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Instability of nonparallel flows

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INSTABILITY OF NONPARALLEL FLOWS

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This is an account of some strongly nonparallel flows and their strongly nonlinear stability (as well as their linear and weakly nonlinear stability), involving mechanisms of instability entirely different from those of parallel flows. First the distinction between parallel, weakly parallel and strongly parallel flows is made. The importance of nonparallelism on mechanisms of instability is shown by examples, first of Jeffery–Hamel flows and flows in a diverging channel, and then of flow in a channel with suction through its porous walls. Lastly, there is a review and preview of work on Long's vortex, another strongly nonparallel flow. We treat this well-known exact solution of the Navier–Stokes equations which represents a class of rotationally symmetric swirling jets, and their instabilities.

1. Nonparallel flows and their instability

To find the stability of parallel flows, researchers in the nineteenth century, notably Helmholtz, Kelvin and Rayleigh, used an exact solution of the Euler equations of motion of an inviscid fluid as a basic flow, considered the stability of that flow by linearizing the Euler equations for small perturbations of the basic flow, and used the method of normal modes. This led to a boundary-value problem with a linear ordinary differential system, which was an eigenvalue problem to determine the stability characteristics of the flow. Researchers in the twentieth century, notably Orr, Sommerfeld, Rayleigh, Tollmien, Schlichting and Chandrasekhar, similarly took an exact (or approximate) solution of the Navier–Stokes equations of a viscous fluid, with or without heat conduction, and found its stability characteristics by solving an eigenvalue problem with a linear ordinary differential system. Researchers later in the twentieth century, notably Landau, Stuart, Watson, Benney, Davey, DiPrima, Eckhaus and Segel, extended this method of linear stability by considering weakly nonlinear perturbations, i.e. by using essentially what is now called centre manifold theory. The stability of the null solution of the linearized problem came to be called linear stability, although definitions of stability, e.g. Liapounov's, refer not to linear, weakly nonlinear or strongly nonlinear stability but just to stability.

The success of all these methods depends on being able to reduce the linear stability problem to an ordinary differential system. This in turn depended upon choosing a basic flow which has a strong symmetry, i.e. is invariant under some continuous group of transformations. It is for this reason that the basic flows considered in the classic works of hydrodynamic stability are all parallel, or rotationally symmetric. Most of the flows treated are in fact unbounded and either states of rest or parallel flows, which are respectively invariant under either a group of translations in space and time, or the group of translations in the direction of the flow. (Of course, the flow must be unbounded or periodic in the directions of the translations for there to be invariance under a group of the translations.) It follows that the linear stability problem may be reduced to an ordinary differential eigenvalue problem by taking independent Fourier wave components. Then the resultant simplicity of the solution of the linearized problem enables the solution to be used as the first approximation to a weakly nonlinear solution of small amplitude.

More recently the development of computational fluid dynamics has allowed researchers to trace the strongly nonlinear development of perturbations of a basic flow, and to describe the transition to turbulence successfully in many cases. Also the qualitative theory of dynamical systems, e.g. that of bifurcations and chaos, has given insight into transition. However, very little is known of the mechanisms of instability of flows which are neither parallel nor approximately parallel, and it is timely to examine the instability of nonparallel flows, which, after all, are the rule rather than the exception in nature. The instability of nonparallel flows poses a challenge to our physical understanding and mathematical skills.

It may be helpful to write of parallel flows, weakly nonparallel flows and strongly nonparallel flows by analogy with linear instability, weakly nonlinear stability and strongly nonlinear stability. Then a parallel flow is a flow whose velocity is of the form $U(\mathbf{x}, t)\mathbf{i}$ for some constant vector $\mathbf{i}$, the unit vector in the x-direction, say. Examples are plane Poiseuille flow with $U = V(1 - y^2/h^2)$ and plane Couette flow with $U = Vy/h$ in the channel $-h < y < h$, where V is the maximum velocity and h the semi-width of the channel. A weakly nonparallel flow is a nearly parallel flow for which the assumption that it is locally parallel permits a successful treatment of the stability characteristics by use of the stability characteristics of the locally parallel flow as a first approximation. This is a part of the theory of hydrodynamic instability which is covered by several papers, some successful and some controversial. Examples of weakly parallel flows of a viscous fluid are the Blasius boundary layer on a flat plate and the Bickley jet. A strongly nonparallel flow is a flow that is neither parallel nor weakly parallel. Examples of strongly nonparallel flow of a viscous fluid are Couette flow between coaxial rotating cylinders, Kármán's flow on a rotating disc, Jeffery–Hamel flows between inclined rigid planes, Berman flows in a long channel with suction through porous walls, and Long's rotationally symmetric swirling vortex. Just as it is often said that it is unfortunate that the negative word 'nonlinear' describes more interesting, more challenging, more general and more realistic phenomena than the positive word 'linear' does, so it may be said that it is unfortunate that 'nonparallel' is a negative word and 'parallel' a positive one.

