.47)a′-c′). Define

$$u(k)\equiv u_y(k,0). \tag{5.52}$$

Then

$$A_{st}(k)\equiv a_{st}(k)u(k)=\frac{i\left(2\alpha\beta\, s_\alpha s_\beta-2c_\alpha c_\beta+\beta^2+1\right)u(k)}{B(k)} \tag{5.53a}$$

$$A_{cl}\equiv a_{cl}u(k)=-\frac{\left(\left(\beta^2+1\right)s_\alpha s_\beta-\alpha\beta\left(1+\beta^2\right)c_\alpha c_\beta+2\alpha\beta\right)u(k)}{\alpha B(k)} \tag{5.53b}$$

$$A_{ct}\equiv a_{ct}u(k)=\frac{2iu(k)}{\left(\beta^2-1\right)k} \tag{5.53c}$$

$$A_{sl}\equiv a_{sl}u(k)=\frac{\left(\beta^2+1\right)u(k)}{\left(\alpha\beta^2-\alpha\right)k} \tag{5.53d}$$

with

$$B(k)=\left(\beta^2-1\right)k\left(c_\alpha s_\beta-\alpha\beta s_\alpha c_\beta\right). \tag{5.53e}$$

We have used the abbreviated notation

$$s_\alpha=\sinh\alpha k;\; s_\beta=\sinh\beta k;\; c_\alpha=\cosh\alpha k;\; c_\beta=\cosh\beta k. \tag{5.54}$$

One now uses the Wiener-Hopf trick. Define

$$\sigma^+=\int_0^\infty\sigma_{yy}(x,0)e^{ikxx}dx;\;\sigma^-=\int_{-\infty}^0\sigma_{yy}(x,0)e^{ikx}dx, \tag{5.55}$$

with u^+ and u^- defined similarly. One should note that σ^- has no poles in the lower half plane, σ^+ no poles in the upper half plane. Similarly u^+ has no poles in the upper half plane, and u^- has no poles in the lower half plane.

One writes that

$$\sigma(k)=\sigma^++\sigma^-=\sigma^+-\frac{\sigma_\infty}{ik}$$
$$u(k)=u^-.$$

Then defining

$$F(k)=-\frac{\sigma}{u} \tag{5.56}$$

we have

$$-F(k)u^-=\sigma^+-\frac{\sigma_\infty}{ik}. \tag{5.57}$$

The point of defining $F(k)$ lies in the fact that since it is a ratio of two quantities expressible in terms of the $A_{cl}\ldots A_{st}$, the unknown function $u(k)$ which appears in all of these does not matter. The function $F(k)$ is

$$F(k)=k\rho c_t^2\frac{s_\alpha s_\beta\left\{\left(\beta^2+1\right)^2+4(\alpha\beta)^2\right\}-c_\alpha c_\beta\alpha\beta\left\{\left(\beta^2+1\right)^2+4\right\}+4\alpha\beta\left(\beta^2+1\right)}{\alpha\left(1-\beta^2\right)\left(\alpha\beta s_\alpha c_\beta-c_\alpha s_\beta\right)}. \tag{5.58}$$

For small k,

$$F\to f_0=\rho c_l^2, \tag{5.59}$$

while for large real k

$$F\to|k|f_\infty=\frac{|k|\rho c_t^2}{\alpha\left(\beta^2-1\right)}\left\{\left(\beta^2+1\right)^2-4\alpha\beta\right\}. \tag{5.60}$$

Let us suppose that we can write $F(k)$ in the following way, as

$$F(k) = \frac{F^-(k)}{F^+(k)}, \tag{5.61}$$

where F^- has no poles in the lower half plane, and F^+ has no poles in the upper half plane. Then we can write

$$-kF^-(k)\,u^- = k\sigma^+ F^+(k) + i\sigma_\infty F^+(k). \tag{5.62}$$

One has set equal an expression with no poles in the upper half plane to one with no poles in the lower half plane. Therefore, both must equal a constant. The constant can be fixed by examining the behavior of the expressions for $k \to 0$. Notice that

$$\lim_{k\to 0} u \sim \frac{\delta}{ik}; \tag{5.63}$$

this statement follows from the fact that u vanishes for large positive x and goes to δ for large negative x. So one has

$$u_y(k,0) = u^-(k) = u(k) = \frac{\delta F^-(0)}{ikF^-(k)}. \tag{5.64}$$

The problem is now solved, apart from the difficulties of decomposing F into F^+ and F^-.

The following equation is frequently useful. We have that

$$F(k) = F(-k) \Rightarrow \frac{F^-(k)}{F^+(k)} = \frac{F^-(-k)}{F^+(-k)}$$

$$\Rightarrow F^-(k)\,F^+(-k) = \text{const.}$$

The constant is arbitrary, since F^+ and F^- can always both be multiplied by any constant and still satisfy all the properties that define them. A simple choice is

$$F^-(k) = \frac{1}{F^+(-k)}, \tag{5.65}$$

which implies that

$$F^-(0) = \sqrt{f_0}. \tag{5.66}$$

For many purposes the limiting forms of F for large and small k are sufficient. One can write

$$F(k) \approx \sqrt{k^2 f_\infty^2 + f_0^2}, \tag{5.67}$$

where f_∞ and f_0 are chosen to get the two limits right for real k. Then one has

$$F^-(k) \approx \sqrt{f_0 + ikf_\infty}, \quad F^+(k) \approx \frac{1}{\sqrt{f_0 - ikf_\infty}}. \tag{5.68}$$

These forms are adequate for studying behavior near the tip, where only large k is important, and interpolate sensibly to large distances, although on the wrong length scale.

5.4. Perturbative solution for accelerating plane stress cracks

Consider a crack in a strip which accelerates slowly. The location of the crack tip is given by $l(t)$, and the dimensionless parameter indicating when acceleration is sufficiently slow is

$$b\ddot{l}/c_l^2 \ll 1,$$

where b is the half-width of the strip. The velocity of the crack is required to change slowly over the timescale in which sound communicates with the boundaries. Then to leading order the potentials for the crack are given by

$$\phi_a(x,y,t) = \phi_a^s(x - l(t), y, v), \tag{5.69}$$

where ϕ_a^s is the steady-state potential appropriate for a crack moving at constant velocity $v = \dot{l}$, and $a = l, t$ can give either the longitudinal or transverse potential. However to be consistent to order $\dot{v} = \ddot{l}$, for an accelerating crack, one must write

$$\phi_a(x,y,t) = \phi_a^s(x - l(t), y, v) + \dot{v}\Delta\phi_a(x - l(t), y, v).$$

In the accelerating frame of reference $x' = x - l(t)$ one has

$$\left.\frac{\partial}{\partial t}\right|_x = \left.\frac{\partial}{\partial t}\right|_{x'} - v(t)\frac{\partial}{\partial x'},$$

so the second time derivative of the potentials is given by

$$\left(\left.\frac{\partial}{\partial t}\right|_{x'} - v(t)\frac{\partial}{\partial x'}\right)^2 \phi_a$$

$$= \frac{\partial^2\phi_a}{\partial t^2} - 2v\frac{\partial^2\phi_a}{\partial t\,\partial x'} - \dot{v}\frac{\partial\phi_a}{\partial x'} + v^2\frac{\partial^2\phi_a}{\partial x'^2}. \tag{5.70}$$

In the accelerating frame, the potentials depend upon time only through their dependence upon $v(t)$. Therefore

$$\frac{\partial}{\partial t} \to \dot{v}\frac{\partial}{\partial v}.$$

Inserting Eq. (5.69) into Eq. (5.70) and working only to lowest order in $\dot{v}$ gives

$$\left.\frac{\partial^2\phi_a}{\partial t^2}\right|_x = v^2\frac{\partial^2}{\partial x'^2}\left[\phi_a^s(x',y,v) + \Delta\phi_a(x',y,v)\right] - \dot{v}\left[2v\frac{\partial^2}{\partial v\,\partial x'} + \frac{\partial}{\partial x'}\right]\phi_a^s(x',y,v) + O(b\dot{v}/c_l^2)^2.$$

We now are ready to write down the wave equation to order $\dot{v}$. Fourier transforming so that $x' \to k$ one finds that

$$\frac{1}{c_a^2}\left\{-v^2k^2\left[\phi_a^s + \Delta\phi_a\right] + ik2v\dot{v}\frac{\partial}{\partial v}\phi_a^s + ik\dot{v}\phi_a^s\right\} = \left[\frac{\partial^2}{\partial y^2} - k^2\right]\left[\phi_a^s + \Delta\phi_a\right].$$

Since $\phi_a^s(k,y,v)$ is defined by the fact that it obeys the wave equation for each velocity v one can simplify this last expression to read

$$\left[\frac{\partial^2}{\partial y^2} - k^2\left(1 - \frac{v^2}{c_a^2}\right)\right]\Delta\phi_a = \frac{ik\dot{v}}{c_a^2}\left(2v\frac{\partial}{\partial v} + 1\right)\phi_a^s. \tag{5.71}$$

One may construct a solution of this inhomogeneous equation as follows: consider

$$\left[\frac{\partial^2}{\partial y^2} - k^2\left(1 - \frac{v^2}{c_a^2}\right)\right]\phi_a^s = 0$$

$$\Rightarrow \frac{\partial^2}{\partial v^2}\left[\frac{\partial^2}{\partial y^2} - k^2\left(1 - \frac{v^2}{c_a^2}\right)\right]\phi_a^s = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial y^2} - k^2\left(1 - \frac{v^2}{c_a^2}\right)\right]\frac{\partial^2}{\partial v^2}\phi_a^s = -2\frac{k^2}{c_a^2}\left(2v\frac{\partial}{\partial v} + 1\right)\phi_a^s.$$

Comparison of this last equation with Eq. (5.71) shows that

$$\Delta\phi_a^i = \frac{1}{2ik}\frac{\partial^2}{\partial v^2}\phi_a^s \tag{5.72}$$

is a solution of the inhomogeneous equation Eq. (5.71). The complete solution of the problem

$$\Delta\phi_a = \Delta\phi_a^i + \Delta\phi_a^h$$

is obtained by adding a function which satisfies the homogeneous wave equation on the left hand side of Eq. (5.71) so as to bring the result into accord with the boundary conditions. The boundary conditions must be written down fairly carefully. Let the sample extend from $-L$ to L in the laboratory frame, and choose a small convergence factor ϵ such that $\epsilon L \ll 1$. The boundary conditions in the accelerating frame are

$$u_x(x',1) = 0 \tag{5.73a}$$
$$u_y(x',1) = 0 \tag{5.73b}$$
$$\sigma_{xy}(x',0) = 0 \tag{5.73c}$$
$$u_y(x',0) = 0, \text{for } x' > 0 \tag{5.73d}$$
$$\sigma_{yy}(x',0) = -\sigma_\infty e^{\epsilon x'}, \text{for } x' < 0. \tag{5.73e}$$

The inclusion of ϵ makes all the Fourier transforms well defined, but causes the fields to decay negligibly within the physical boundaries of the sample.

All of the fields u and σ can be obtained from the potentials ϕ_a by action with linear operators. Therefore, the fields u and σ may be written in the form

$$u_\alpha = u_\alpha^s + \dot{v}\Delta u_\alpha$$

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^s + \dot{v}\Delta\sigma_{\alpha\beta},$$

where the first term is the steady state result appropriate for velocity v, and the second is derived from the $\Delta\phi_a$'s. Since the steady state fields already obey the boundary conditions Eq. (5.73), the perturbations must obey

$$\Delta u_x(x',1) = 0 \tag{5.74a}$$
$$\Delta u_y(x',1) = 0 \tag{5.74b}$$
$$\Delta\sigma_{xy}(x',0) = 0 \tag{5.74c}$$
$$\Delta u_y(x',0) = 0, \text{for } x' > 0 \tag{5.74d}$$
$$\Delta\sigma_{yy}(x',0) = 0, \text{for } x' < 0. \tag{5.74e}$$

Since the inhomogeneous solution Eq. (5.72) is formed from the steady state solution by the action of a linear operator, one can find all the fields Δu_α^i, $\Delta\sigma_{\alpha\beta}^i$ which result from the inhomogeneous solution merely by acting on the steady state solutions with the same operator:

$$\Delta u_\alpha^i = \frac{1}{2ik}\frac{\partial^2}{\partial v^2}u_\alpha^s$$
$$\Delta\sigma_{\alpha\beta}^i = \frac{1}{2ik}\frac{\partial^2}{\partial v^2}\sigma_{\alpha\beta}^s. \tag{5.75}$$

These fields are close to being the needed solution. The steady-state solutions satisfy all of the boundary conditions Eq. (5.73), and it is easy to verify that the solutions Eq. (5.75) satisfy the first three of Eq. (5.74). However they do not obey the last two equations, so we will need to add some solution of the homogeneous problem.

