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Workshop on Fluid Mechanics

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Porous Media: Ground Water Flows

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Ground water Flows (AWW)

- Chemical budget of land and sea
- Dispersion of pollutants
- Diagenetic reactions causing post-depositional mineralisation / alteration of sedimentary deposits.
- Heat budget - hot springs, hydrothermal vent
- Formation of hydrocarbons

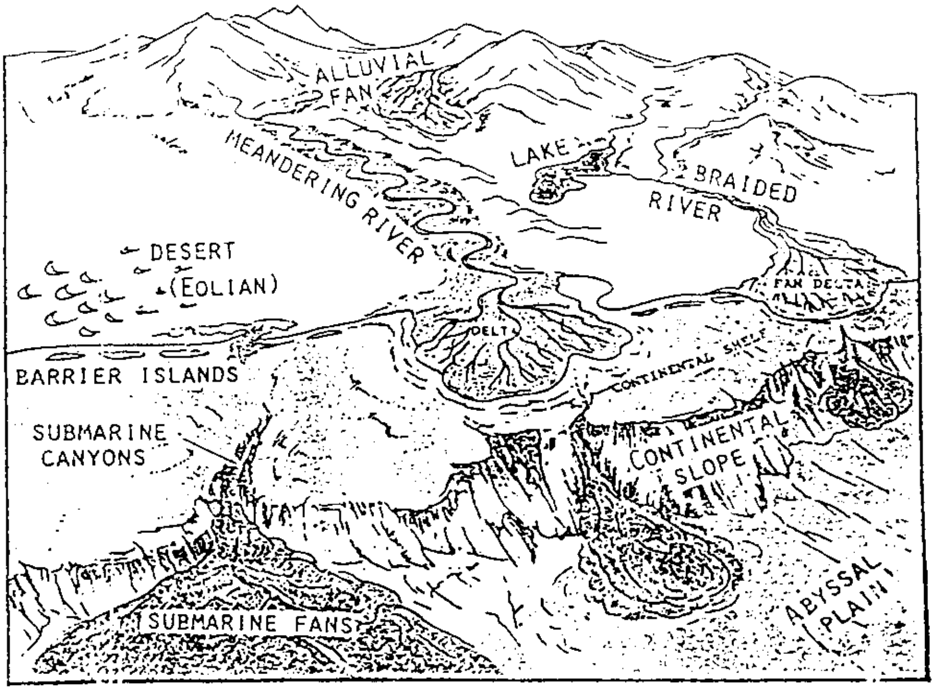


Fig. 5.1. Schematic representation of sedimentary environments on a passive margin

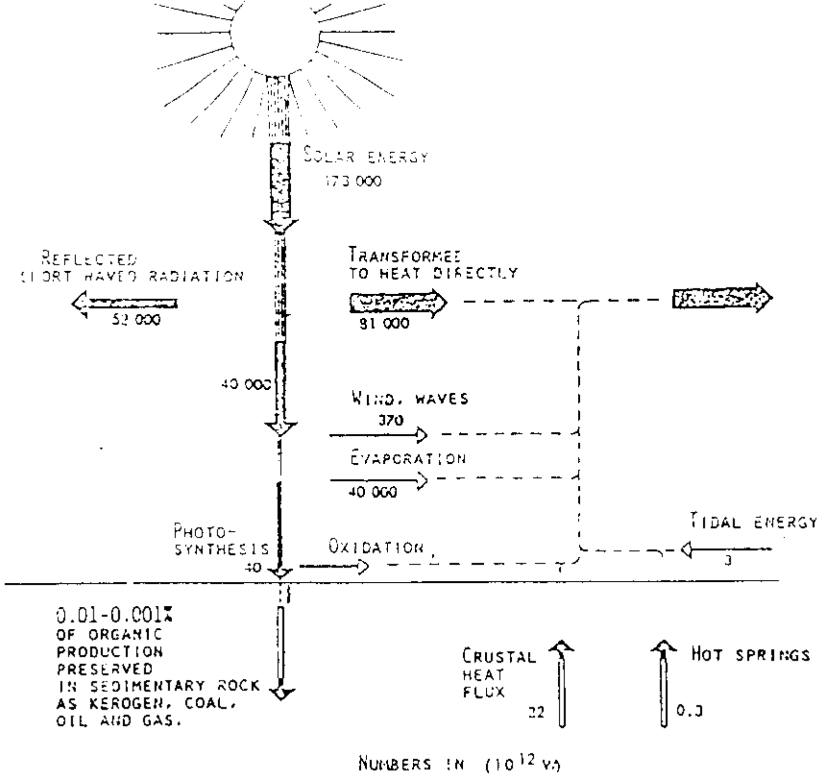


Fig. 13.1. Diagrammatic representation of the energy budget on the earth. We see that only a very small percentage of light energy is used for photosynthesis, and most of this (more than 99%) is broken down, with only a few ppm being converted into fossil energy