2. Jeffery–Hamel and Berman flows

For our first example to show the importance of nonparallelism, consider the stability of Jeffery–Hamel flows. Recall that these are steady two-dimensional flows of a uniform viscous incompressible fluid between two rigid inclined planes, say $\phi = \pm\alpha$, driven by a uniform line source, of strength Q say, at the line of intersection $r = 0$ of the planes, on use of cylindrical polar coordinates r, ϕ, z. The Reynolds number may be defined as $R = Q/2\nu$, where ν is the kinematic viscosity of the fluid. Then the solution of the Navier–Stokes equations may, by seeking a similarity solution for which the streamfunction depends only on ϕ, be reduced to the solution of a nonlinear ordinary differential boundary-value problem. It has been shown that there is an infinite number of such flows for any given pair of the dimensionless parameters R, α. Banks, Drazin & Zaturska [1] treated the stability of these flows in some detail, and reviewed the literature. They found that almost all types of Jeffery–Hamel flow are unstable always. They also found that if the semi-angle between the planes $\alpha < 0.07°$ or thereabouts then a Jeffery–Hamel flow of one type is nearly parallel in the sense that it is approximately plane Poiseuille flow and so unstable to what are called Tollmien–Schlichting waves. (It is well known that plane Poiseuille flow is *linearly* unstable to Tollmien–Schlichting waves when another Reynolds number $R' > 5772$, where $R' = Vh/\nu$, this having been found by solving what is called the Orr–Sommerfeld problem.) The mechanism of instability, with energy transfer to the perturbation from the basic flow at the 'critical layer', gives a subcritical Hopf bifurcation at the critical value of R (which is two thirds of the critical value 5772 of R'). However, if $\alpha > 0.07°$ then the mechanism of instability is entirely different, and the Jeffery–Hamel flow may be said to be strongly nonparallel. This mechanism gives a subcritical pitchfork bifurcation at the critical value of R, which depends on α. These results may be applied to flow in a diverging channel by assuming that it is either weakly nonparallel or nearly plane walled, so that its stability characteristics may be found by approximating the flow locally by either plane Poiseuille flow or a Jeffery–Hamel flow. The smallness of the angle 0.07° indicates that nonparallelism is very important in practice, and that the weakly nonparallel approximation is useless except for very well machined and aligned planes in a careful laboratory experiment.

For the second example of strong nonparallelism, consider the stability of Berman flows. For these, a viscous incompressible fluid is driven along a long channel by uniform suction through the plane rigid porous walls. It is convenient to choose the x-axis along the channel and to take the channel walls as $y = \pm h$. Berman [3] considered steady symmetric two-dimensional flow in the channel by use of Hiemenz's form of similarity solution, assuming that the streamfunction is proportional to x; thus he reduced the Navier–Stokes equations to a fourth-order ordinary differential equation with y as the independent variable. This ordinary differential equation and the boundary conditions of suction and no slip at the two walls comprise a nonlinear boundary-value problem. In the 1960s several papers on solutions of this problem were published, but Zaturska, Banks & Drazin [16] enlarged the problem by treating both asymmetric and unsteady flows in the channel.

We shall make only one point about the nonparallelism. Berman's symmetric steady flow in the channel is [16] unstable if $R > 6$, where the Reynolds number is here defined as $R = Vh/\nu$, i.e. it is based on the suction velocity and semi-width of the channel. However, Hocking [10] calculated the stability characteristics of the asymptotic suction profile by solving an Orr–Sommerfeld-like problem, and found that the flow is unstable if $R_x > 54370$, where the local Reynolds number is defined as $R_x = Uh/\nu$ and $U(x)$ is the velocity of Berman's flow outside a boundary layer on a wall for large R, i.e. the definition is based on the local velocity in the middle of the channel and the semi-width of the channel. Now the local velocity $U \propto x$ in Berman's similarity solution, so we have the following situation in a channel of *finite* length l as the suction velocity V is slowly increased. If l is so large that the maximum of U in the channel is large enough, then R_x will exceed its critical value 54370 before R exceeds its critical value 6, and Tollmien–Schlichting waves will occur in the flow locally near the walls where $|x|$ is large; in this event the flow might be said to be weakly nonparallel. However, if l is so small that R exceeds its critical value 6 before R_x exceeds its critical value 54370 then there will be a global instability over the whole channel [16]; in this event the flow might be said to be strongly nonparallel.