Before doing so, I would like to recall the solution of the steady-state problem, but taking care with ϵ. After transforming the final two boundary conditions of Eq. (5.73) into Fourier space one finds that for fields as a function of k and evaluated at $y = 0$,

$$-F(k)\,u_y^{s\,-} = \frac{-\sigma_\infty}{ik+\epsilon} + \sigma_{yy}^{s\,+}$$

$$\Rightarrow -F^-(k)\,u_y^{s\,-} + \frac{F^+(i\epsilon)\,\sigma_\infty}{ik+\epsilon} = \frac{\left[F^+(k) - F^+(i\epsilon)\right]}{ik+\epsilon}(-\sigma_\infty) + F^+(k)\,\sigma_{yy}^{s\,+}.$$

The left hand side is free of poles in the lower half plane; the right hand side is free of poles in the upper half plane, so the two must equal a constant. Checking the asymptotic behavior of either side as $k \to 0$ one finds the constant to be zero. So

$$u_y^s = u_y^{s\,-} = \frac{F^+(i\epsilon)\,\sigma_\infty}{(ik+\epsilon)\,F^-(k)}$$

$$= \frac{\delta}{(ik+\epsilon)}\frac{F^-(i\epsilon)}{F^-(k)} \tag{5.76}$$

to order ϵ. With this background we now return to the problem of finding Δu_y. As before, we evaluate all functions at $y = 0$. We have that

$$\Delta u_y = \Delta u_y^i + \Delta u_y^h.$$

Since fields with superscipt h obey the homogeneous equations we must have that

$$-F(k)\,\Delta u_y^{h\,-} = -F(k)\,\Delta u_y^h = \Delta\sigma_{yy}^{h\,+} = \Delta\sigma_{yy}^h.$$

Thus we find immediately that

$$-F(k)\,k\left[\Delta u_y^{-} - \Delta u_y^{i\,-}\right] = k\left[\Delta\sigma_{yy}^{+} - \Delta\sigma_{yy}^{i\,+}\right].$$

Multiplication by k on both sides is necessary to get rid of the pole at $k = 0$ which appears in Eq. (5.75). As before we find that

$$kF^-(k)\,\Delta u_y^- - kF^-(k)\,\Delta u_y^{i\,-} = C$$

$$\Rightarrow \Delta u_y^- = \Delta u_y^{i-} + \frac{C}{kF^-(k)}.$$

Using Eq. (5.76) and Eq. (5.75) one has that

$$\Delta u_y = \frac{1}{2ik}\frac{\partial^2}{\partial v^2}\frac{\delta}{(ik+\epsilon)}\frac{F^-(i\epsilon)}{F^-(k)} + \frac{C}{kF^-(k)}.$$

To find the constant C, one can look at the behavior of Δu_y as $k \to 0$. There can be no pole there, so

$$\lim_{k\to 0}\Delta u_y = \lim_{k\to 0}\frac{1}{2ik}\frac{\partial^2}{\partial v^2}\frac{\delta}{(\epsilon)}\frac{F^-(i\epsilon)}{F^-(0)} + \frac{C}{kF^-(0)}$$

$$\Rightarrow C = -\frac{\delta}{2i}\frac{\partial^2}{\partial v^2}\left[\frac{F^-(i\epsilon)}{\epsilon}\right]$$

Since $F^-(0)$ is independent of v, one can write to order ϵ that

$$\frac{\partial^2}{\partial v^2}\left[\frac{F^-(i\epsilon)}{\epsilon}\right] = F^-(0)\frac{\partial^2}{\partial v^2}H(v),$$

with

$$H(v) = i\frac{\partial}{\partial k}\ln F^-(k)\Big|_{k=0}. \tag{5.77}$$

The general rule concerning ϵ is that it must be kept in any term such that for some value of k the term can become infinite Finally we have that

$$\Delta u_y(k,0) = \frac{1}{2ik}\frac{\partial^2}{\partial v^2}u_y^s(k,0) - \frac{F^-(0)}{F^-(k)}\frac{\delta}{2ik}\frac{\partial^2}{\partial v^2}H(v) \tag{5.78}$$

$$= \mathcal{M}u_y^s(k,0),$$

where

$$\mathcal{M} = \frac{1}{2ik}\frac{\partial^2}{\partial v^2} - \frac{F^-(0)}{F^-(i\epsilon)}\frac{ik+\epsilon}{2ik}\left[\frac{\partial^2}{\partial v^2}H(v)\right].$$

$$= \frac{1}{2ik}\frac{\partial^2}{\partial v^2} - \frac{ik+\epsilon}{2ik}\left[\frac{\partial^2}{\partial v^2}H(v)\right] \tag{5.79}$$

to relevant order in ϵ. In general one can find any field by application of the operator $\mathcal{M}$ to the appropriate steady state field. For example,

$$\sigma_{yy}(k,y) = \sigma_{yy}^s(k,y) + \dot{v}\mathcal{M}\sigma_{yy}^s(k,y)$$

gives the full stress field to first order in acceleration.

5.4.1. Energy Flux

We now turn to the task of computing the total energy flux into a strip containing a slowly accelerating crack, beginning with problem A. The result comes straight from the stress intensity factor. For large k, Eq. (5.78) shows that

$$u_y \sim u_y^s\left(1 - \frac{\dot{v}}{2}\frac{\partial^2}{\partial v^2}H(v)\right)$$

$$\sigma_{yy} \sim \sigma_{yy}^s\left(1 - \frac{\dot{v}}{2}\frac{\partial^2}{\partial v^2}H(v)\right).$$

Therefore the energy flux from the tip of an accelerating crack must be

$$\left(1 - \frac{\dot{v}}{2}\frac{\partial^2}{\partial v^2}H(v)\right)^2$$

times greater than the flux from the non-accelerating crack at the same velocity. The steady state energy flux is

$$-\delta\sigma_\infty v,$$

so the flux in the presence of acceleration is

$$-\delta\sigma_\infty v\left[1 - \dot{v}\frac{\partial^2}{\partial v^2}H(v)\right]$$

$$= -\delta\sigma_\infty v + \delta\sigma_\infty \dot{v}\frac{\partial}{\partial v}\left[v\frac{\partial}{\partial v} - 1\right]H(v)$$

which implies that the total energy of the plate is

$$-l(t)\delta\sigma_\infty + \delta\sigma_\infty\left[v\frac{\partial}{\partial v} - 1\right]H(v).$$

Of course, one can also go through a long and painful computation to verify that this works. It is easier to begin with problem (B). In the accelerating reference frame one has to compute

$$\lim_{y\to 0} -2\int_{-L-l(t)}^{L-l(t)} dx'\left[\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t}\right)u_y(x',y,t)\right]\sigma_{yy}(x',y,t).$$

In passing from Eq. (4.45) note first that $\sigma_{xy}(x',0) = 0$, and passing to the limit $y \to 0$ carefully shows that this term can in fact be neglected. Second the factor of two comes from the fact that we are integrating over the surface of only the upper half of the plate. Third, the contributions from the far vertical boundaries at $\pm L$ can be neglected, since u_y is exponentially close to δ at the back and to 0 at the front regardless of crack velocity. The last equation can be rewritten

$$\begin{aligned}&2\int_{-\infty}^{-L-l(t)} dx'\left[\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t}\right)u_y(x',y,t)\right]\sigma_{yy}(x',y,t)\\ &-\lim_{y\to 0}2\int_{\infty}^{\infty} dx'\left[\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t}\right)u_y(x',y,t)\right]\sigma_{yy}(x',y,t)\end{aligned} \tag{5.80}$$

Inserting the asymptotic behavior of the fields into the first term gives

$$2\int_{-\infty}^{-L-l(t)} dx' \left[\left(-v\frac{\partial}{\partial x'}+\frac{\partial}{\partial t}\right)\delta e^{\epsilon x'}\right](-\sigma_\infty)\, e^{\epsilon x'}$$

$$= v\sigma_\infty\delta \tag{5.81}$$

since $\epsilon L \ll 1$. Writing the second term of Eq. (5.80) in k space, one has

$$-2\lim_{y\to 0}\int\frac{dk}{2\pi}\left[\left(ikv+\frac{\partial}{\partial t}\right)u_y(k,y)\right]\sigma_{yy}(-k,y)$$

$$= -2\lim_{y\to 0}\int\frac{dk}{2\pi}\left[\left(ikv+\frac{\partial}{\partial t}\right)u_y^s(k,y)\right]\sigma_{yy}^s(-k,y)$$

$$-2\lim_{y\to 0}\int\frac{dk}{2\pi}ikv\,u_y^s(k,y)\left[-\frac{\dot v}{2ik}\frac{\partial^2}{\partial v^2}\sigma_{yy}^s(-k,y)-\frac{\dot v}{2}\left(1+\frac{i\epsilon}{k}\right)\sigma_{yy}^s(-k,v)\frac{\partial^2}{\partial v^2}H(v)\right].$$

$$-2\lim_{y\to 0}\int\frac{dk}{2\pi}ikv\,\sigma_y^c y(-k,y)\left[\frac{\dot v}{2ik}\frac{\partial^2}{\partial v^2}u_y^s(k)-\frac{\dot v}{2}\left(1-\frac{i\epsilon}{k}\right)u_y^s(k,y)\frac{\partial^2}{\partial v^2}H(v)\right]$$

There are six terms to be calculated. They are

$$-2\lim_{y\to 0}\int\frac{dk}{2\pi}ikv\,u_y^s(k,y)\,\sigma_{yy}^s(-k,y)$$

$$= -2\int\frac{dk}{2\pi}ikv\,u_y^s(k,0)\,\sigma_{yy}^s(-k,0)$$

$$= -2\int\frac{dk}{2\pi}ikv\,u_y^s(k,0)\,u_y^s(-k,0)(-F(k))$$

$$= -2\int\frac{dk}{2\pi}ikv\frac{\delta\sigma_\infty}{k^2+\epsilon^2} = 0. \tag{5.82}$$

In fact, I have checked that this integral vanishes by symmetry for all values of y. This integral is the only one for which their is any difficulty involved in passing to the limit $y \to 0$. In what follows, the limit will be taken immediately, and

$$\int\frac{dk}{2\pi}ik\,u_y^s(k,0)\,\sigma_{yy}^s(-k,0)$$

will be set to zero when it appears again. Next:

$$-2\int\frac{dk}{2\pi}\left[\frac{\partial}{\partial t}u_y^s(k,0)\right]\sigma_{yy}^s(-k,0)$$

$$= -2\int\frac{dk}{2\pi}\left[\dot v\frac{\partial}{\partial v}u_y^s(k,0)\right]\sigma_{yy}^s(-k,0)$$

$$= -2\int\frac{dk}{2\pi}\left[\dot v\frac{\partial}{\partial v}\frac{\delta F^-(i\epsilon)}{(ik+\epsilon)F^-(k)}\right]\left[\frac{-\sigma_\infty F^-(k)}{(-ik+\epsilon)F^-(-i\epsilon)}\right].$$

Here I have used $F^+(-k) = 1/F^-(k)$. Continuing, the integral has one pole in the lower half plane, at $k = -i\epsilon$. Evaluating the residue at this pole gives

$$-2\frac{(-2\pi i)}{2\pi}\dot v\frac{\partial}{\partial v}\left[\frac{\delta F^-(i\epsilon)}{2\epsilon F^-(-i\epsilon)}\right]\frac{-\sigma_\infty}{-i}$$

$$= 2\delta\sigma_\infty\dot v\frac{\partial}{\partial v}H \tag{5.83}$$

Next:

$$\int\frac{dk}{2\pi}u_y^s\dot v v\frac{\partial^2}{\partial v^2}\sigma_{yy}(-k) = \int\frac{dk}{2\pi}\dot v v\frac{\delta F^-(i\epsilon)}{(ik+\epsilon)F^-(k)}\frac{\partial^2}{\partial v^2}\frac{-\sigma_\infty F^-(k)}{(-ik+\epsilon)F^-(-i\epsilon)}$$

$$\sim \frac{\partial^3}{\partial v^2}\frac{F^-(-i\epsilon)}{F^-(-i\epsilon)} = 0. \tag{5.84}$$

Next:

$$-\int\frac{dk}{2\pi}u_y^s(k,0)\,\dot v\,(-)\,\sigma_{yy}^s(-k,0)\,ikv\left(1+\frac{i\epsilon}{k}\right)\frac{\partial^2}{\partial v^2}H(v)$$

$$= -\int\frac{dk}{2\pi}\epsilon v\,u_y^s(k,0)\,\sigma_{yy}^s(-k,0)\frac{\partial^2}{\partial v^2}H(v)$$

$$= \int\frac{dk}{2\pi}\frac{\epsilon\delta\sigma_\infty}{k^2+\epsilon^2}\frac{\partial^2}{\partial v^2}H(v)$$

$$= \frac{1}{2}v\dot v\delta\sigma_\infty\frac{\partial^2}{\partial v^2}H(v). \tag{5.85}$$

Next:

$$-\int\frac{dk}{2\pi}\dot v\left[v\frac{\partial^2}{\partial v^2}\frac{\delta F^-(i\epsilon)}{(ik+\epsilon)F^-(k)}\right]\frac{-\sigma_\infty F^-(k)}{(-ik+\epsilon)F^-(-i\epsilon)}$$

$$= -\frac{(-2\pi i)}{2\pi}v\dot v\frac{\delta(-\sigma_\infty)}{-i}\frac{\partial^2}{\partial v^2}\frac{F^-(i\epsilon)}{2\epsilon F^-(-i\epsilon)}$$

$$= \dot v v\delta\sigma_\infty\frac{\partial^2}{\partial v^2}H(v). \tag{5.86}$$

Finally:

$$-\int\frac{dk}{2\pi}u_y^s(k,0)\,\dot v\,(-)\,\sigma_{yy}^s(-k,0)\,ikv\left(1-\frac{i\epsilon}{k}\right)\frac{\partial^2}{\partial v^2}H(v)$$

$$= \int\frac{dk}{2\pi}\epsilon v\,u_y^s(k,0)\,\sigma_{yy}^s(-k,0)\frac{\partial^2}{\partial v^2}H(v)$$

$$= -\int\frac{dk}{2\pi}\frac{\epsilon\delta\sigma_\infty}{k^2+\epsilon^2}\frac{\partial^2}{\partial v^2}H(v)$$

$$= -\frac{1}{2}v\dot v\delta\sigma_\infty\frac{\partial^2}{\partial v^2}H(v). \tag{5.87}$$