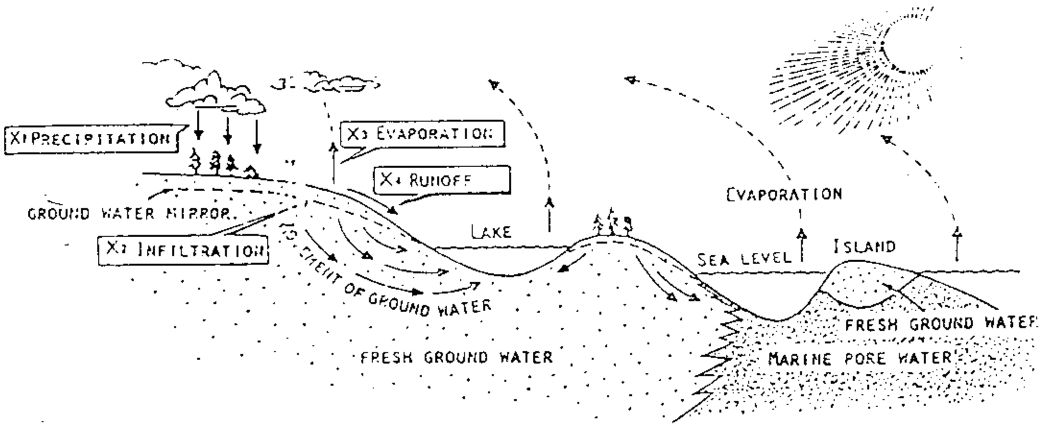
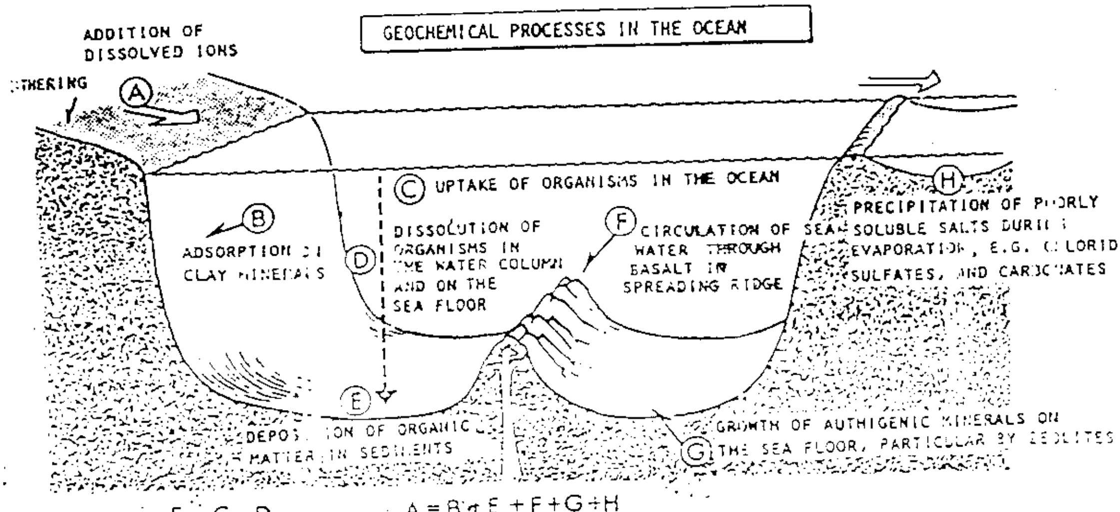


Fig. 5.4. Diagram showing the circulation of groundwater on land and under marine basins



$$E = C - D \qquad A = B + E + F + G + H$$

Fig. 7.8. Geochemical processes in the ocean. The addition of ions from the continents, plus possible [illegible] correspond to what is removed from the sea water by biological

Introduction to Porous Media

Fluid flows thro' the pore spaces

if ϕ = void fraction

v = velocity of fluid in x-direction

mass flux in x direction through an area A is give by $\int_A v \, \phi \, dA$

If ϕ is uniform, flux per unit area at any point is $v\phi$

$v\phi$ is called the Darcy velocity, u, a measure of the local volume flux. [$\underline{\nabla}.\underline{u} =$

If ϕ is uniform on the scale of the pore spacing, but varies over longer length scales, we may use u to describe the flow macroscopically

Figure 2.1. Extremely porous limestone from a Bermuda coral reef, approximately half-scale, containing many interstices with scales of up to a centimeter or so.

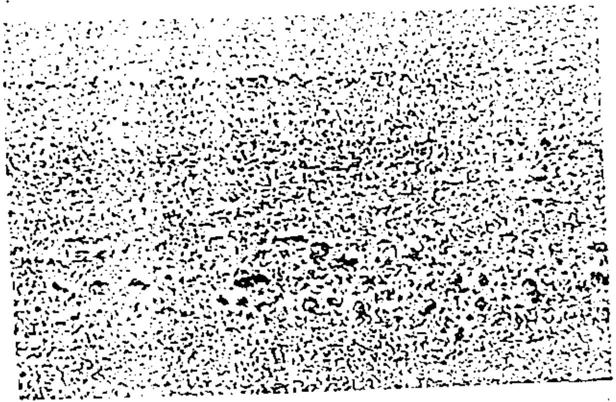


Figure 2.2. A highly porous, partially cemented sandstone, approximately full scale.

