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QUASICONFORMAL DEFORMATIONS IN SEVERAL VARIABLES

A. PIERZCHALSKI

Institute of Mathematics
Polish Academy of Sciences
Łódź Branch
Kilinskiego 86
PL-90-012 Łódź
Poland

§ 1 Quasiconformal mappings

Every 1-1 holomorphic mapping on the plane $\mathbb{C} = \mathbb{R}^2$ is a conformal mapping. Geometrically, at every point of the domain of its definition, it maps infinitesimal circles onto infinitesimal circles. If we assume that a ~~mapping~~ maps infinitesimal circles onto infinitesimal ellipses (or spheres onto ellipsoids in a multidimensional case) with a bounded ratio of semiaxes, we arrive at the notion of a quasiconformal mapping.

More precisely:

Consider at first the complex plane $\mathbb{C} = \mathbb{R}^2$. Let f be a diffeomorphism ($= f$ and f^{-1} are C^1-mappings) from a domain D onto another domain D'. At every point $z = x+iy \in D$ the differential Df_z of f is an $\mathbb{R}$-linear isomorphism of T_zD onto $T_{f(z)}D'$. Using the complex notation we can write

(1)
$$Df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z$$

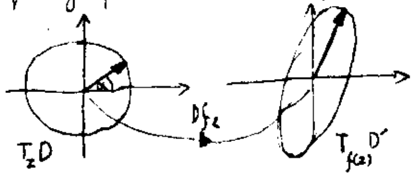
where $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$, $dz = dx + i dy$. If we write $f = u + iv$, then the Jacobian J_f of f is of the form

$$J_f(z) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix} = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar z}\right|^2.$$

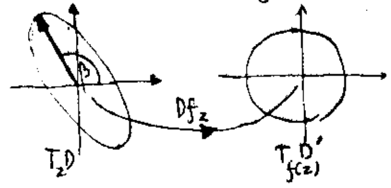
Assume now that f is sense preserving, i.e. $\left|\frac{\partial f}{\partial z}\right| > \left|\frac{\partial f}{\partial \bar z}\right|$, then it follows from (1) that

$$\left(\left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \bar z}\right|\right)|dz| \leq |Df| \leq \left(\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar z}\right|\right)|dz|$$

One may check that both limits can be attained, namely, the right-hand side equality holds for vectors with arguments $\alpha = \frac{1}{2}\arg\left(\frac{\partial f}{\partial \bar z} \Big/ \frac{\partial f}{\partial z}\right)$ and the left-hand equality for vectors with arguments $\beta = \alpha + \frac{\pi}{2}$. We have then the following illustrations.



or equivalently



In this case the ratio

$$K(z) = \frac{\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar z}\right|}{\left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \bar z}\right|} \geqslant 1$$

of the major axis of the ellipse to the minor one is called the dilatation of f at the point z. If the function is bounded in the whole D then the mapping f is said to be quasiconformal. For f is holomorphic, i.e. for $\frac{\partial f}{\partial \bar z} = 0$ we get $K(z) = 1$ and, therefore f is a conformal mapping.

Consider now a generalization to the multidimensional case $\mathbb{R}^n$, $n \geqslant 2$.

Let $x \in \mathbb{R}^n$ and let $G(x)$ be a symmetric positive definite matrix normalised by the condition $\det G(x) = 1$. Then the equation

$$\langle G(x)X, X\rangle = \lambda_1^{4/n} \cdot \ldots \cdot \lambda_{n-1}^{2/n} h^2, \quad X \in T_x\mathbb{R}^n, \quad h \in (0, \infty)$$

where $\langle\ ,\ \rangle$ denote the standard scalar product, describes a family of similar ellipsoids in the tangent space $T_x\mathbb{R}^n$ with semiaxes $\lambda_1 h \geqslant \ldots \geqslant \lambda_{n-1} h \geqslant h$. This family we call the infinitesimal ellipsoid at x with characteristic $G(x)$. The ratio λ_1 of the maximal semiaxis to the minimal one is called the dilatation of the ellipsoid.