Adding all of the terms of (9.4) together gives

$$v\delta\sigma_\infty + \dot{v}\delta\sigma_\infty\left[v\frac{\partial^2}{\partial v^2} + 2\frac{\partial}{\partial v}\right]H(v)$$
$$= v\delta\sigma_\infty + \dot{v}\delta\sigma_\infty\left[\frac{\partial}{\partial v}\right]\left[v\frac{\partial}{\partial v} + 1\right]H(v). \tag{5.88}$$

So the total energy of the plate for problem (B) is

$$l(t)\,\delta\sigma_\infty + \delta\sigma_\infty\left[v\frac{\partial}{\partial v} + 1\right]H(v),$$

if $H(v)$ is chosen so that $H(0) = 0$. Comparison with previous work shows that

$$\Sigma(v) - \Sigma(0) = \delta\sigma_\infty H(v).$$

This implies that

$$K(v) = \frac{1}{2}\delta\sigma_\infty v\frac{\partial}{\partial v}H(v).$$

Both of these relations are borne out numerically. It is not necessary to carry out the Wiener-Hopf decomposition in order to calculate $H(v)$. Consider

$$\int\frac{dk}{2\pi}\left[\frac{\partial}{\partial v}\frac{F^-(0)}{(ik+\epsilon)F^-(k)}\right]\frac{F^+(0)}{(-ik+\epsilon)F^+(-k)}$$
$$= \frac{-2\pi i}{2\pi}\left[\frac{\partial}{\partial v}\frac{F^-(0)}{2\epsilon F^-(-i\epsilon)}\right]\frac{F^+(0)}{(-i)F^+(i\epsilon)} = \frac{1}{2}\frac{\partial}{\partial v}H(v).$$

On the other hand one can also write this integral as

$$\int\frac{dk}{2\pi}\left[\frac{\partial}{\partial v}\frac{F^-(0)}{(ik+\epsilon)F^-(k)}\right]\frac{F(k)F^+(0)}{(-ik+\epsilon)F^-(-k)}$$
$$= \int\frac{dk}{2\pi}\frac{1}{k^2+\epsilon^2}\frac{1}{2}F(k)\frac{\partial}{\partial v}\left[\frac{F^-(0)F^+(0)}{F^-(k)F^-(-k)}\right]$$
$$= \int\frac{dk}{2\pi}\frac{1}{k^3}\frac{1}{2}F(k)\frac{\partial}{\partial v}\left[\frac{1}{F(k)}\right]$$
$$= \frac{1}{2}\frac{\partial}{\partial v}\int\frac{dk}{2\pi}\frac{1}{k^2}\ln\frac{1}{F(k)}$$

so that

$$H(v) = \int\frac{dk}{2\pi}\frac{1}{k^2}\ln\left[\frac{F(k,0)}{F(k,v)}\right]. \tag{5.89}$$

To move to problem A one must add a uniform stress to σ_{yy}. Therefore, to the results for problem B one must add

$$-2\lim_{\substack{y\to 0\\k\to 0}}\left(ikv + \frac{\partial}{\partial v}\right)u_y(k,y)\,\sigma_\infty + 2\int_{-\infty}^{-L-l(t)}dx'\left[\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t}\right)u_y(x',y)\right]\sigma_\infty$$

$$= \lim_{\substack{y\to 0\\k\to 0}}\left(ikv + \frac{\partial}{\partial t}\right)(-2)\,\delta\sigma_\infty\frac{F^-(i\epsilon)}{(ik+\epsilon)F^-(k)} + 2\int_{-\infty}^{-L-l(t)}dx'\sigma_\infty\left(-v\frac{\partial}{\partial x'}\delta e^{\epsilon x'}\right).$$
$$= \dot{v}\frac{\partial}{\partial v}(-2)\,\delta\sigma_\infty\frac{F^-(i\epsilon)}{\epsilon F^-(k)} - 2\sigma_\infty\delta v$$
$$= -2\sigma_\infty\delta\left[\dot{v}\frac{\partial}{\partial v}H(v) + v\right]$$

So the total energy for problem A is

$$-l(t)\,\delta\sigma_\infty + \delta\sigma_\infty\left[v\frac{\partial}{\partial v} - 1\right]H(v),$$

in agreement with the previous result.

6. A One-dimensional Model

Consider the one-dimensional model

$$\ddot{u} = \frac{\partial^2 u}{\partial x^2} - \epsilon^2(u-\Delta) - u\theta(1-u) - f(x,t). \tag{6.1}$$

This can be considered the limit of a two-dimensional strip model, where a very thin and very massive bar is pasted right over the region which will fracture. The function $f(x)$ represents the cohesive force at the crack tip, and it may be chosen to include complicated dependence upon the field configuration u. The cohesive force must be taken nonzero if steady states are to be possible.

6.1. Steady States

The first task is to find the steady states of the model. The particularly simple cohesive force will be used, one which makes dissipation proportional to the speed of the crack, and depends upon temperature (for the moment held constant) but acting over an exceedingly small range:

$$f(x,t) = T\dot{u}\delta(1-u). \tag{6.2}$$

In a frame moving at velocity v, Eq. (6.1) becomes

$$\frac{\partial^2 u}{\partial x^2}\left(1-v^2\right) - \epsilon^2(u-\Delta) - u\theta(1-u) + v\frac{\partial u}{\partial x}T\delta(1-u) = 0. \tag{6.3}$$

Taking the fracture to move from left to right, consider the solution to the left of the point where $u = 1$, which may be taken to be $x = 0$. In the left region

$$u = \Delta + A_l e^{xq_l};\ q_l = \epsilon\gamma, \tag{6.4}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}. \tag{6.5}$$

In the right hand region,

$$u = u_\infty + A_r e^{xq_r}; q_r = -\sqrt{1+\epsilon^2}\gamma, \tag{6.6}$$

with

$$u_\infty = \frac{\Delta\epsilon^2}{1+\epsilon^2}. \tag{6.7}$$

By assumption one has that

$$u(0) = 1 \tag{6.8}$$

so that

$$A_l = 1 - \Delta \tag{6.9}$$

and

$$A_r = 1 - u_\infty. \tag{6.10}$$

By integrating Eq. (6.3) from a little before $x = 0$ to a little after, one finds that

$$(A_r q_r - A_l q_l)(1 - v^2) = vT, \tag{6.11}$$

which may be solved to give

$$v = \frac{(\Delta - \Delta_c)\left[\sqrt{\epsilon^2+1}+\epsilon\right]}{\sqrt{\Delta_c^2 T^2 + (\Delta-\Delta_c)^2\left[\sqrt{\epsilon^2+1}+\epsilon\right]^2}} \tag{6.12a}$$

$$\Delta_c = \sqrt{1+\epsilon^{-2}} \tag{6.12b}$$

Notice that if $T \to 0$, v tends to the wave speed 1 except very near the critical strain Δ_c.

As a next task, consider the linear stability of these steady states. Adopt a coordinate

$$x' = x - vt - \alpha e^{\omega t} \tag{6.13}$$

which is defined to be the location of the crack tip, $u = 1$. In this coordinate system, dropping the primes, one has

$$\begin{aligned}&\left\{\frac{\partial^2}{\partial t^2} - 2\left[v + \alpha e^{\omega t}\omega\right]\frac{\partial^2}{\partial x \partial t} - \alpha\omega^2 e^{\omega t}\frac{\partial}{\partial x} + \left(v^2 + 2v\alpha\omega e^{\omega t}\right)\frac{\partial^2}{\partial x^2}\right\}u \\ &= \frac{\partial^2 u}{\partial x^2} - \epsilon^2(u-\Delta) - u\theta(1-u) - \left(\frac{\partial}{\partial t} - \left[v + \omega\alpha e^{\omega t}\right]\frac{\partial}{\partial x}\right)uT\delta(1-u)\end{aligned} \tag{6.14}$$

Let

$$u = u_s + \alpha\tilde{u}e^{\omega t}, \tag{6.15}$$

where u_s is the steady state solution at some velocity v. Then using the fact that the coordinate system has been defined to keep the crack tip at $x = 0$, one has

$$\begin{aligned}&\left[2v\omega\frac{\partial}{\partial x} - \omega^2\right]\left[\frac{\partial u_s}{\partial x} - \tilde{u}\right] \\ &= \frac{\partial^2\tilde{u}}{\partial x^2}(1-v^2) - \epsilon^2\tilde{u} - \tilde{u}\theta(-x) - \omega T\delta(x)\end{aligned} \tag{6.16}$$

It is clear by inspection that $\tilde{u} = \frac{\partial u_s}{\partial x}$ is a solution except for difficulties near the crack tip, where $\tilde{u}$ must be zero, but is not. The correct solution is therefore seen to be

$$\tilde{u}_{l,r} = A_{l,r}q_{l,r}\left[e^{xq_{l,r}} - e^{x\tilde{q}_{l,r}}\right] \tag{6.17}$$

where

$$\tilde{q}_l = \frac{\sqrt{\omega^2 + \epsilon^2(1-v^2)} - v\omega}{1-v^2} \tag{6.18a}$$

$$\tilde{q}_r = -\frac{\sqrt{\omega^2 + (\epsilon^2+1)(1-v^2)} + v\omega}{1-v^2}. \tag{6.18b}$$

The discontinuity of the derivative of $\tilde{u}$ at $x = 0$ is determined by

$$\left[A_r q_r(q_r - \tilde{q}_r) - A_l q_l(q_l - \tilde{q}_l)\right](1-v^2) = \omega T + 2v\omega(q_r A_r - q_l A_l). \tag{6.19}$$

Any solutions of Eq. (6.19) where ω has positive real part will correspond to unstable modes. However, the solutions always occur for ω negative and real, so the steady state cracks are quite stable. There is at most one such solution. This fact may seem to present a problem, since solutions of the linearized problem about the steady states should constitute a complete set of functions. The resolution is that when ω is purely imaginary and sufficiently large, one or both of $\tilde{q}_{l,r}$ will be purely imaginary, in which case the solution Eq. (6.17) is not sufficiently general, since both signs of $\tilde{q}_{l,r}$ become acceptable. Thus an arbitrary perturbation about the steady state is resolved in terms of undamped traveling waves. These waves correspond to vertical oscillations of the whole strip, and cannot be regarded as characteristic of the crack motion, although they couple to it.

6.2. Coupling in Temperature

The model of the previous section may be generalized slightly by allowing temperature to evolve dynamically. The idea of the calculation is that if the crack speeds up, it should heat up the region in front of it (actually, experiment shows precisely the opposite–see Fuller, Fox, and Field! In fact, there is a temperature drop lasting for about 5 μs when the crack approaches, which corresponds to a frequency of 200 kHz. The drop may occur for the same reason that rubber cools when you stretch it. Could this be the answer? It would be disappointing, since it would be particular to polymers....) The equation for the crack is

$$\ddot{u} = \frac{\partial^2 u}{\partial x^2} - \epsilon^2(u-\Delta) - u\theta(1-u) - g(T)\dot{u}\delta(1-u), \tag{6.20a}$$

where g is an arbitrary function of temperature, and temperature evolves according to

$$\frac{\partial T}{\partial t} = (\dot{u})^2. \tag{6.21}$$

Analysis of this system of equations is little changed from the previous case. The velocity is now determined by

$$(A_r q_r - A_l q_l)(1-v^2) = vg(T_0), \tag{6.22a}$$

where

$$T_0 = -\frac{v}{2}q_r A_r^2, \tag{6.23}$$

and stability of the solutions is determined by

$$[A_r q_r (q_r - \tilde{q}_r) - A_l q_l (q_l - \tilde{q}_l)](1 - v^2) = \omega g(T_0) + v\tilde{g}'(T_0)\tilde{T}_0 2v\omega (q_r A_r - q_l A_l), \quad (6.24a)$$

with

$$\tilde{T}_0 = vA_r^2 q_r^2 \left\{ \frac{\omega + (q_r - \tilde{q}_r)v}{\omega - (q_r + \tilde{q}_r)v} \right\}. \quad (6.25)$$

The solutions of Eq. (6.24a) only occur for ω with negative real part, unless $g'(T_0)$ is negative. In the case where g' is negative, it means that fracture energy is a decreasing function of temperature, so one has settled on a steady state which is unstable against a continually accererating (brittle) solution. The algebra is manageable, although complicated when ω is very small and is set up in `crack/langer/baby_therm2.mc`.

The lesson seems to be that when a crack accelerates the biggest effect is immediately in front of it, not at some distance, causing it to start slowing down immediately and killing off oscillations.

6.3. Alternate Version

Instead of including dissipation only at one point at the tip of the crack, one can include it throughout. In this case, one has the equation

$$\ddot{u} = \frac{\partial^2 u}{\partial x^2} - \epsilon^2 (u - \Delta) - u\theta(1 - u) - b\dot{u}, \quad (6.26a)$$

which has a steady state solution given by Eq. (6.4) and Eq. (6.6) as before, except that now

$$q_l = -vb + \sqrt{\frac{(vb)^2 + 4\epsilon^2(1 - v^2)}{2(1 - v^2)}} \quad (6.27)$$

and

$$q_r = -vb - \sqrt{\frac{(vb)^2 + 4(\epsilon^2 + 1)(1 - v^2)}{2(1 - v^2)}}. \quad (6.28)$$

Now the derivative of u must be continuous across the crack tip, so the velocity is determined by

$$q_l A_l = q_r A_r, \quad (6.29)$$

which is too easy to solve numerically to be worth further analytical struggles.