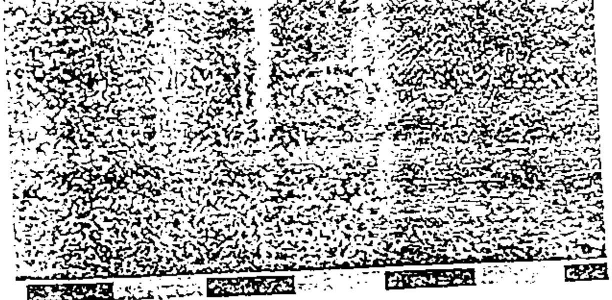


Figure 2.3. Pores in dolomite from the Latemar Massif in northern Italy. The blocks in the scale are 1 cm long. (Photograph courtesy of Dr. E. N. Wilson.)

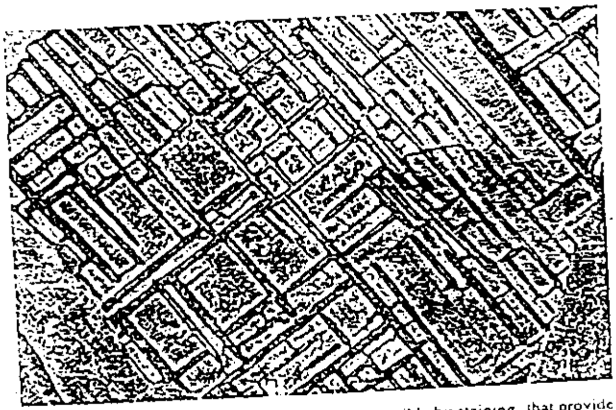


Figure 2.4. A network of cracks, made visible by staining, that provide pathways for flow along a sandstone cleavage plane. Approximately full scale.

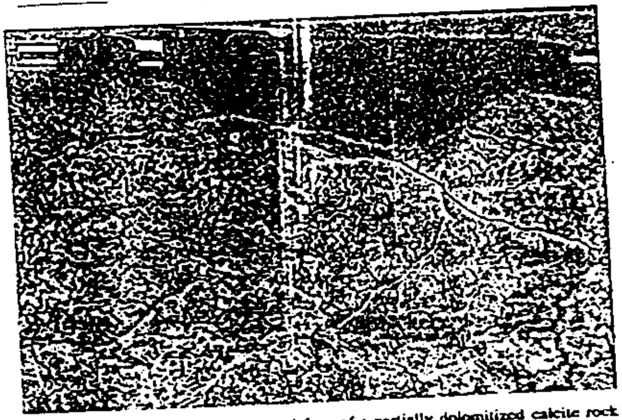


Figure 2.5. The weathered face of a partially dolomitized calcite rock in which veinlets of dolomite produced along fluid pathways stand out in relief in a dense, fine tracery on the left and along a crack on the right. (Photograph courtesy of Dr. E. N. Wilson.)

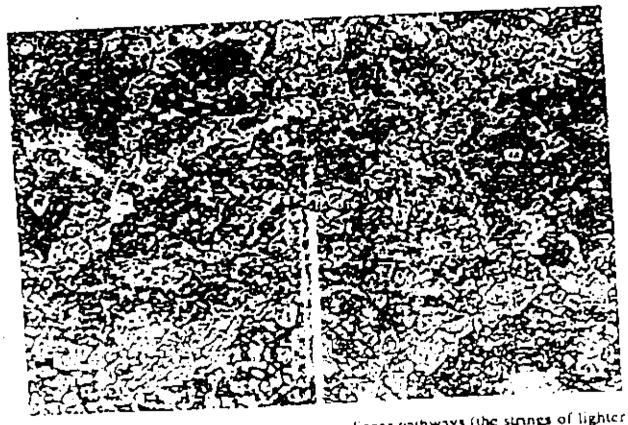


Figure 2.6. Dolomitization along linear pathways (the strings of lighter rhombic crystals) in a calcite matrix. Magnification ×20. (Photograph courtesy of Dr. E. N. Wilson.)

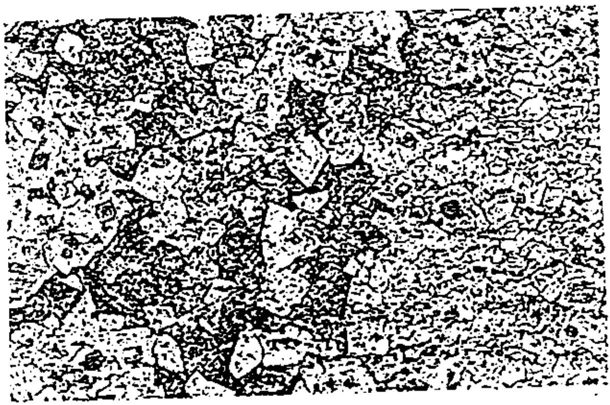


Figure 2.8. Dolomite of lighter shade has replaced calcite almost completely on the right, except in the cores of some of the grains; the reaction is only partially complete in the center. Magnification ×40. (Photograph courtesy Dr. E. N. Wilson.)