Let D, D' be domains in $\mathbb{R}^n$ and $f\ D \to D'$ a sense preserving diffeomorphism. At every point $x \in D$ the differential Df_x is an isomorphism between the tangent spaces T_xD and $D_{f(x)}D'$ (In this case $T_xD \simeq T_{f(x)}D' \simeq \mathbb{R}^n$). The condition that the differential Df_x maps an infinitesimal ellipsoid with characteristic $G(x)$ onto an infinitesimal ellipsoid with characteristic $G'(f(x))$ has the form

$$\langle G'(f(x))Df_xX, Df_xX\rangle = C \cdot \langle G(x)X, X\rangle$$

with some constant C, or, more concisely,

$$Df^* G' Df^t = C \cdot G.$$

This, by the normalizing conditions $\det G' = \det G = 1$, gives

(2)
$$Df^* G' Df = J_f^{2/n} G,$$

where J_f is the Jacobian of f, cf. [B-I]. In coordinates the last equation takes the form

(2')
$$\sum_{j,k=1}^{n} G'_{jk}\frac{\partial f_j}{\partial x_r}\frac{\partial f_k}{\partial x_s} = J_f^{2/n} G_{rs}, \qquad r,s = 1,\ldots,n,$$

where $G = (G_{rs})$, $G' = (G'_{jk})$. In the particular case $G' = I$ ~~we get~~ (such normalization is always possible) we get

(3) $$J^{-2/n} Df^{*} Df = G.$$

or, in the coordinal system

(3′) $$J^{-2/n} \sum_{j} \frac{\partial f_j}{\partial x_r} \frac{\partial f_j}{\partial x_s} = G_{rs}$$

In this case, at every point $x \in D$, the differential maps the infinitesimal ellipsoid with characteristic $G(x)$ onto the infinitesimal sphere. The dilatation of this ellipsoid will be called the linear dilatation of f at x. If $G = I$, i.e. if f maps infinitesimal spheres onto infinitesimal spheres, then f is a conformal mapping. We may then say that the deviation of the normalized matrix

(4) $$J_f^{-2/n} Df^{*} Df$$

from the unit matrix I is a measure of the deviation of f from conformality, i.e. a measure of quasiconformality. Then by quasiconformal mappings we shall mean such mappings that G is in some sense „not far" from the unit matrix. There are several definitions of quasiconformality. We choose one of them motivated by the above consideration.

By a K-quasiconformal diffeomorphism, $K \in [1, \infty)$ we shall mean a diffeomorphism $f : D \to D'$ such that

$$\| Df_x \|^n \leq n^{n/2} K^n J_f(x), \qquad x \in D$$

for every $x \in D$, where $\| Df_x \|$ is the norm of the matrix Df_x $\left(\|(a_{ij})\| = \left(\sum_{ij} a_{ij}^2\right)^{1/2} \right)$.

It may be shown that 1-quasiconformal mappings are conformal.

Because the length of semiaxes of an infinitesimal ellipsoid may be expressed by eigenvalues of the characteristic G and those eigenvalues as roots of an algebraic equation estimated by coefficients of the equation, we see that the linear dilatation of f has a uniform estimation by the norm of the matrix G. Therefore a quasiconformal mapping is a mapping with a bounded linear dilatation in the whole domain.

It arises the problem of existence of mappings with prescribed characteristics, i.e the problem of solvability of the equation (3), or more precisely, of the system of differential equations (3′) with $G = (G_{rs})$ given.

The problem of local integrability of the system (3′) may be treated as a problem of existence of isothermal coordinates on the Riemannian manifold D with Riemannian scalar product G. We have then

Theorem 1 A necessary and sufficient condition for a system (3′) to be locally integrable is that D treated as a Riemannian manifold with metric G is conformally flat. In the case of C^3-manifolds it is equivalent to the condition

$$0 = 0 \qquad \text{for } n = 2$$
$$B = 0 \qquad \text{for } n = 3$$
$$C = 0 \qquad \text{for } n > 3$$

where B is the tensor field defined by

$$D(X,Y,Z) = (\nabla_X L)(Y,Z) - (\nabla_Y L)(Y,Z), \qquad X,Y,Z \in TD$$

and C the Weil conformal curvature tensor defined by

$$C(X,Y)Z = K(X,Y)Z - L(Y,Z)X - L(X,Z)Y + g(Y,Z)NX - g(X,Z)NY, \qquad X,Y,Z \in TD.$$

Here K is the curvature tensor of the metric G, $\cancel{L = EX}$

$L(X,Y) = \frac{1}{n-1}\left(-R(X,Y) + R(X,Y)\right)$, R - the Ricci tensor, and $g(NX,Y) = L(X,Y)$.