6.4. Using the Wiener-Hopf Technique

It is a good exercise to solve Eq. (6.3) using the Wiener-Hopf technique. To make sure that the Fourier transform is well defined, it is best to rewrite the equation as

$$\frac{\partial^2 u}{\partial x^2}(1 - v^2) - \epsilon^2\left(u - \Delta e^{-\alpha|k|}\right) - u\theta(1 - u) + v\frac{\partial u}{\partial x}T\delta(1 - u) = 0, \quad (6.30a)$$

with α taken to zero at the end of the calculation. Fourier transforming Eq. (6.30a), one has that

$$-k^2(1 - v^2)u - \epsilon^2 u + \epsilon^2\Delta\left\{\frac{1}{\alpha + ik} + \frac{1}{\alpha - ik}\right\} - u^+ - vT = 0, \quad (6.31)$$

where

$$u^+ = \int_0^\infty u(x)e^{ikx} \quad (6.32)$$

and has no poles in the upper k plane. Defining u^- similarly, one has that

$$u^+\left[k^2(1 - v^2) + \epsilon^2 + 1\right] + u^-\left[k^2(1 - v^2) + \epsilon^2\right] = \epsilon^2\Delta\left\{\frac{1}{\alpha + ik} + \frac{1}{\alpha - ik}\right\} - vT \quad (6.33)$$

$$\Rightarrow u^+ + F(k)u^- = \frac{\epsilon^2}{1 + \epsilon^2}\left\{\frac{1}{\alpha + ik} + \frac{1}{\alpha - ik}\right\} - \frac{vT}{k^2(1 - v^2) + \epsilon^2 + 1}, \quad (6.34)$$

where

$$F(k) = \frac{(k + iq_l)(k - iq_l)}{(k + iq_r)(k - iq_r)}, \quad (6.35)$$

the roots q_r and q_l having been defined by Eq. (6.4) and Eq. (6.6). The secret of the Wiener-Hopf procedure lies in transforming Eq. (6.34) so that on the left hand side are functions which are completely regular in the lower k plane, while on the right hand side are functions which are completely regular in the upper k plane. Two such functions can be equal only if they are both equal to a constant K, and once that constant is found the problem is solved. In the present case, this whole procedure can be carried out explicitly.

One first writes that

$$F(k) = \frac{F^-(k)}{F^+(k)}, \quad (6.36)$$

where F^- is regular in the lower half plane, and F^+ is regular in the upper half plane. Such a decomposition always exists. One then has that

$$F^+u^+ + F^-u^- = G(k), \quad (6.37)$$

with

$$G(k) = \frac{\epsilon^2}{1 + \epsilon^2}F^+(0)\left\{\frac{1}{\alpha + ik} + \frac{1}{\alpha - ik}\right\} - F^+(k)\frac{vT}{(1 - v^2)(k + iq_r)(k + iq_l)}. \quad (6.38)$$

The terms involving α form a delta function, so it is legitimate to evaluate anything which multiplies them at $k = 0$, and the last term was rewritten using Eq. (6.4) and Eq. (6.6). Finally, one has to write

$$G(k) = G^+(k) + G^-(k), \quad (6.39)$$

where as usual G^+ is regular in the upper half plane, and G^- is regular in the lower half plane. Then Eq. (6.37) can be written as

$$F^-u^- + G^- = -F^+u^+ - G^+ \tag{6.40}$$

$$\Rightarrow F^-u^- + G^- = K, \tag{6.41}$$

with K, as announced, an unknown constant. The necessary decompositions can be carried out easily after one recalls that q_r has negative real part and q_l has positive real part:

$$F^- = \frac{(k-iq_l)}{(k+iq_r)} \tag{6.42a}$$

$$F^+ = \frac{(k-iq_r)}{(k+iq_l)} \tag{6.42b}$$

$$G^- = \frac{\epsilon^2\Delta}{\epsilon^2+1}\left(\frac{q_r}{q_l}\right)\frac{1}{\alpha+ik} - \frac{vT}{i(q_r-q_l)}\frac{1}{k+iq_r}\frac{1}{1-v^2} \tag{6.42c}$$

$$G^+ = \frac{\epsilon^2\Delta}{\epsilon^2+1}\left(\frac{q_r}{q_l}\right)\frac{1}{\alpha-ik} - \frac{vT}{i(q_l-q_r)}\frac{1}{k+iq_l}\frac{1}{1-v^2} \tag{6.42d}$$

Therefore

$$u^-(k) = -\frac{\epsilon^2\Delta}{\epsilon^2+1}\left(\frac{q_r}{q_l}\right)\frac{1}{\alpha+ik}\frac{(k+iq_r)}{(k-iq_l)} + \frac{vT}{i(q_r-q_l)}\frac{1}{k-iq_l}\frac{1}{1-v^2} + K\frac{(k+iq_r)}{(k-iq_l)}. \tag{6.43a}$$

For large k, Eq. (6.43) goes as K plus terms that drop to zero. Therefore K must be zero, or else $u(x)$ will have a delta function at the origin, and one has finally that

$$u^-(k) = -\frac{\epsilon^2\Delta}{\epsilon^2+1}\left(\frac{q_r}{q_l}\right)\frac{1}{\alpha+ik}\frac{(k+iq_r)}{(k-iq_l)} + \frac{vT}{i(q_r-q_l)}\frac{1}{k-iq_l}\frac{1}{1-v^2}. \tag{6.43b}$$

Inverting the Fourier transform and requiring that $u(x)=1$ at $x=0$ one finds that

$$\Delta\left(1-\frac{q_l}{q_r}\right) + \frac{vT}{q_r-q_l}\frac{1}{1-v^2} = 1, \tag{6.44}$$

which after a certain amount of manipulation reproduces Eq. (6.22).

While the Wiener-Hopf technique can hardly be recommended for solving this problem, it is good to have a list a cases where the technique can be carried through explicitly

6.5. Green's Function Approach

The goal of this subsection is to find a general equation of motion for cracks in this one-dimensional model. It will be necessary to make an approximation similar in spirit to the small-scale yielding hypothesis, but otherwise all results will be exact.

To begin, I need to find the Green function

$$\left[\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} + b\frac{\partial}{\partial t} + \epsilon^2\right] G_\epsilon(x-x', t-t') = \delta(x-x', t-t'). \tag{6.45}$$

The damping constant b is assumed small in comparison with the elastic constant ϵ. To find this function, one has to do the integral

$$\int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\frac{e^{-ikx-i\omega t}}{k^2+\epsilon^2-\omega^2-ib\omega}. \tag{6.46}$$

Doing the integral over ω gives

$$-\int \frac{dk}{2\pi} e^{-ikx-bt/2}\frac{\sin t\sqrt{k^2+\epsilon^2-b^2/4}}{\sqrt{k^2+\epsilon^2-b^2/4}}\theta(t), \tag{6.47}$$

and changing variables to

$$k = \epsilon' \sinh\sigma \tag{6.48}$$

with

$$\epsilon' = \sqrt{\epsilon^2 - b^2/4} \tag{6.49}$$

gives

$$-\int \frac{d\sigma}{2\pi}\theta(t)\,e^{-bt/2}\frac{e^{-ix\epsilon'\sinh\sigma+it\epsilon'\cosh\sigma} - e^{-ix\epsilon'\sinh\sigma-it\epsilon'\cosh\sigma}}{2i} \tag{6.50}$$

Using the identities

$$t\cosh\sigma - x\sinh\sigma = \sqrt{t^2-x^2}\cosh(\sigma-\sigma_0) \tag{6.51a}$$

$$t\cosh\sigma + x\sinh\sigma = \sqrt{t^2-x^2}\cosh(\sigma+\sigma_0) \tag{6.51b}$$

if $|t|>|x|$ and

$$\sigma_0 = \ln\sqrt{\frac{t+x}{t-x}}, \tag{6.52}$$

while

$$t\cosh\sigma - x\sinh\sigma = \sqrt{x^2-t^2}\sinh(\sigma-\sigma_0') \tag{6.52c}$$

$$t\cosh\sigma + x\sinh\sigma = \sqrt{t^2-x^2}\sinh(\sigma+\sigma_0') \tag{6.52d}$$

if $|x|>|t|$ and

$$\sigma_0' = \ln\sqrt{\frac{t+x}{x-t}} \tag{6.53}$$

gives

$$-\theta(t)\,\theta(t-|x|)\,e^{-bt/2}\int\frac{d\sigma}{2\pi}\sin\sqrt{t^2-x^2}\,\epsilon'\cosh\sigma \tag{6.54}$$

so that

$$G_\epsilon(x,t) = -\frac{e^{-bt/2}}{2}\theta(t)\,\theta(t-|x|)\,J_0\left(\sqrt{t^2-x^2}\sqrt{\epsilon^2-b^2/4}\right). \tag{6.55}$$

6.6. Solution of Boundary Value Problem

The goal is now to solve Eq. (6.26) in as much generality as possible. Note first of all that

$$\int_{-\infty}^{l(t')} dx' G\left(x-x',t-t'\right) \frac{\partial^2 u}{\partial x'^2} u\left(x'\right)$$

$$= \left[G\left(x-x',t-t'\right)\frac{\partial u}{\partial x'} - \frac{\partial G}{\partial x'} u\left(x',t'\right)\right]\Big|_{x'=l(t')} + \int_{-\infty}^{l(t')} dx' u\left(x',t'\right)\frac{\partial^2}{\partial x'^2} G\left(x-x',t-t'\right). \tag{6.56}$$

Next define $T'(x')$ by

$$T'\left(x'\right) = \max\left(l^{-1}\left(x'\right),0\right). \tag{6.57}$$

Then

$$\int_{T'}^{\infty} dt' G\left(x-x',t-t'\right)\frac{\partial^2 u}{\partial t'^2}$$

$$= -G\left(x-x',t-T'\right)\frac{\partial u}{\partial t'} + \frac{\partial}{\partial t'} G\left(x-x',t-T'\right) + \int_{T'}^{\infty} dt' u\left(x'\right)\frac{\partial^2}{\partial t'^2} G\left(x-x',t-t'\right). \tag{6.58}$$

Also

$$\int dx' \int_{T'}^{\infty} G\left(x-x',t-t'\right)\frac{\partial^2 u}{\partial t'^2}$$

$$\begin{aligned} = &-\int_0^{\infty} dt' \dot{l}\left(t'\right)\left[G\left(x-l\left(t'\right),t-t'\right)\frac{\partial u}{\partial t'} - \frac{\partial G}{\partial t'} u\left(l\left(t'\right),t'\right)\right] \\ &- \int_{-\infty}^{l(0)} dx' G\left(x-x',t\right)\left.\frac{\partial u}{\partial t'}\right|_{t'=0} - \left.\frac{\partial G}{\partial t'}\right|_{t'=0} u\left(x',0\right) \\ &+ \int dx' \int_{T'(x')}^{\infty} dt' u\left(x'\right)\frac{\partial}{\partial t'} G\left(x-x',t-t'\right) \end{aligned} \tag{6.59}$$

Therefore, taking Eq. (6.26), multiplying by $G(x-x',t-t')\theta(l(t')-x')\theta(t')$ and integrating over space and time gives

$$\begin{aligned} &\int dx'dt'\theta\left(t'\right)\theta\left(l\left(t'\right)-x'\right)G\left(x-x',t-t'\right)\left[\frac{\partial^2 u}{\partial t'^2} - \frac{\partial^2 u}{\partial x'^2} + \epsilon^2 u\right] \\ &= \int dx'dt'\theta\left(l\left(t'\right)-x'\right)\theta\left(t'\right)G\left(x-x',t-t'\right)\left(f\left(x',t'\right)+\epsilon^2\Delta\right) \end{aligned} \tag{6.60}$$

$$= u(x,t)\,\theta(l(t)-x) \begin{aligned} &-\int dt'dx'\delta\left(x'-l\left(t'\right)\right)\left\{\dot{l}\left(t'\right)\left[G\frac{\partial u}{\partial t'} - \frac{\partial G}{\partial t'} u\right] + G\frac{\partial u}{\partial x'} - \frac{\partial G}{\partial x'} u\right\} \\ &-\int dx'dt'\theta\left(l\left(t'\right)-x'\right)\delta\left(t'\right)\left[G\frac{\partial u}{\partial t'} - \frac{\partial G}{\partial t'} u\right] \end{aligned} \tag{6.61}$$

One now uses that $u(l(t'),t') = 1$ is the definition of l to find that for $x < l(t)$,

$$u(x,t) - \frac{\Delta}{2} = \begin{aligned} &\int dt'\left\{\left(1-\dot{l}^2\right)G\frac{\partial u}{\partial x'} - \frac{\partial G}{\partial x'} - \dot{l}\left(t'\right)\frac{\partial G}{\partial t'}\right\} \\ &+ \int_{-\infty}^{l_0} dx'\left\{G\frac{\partial u_0}{\partial t'} - \frac{\partial G}{\partial t'} u_0\right\} \\ &+ \int dx'dt'\theta\left(l\left(t'\right)-x'\right) f\left(x',t'\right) G. \end{aligned} \tag{6.62}$$

The only unknown quantity in Eq. (6.62) is $\partial u/\partial x'$, which needs to be known at at the crack tip. To solve for it is an unpleasant numerical task. Insteady, I will invoke a fracture mechnics type assumption. The time scale for material ahead of the crack tip is 1, whereas the time scale behind it is $1/\epsilon$; these are the typical times needed to establish steady states. Therefore, I will assume that the field u always adopts the steady state configuration ahead of the crack tip, and that one can take

$$\frac{\partial u}{\partial x}\left(l\left(t\right),t\right) = A_r q_r, \tag{6.63}$$

as given by Eq. (6.10).