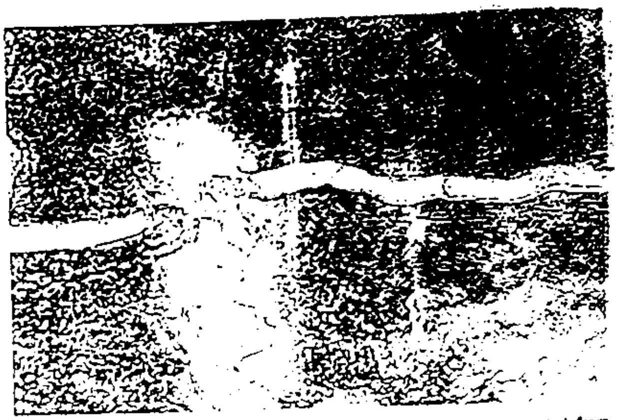


Figure 2.9. A veinlet in which calcite cement has precipitated from solution, intersecting and filling a pore. Magnification ×100. (Photograph courtesy Dr. E. N. Wilson.)

Darcy's Law

- Pressure gradients drive a flow
- Inertia + Friction balance [illegible]

In each pore, of width δ say, the viscous stresses exert a force

① $$\underline{F_v \propto \frac{\mu v}{\delta^2}}$$

μ = dynamic viscosity
δ = pore size
v = local velocity

and the inertia of the fluid generates pressure gradients of magnitude

② $$\underline{\frac{\rho v^2}{\delta}}$$ if the path changes directio on the lengthscale δ

$\frac{\delta \rho v}{\mu} << 1$

LOW REYNOLDS NUMBER

If $$\frac{\rho v^2}{\delta} << \frac{\mu v}{\delta^2} \Rightarrow \underline{v << \frac{\mu}{\delta \rho}}$$

then the viscous drag dominates the inertia

$$u = v\phi \Rightarrow \boxed{u << \frac{\mu \phi}{\rho \delta}}$$

"Slow" Darcy Velocity

⇒ Friction dominates

? When friction dominates inertia ?

(e.g. $u \sim 10^{-3} - 10^{-6}$ cm/s

$\frac{\mu}{\rho} \sim 10^{-2}$ cm^2/s

requires $\frac{\phi}{\delta} >> 10^{-1} - 10^{-4}$

$\phi \sim 10^{-2} \Rightarrow \delta << 10^{-1} - 10^{-2}$ cm

Void Fraction

Pore Spacing

$\uparrow$ fast flow

$\uparrow$ slow flow

$\Rightarrow$ nearly always satisfied in sediment layers

In this case

$$\left(\frac{\mu}{\delta^2\phi}\right) \underline{u} = - \underline{\nabla} P$$

$\uparrow$ friction

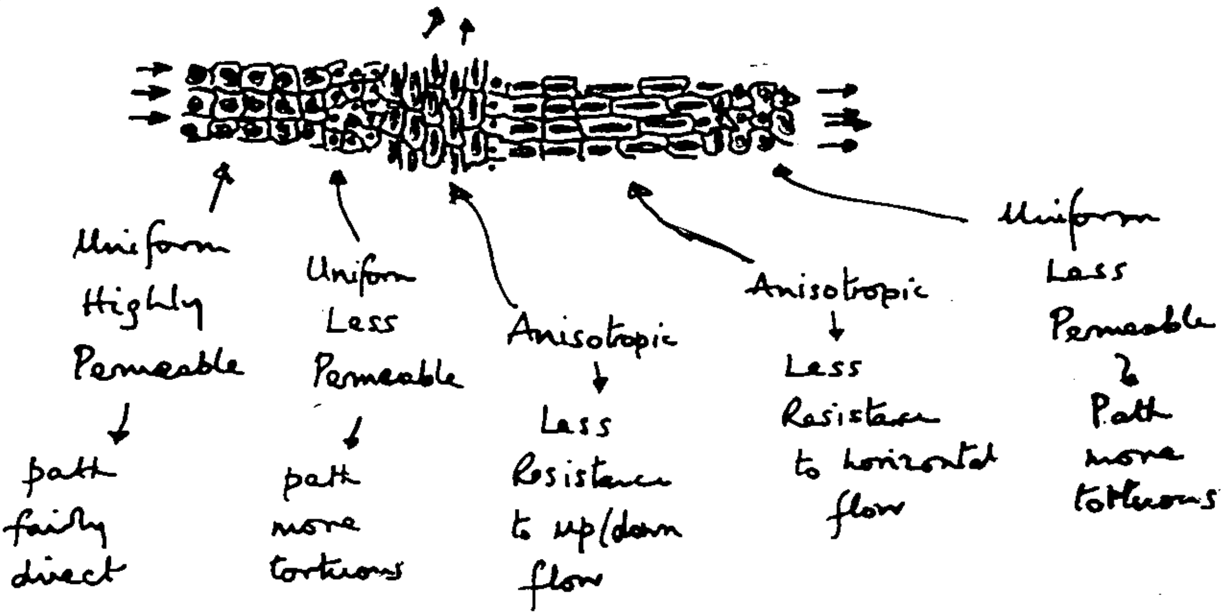
Experiments by Darcy + others, show that, to good approximation, in a uniform layer

$$\underline{u} \propto -\left(\frac{\delta^2\phi}{\mu}\right) \nabla P$$

define $K \propto \delta^2\phi$ $\boxed{u = -\frac{K}{\mu} \nabla P}$ K = permeabilit precise form of k is not simple in many situations — typically, pore spaces are preferentially oriented in one direction ⇒ k may be anisotropic

$$\text{Tensor} \quad \underline{\underline{k}} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \qquad \left(\mu \underline{u} = -\underline{\underline{k}} \cdot \underline{\nabla} p \right)$$

i.e. resistance to motion differs in different directions (and may differ through space as well !)