3. Long's vortex

A third of a century has passed since Long [12, 13] found a remarkable class of exact solutions of the Navier–Stokes equations. They are similarity solutions representing steady rotationally symmetric swirling jets of an incompressible viscous fluid. They are quintessentially nonparallel, having helicity density $\mathbf{u}.(\nabla \times \mathbf{u})$ which is not identically zero, where $\mathbf{u}$ is the velocity of the fluid. So they are flows of great interest in the theory of nonparallel flows and their stability, allowing us to study some strongly nonparallel flows by analytic as well as numerical methods. They are also of practical value as models of some jets and vortices, and, as we shall see, provide yet another possible mechanism for vortex breakdown.

Long [12] assumed a steady flow of a uniform incompressible fluid of density ρ and pressure p with a Stokes streamfunction $\psi(r,z)$ such that the radial and axial velocity components are

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{3.1}$$

respectively. He also assumed that there is a flow of similarity form with

$$\psi(r,z) = KzF(x), \;\; u_\phi(r,z) = KG(x)/r, \;\; p(r,z) = -\rho K^2 H(x)/z^2, \tag{3.2}$$

where the independent variable is $x = r/z$, and K is the constant swirl of the vortex at infinity, i.e. $ru_\phi \to K \quad$ as $r \to \infty$. The Reynolds number is defined as

$$R = K/\nu, \tag{3.3}$$

being based on the swirl. Another fundamental parameter is the dimensionless 'flow force', defined as

$$M = K^{-2}\int_0^{2\pi}\int_0^\infty (u_z^2 + p/\rho)\ r\mathrm{d}r\ \mathrm{d}\phi \tag{3.4}$$

$$= 2\pi\int_0^\infty (F'^2 - x^2H)/x\ \mathrm{d}x, \tag{3.5}$$

in fact, independently of the value of z . This flow is a combined vortex and jet with axis $r=0$ and velocity components

$$u_r = K(xF' - F)/r, \;\; u_\phi = KG/r, \;\; u_z = KF'/r, \tag{3.6}$$

where a prime denotes differentiation with respect to the similarity variable x . Substitution into the Navier–Stokes equations and a little manipulation gives

$$x^3(1+x^2)H' + 3x^4H = -(F^2 - 2xFF' + G^2), \tag{3.7}$$

$$x(1+x^2)G'' + (2x^2-1)G' = -RFG', \tag{3.8}$$

$$x(1+x^2)F'' - F' = R(x^3H - FF'). \tag{3.9}$$

The boundary conditions are

$$F(0) = F'(0) = G(0) = 0, \;\; F'(\infty) = 2^{-\frac{1}{2}}, \;\; G(\infty) = 1, \;\; H(\infty) = 0. \tag{3.10}$$

This nonlinear boundary-value problem specifies the family of Long's vortices in terms of given dimensionless parameters R, M. Note that M is determined by equation (3.5) *after* the solution has been calculated.

Long [12] rescaled this problem, re-posed it in the boundary-layer limit as $R \to \infty$, solved it, and found then that there are dual solutions for $M > M_c$ and no solution for $M < M_c$, where M_c is a critical value, in fact 3.75. His results are indicated in the broken curve of Figure 1 for $R = \infty$.

Some years later it was found [6, 15, 8, etc.] that both these flows of an inviscid fluid are unstable to helical modes, i.e. to modes which are not axisymmetric. In view of this fact that the vortices are always unstable in the boundary-layer limit of small viscosity which Long considered, it is surprising that work on Long's problem has been largely confined to this limit. An exception to the rule is the paper by Foster & Jacqmin [7], who generalized the asymptotic solutions of Foster & Smith [8] for large flow force M and $R = \infty$, by considering large M for arbitrary R. Burggraf & Foster [4] considered some small spatial perturbations of Long's boundary-layer flow at large values of R. All this work has slowly built up an understanding of the class of vortices and their instabilities, but much remains to be discovered.