As a check of these results, it is good to reproduce the steady state solutions. Using the indentity

$$\int dt' G\left(x-vt',t-t'\right) = \frac{1}{2\left(1-v^2\right)q_l} e^{(x-vt)q_l} \tag{6.64a}$$

$$\int dt' \frac{\partial}{\partial x'} G\left(x-vt',t-t'\right) = -\frac{1}{2\left(1-v^2\right)} e^{(x-vt)q_l} \tag{6.64b}$$

$$\int dt' \frac{\partial}{\partial t'} G\left(x-vt',t-t'\right) = \frac{v}{2\left(1-v^2\right)} e^{(x-vt)q_l} \tag{6.64c}$$

$$\int dx'dt' v\delta\left(vt'-x'\right) G\left(x-vt',t-t'\right) = \frac{1}{\left(1-v^2\right)q_l} e^{(x-vt)q_l} \tag{6.64d}$$

Putting these results into Eq. (6.62), ignoring the terms evaluated at $t' = 0$ since their influence eventually decays to zero, using Eq. (6.2) for f, and setting $u(l(t),t) = 1$ gives

$$u\left(l\left(t\right),t\right) - \frac{\Delta}{2} = \left[\frac{qrAr}{2q_l} + \frac{1}{2} + \frac{Tv}{q_l\left(1-v^2\right)}\right] \tag{6.65}$$

$$\Rightarrow \left(A_r q_r - (1-\Delta) q_l\right)\left(1-v^2\right) = vT. \tag{6.66}$$

Equation Eq. (6.66) is precisely the same as Eq. (6.22).

7. Discrete Models

7.1. Discrete One-Dimensional Model

As shown by Slepyan, it is possible to find steady state solutions of discrete fracture models. In this section, I will find steady states for the discrete analog of the one-dimensional model, Eq. (6.26a). The physical idea is that the passage of a crack through a lattice involves a periodic forcing, applied to successive lattice sites, and at time intervals $1/v$. This periodic forcing will always excite at least one resonant mode of the lattice, which will create a wave either ahead of or behind the crack, depending on whether the group velocity of the wave is less than or greater than the velocity of the crack. The resonant modes appear in the formalism as roots of the lattice dispersion relation, and the group velocity appears in the course of deciding which half of the complex plane the roots belong to. The starting point is the equation

$$\ddot{u}_m = u_{m+1} - 2u_m + u_{m-1} - \epsilon^2 (u_m - \Delta) - u_m \theta (1 - u_m) - b\dot{u}, \tag{7.1}$$

the lattice version of Eq. (6.26a). In one respect, the lattice equation must be taken to be simpler than the continuum equation, since the dissipation b must always be assumed to be infinitesimal for the following arguments to go through.

There is no unique way to define a steady state for a lattice model, but the simplest possible assumption is that a state traveling at velocity v completely reproduces itself, apart from translation of one lattice spacing, at time intervals of $1/v$. Stated mathematically, one has that

$$u_{m+1}(t + 1/v) = u_m(t) \Rightarrow u_m(t) = u_0(t - m/v) \equiv u(\tau). \tag{7.2}$$

All information about this state is contained in the behavior over time of any single lattice site. More complicate candidates for steady states could be constructed, in which for example the state reconstructs itself only after passing two lattice sites, but I have been unable to solve them. The equation for the steady state takes the form

$$\ddot{u}_m = u(\tau + 1/v) - 2u + u(\tau - 1/v) - \epsilon^2 u + \epsilon^2 \Delta e^{-\alpha|\tau|} - u\theta(-\tau) - b\dot{u}, \tag{7.3}$$

with α to be taken to zero at the end of the calculation. It should be mentioned that Eq. (7.3) does not quite follow from Eq. (7.1). For τ very large compared with $1/\alpha$, u falls back towards zero, and the step function should jump back up to one at this point. In the limit as $\alpha \to \infty$, this second crack tip can be neglected; or one may choose to view Eq. (7.2) as the starting point.

Taking the Fourier transform of Eq. (7.3), by

$$u(\omega) = \int d\tau\, u(\tau) e^{i\omega\tau}, \tag{7.4}$$

and defining

$$u^{\pm}(\omega) = \int d\tau\, \theta(\pm\tau) u(\tau) e^{i\omega\tau}, \tag{7.5}$$

one obtains

$$F(\omega) u^+ + G(\omega) u^- = -\Delta\epsilon^2 \left\{ \frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega} \right\}, \tag{7.6}$$

with

$$F(\omega) = w^2 - 4\sin^2(\omega/2v) - \epsilon^2 + i\omega b, \tag{7.7a}$$

and

$$G(\omega) = w^2 - 4\sin^2(\omega/2v) - \epsilon^2 - 1 + i\omega b. \tag{7.7b}$$

Letting

$$F = \frac{F^-}{F^+}, \text{ and } G = \frac{G^-}{G^+}, \tag{7.8}$$

where a superscript $\pm$ indicates whether it is in the lower or upper half plane that the function is regular, one can use the argument of Wiener and Hopf to see that

$$u^-(\omega) = \frac{F^-(\omega)}{G^-(\omega)} \frac{G^+(0)}{F^+(0)} \frac{\Delta}{\alpha + i\omega}; \tag{7.9a}$$

$$u^+(\omega) = \frac{F^+(\omega)}{G^+(\omega)} \frac{G^+(0)}{F^+(0)} \frac{\Delta}{\alpha - i\omega}. \tag{7.9b}$$

The feature of the discrete model which is quite different from the continuous model lies in the fact that F and G have roots lying almost exactly on the real axis, pushed off it only by the infinitesimal damping b. These roots turn into real poles of u, and therefore correspond to traveling waves, and because of this special significance, one should separate them out. Let $f_i^{\pm}$ be the real roots of F, with the $\pm$ sign indicating whether the root will belong to F^- or F^+ (f_i^- will have an infinitesimal imaginary part *above* the real ω axis), and let $g_i^{\pm}$ be the corresponding real roots of G. One can tell which camp a root belongs to by computing, for example,

$$\frac{1}{f_i} \frac{dF(f_i)}{dw}. \tag{7.10}$$

If this quantity is positive, the root in question belongs with F^+, and otherwise it belongs with F^-. Next, define

$$\tilde{F}(\omega) = \frac{F(\omega)}{\prod_i (\omega - f_i^{\pm})}, \text{ and } \tilde{G}(\omega) = \frac{G(\omega)}{\prod_i (\omega - g_i^{\pm})}. \tag{7.11}$$

The formal reason to divide out the roots in this manner has to do with the identities

$$F(\omega; b) = F(-\omega; -b); \tag{7.12a}$$

$$G(\omega; b) = G(-\omega; -b). \tag{7.12b}$$

The roots of F and G will only move infinitesimally if b changes sign, and so long as the roots are away from the real axis, this will not matter. So one can write

$$\tilde{F}(\omega) = \tilde{F}(-\omega); \tag{7.13a}$$

$$\tilde{G}(\omega) = \tilde{G}(-\omega). \tag{7.13b}$$

These last identities are valuable because employing Eq. (7.8) one can see immediately that

$$\frac{\tilde{F}^-(\omega)}{\tilde{F}^+(\omega)} = \frac{\tilde{F}^-(-\omega)}{\tilde{F}^+(-\omega)} \tag{7.14}$$

$$\Rightarrow \tilde{F}^-(\omega)\tilde{F}^+(-\omega) = \tilde{F}^+(\omega)\tilde{F}^-(-\omega). \tag{7.15}$$

Since $F^+(-\omega)$ is regular in the lower half plane, on the left side of Eq. (7.15) is a function which is regular in the lower half plane, on the right hand side a function which is regular in the upper half plane, and both sides must equal a constant. The most convenient form in which to express this relation is

$$\tilde{F}^-(\omega)\tilde{F}^+(-\omega) = \tilde{F}^-(0)\tilde{F}^+(0); \tag{7.16a}$$

$$\tilde{G}^-(\omega)\tilde{G}^+(-\omega) = \tilde{G}^-(0)\tilde{G}^+(0). \tag{7.16b}$$

Similar identities do not hold for F and G because when b changes sign, the real roots flip between belonging to F^- and F^+, and so have to be treated separately.

The most interesting calculation to perform, as emphasized by Slepyan, is of the amount of energy emanating from the crack tip in the form of traveling waves. Far ahead of the crack, there is an energy per site

$$E_{\text{ahead}} = \frac{1}{2}u_\infty^2 + \frac{\epsilon^2}{2}(\Delta - u_\infty)^2 = \frac{1}{2}\Delta u_\infty, \tag{7.17a}$$

using Eq. (6.7), while far ahead of the crack, there is an energy per site of

$$E_{\text{behind}} = \frac{1}{2}, \tag{7.17b}$$

which is the total energy needed to bring the lower spring from zero to failure. Any difference between these two quantities

$$E_{\text{radiation}} = \frac{1}{2}[\Delta u_\infty - 1] \tag{7.17c}$$

must be energy carried by traveling waves.

The way to proceed is to calculate $u(\tau = 0)$, since for the steady state equation Eq. (7.2) to follow from the original model Eq. (7.1), one must have

$$u(\tau = 0) = 1. \tag{7.18}$$

Imposing Eq. (7.18) will force a particular choice of Δ, which when used with Eq. (7.17) will give the desired result. One needs only to find the behavior of u^- for large ω, since at $\tau = 0$, $u^-(\tau)$ jumps from 0 up to $u(0)$. Such a discontinuity is produced by Fourier transforms that fall off as $1/i\omega$ for large ω, so that comparing with Eq. (7.9) one sees immediately

$$u(\tau = 0) = \frac{F^-(\infty)}{G^-(\infty)}\frac{G^+(0)}{F^+(0)}\Delta. \tag{7.19}$$

Since from Eq. (7.7) one has

$$\frac{G(\infty)}{F(\infty)} = 1, \tag{7.20}$$

using the identities Eq. (7.16) it is not hard to show that

$$\frac{F^-(\infty)}{G^-(\infty)} = \sqrt{\frac{\tilde{F}^-(0)\tilde{F}^+(0)}{\tilde{G}^-(0)\tilde{G}^+(0)}}. \tag{7.21}$$

Using this expression together with the definitions of $\tilde{F}$ and $\tilde{G}$ in Eq. (7.11) gives

$$u(\tau = 0) = \Delta\sqrt{\frac{F(0)}{G(0)}}\sqrt{\prod \frac{f_i^+ g_i^-}{f_i^- g_i^+}}, \tag{7.22}$$

so that finally the energy involved in snapping the springs is

$$\frac{1}{2}u^2(\tau = 0) = \frac{1}{2}\Delta u_\infty \prod \frac{f_i^+ g_i^-}{f_i^- g_i^+} = \frac{1}{2}, \tag{7.23}$$

and the energy carried off in radiation is

$$\frac{1}{2}\left[\prod \frac{f_i^- g_i^+}{f_i^+ g_i^-} - 1\right] = E_{\text{radiation}} \tag{7.24}$$

7.1.1. Physical Interpretation of Lattice Results

In order to understand these results, it is useful to consider the modes that are excited in a picket fence by a boy dragging a stick across it. In other words, one considers periodic forcing which jumps ahead by one lattice point during each period. As the basic equation, consider

$$\ddot{u}_m = u_{m+1} - 2u_m + u_{m-1} - 4A^2u_m + \sum_n \delta_{m,n}\delta(t - n/v). \tag{7.25}$$

Defining

$$u_k = \sum_m e^{ikm}, \tag{7.26}$$

and Fourier transforming in time as well, one has that

$$[4A^2 + 2 - 2\cos k - w^2]\,u_k = \sum_n e^{ikn + i\omega n/v} \tag{7.27}$$

$$\Rightarrow u_k = \frac{2\pi\delta(k + \omega/v)}{4A^2 + 2 - 2\cos k - w^2}. \tag{7.28}$$

From Eq. (7.28) it is apparent that one can excite at least one resonance in the system at any velocity v. The resonances are given by the roots of $F(\omega)$, which has already appeared in Eq. (7.7). Of course, at low velocities it is possible to have large numbers of poles in Eq. (7.28). The wavelengths of the oscillations may be quite large, since ω ranges roughly between A and $\sqrt{A^1 + 1}$, but when v is small k may end up being mapped almost anywhere in the Brillouin zone. In particular, the group velocities of the excited waves can assume all allowed values. The group velocities have particular physical importance because they determine whether the resonant wave will form before or behind the banging stick. At wave numbers where the group velocity is greater than v, the resonant wave will form ahead of the forcing function, otherwise it will form behind. The group velocity is

$$v_g(\omega) = \frac{\sin k}{\omega} = \frac{\sin \omega/v}{\omega}, \tag{7.29}$$

and so requiring the group velocity to be greater than v gives

$$\frac{\sin \omega/v}{\omega} - v > 0. \tag{7.30}$$

This equation is identical with Eq. (7.10), which was used before determine whether certain roots belong with F^+ or F^-. It is now apparent that the roots were being assigned according to the value of the relevant group velocity.