However, it is useful to study the case of uniform permeability first, and then to address anisotropic permeabilities — anisotropies important for mixing

Dispersion + Diffusion in Porous Flows

Two marked fluid elts. start close together

Elements move apart as a result of dispersion by porous matrix.

Path of fluid from $\underline{a}$ to $\underline{x}(\underline{a},t)$

If $\underline{v}(a,t) = \frac{d}{dt}\left[\underline{x}(\underline{a},t)\right]$ where $\underline{x}(\underline{a},0) = \underline{a}$

then
$$\underline{x}(\underline{a},t) = \int_0^t \underline{v}(\underline{a},t')\,dt' + \underline{a}$$

We seek to describe motion on macroscopic scale, larger than the pore size l_0. If the starting point of our fluid element changes by an amount of order εl_0 (i.e. we start at $\underline{a} + \varepsilon \underline{l}_0$) then the path may also change (blue line). – macroscopic description → we average over these possible starting values

If the mean velocity is $\bar{v}(\underline{a})$, then

mean position $\bar{x}(a,t) - \underline{a} = \int_0^t \overline{v(a,t')}\,dt' = \bar{v}(\underline{a})\,t$

So $$\underline{x}(\underline{a},t) - \bar{\underline{x}}(\underline{a},t) = \int_0^t \left[\underline{v}(\underline{a},t') - \bar{v}(\underline{a})\right] dt'$$

$$\Rightarrow \left[\frac{d}{dt}\left(\underline{x}(\underline{a},t) - \bar{\underline{x}}(\underline{a},t)\right)\right]\left(\underline{x}(\underline{a},t) - \bar{\underline{x}}(\underline{a},t)\right) = \int_0^t \left(\underline{v}(\underline{a},t) - \bar{v}(\underline{a})\right)\left(\underline{v}(\underline{a},t') - \bar{v}(\underline{a}\right.$$

Hence by averaging over a small range of starting ~~values~~ positions we have

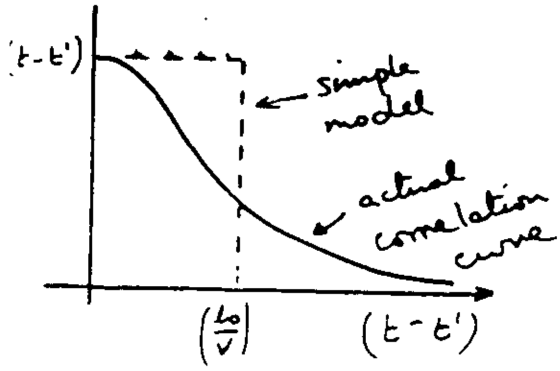
$$\frac{1}{2}\frac{d}{dt}\left\{\overline{\left(x(\underline{a},t)-\bar{x}(\underline{a},t)\right)^2}\right\} = \int_0^t dt'\,\overline{\left(v(a,t)-\bar{v}(a)\right)\left(v(a,t')-\bar{v}(a)\right.}$$

$\overline{\left(v(a,t)-\bar{v}(a)\right)\left(v(a,t')-\bar{v}(a)\right)}$ correlates the velocity fluctuations at time t and t'. As $(t-t')$ increases, the velocity fluctuations become ~~more~~ decorrelated – individual interstices are convoluted on the lengthscale of the interstices, and hence the flow velocity fluctuation becomes decorrelated when $(t-t') \sim \left(\frac{l_0}{\bar{v}}\right)$

Hence $\overline{\left(v(a,t)-\bar{v}(a)\right)\left(v(a,t')-\bar{v}(a)\right)} = \bar{v}^2 f(t-t')$

and $\int_0^t d(t-t')\left[\bar{v}^2 f(t-t')\right] \sim \bar{v}^2\left(\frac{l_0}{\bar{v}}\right)$

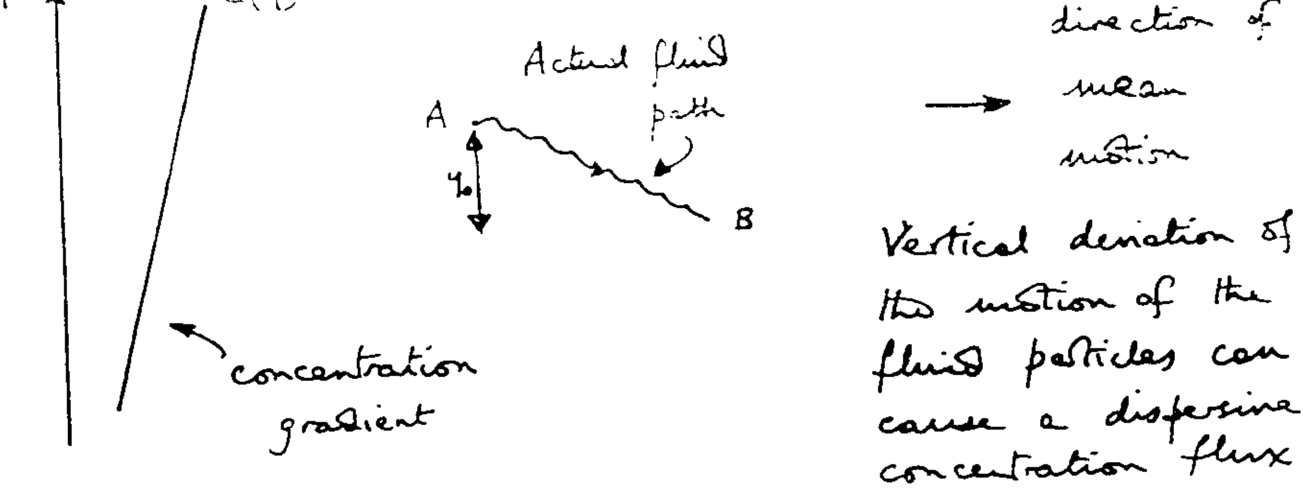
(note, decorrelation time may differ in an anisotropic medium)