For $n = 2$ the result is due to Korn - Lichtenstein (in the analytic case) and to Chern (in the smooth case), for $n \geq 3$ it is due to Weil. Moreover, in the case $n = 2$ the system (3′) with bounded norm of G has always a solution. This case has been investigated by several authors (also under very weak regularity assumptions on G), among them by Ahlfors [A-1], Bers [Be], Bojarski [Bo], Vekua [V]. For $n \geq 3$ the problem of existence of a global solution of (3′) is still open and seems to be difficult, but the regularity and stability theorems for solutions of (3′) hold cf [I], [Rk], [I-K].

§ 2 Quasiconformal deformations

The parametrization problem for quasiconformal mappings may be formulated as follows.

Given a quasiconformal mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, find a smooth ($= C^1$) mapping $F : \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$ such that

1° for every $t \in [0,1]$ the mapping $F(\cdot, t)$ is quasiconformal and for $t_1 \le t_2$ the linear dilatation of $F(\cdot, t_1)$ is less or equal to the dilatation of $F(\cdot, t_2)$,

2° $F(x, 0) = x$,

3° $F(x, 1) = f(x)$.

In the case $n = 2$ the parametrical method was investigated by Shah Tao-shing. The main results with sketch of proofs may be found in the Ahlfors' book [A-1]. A precise description of the parametrical methods for the case $n=2$ may be found in the book of Lawgnowa and Kuzi [L-K].

For $n > 2$ the parametrization is still an open problem. The first step to solve this problem are quasiconformal deformations.

Consider an one-parameter family of diffeomorphism of $\mathbb{R}^n$

$$f_t(x) = x + tZ(x) + \sigma(t), \qquad |t| < \varepsilon$$

For every such t let $G_t(x)$ be the characteristic matrix of f_t at x, i.e.

$$G_t(x) = J^{-2/n} Df_t^* Df_t \tag{5}$$

Then we have

$$G_t(x) = I + t\nu(x) + \sigma(t),$$

where $\nu(t)$ is a matrix which we have to determine. By (5), we have that $\det G_t(x) = 1$. But $\det G_t(x) = \det(I + t\nu(x) + \sigma(t)) = 1 + t \operatorname{tr} \nu + \sigma(t)$. Consequently, $\operatorname{tr} \nu = 0$. Moreover, $\nu(x)$ is a symmetric matrix, because so is $G_t(x)$. The equation (3) takes the form

$$[1 + t \operatorname{tr} DZ + \sigma(t)]^{-2/n} (I + tDZ + \sigma(t))^* (I + tDZ + \sigma(t)) = I + t\nu + \sigma(t),$$

and this gives

$$DZ + DZ^* - \frac{2}{n} \operatorname{tr} DZ \cdot I = \nu.$$

Consequently, following Ahlfors [A-2] we introduce the operator S as follows:

$$SZ = DZ + DZ^* - \frac{2}{n} \operatorname{tr} DZ \, I \tag{6}$$

for every vector field Z on $\mathbb{R}^n$ (the original Ahlfors' operator equals $\frac{1}{2} SZ$).
In the coordinate system

$$SZ_{jk} = \frac{\partial Z^j}{\partial x_k} + \frac{\partial Z^k}{\partial x_j} - \frac{2}{n} \delta_{jk} \sum_s \frac{\partial Z^s}{\partial x_s} \tag{6'}$$

Notice that SZ is a symmetric matrix with zero trace.