7.1.2. Asymptotic Results

It is possible to evaluate the product

$$P(A) = \prod \frac{w_i^+}{w_i^-}, \tag{7.31}$$

where for generality w_i is the root of

$$\omega^2 - 4\sin^2(\omega/2v) - 4A^2 \tag{7.32}$$

analytically in certain limits. In the limit of low velocity, it is helpful first to write the condition that Eq. (7.32) have a root as

$$\omega = 2v \sin^{-1} \sqrt{\omega^2/4 - A^2}. \tag{7.33}$$

In the limit of low velocity, by looking at a graph of Eq. (7.32), one sees that the roots are given approximately by

$$w_i^+ = 2v\left[(n_0+i)\pi - \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)\right], \tag{7.34a}$$

$$w_i^- = 2v\left[(n_0+i)\pi + \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)\right], \tag{7.34b}$$

and that there is always one more positive root w_i^+ than there are positive roots w_i^-. The starting integer n_0 is given by

$$A = 2vn_0\pi \tag{7.35a}$$

(one will have to fiddle around with v a bit to make this precisely true) and the largest value of i is n_f, given by

$$\sqrt{A^2+1} = (n_0+n_f)\,v\pi \tag{7.35b}$$

(this comes close to the truth in the limit of small v).

For the moment, let us restrict ourselves just to the positive roots of Eq. (7.32). Then

$$\frac{1}{2}\ln P(A) = \sum_{i=0}^{n_f-1} \ln\left[\frac{(n_0+i)\pi - \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)}{(n_0+i)\pi + \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)}\right] + \ln\left[(n_0+n_f)\pi\right] \tag{7.36}$$

$$= \sum_{i=0}^{n_f-1} \ln\left[\frac{1 - \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)/(n_0+i)\pi}{1 + \sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)/(n_0+i)\pi}\right] + \ln\left[(n_0+n_f)\pi\right] \tag{7.37}$$

$$= \sum_{i=0}^{n_f-1} (-2)\sin^{-1}\left(\sqrt{v^2[n_0+i]^2\pi^2 - A^2}\right)/(n_0+i)\pi + \ln\left[(n_0+n_f)\pi\right] \tag{7.38}$$

$$= \int_A^{\sqrt{A^2+1}} dx \left(\frac{-2}{\pi}\right) \frac{\sin^{-1}\sqrt{x^2-A^2}}{x} + \ln\left(\sqrt{A^2+1}/v\right) \tag{7.39}$$

and integrating by parts gives

$$\frac{1}{2}\ln P(A) = \ln\frac{1}{v} + \frac{2}{\pi}\int_A^{\sqrt{A^2+1}} \frac{x\ln x\,dx}{\sqrt{x^2-A^2}\sqrt{1+A^2-x^2}}, \tag{7.40}$$

which after the change of variables $x^2 = A^2 + \sin^2\theta$ becomes

$$P(A) = \frac{1}{v^2}\exp\left[\frac{1}{\pi}\int_0^\pi \ln\left[A^2+\sin^2\theta\right]\right]d\theta. \tag{7.41}$$

In the particular case of Eq. (7.24), one wants to evaluate

$$\left(\frac{\Delta}{\Delta_c}\right)^2 = \frac{P\left(\sqrt{1+\epsilon^2}/2\right)}{P(\epsilon/2)} = \exp\left[\frac{1}{\pi}\int_0^\pi \ln\left[\frac{\epsilon^2/4 + 1/4 + \sin^2\theta}{\epsilon^2/4 + \sin^2\theta}\right]\right]d\theta. \tag{7.42}$$

Gradshteyn and Rizhyk have the integral (4.399)

$$\int_0^\pi dx \ln(1 + a\sin^2 x) = 2\pi \ln\left(\frac{1+\sqrt{1+a}}{2}\right). \tag{7.43}$$

This gives finally

$$\frac{\Delta}{\Delta_c} = \frac{\sqrt{\epsilon^2/4+1/4} + \sqrt{1+\epsilon^2/4+1/4}}{\epsilon/2 + \sqrt{1+\epsilon^2/4}}. \tag{7.44}$$

In the limit $\epsilon \to 0$, one has that

$$\frac{\Delta}{\Delta_c} = \frac{1+\sqrt{5}}{2} = 1.6180\ldots, \tag{7.45}$$

the golden mean, in agreement to three places with the direct evaluation of the roots in Eq. (7.24). One has that a stationary lattice crack in a noiseless environment will not begin to move until the driving strain exceeds by this amount the strain that would be predicted in a continuum model.

At velocities that approach 1, there is one root f^+ and one root g^+. In the limit of small ϵ, and for $v = 1$, $g^+ = 1.91892$. One finds $f^+ = \sqrt{2\sqrt{3}\epsilon + 12(1-v)}$. Therefore

$$\Delta/\Delta_c \to \sqrt{\frac{1.91892}{\sqrt{2\sqrt{3}\epsilon + 12(1-v)}}}. \tag{7.46}$$

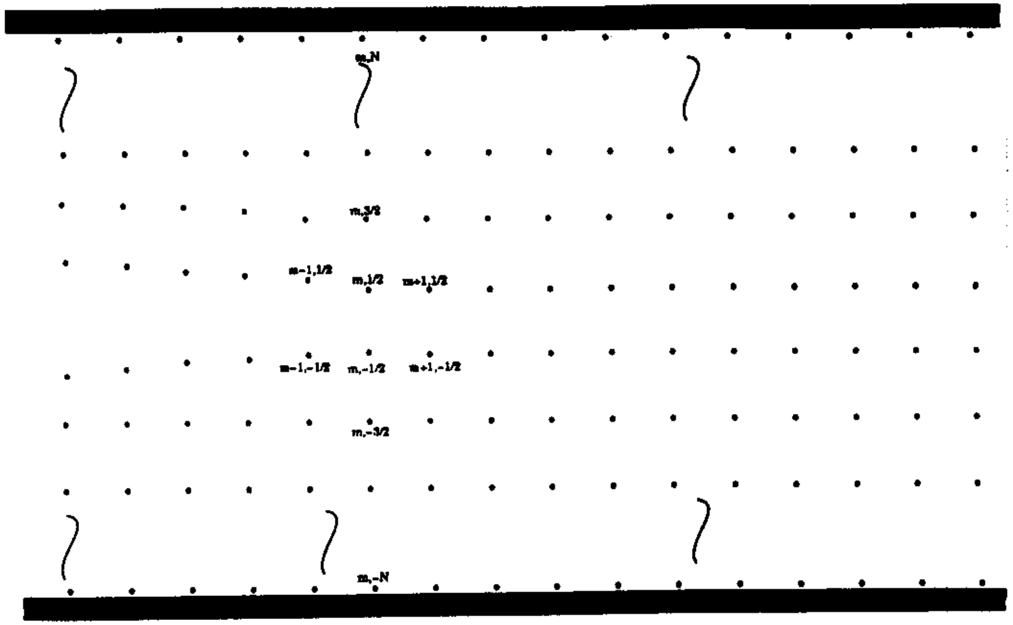


Figure 7.1: Diagram showing the lattice considered in this section, and the way its points are indexed.

7.2. Crack in a Square Lattice: Antiplane Shear

Consider a strip of material made from a square lattice of mass points. The motions of the mass points will be constrained to a single direction, and therefore describe mode III fracture. Letting $u(m,n)$ describe the displacement of the mass point indexed by m and n, the equation of motion for $u(m,n,t)$ is

$$\ddot{u}(m,n) = \begin{bmatrix} & +u(m,n+1) & \\ +u(m-1,n) & -4u(m,n) & +u(m+1,n) \\ & +u(m,n-1) & \end{bmatrix} - b\dot{u}(m,n) \tag{7.47a}$$

if $n > 1/2$, and

$$\ddot{u}(m,n) = \begin{bmatrix} & +u(m,n+1) & \\ +u(m-1,n) & -3u(m,n) & +u(m+1,n) \\ & +\mathcal{F}[u(m,n-1)-u(m,n)] & \end{bmatrix} - b\dot{u}(m,n) + \sigma(m,t), \tag{7.47b}$$

for $n = 1/2$, where $\sigma(m,t)$ is the external force causing the crack to propagate, and

$$\mathcal{F}[u] \equiv u\theta(u_c - |u|) \tag{7.48}$$

describes a bond which is linear until it fails at extension u_c. At the top of the strip there is the boundary condition

$$u(m, N+1/2) = 0. \tag{7.49}$$

The goal of this section will be to calculate properties of steady state cracks moving in this lattice. This calculation is very close to the one originally performed by Slepyan, although his calculation was in an infinite plate, rather than a strip. Apart from this minor difference, however, it appears that he neglected the contribution of a branch cut that arises in the course of the calculation.

In the case of steady state motion, one has

$$u(m,n,t) = u(m+1,n,t+1/v). \tag{7.50}$$

$$\Rightarrow u(m,n,t) = u(0,n,t-m/v). \tag{7.51}$$

It will only be necessary to carry out the calculation for $n > 0$ because of the symmetry

$$u(m,n) = -u(m,-n). \tag{7.52}$$

Because of this relation, one needs only to know $u(0,n,t) = u_n(t)$, and the index m will be dropped from now on. Of course, a steady state will only come to pass if the applied force is of the proper form too:

$$\sigma(m,t) = \sigma(m - t/v). \tag{7.53}$$

Rewriting Eq. (7.47) for the case of steady motion, one has

$$\ddot{u}_n(t) = \begin{bmatrix} & u_{n+1}(t) & \\ +u_n(t+1/v) & -4u_n(t) & +u_n(t-1/v) \\ & +u_{n-1}(t) & \end{bmatrix} - b\dot{u}_n(t) \tag{7.54a}$$

for $n > 1/2$ and

$$\ddot{u}_n(t) = \begin{bmatrix} & u_{n+1}(t) & \\ +u_n(t+1/v) & -3u_n(t) & +u_n(t-1/v) \\ & -\mathcal{F}[2u_n(t)] & \end{bmatrix} - b\dot{u}_n(t) + \sigma_\infty e^{-\alpha|t|} \tag{7.54b}$$

for $n = 1/2$. The symmetry Eq. (7.52) has been used to simplify part of the expression and the force is taken to be of a particularly simple form, with α assumed to be very small. This problem is equivalent to one in which no force is applied to the crack faces, but the top and bottom of the strip displaced instead. One moves from the first problem to the second by adding the solution of a problem where a static strip has its top and bottom displaced, but forces $-\sigma_\infty$ are applied to the layers at $\pm 1/2$ so that these layers end up not moving at all. That is, let

$$0 = \begin{bmatrix} & +v(m,n+1) & \\ +v(m-1,n) & -4v(m,n) & +v(m+1,n) \\ & +v(m,n-1) & \end{bmatrix} \tag{7.55a}$$

if $n > 1/2$, and

$$0 = \begin{bmatrix} & +v(m,n+1) & \\ +v(m-1,n) & -3v(m,n) & +v(m+1,n) \\ & +0 & \end{bmatrix} - \sigma_\infty , \tag{7.55b}$$

for $n = 1/2$ so that

$$v(m,n) = \sigma_\infty m - \frac{\sigma_\infty}{2}. \tag{7.55c}$$

Then $u(m,n)+v(m,n)$ can replace $u(m,n)$ everywhere in Eq. (7.54), and one will still have a solution, but the force on the crack face has disappeared and the top strip of particles is displaced by an amount $N\sigma_\infty$. This procedure works because $v(m,1/2) = v(m,-1/2) = 0$. The force on the crack surface actually only disappears within a distance v/α of the crack tip, but in the limit that $\alpha \to 0$, this technical point will not be important.

Fourier transforming Eq. (7.54) by

$$u_n(\omega) = \int e^{i\omega t} u_n(t) \tag{7.56}$$

turns the steady state equations to

$$u_{n+1}(\omega) + u_{n-1}(\omega) + [2\cos\omega/v - 4 + ib\omega]\, u_n(\omega) = -\omega^2 u_n(w) \tag{7.57}$$

for $n > 1/2$. Define

$$u(\omega) = u_{1/2}(\omega). \tag{7.58}$$

Then since displacements must vanish for $n = N + 1/2$, where N is the number of vertical layers making up the strip, one has from Eq. (7.57) that

$$u_n(\omega) = u(\omega)\left[\frac{\sinh k(N+1/2-n)}{\sinh kN}\right] \tag{7.59}$$

provided that k is chosen to satisfy

$$\omega^2 + ib\omega + 2\cosh k + 2\cos\omega/v - 4 = 0. \tag{7.60}$$

Instead of using k, one can alternatively define

$$z = 2 - \cos\omega/v - \omega^2/2 - i\omega b/2 \tag{7.61}$$

and

$$y = e^k. \tag{7.62}$$

so that

$$y = z + \sqrt{z^2 - 1} \tag{7.63}$$

and

$$u_n(\omega) = u(\omega)\left[\frac{y^{[N+1/2-n]} - y^{-[N+1/2-n]}}{y^N - y^{-N}}\right]. \tag{7.64}$$

In using the variable y, I will stick to the convention that $\sqrt{z^2-1}$ is positive if it is real, and that its imaginary part is positive otherwise. Using this solution for u for $n > 1/2$, one can write the equation governing $u(1/2,\omega) = u(\omega)$ as

$$u(\omega)\left[\frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z + 1\right] - 2u^-(\omega) = -\sigma_\infty\left[\frac{2\alpha}{\omega^2 + \alpha^2}\right]. \tag{7.65}$$

This expression follows from Eq. (7.54b) after decreeing that $t = 0$ is the time at which the bond at $u(0,n,t)$ breaks, defining

$$u^-(\omega) = \int_{-\infty}^{0} u(t)\, e^{i\omega t}, \tag{7.66}$$

and specializing to an external force of the form

$$\sigma(t) = \sigma_\infty e^{-\alpha|t|}. \tag{7.67}$$

I will always work in the limit where $\alpha \ll 1$. Since $t = 0$ is the time at which the bond at $m = 0$ snaps, later on when the solution is found I will have to choose σ_∞ properly so as to insure that in fact $2u(1/2,t)$ did equal u_c at the critical instant.