Thus $\frac{d}{dt}\left(\overline{\left(x(\underline{a},t)-\bar{x}(\underline{a},t)\right)^2}\right) = \left(2\bar{v}L_0\right)$

$\Rightarrow \overline{\left(x(\underline{a},t)-\bar{x}(\underline{a},t)\right)^2} = \left(2\bar{v}L_0\right)t$

Hence fluid elements dispersed about mean position by an amount $\left[(2\bar{v}L_0)t\right]^{1/2}$ in time t.



Vertical deviation of the motion of the fluid particles can cause a dispersive concentration flux

Effect of particle moving from A to B

$$c(y_0) = c(0) + y_0 \frac{\overline{dc}}{dy}$$ ← difference in concentration at A and B

If path deviates by amount y_0, then

$$-y_0 = \int_{t_0}^{t_\phi} \left(\underline{v}(a,t') - \bar{\underline{v}}\right) dt'$$ (downward deviation

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⇒ Net dispersive flux in fluid arriving at B is $\overline{c\left(\underline{v}(\underline{a},t_\phi)-\bar{\underline{v}}\right)}_y$

averaging over all $\underline{a}$, $\overline{\left(\underline{v}(\underline{a},t_\phi)-\bar{\underline{v}}\right)}_y = 0$

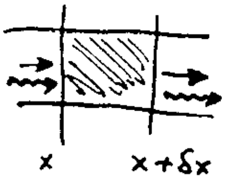
hence $\overline{c\left(\underline{v}(\underline{a},t_\phi)-\bar{\underline{v}}\right)} = \overline{c'\left(v(\underline{a},t_\phi)-\bar{\underline{v}}\right.}$

and $c' = y_0 \frac{\overline{dc}}{dy} = -\int_{t_0}^{t_\phi}\left(\underline{v}(a,t')-\bar{\underline{v}}\right) dt' \frac{\overline{dc}}{dy}$

⇒ $\overline{c'\left(v(a,t_\phi)-\bar{\underline{v}}\right)} = -\frac{\overline{dc}}{dy}\int_{t_0}^{t_\phi} \overline{\left(\underline{v}(a,t_\phi)-\bar{\underline{v}}\right)\left(\underline{v}(a,t')-\bar{\underline{v}}\right)}\, d$

⇒ Dispersive flux per unit area $= -\phi(\bar{v}\, l_0)\left(\frac{\overline{dc}}{dy}\right)$ ⇒ Dispersion enhanc diffusion coefficient ~~flux~~ by $(\bar{v} l$

Heat Transfer



Assume rock + liquid are locally in thermal $\equiv$'m

$\Rightarrow$ $\kappa \gg \frac{u\delta}{\phi}$ Typically True

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) → Advective Heat flux into shaded box $= \rho C_p|_l \, u T(x)$

→ Advective heat flux out of shaded box $= \rho C_p|_l \, u T(x+\delta x)$

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2) Diffusive heat flux $= -\bar{k} \; T_x|_x$ into

$= -\bar{k} \; T_x|_{x+\delta x}$ out of

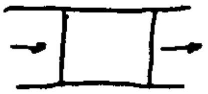
---

3) Hence $\overline{\rho C_p} \frac{\partial T}{\partial t} = -\rho C_{p_L} \; \underline{\nabla}.(\underline{u}T) \; - \left(-\bar{k} \; \nabla^2 T\right)$

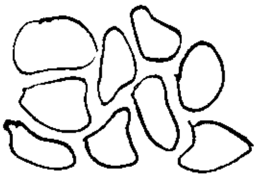
$$\Rightarrow \boxed{\overline{\rho C_p}\left(\frac{\partial T}{\partial t}\right) + \rho C_{p_L} \; (\underline{u}.\underline{\nabla})T = \bar{k} \; \nabla^2 T}$$

$\overline{x}$ mea
the aver
e.g. $(1-\Phi)c\rho_s +$

Here we use result that $\underline{\nabla}.\underline{u} = 0$ since the net [illegible] flux of liquid does not change

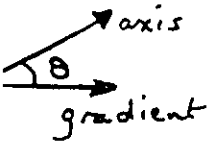


$\underline{u}$ = Darcy velocity
= flux of liquid per unit area



Molecular Diffusion acts along pore axes

Molecular flux $= -D_m \dfrac{\partial c}{\partial s}$ s is in alongaxis direction