There is a considerable literature on laboratory and numerical experiments on swirling jets, not least the paper by Long [12] himself. Experimental work is perhaps best traced by use of reviews [9, 11]. A recent paper [12] on numerical integrations of swirling jets of a viscous fluid gives access to the numerical literature on the problem. However, let us add that experience of integrating the slow algebraic decay of the basic Long vortex and its perturbations suggests that the influence of the walls in a laboratory or numerical experiment may depend crucially on the location and nature of the walls, inlet and outlet in practice. Of course, walls, inlet and outlet always occur in experiments, even though they do not in Long's similarity solutions.

Next we preview some forthcoming work [5] on Long's vortex. The property of dual solutions found by Long [13] is characteristic of a turning point, known so widely now when bifurcation theory is fashionable but little known when Long found his solutions. This knowledge leads us at once to conclude from the generic case of a turning point that the bifurcation where the flow force $M = M_c$ corresponds to a change in the sign of the real eigenvalue of one mode of the linearized stability problem of the flow, so that either one of the dual flows is stable and the other unstable, or both are unstable to a different eigenmode. Similar ideas of bifurcation theory are relevant when the Reynolds number R is finite.

Drazin, Banks & Zaturska [5] report results for all values of R, not just large ones. In particular they have solved the problem asymptotically in the limit of small R for arbitrary flow force M, and linked their results with those of Long for large R by direct numerical integration of the system. This gives an overall picture of the results, the occurrence of multiple solutions and their instabilities. Mostly rotationally symmetric steady perturbations consistent with the similarity form were treated, although, of course, limitation of the class of perturbations of a basic flow permits a demonstration of its instability but not of its stability. This excludes consideration of Hopf bifurcations to time-periodic flows, which in general occur when the least unstable mode, i.e. the 'most dangerous' mode, is not governed by the principle of exchange of stabilities.

Below we write of 'spatial stability' when considering only steady perturbations. Note that by considering only steady perturbations one can treat instability only indirectly, because stability is defined in terms of temporal evolution of perturbations. However, the close relationship between spatial and temporal modes, and between instability and bifurcation of steady solutions, is well known. So spatial instability informs us indirectly of the instability of Long's vortex.

Benjamin's mechanism of vortex breakdown [cf. 9, 11], namely the sudden change of one vortex regime to another, depends on the coexistence of two conjugate flows, i.e. equilibria, so that a disturbance may lead to an abrupt change of equilibrium, as in a hydraulic jump or a 'catastrophe'. Other proposed mechanisms of breakdown involve the hydrodynamic instability of a vortex. These possibilities were considered in the context of Long's model, limited though that context is: the occurrence of multiple solutions, losing and gaining stability as the governing parameters vary, is a natural framework for sudden changes of flow regime.

The work of Burggraf & Foster [4, §3] was generalized to reveal the importance of its context by treating not only all values of R but also more general dependence of the steady perturbations of the flow on the axial coordinate z; this leads to the following spatial eigenvalue problem which governs the decay (or growth) of the flow downstream as it approaches (or leaves, respectively) Long's flow. Write

$$\psi = K(\psi_0 + \psi_1), \quad u_\phi = K(v_0 + v_1)/r, \quad p = \rho K^2(p_0 + p_1), \tag{3.11}$$

where $\psi_0(r,z) = zF(x)$, $v_0(r,z) = G(x)$, $p_0(r,z) = -H(x)/z^2$, give the basic flow as in equation (2), and then linearize the Navier–Stokes equations for small perturbations ψ_1, v_1, p_1. It can then be seen that the variables may be separated by taking solutions of the form

$$\psi_1(r,z) = z^\lambda f(x), \quad v_1(r,z) = z^{\lambda-1}g(x), \quad p_1(r,z) = -z^{\lambda-3}h(x), \tag{3.12}$$

where λ is the separation constant, possibly complex. The linearized Navier–Stokes equations imply at length that

$$x^3(1+x^2)f''' - x^2[(\lambda-1) + 3(\lambda-2)x^2]f'' + (\lambda-1)x[1+3(\lambda-2)x^2]f'$$