Writing

$$u(\omega) = u^+(\omega) + u^-(\omega), \tag{7.68}$$

rewrite Eq. (7.65) as

$$u^+(\omega)F(\omega) + u^-(\omega)G(\omega) = -\sigma_\infty\left\{\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega}\right\}, \tag{7.69}$$

with

$$F(\omega) = \frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z + 1 \tag{7.70a}$$

and

$$G(\omega) = \frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z - 1. \tag{7.70b}$$

Eq. (7.69) is identical with Eq. (7.6), although the particular forms of F and G are different. Therefore, all of the formal development that followed Eq. (7.6) can immediately be transferred. The only difference between the present case and the former one is that now both F and G have poles as well as roots along the real axis, and these must be divided out as well. Therefore, letting $\omega_{pj}^{F\pm}$ be the real poles belonging to $F^\pm$ respectively, $\omega_{rj}^F\pm$

be the real roots belonging to $F^{\pm}$, and with analogous notation for the real roots and poles of G, one has following the path that led to Eq. (7.22)

$$u(t=0) = \sigma_\infty \sqrt{\frac{1}{F(0)G(0)}} \sqrt{\prod_j \frac{\omega_{pj}^{F-} \omega_{rj}^{F+} \omega_{pj}^{G+} \omega_{rj}^{G-}}{\omega_{rj}^{F-} \omega_{pj}^{F+} \omega_{pj}^{G-} \omega_{rj}^{G+}}}. \tag{7.71}$$

First, the prefactor in this expression can be understood by appealing to energy balance. Far ahead of the crack, mass points are located in the vertical direction by

$$u_n = \frac{\sigma_\infty}{2} - \frac{\sigma_\infty}{2N+1} n, \tag{7.72}$$

while far behind they obey

$$u_n = \sigma_\infty \left(N + \frac{1}{2}\right) - \sigma_\infty n \tag{7.73}$$

for $n > 0$. The bond between $u_{1/2}$ and $u_{-1/2}$ is stretched much further than all the rest, to a distance

$$u_{1/2} - u_{-1/2} = \sigma_\infty N/(2N+1). \tag{7.74}$$

The energy per bond length before the crack, counting both halves of the strip, is then

$$E_{\text{Before}} = 2\frac{N}{2}\left[\frac{\sigma_\infty}{2N+1}\right]^2 + \frac{1}{2}\left[\sigma_\infty \frac{2N}{2N+1}\right]^2, \tag{7.75}$$

while the energy per bond length after the crack is

$$E_{\text{After}} = 2\frac{1}{2}\sigma_\infty^2 N. \tag{7.76}$$

The energy supplied by the external forces on the faces of the strip is

$$E_{\text{External}} = 2\left[\sigma_\infty N - \frac{\sigma_\infty}{2}\left(\frac{2N}{2N+1}\right)\right]\sigma_\infty. \tag{7.77}$$

The maximum energy (for given σ_∞) that could possibly be stored in the bond which breaks at $t = 0$ is therefore

$$E_{\text{Ideal}} = E_{\text{Before}} + E_{\text{External}} - E_{\text{After}} = \frac{2N^2\sigma_\infty^2}{2N+1}. \tag{7.78}$$

This energy corresponds to an ideal case where no energy is lost to dissipation or traveling waves, and is the energy of a bond that breaks when $u(1/2)$ reaches the ideal length

$$u_{\text{Ideal}} = \frac{N\sigma_\infty}{\sqrt{2N+1}} \tag{7.79}$$

Using Eq. (7.70), one can therefore write that

$$u(t=0) = u_{\text{Ideal}} \sqrt{\prod_j \frac{\omega_{pj}^{F-} \omega_{rj}^{F+} \omega_{pj}^{G+} \omega_{rj}^{G-}}{\omega_{rj}^{F-} \omega_{pj}^{F+} \omega_{pj}^{G-} \omega_{rj}^{G+}}}. \tag{7.80}$$

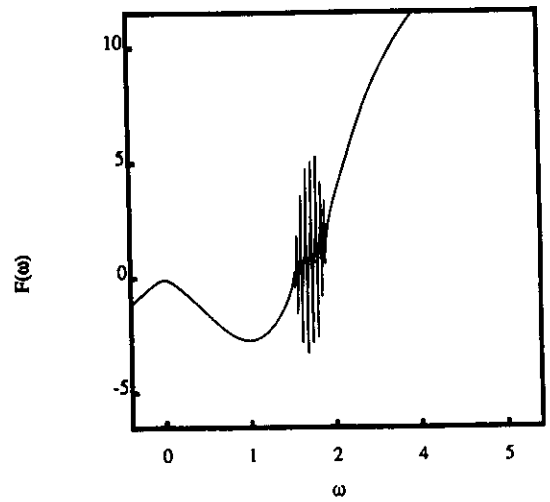


Figure 7.2: The function $F(\omega)$, for $v = .5$ $N = 10$, and $b = .01$. For lower values of v, the structure can become more complicated, with more than one group of positve poles, and for larger values of N the poles become more finely spaced.

The product of roots and poles contains complete information about energy carried by traveling waves. In fact, the poles of F and G will occur at precisely the same spot, so they will cancel out and turn out not to have been necessary in Eq. (7.80) after all. However, they are very useful in organizing the product, and will not drop out of the argument altogether. Second, the question of precisely which roots to include in the product on the right hand side of Eq. (7.71) hinges on the relationship between the damping constant b and the strip height N. In the limit $b \ll 1/N$, $F(\omega)$ looks as in Fig. 7.2. However, in the limit where $b \gg 1/N$, the poles now visible along the real axis merge into a branch cut. On physical grounds, the relationship between b and $1/N$ is important. When b is small, physically one has a situation where waves travel to the top boundary, reflect and return to the crack line many times. When b is larger than $1/N$, one has by contrast a situation where waves damp out before reflecting off the top boundary: it is this case which corresponds to the infinite plate considered by Slepyan. It is fairly easy to calculate the contribution of the many roots in the case where $b \ll 1/N$, but in the opposite limit I do not now see how to carry through the calculation, and Slepyan seems to have ignored the matter altogether, just focusing upon the real roots.

Physically, whether a root belongs to F^+ or F^- is determined by whether the group velocity of the corresponding wave is larger or smaller than the velocity of the crack. Mathematically, note that as b increases from zero, the change in the location of a root ω

is given by

$$\delta\omega = b\left.\frac{\partial\omega}{\partial b}\right|_F. \tag{7.81}$$

Since F is a function of z alone, one may rewrite Eq. (7.81) as

$$\delta\omega = b\left.\frac{\partial\omega}{\partial b}\right|_z = -b\frac{\partial z/\partial b}{\partial z/\partial\omega} \tag{7.82}$$

$$= \frac{i\omega b/2}{\partial z/\partial\omega}. \tag{7.83}$$

So if $\omega\partial z/\partial\omega$ is positive, the imaginary part of the root is positive, and the root belongs with F^-, while otherwise it belongs with F^+. Whether a pole belongs in the + or − group is determined by precisely the same condition. Since F and G have the same poles, they will from now on be denoted simply by ω_{pj}, and since ω_{pj} will be very close to ω^F_{rj} and ω^G_{rj}, one can evaluate Eq. (7.83) at ω_{pj} to determine whether roots belong to F^+ or F^-. Considering the limit where $N \gg 1$, one can then rewrite Eq. (7.80) as

$$u(t=0) = u_{\text{Ideal}}\exp\left[\frac{1}{2}\sum_j \frac{\omega_{pj}\partial z/\partial\omega_{pj}}{|\omega_{pj}\partial z/\partial\omega_{pj}|}\log\left(\frac{\omega^G_{rj}}{\omega^F_{rj}}\right)\right]. \tag{7.84}$$

$$= u_{\text{Ideal}}\exp\left[\frac{1}{2}\sum_j \frac{\omega_{pj}\partial z/\partial\omega_{pj}}{|\omega_{pj}\partial z/\partial\omega_{pj}|}\frac{\left(\omega^G_{rj}-\omega^F_{rj}\right)}{\omega_{pj}}\right]. \tag{7.85}$$

$$= u_{\text{Ideal}}\exp\left[\frac{1}{2}\sum_j \frac{\left(z^G_{rj}-z^F_{rj}\right)}{|\omega_{pj}\partial z/\partial\omega_{pj}|}\right]. \tag{7.86}$$

In Eq. (7.86), z^F_{rj} means $z(\omega^F_{rj}$; that is, the value of z for which ω is the j'th real root of F. It was assumed that ω^F_{pj} is very near to ω^G_{pj}, and this will soon be shown to be true. Expression Eq. (7.86) is valuable because it remains true in more complicated lattices, where the dispersion relation is different.

Dealing with discrete real roots as they occur is relatively trivial, but in the limit $N \to \infty$ one needs to deal with the forest of roots that is apparent in Fig. 7.2. To organize them, it is best to look first at the locations of the poles. There is a pole both in F and in G every time

$$y^N = y^{-N} \tag{7.87}$$

$$\Rightarrow y_{pj} = e^{ij\pi/N}, \tag{7.88}$$

with j varying between 1 and $N-1$. There is no pole when $j=0$ or $j=N$ because of a cancellation between the numerator and denominator in F or G. Each value of y_j corresponds to

$$z_{pj} = \sin j\pi/N, \tag{7.89}$$

and also to ω_{pj}, which however cannot be written down in simple form.

The roots of F and G can also be found fairly explicitly. Solving Eq. (7.70a) for y when $F=0$ gives that

$$y^{2N} = \frac{2z-1-y}{2z-1-y^{-1}} \tag{7.90}$$

$$\Rightarrow y^{2N} = \frac{y^{-1}-1}{y-1} \tag{7.91}$$

from Eq. (7.63)

$$\Rightarrow y^{2N+1} = -1 \tag{7.92}$$

By convention, the imaginary part of y is positive, so

$$y^F_{rj} = e^{i(2j-1)\pi/(2N+1)} \tag{7.93}$$

for $1 \le j \le N$

$$\Rightarrow z^F_{rj} = \cos\left[(2j-1)\pi/(2N+1)\right]. \tag{7.94}$$

Similarly considering G one finds instead of Eq. (7.92) that

$$y^{2N+1} = 1 \tag{7.95}$$

$$\Rightarrow y^G_{rj} = e^{2ij\pi/(2N+1)} \tag{7.96}$$

for $1 \le j \le N$

$$\Rightarrow z^G_{rj} = \cos\left[2j\pi/(2N+1)\right]. \tag{7.97}$$

Since

$$z^G_{rj} - z^F_{rj} \approx \frac{1}{2}\frac{\partial z^G_{rj}}{\partial j} = -\frac{1}{2}\left|\frac{\partial z^G_{rj}}{\partial j}\right| \tag{7.98}$$

one can finally write for Eq. (7.86) that

$$u(t=0) = u_{\text{Ideal}}\exp\left[-\frac{1}{4}\int\frac{d\omega}{|\omega|}\theta(1-|z|)\right]. \tag{7.99}$$

This expression bears an interesting relationship to the one found by Slepyan. In the limit $N \to \infty$, F and G appear to pass through zero just as the branch cuts begin; these points correspond exactly to the places where $|z|=1$. So treating these points as real roots, and using them in Eq. (7.86), one would obtain precisely Eq. (7.99), except that the factor of 1/4 would need to be replaced by a factor of 1/2. Accounting for the branch cuts properly gives a result that is the square root of the apparent result.

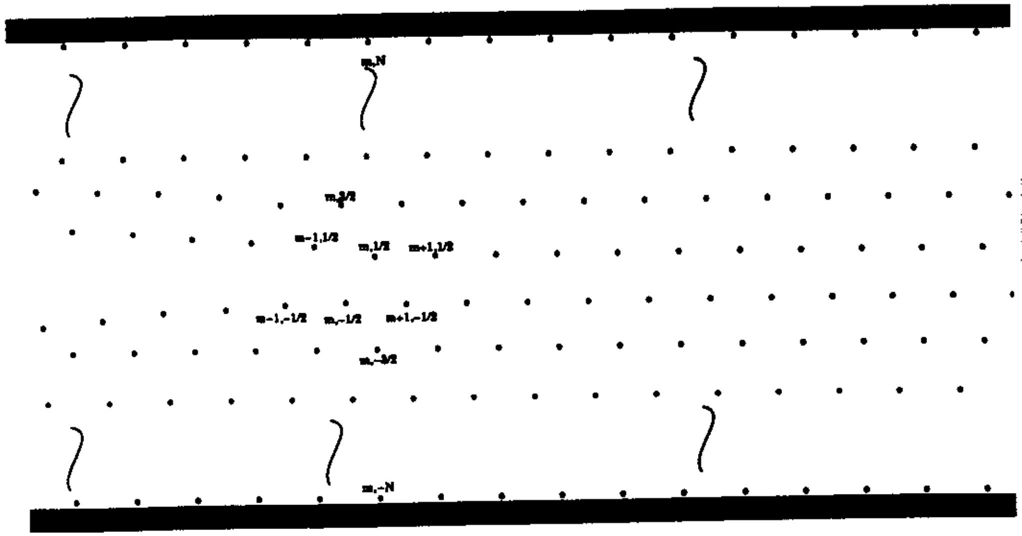


Figure 7.3: Diagram showing the lattice considered in this section, and the way its points are indexed.