$\dfrac{\partial c}{\partial s}$ has a component $\cos\theta \dfrac{\partial c}{\partial s}$ in direction of gradient

So molecular flux $= -D_m \cos\theta \dfrac{\partial c}{\partial s} = -D_m \cos^2\theta \dfrac{\partial c}{\partial y}$

as $ds = dy / \cos\theta$

Net molecular flux, averaged macroscopically is therefore

$$-\phi D_m \overline{\cos^2\theta \dfrac{\partial c}{\partial y}} \simeq -\phi D_m \overline{\cos^2\theta} \dfrac{\partial \bar{c}}{\partial y}$$

(Approximation)

($\overline{\cos^2\theta}$ called tortuosity)

Total diffusive + dispersive transport is

$$-\phi \left\{ D_m \overline{\cos^2\theta} + \bar{V} l_0 \right\} \frac{\partial \bar{c}}{\partial y}$$

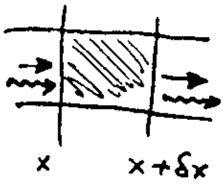
($\bar{v}$ is the mean interstitial velocity)

or $$-\left\{ \phi D_m \overline{\cos^2\theta} + \bar{u} l_0 \right\} \frac{\partial \bar{c}}{\partial y}$$

Diffusion Dominates i

u = Darcy velocity. Typically $\bar{V} \sim 10^{-5} \frac{cm}{s}$; $D_m \sim 10^{-5}\ cm^2/s \Rightarrow l_0 <$

Heat Transfer



Assume rock + liquid are locally in thermal $\equiv$'m
➡ $K \gg \frac{u\delta}{\phi}$ Typically True

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① → Advective Heat flux into shaded box = $\rho C_p|_l \, u T(x)$

→ Advective heat flux out of shaded box = $\rho C_p|_l \, u T(x + \delta x)$

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② ⇝ Diffusive heat flux $= -\bar{k} \, T_x|_x$ into

$= -\bar{k} \, T_x|_{x+\delta x}$ out of

---

③ Hence $\overline{\rho C_p} \frac{\partial T}{\partial t} = - \rho C_{p_L} \, \underline{\nabla} \cdot (\underline{u} T) - (-\bar{k} \, \nabla^2 T)$

$$\Rightarrow \quad \boxed{\overline{\rho C_p} \left(\frac{\partial T}{\partial t}\right) + \rho C_{p_L} (\underline{u} \cdot \underline{\nabla}) T = \bar{k} \, \nabla^2 T}$$

$\bar{x}$ mea
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e.g. $(1-\phi) C\rho_s + ($

Here we use result that $\underline{\nabla} \cdot \underline{u} = 0$ since the net flux of liquid does not change

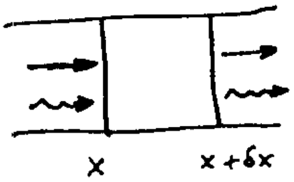
$\underline{u}$ = Darcy velocity
= flux of liquid per unit area

Tracer Transport – e.g. Dye

Salt

Radio-activity

nsider Non-Reacting System



Let c be the concentration of tracer in the liquid

In the volume between x and $x+\delta x$, the amount of tracer is

$c\ \phi\,\delta x$ per unit X-sectional area

The advective flux of tracer is $-\left\{ uc\big|_{x+\delta x} - uc\big|_{x} \right\}$

$-(\underline{u}\cdot\underline{\nabla})\,c$

The diffusive flux of tracer is $-\left\{ -\phi D \frac{\partial c}{\partial x}\Big|_{x+\delta x} + \phi D \frac{\partial c}{\partial x}\Big| \right.$

$\phi D \nabla^2 c$ remember $D = \left(D_m \overline{\cos^2\theta} + \frac{\bar{u} b}{\phi} \right)$

Conservation of tracer is

$$\boxed{\phi \frac{\partial c}{\partial t} + (\underline{u}\cdot\underline{\nabla})\,c = \phi D\, \nabla^2 c}$$

Important Result : Tracer travels faster than heat anomaly

Physics : { Tracer moves thro' liquid phase only and is carried by flow

{ Thermal anomaly must heat up rock as it is swept through the material by the liquid ➔ <u>thermal inertia</u>

Heat : $$\frac{\partial T}{\partial t} + \left(\frac{\rho c_{p\ell}}{\overline{\rho c_p}}\right)(\underline{u}\cdot\underline{\nabla})\,T = \left(\frac{\bar{k}}{\overline{\rho c_p}}\right)\nabla^2 T$$

Tracer : $$\frac{\partial c}{\partial t} + \left(\frac{1}{\phi}\right)(\underline{u}\cdot\underline{\nabla})\,c = D\,\nabla^2 c$$

$\Rightarrow$ Effective advection speed of tracer is $\left(\frac{u}{\phi}\right)$

(i.e. the real velocity)

Effective advection speed of heat is $\left[\left(\frac{\rho c_{p\ell}}{\overline{\rho c_p}}\right) u\right] \sim 1$

Ratio $$\frac{\text{Tracer Speed}}{\text{Thermal Speed}} \sim \frac{1}{\phi} \gg 1 \qquad (\phi \sim 10^{-}$$