$$-\lambda(\lambda-1)(\lambda-2)x^2 f = R[-x^3 h' - x^2 F f'' + (\lambda+1)x F f'$$

$$+(\lambda-3)x^2 F' f' - \lambda x^2 F'' f - \lambda(\lambda-3)x F' f - 2\lambda F f - 2Gg], \tag{3.13}$$

$$x(1+x^2)g'' - [1+2(\lambda-2)x^2]g' + (\lambda-1)(\lambda-2)xg$$

$$= R[-\lambda G' f - F g' + (\lambda-1)F' g], \tag{3.14}$$

$$x^2(1+x^2)f''' - x[1+2(\lambda-2)x^2]f'' + [1+(\lambda-1)(\lambda-2)x^2]f'$$

$$= R[x^4 h' - (\lambda-3)x^3 h - x F f'' + (\lambda-3)x F' f' + F f' - \lambda(x F'' - F')f], \tag{3.15}$$

and the linearized boundary conditions that

$$f(0) = f'(0) = g(0) = f'(\infty) = g(\infty) = h(\infty) = 0. \tag{3.16}$$

In addition, there is a linearized equation to ensure that M is fixed. This poses a problem to determine eigenvalues λ and corresponding eigenfunctions f, g, h. There is spatial instability, with some steady disturbances growing faster than the basic flow as $z \to \infty$, when there is at least one eigenvalue, of three sequences of eigenvalues whose real parts decrease to $-\infty$, such that $\Re(\lambda) > 1$.

It can be verified that a special solution is given by

$$\lambda = 0, \quad f = xF' - F, \quad g = xG', \quad h = xH' + 2H. \tag{3.17}$$

This solution is a simple generalization of a result of Burggraf & Foster [4, §3] for the boundary-layer case. However, it is an eigensolution for all M, R because it in fact also satisfies conditions (3.17) and the linearized form of equation (3.5). Drazin, Banks & Zaturska [5] have solved this eigenvalue problem completely in terms of special functions in the Stokes limit as $R \to 0$; the solutions are, not surprisingly, spatially stable.

Further, the spatial stability of Long's vortices with respect to asymmetric perturbations was treated similarly, since both variables ϕ, z may be separated by assuming that the perturbations are resolved into modes proportional to $e^{in\phi}$ so that $v_1(r,\phi,z) = z^{\lambda-1}e^{in\phi}\hat{v}(x)$ etc. In this way the linear stability of the flow to spatial helical modes may be treated without recourse to solving a partial differential system. Thus we express

$$(u_r, u_\phi, u_z) = K(u_0 + u_1, v_0 + v_1, w_0 + w_1)/r, \quad p = \rho K^2(p_0 + p_1), \tag{3.18}$$

where $(u_0, v_0, w_0) = (xF' - F, G, F')$, $p_0 = -H/z^2$, as in equation (3.2). Then, on linearizing the Navier–Stokes equations for small perturbations

u_1, v_1, w_1, p_1 , it can be seen that the variables may be separated by taking spatial modes of the form

$$(u_1, v_1, w_1) = z^{\lambda-1} e^{in\phi}(\hat{u}(x), \hat{v}(x), \hat{w}(x)), \quad p_1 = -z^{\lambda-3} e^{in\phi}\hat{p}(x), \qquad (3.19)$$

where the non-negative integer n is the azimuthal wavenumber and the complex eigenvalue λ gives the rate of spatial growth or decay in the axial direction. The linearized equations and boundary conditions may now be found at length. This leads to the spatial eigenvalue problem for general asymmetric steady perturbations.

Finally we shall give a few snapshots of the numerical results of Drazin, Banks & Zaturska [5]. The aim is to give an overall view of the multiple solutions of Long's form, their qualitative properties as the parameters R, M vary, and, in particular, their stability characteristics. The methods of bifurcation theory are suitable for this task.