7.3. Crack in a Triangular Lattice: Antiplane Shear

The triangular lattice can be treated by methods very similar to those which solve the square lattice. In this section, I will solve the problem of a triangluar lattice in antiplane shear. Let $u(m,n)$ describe the displacement of the mass point indexed by m and n, as in Fig. 7.3, and let

$$g_n = \begin{cases} 0 & \text{if } n = 1/2, 5/2\ldots \\ 1 & \text{if } n = 3/2, 7/2\ldots. \\ \mathrm{mod}\,(n-1/2,2) & \text{in general} \end{cases} \tag{7.100}$$

Then the equation of motion for $u(m,n,t)$ is

$$\ddot{u}(m,n) = \frac{1}{2}\begin{bmatrix} +u(m+g_{n+1}-1,n+1)+u(m+g_{n+1},n+1) \\ +u(m-1,n)-6u(m,n)+u(m+1,n) \\ +u(m+g_{n-1}-1,n-1)+u(m+g_{n-1},n-1) \end{bmatrix} - b\dot{u}(m,n) \tag{7.101a}$$

if $n > 1/2$, and

$$\ddot{u}(m,n) = \frac{1}{2}\begin{bmatrix} +u(m+g_{n+1}-1,n+1)+u(m+g_{n+1},n+1) \\ +u(m-1,n)-4u(m,n)+u(m+1,n) \\ +\mathcal{F}[u(m+g_{n-1}-1,n-1)-u(m,n)]-\mathcal{F}[u(m+g_{n-1},n-1)-u(m,n)] \end{bmatrix} - b\dot{u}(m,n)+\sigma(m,t) \tag{7.101b}$$

if $n = 1/2$, with $\mathcal{F}$ given as before by Eq. (7.48). The boundary condition on the top of the strip is that

$$u(m,(N+1/2),t) = 0. \tag{7.102}$$

In steady state, one has the symmetry

$$u(m,n,t) = u(m+1,n,t+1/v) \tag{7.103a}$$

and also

$$u(m,n,t) = -u(m,-n,t-[1/2-g_n]/v) \tag{7.103b}$$

which implies in particular that

$$u(m,1/2,t) = -u(m,-1/2,t-1/2v). \tag{7.103c}$$

Specializing to the force given in Eq. (7.53), one can eliminate the variable m, defining $u_n(t) = u(0,n,t)$ and write the system of equations in steady state as

$$\ddot{u}_n(t) = \frac{1}{2}\begin{bmatrix} +u_{n+1}(t-(g_{n+1}-1)/v)+u_{n+1}(t-g_{n+1}/v) \\ +u_n(t+1/v)-6u_n(t)+u_n(t-1/v) \\ +u_{n-1}(t-(g_{n-1}-1)/v)+u_{n-1}(t-g_{n-1}/v) \end{bmatrix} - b\dot{u}_n \tag{7.104a}$$

if $n > 1/2$, and

$$\ddot{u}_{1/2}(t) = \frac{1}{2}\begin{bmatrix} +u_{3/2}(t)+u_{3/2}(t-1/v) \\ +u_{1/2}(t+1/v)-4u_{1/2}(t)+u_{1/2}(t-1/v) \\ +[u_{-1/2}(t)-u_{1/2}(t)]\,\theta(-t)+[u_{-1/2}(t-1/v)-u_{1/2}(t)]\,\theta(1/(2v)-t) \end{bmatrix} - b\dot{u}_{1/2}+\sigma_\infty e^{-\alpha|t|}. \tag{7.104b}$$

The time at which the bond between $u(0,1/2,t)$ and $u(0,-1/2,t)$ breaks has been chosen to be $t=0$, so that by symmetry the time the bond between $u(0,1/2,t)$ and $u(1,-1/2,t)$ breaks is $1/2v$. The problem described by Eq. (7.104) is equivalent to one in which the top and bottom of the strip are held at displacements $N\sigma_\infty$, since $u(m,n,t)+v(m,n)$ still solves Eq. (7.104), where v_n is given by Eq. (7.55).

For $n > 1/2$ it is easy to solve the linear set of equations Eq. (7.101). Using Eq. (7.103a) to eliminate the variable m and Fourier transforming in time gives

$$-\omega^2 u_n(\omega) = ib\omega + \begin{matrix} \frac{1}{2}u_{n+1}(\omega)\left[e^{i\omega(g_{n+1}-1)/v} + e^{i\omega(g_{n+1})/v}\right] \\ \frac{1}{2}u_n(\omega)\left[e^{i\omega/v} - 6 + e^{-i\omega/v}\right] \\ \frac{1}{2}u_{n-1}(\omega)\left[e^{i\omega(g_{n-1}-1)/v} + e^{i\omega(g_{n-1})/v}\right] \end{matrix} \tag{7.105}$$

Defining

$$\tilde{u}(t) = u_{1/2}(t) \tag{7.106}$$

let

$$u_n(\omega) = \tilde{u}(\omega)\, e^{k(n-1/2)-i\omega g_n/(2v)}. \tag{7.107}$$

Substituting this expression into Eq. (7.105), and noticing that $g_n + g_{n+1} = 1$ gives

$$-\omega^2 \tilde{u}(\omega) = ib\omega\tilde{u}(\omega) + \begin{matrix} \frac{1}{2}\tilde{u}(\omega)\, e^{k}\left[e^{i\omega(g_{n+1}+g_n-2)/(2v)} + e^{i\omega(g_{n+1}+g_n)/(2v)}\right] \\ \frac{1}{2}\tilde{u}(\omega)\left[e^{i\omega/v} - 6 + e^{-i\omega/v}\right] \\ \frac{1}{2}\tilde{u}e^{-k}(\omega)\left[e^{i\omega(g_{n-1}+g_n-2)/(2v)} + e^{i\omega(g_{n-1}+g_n)/(2v)}\right] \end{matrix} \tag{7.108}$$

$$\Rightarrow \omega^2 + ib\omega + 2\cosh(k)\cos(\omega/(2v)) + \cos(\omega/v) - 3 = 0. \tag{7.109}$$

Defining

$$z = \frac{3 - \cos(\omega/v) - \omega^2 - ib\omega}{2\cos(\omega/2v)} \tag{7.110}$$

one has equivalently that

$$y = z + \sqrt{z^2 - 1}, \tag{7.111}$$

with

$$y = e^{k}. \tag{7.112}$$

Therefore,

$$u_n(\omega) = \tilde{u}(\omega)\, e^{-i\omega g_n/2v}\left[\frac{y^{[N+1/2-n]} - y^{-[N+1/2-n]}}{y^N - y^{-N}}\right]. \tag{7.113}$$

In order to Fourier transform the line of points where $n = 1/2$, define

$$u(t) = \frac{u_{1/2}(t) - u_{-1/2}(t)}{2} = \frac{u_{1/2}(t) + u_{1/2}(t + 1/2v)}{2}. \tag{7.114}$$

Then rewrite Eq. (7.104b) as

$$\ddot{\tilde{u}}(t) = \frac{1}{2}\left[\begin{matrix} +u_{3/2}(t) + u_{3/2}(t - 1/v) \\ +\tilde{u}(t + 1/v) - 4\tilde{u}(t) + \tilde{u}(t - 1/v) \\ -2u(t)\theta(-t) - 2u(t - 1/2v)\theta(1/(2v) - t) \end{matrix}\right] - b\dot{\tilde{u}} + \sigma_\infty e^{-\alpha|t|}. \tag{7.115}$$

Fourier transforming this expression using Eq. (7.113) and Eq. (7.66) now gives

$$\tilde{u}(\omega)\left[\left\{\frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z\right\}\cos(\omega/2v) + 1\right] - \left(1 + e^{i\omega/2v}\right)u^-(\omega) = -\sigma_\infty\frac{2\alpha}{\omega^2 + \alpha^2}, \tag{7.116}$$

Finally, use

$$u(\omega) = \frac{\left(1 + e^{-i\omega/2v}\right)}{2}\tilde{u}(\omega) \tag{7.117}$$

to obtain

$$u(\omega)\left[\left\{\frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z\right\}\cos(\omega/2v) + 1\right] - 2\left(\cos^2\omega/4v\right)u^-(\omega) = -\sigma_\infty\frac{2\alpha}{\omega^2 + \alpha^2}. \tag{7.118}$$

Decomposing u as in Eq. (7.68) gives

$$u^+(\omega)F(\omega) + u^-(\omega)G(\omega) = -\sigma_\infty\left\{\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega}\right\}, \tag{7.119}$$

with

$$F(\omega) = \left\{\frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z\right\}\cos(\omega/2v) + 1 \tag{7.120a}$$

and

$$G(\omega) = \left\{\frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z - 1\right\}\cos(\omega/2v). \tag{7.120b}$$

Eq. (7.120) is identical with Eq. (7.69), except for the fact that F, G, and z have been redefined. In the limit where N is large, the derivatives of F are completely dominated by the portion of F which depends upon z, and one has in analogy with Eq. (7.83) that if

$$w\cos(\omega/2v)\,\partial z/\partial w \tag{7.121}$$

is positive, the corresponding root belongs to F^- or G^-, and to the other camp in the opposite case. Since $F(0)$ and $G(0)$ are exactly the same as previously, one can proceed to write as before that

$$u(t = 0) = \frac{u_{1/2}(t = 0) - u_{-1/2}(t = 0)}{2} \tag{7.122}$$

$$= u_{\text{Ideal}}\exp\left[\frac{1}{2}\sum_j \frac{\cos(\omega/2v)}{|\cos(\omega/2v)|}\frac{\left(z^G_{rj} - z^F_{rj}\right)}{|\omega_{pj}\partial z/\partial\omega_{pj},|}\right], \tag{7.123}$$

essentially copying Eq. (7.86), with Eq. (7.121) in mind, and retaining the definition of u_{Ideal} given in Eq. (7.79). Eq. (7.123) compared to Eq. (7.86) The poles of both F and

G are in the same location as for the square lattice, and the roots of G are in the same location as well. However, the roots of F are now determined by the implicit equation

$$y_{rj}^F = e^{i(2j-1)\pi/(2N+1)+\ln\left[\frac{y_{rj}^F - \cos(\omega/2v)}{y_{rj}^F \cos(\omega/2v)-1}\right]/(2N+1)} . \tag{7.124}$$

Therefore

$$z_{rj}^F = \cos\left[\frac{(2j-1)\pi}{2N+1} + \frac{1}{i(2N+1)}\ln\left\{\frac{y_{rj}^F - \cos\omega/2v}{y_{rj}^F(\cos\omega/2v)-1}\right\}\right] \tag{7.125a}$$

$$z_{rj}^G = \cos\left[\frac{2j\pi}{2N+1}\right], \tag{7.125b}$$

and

$$z_{rj}^G - z_{rj}^F \approx -|\frac{\partial z_r^G}{\partial j}|\left[\frac{1}{2} - \frac{1}{2\pi i}\ln\left\{\frac{y_{rj}^F - \cos\omega/2v}{y_{rj}^F(\cos\omega/2v)-1}\right\}\right] \tag{7.126}$$

This expression is really the farthest one can go in finding the energy flux for the triangular lattice. It would seem that one could consider the large N limit and write

$$u(t=0) = u_{\text{Ideal}}\exp\left[-\frac{1}{4}\int\frac{d\omega}{|\omega|}\frac{\cos(\omega/2v)}{|\cos(\omega/2v)|}\theta(1-|z|)\left(1-\frac{1}{\pi i}\ln\left\{\frac{y-\cos\omega/2v}{(y\cos\omega/2v)-1}\right\}\right)\right]. \tag{7.127}$$

Unfortunately this expression is false because not all roots of F occur for $|z| < 1$, and there is no simple rule describing when some root – corresponding to an isolated surface mode – will miraculously appear. However, Eq. (7.125a) can be made the basis of a very fast numerical procedure for finding all the roots. One first finds all the places where $|z| = 1$. Next, in each subregion where $|z| > 1$, one checks to see if F has a zero; F has no rapid oscillations in these subregions, so the search is rapid. Finally, one finds the roots in each subregion where $|z| < 1$. The way to do this is to look for the roots of

$$Q(\omega) = z - \cos\left[\frac{(2j-1)\pi}{2N+1} + \frac{1}{i(2N+1)}\ln\left\{\frac{y-\cos\omega/2v}{y(\cos\omega/2v)-1}\right\}\right] \tag{7.128a}$$

for each j ranging from 0 to $N+1$. For a given j, and in each connected region with $|z| < 1$, there is at most one root of $Q(\omega)$. This procedure is vastly faster than a direct one which attempts to find the roots of F.