In the particular case we get

$$SZ = 2 \begin{pmatrix} \operatorname{Re} \frac{\partial Z}{\partial \bar z} & \operatorname{Im} \frac{\partial Z}{\partial \bar z} \\ \operatorname{Im} \frac{\partial Z}{\partial \bar z} & -\operatorname{Re} \frac{\partial Z}{\partial \bar z} \end{pmatrix}$$

where $Z = Z^1 + iZ^2$, $z = x_1 + ix_2$, and, therefore, SZ may be considered as a generalization of the derivative $\frac{\partial Z}{\partial \bar z}$.

Obviously in the case $n = 2$ the fields Z, for which $SZ = 0$, are holomorphic. For $n \geqslant 2$ transformations Z, for which $SZ = 0$, are of the form

$$Z^k(x) = a^k + \sum_j b^k_j x^j + 2\langle c, x\rangle x^k - |x|^2 c^k,$$

where a and c are constant vectors and b is a constant matrix with $Sb = 0$. This may be derived by partial differentiation of $SZ = 0$ and utilizing the symmetry of SZ, cf. [A-2].

Notice that SZ is an infinitesimal version of G and, consequently, it „measures" the quasiconformality of Z. This motivates the following definition.

A transformation (vector field) $Z : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be a k-quasiconformal deformation if $\|SZ\| \le k$ in $\mathbb{R}^n$.

An interesting question is the problem of solvability of the equation

$$SZ = \nu, \tag{7}$$

where $\nu = \nu(x)$ is a given symmetric matrix with zero trace. This equation may be treated as a linearization of the equation (3).

Bojarski and Iwaniec [B-I] showed that the local integrability conditions for (7) are linearization of the local integrability conditions in Theorem 1.

Another answer for this question is given by the following result of Ahlfors [A-2].

Theorem 2 If ν has a compact support, then the necessary and sufficient condition that (7) has a solution is that ν satisfies the integral equation.

$$\nu_{ij}(y) = \frac{n(n+2)}{2(n+1)(n-2)\omega_n} \int_{\mathbb{R}^n} \operatorname{tr}\left(S\gamma_{ij}(x-y)\,\nu(x)\right) dx$$

where the matrices γ^k, $k = 1, \dots, n$ are linearly independent solutions of the equation $S^*\gamma = 0$, they are of the form

$$\gamma^k_{ij}(x) = \frac{\delta_{ik}x^j + \delta_{jk}x^i - \delta_{ij}x^k}{|x|^n} + (n-2)\frac{x^i x^j x^k}{|x|^{n+2}} \ , \ .$$

Here S^* is the operator formally adjoind to S, i.e the operator defined by $\langle SX, \varphi\rangle = \langle X, S^*\varphi\rangle$, where $\langle X, Y\rangle = \int \langle X(x), Y(x)\rangle\, dx$ for vector field X, Y and $\langle \varphi, \psi\rangle = \int \operatorname{tr}(\varphi(x)\psi(x))\, dx$ for matrix fields φ, ψ.

Remark. It can be checked that

$$(S^*\varphi)^k = \sum_j \frac{\partial}{\partial x_j}\varphi_{kj} \ , \qquad \varphi = (\varphi_{kj}) .$$

Another question of the theory of quasiconformal deformations is to find relations between quasiconformality of a deformation and quasiconformality of the one-parameter family of diffeomorphisms generated by this deformation.

Let Z be a quasiconformal deformation of $\mathbb{R}^n$, i.e. let Z be a vector field on $\mathbb{R}^n$ such that $\|SZ\|$ is a bounded function. Then it generates the one-parameter group f_t, $t \in R$ of diffeomorphisms of $\mathbb{R}^n$, being solutions of the differential equation

$$\frac{d}{dt} f_t(x) = Z(f_t(x)) \ , \quad f_0(x) = x ,$$

cf [A-3]. Moreover Reimann [Rn] proved the following

Theorem 3 Every k-quasiconformal deformation $Z : \mathbb{R}^n \to \mathbb{R}^n$ generates a group of quasiconformal diffeomorphisms of $\mathbb{R}^n$ such that for every t the mapping f_t is $\exp(c \cdot k \cdot |t|)$-quasiconformal, where c is a constant.

Remark All the above results hold also under weaker regularity assumptions

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