First, look again at the bifurcation diagram in the $(M, w/R)$-plane for $R = 15$ in Figure 1, where we define a scaled axial velocity at the centre of the vortex as $w = [zu_z/K]_{r=0} = [F'/x]_{x=0}$. Then look at Figure 2 to see the bifurcation diagram in the $(R, w/R)$-plane for $M = 4, 6$. Incidentally, note how well the asymptotic and numerical results agree. Next picture the solution surface in the three-dimensional space of $w/R, R, M$. Figures 1, 2 indicate this very imperfectly: they are related to a few perpendicular sections of the two-surface in the three-space. Long's asymptotic results for large R and the asymptotic results for small R fill the picture a bit. The preliminary numerical results suggest that there is more than one basic solution, and more than one stable steady rotationally symmetric flow in various various parts of (R, M)-space, and that various kinds of catastrophes arise. Perhaps the most noteworthy result of the solution curves found in the $(R, w/R)$-plane for various fixed values of M is the occurrence of both a simple curve for $M = 4$ and a looped curve for $M = 6$; it may be inferred that there is a cusp for $M = M^*, R = R^*$, where $M^* \approx 5.3, R^* \approx 17$. (There is, in fact, no bifurcation where the looped curve crosses itself.) This may be described *topologically* by views of a non-planar smooth curve as it rotates, so that the cusp appears as a singular view but not a singularity of the curve; again it is topologically described by the conchoid of Nicomedes, with equation $(R - R^*)^2 w^2 = (M^* - M + 1 + w)^2(1 - w^2)$. However, it is plausible that solutions in the neighbourhood of the loop are spatially unstable, though two stable steady solutions coexist for some values of M, R. In summary, the set of solutions has a rich structure in need of careful and extensive investigation.

Drazin, Banks & Zaturska [5] will publish full details. At a late stage in the preparation of their paper they learned of independent work of Shtern & Hussain [14] on a closely related problem. Shtern & Hussain used the same similarity form of solution of the Navier–Stokes equations, but used spherical polar coordinates, whereas Long and the other authors have used cylindrical polar coordinates. The former coordinates lead to a simpler form of the ordinary differential system, which seems to be make both the expression of the problem and its solution somewhat easier. However, whereas Long and the others have imposed the boundary condition $F'(\infty) = 2^{-1/2}$, Shtern & Hussain have imposed (in our notation) $\lim_{x\to\infty} x^{-1}F(x) = 0$. They also used a slightly different definition of the flow force, M, from Long's, namely

$$M_{SH} = K^{-2}\int_0^{2\pi}\int_0^{\infty}(u_z^2 + p/\rho - 2\nu\partial u_z/\partial z)\ r\mathrm{d}r\ \ \mathrm{d}\phi, \tag{3.20}$$

$$= M + 2\nu K^{-2}\int_0^{2\pi}\int_0^{\infty}(\partial(ru_r)/\partial r + \partial u_\phi/\partial\phi)\ \mathrm{d}r\ \ \mathrm{d}\phi$$

$$= M + 4\pi\nu K^{-2}[ru_r]_0^{\infty} = M + 4\pi\nu R^{-1}[xF' - F]_0^{\infty}.$$

Both definitions coincide in the boundary-layer limit. By properly accounting for the viscous normal stress, it would seem that Shtern & Hussain's definition is preferable on physical grounds. However, the long history of the problem gives some authority to Long's definition, and changing the definition changes the method of describing the results of the problem but not the results themselves. Some of the results of Drazin, Banks & Zaturska [4] are qualitatively similar to those of Shtern & Hussain [14], although the latter did not treat stability at all. It may be said that the two sets of results are complementary, illuminating the rich structures of two closely related families of strongly nonparallel flows of a viscous fluid.

Long's vortex is a strongly nonparallel flow par excellence. It and the well-known flows due to a rotating disc are rare examples of a flow with nonzero helicity density which can be expressed in fairly simple mathematical terms. It can easily be shown that the helicity density of Long's vortex is in general nonzero, although it is singular at the origin $z = 0$ of the jet because the velocity is singular there, so the helicity itself is unbounded. Thus the stability of Long's vortex is of importance both for its practical applications and its fundamental theoretical properties.

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CAPTIONS

Figure 1. The bifurcation diagram in the $(M, w/R)$-plane for $R = 15, \infty$, where the scaled axial velocity at the centre of the vortex is $w = [zu_z/K]_{r=0} = [F'/x]_{x=0}$. The broken curve is for $R = \infty$ and the continuous curve for $R = 15$.

Figure 2. The bifurcation diagram in the $(R, w/R)$-plane for $M = 4, 6$. Broken lines denote asymptotic results due to two terms of the Stokes expansion for small R and dot-dashed lines denote asymptotes taken from the boundary-layer solution for large R. (a) The continuation of the solutions of type I. Note that w is plotted for $R < 1$ and w/R for $R > 1$. (b) The continuation of the solutions of type II.