➔ Pump green + hot water into an aquifer : cold green water appears

Consider dimensional flows :

$$\underline{u} = -\frac{k}{\mu} \underline{\nabla} P$$

In a region of constant, k , permeability,

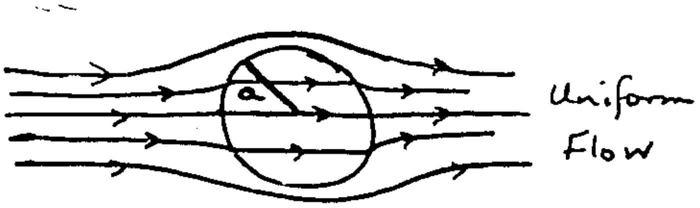
$$\underline{\nabla} \cdot \underline{u} = -\frac{k}{\mu} \nabla^2 P \quad \text{but} \quad \underline{\nabla} \cdot \underline{u} = 0$$

mass conservation

$\Rightarrow$ $\boxed{\nabla^2 P = 0}$ and $\frac{k}{\mu} P$ is a velocity potential for the flow

$$\left(2\text{-D} \Rightarrow \nabla^2 P = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} \right)$$

e.g. Consider flow round a cylinder of different permeability



Cf. heat transfer around a cylinder of different thermal conductivity

if outside permeability is k_o
interior " " is k_i

then outside $p \rightarrow -w \, r\cos\theta$ as $r \rightarrow \infty$
"$(r\cos\theta, x)$"

so expect $p_o = -\left(1 - \frac{B}{r^2}\right) w \, r \cos\theta$ $r > a$ Outside

and $p_i = -C \, w \, r \cos\theta$ $r < a$ Inside

with $p_o(a) = p_i(a)$ ← pressure continuous across boundary

$-\frac{k_o}{\mu} \left.\frac{\partial p_o}{\partial r}\right|_a = -\frac{k_i}{\mu} \left.\frac{\partial p_i}{\partial r}\right|_a$ ← mass flux continuous across boundary

Matching $\Rightarrow$ $$\left(1 - \frac{B}{a^2}\right) = C$$

$$k_o\left(1 + \frac{B}{a^2}\right) = k_i C$$

$$\Rightarrow \boxed{\begin{aligned} C &= \frac{2k_o}{k_o + k_i} \\ B &= \left(\frac{k_i - k_o}{k_i + k_o}\right) a^2 \end{aligned}}$$

<u>Physics</u> : $k_o >> k_i$ ➜ exterior more permeable than interior of cylinder

➜ flow tried to go round cylinder rather than thro' cylinder

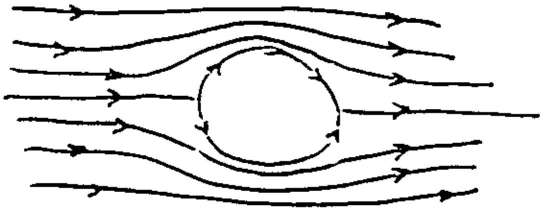
<u>Check</u> : in far field flow $\underset{\sim}{u}_o \sim \frac{k_o}{\mu} w$

inside cylinder flow is uniform $\underset{\sim}{u}_i \sim \frac{k_i}{\mu}\left(\frac{2k_o}{k_o+k_i}\right)$,

ratio $$\boxed{\frac{\underset{\sim}{u}_o}{\underset{\sim}{u}_i} \sim \frac{k_o(k_o+k_i)}{2k_o k_i}}$$

If $k_o >> k_i$, $\frac{u_o}{u_i} \sim \frac{k_o}{2k_i} >> 1$

➜ all flow is outside



Hydrodynamic Dispersion – Macro-Scopic

We have seen dispersion acts on the scale of the interstices to mix up the fluid with an effective "local" diffusion coefficient

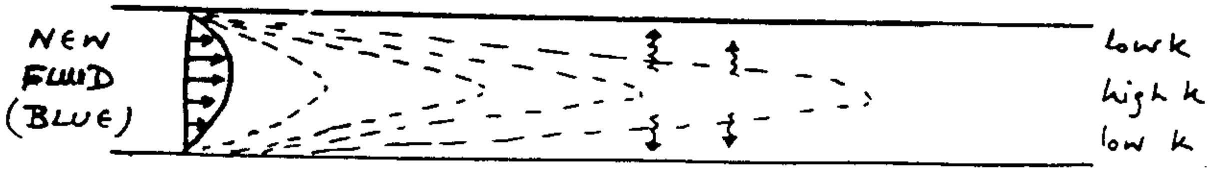
$$\left\{ D_m \overline{\cos^2\theta} + \frac{u}{\phi} l_0 \right\}$$

However, larger-scale shear in the flow can induce mixing across the flow by intensifying concentration gradients

We may study the larger scale "shear-induced" dispersion as a process of

(1) Shearing + Intensifying gradients

(2) Diffusing across gradients



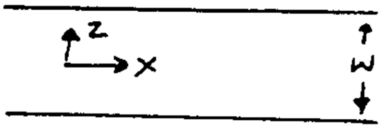
→ assume D_m domin

In absence of diffusion cross-flow, a tongue of blue fluid advances into aquifer. – Dispersion causes tracer to mix cross flow → first arrival time increases

If we move to the frame of the centre of mass of the new fluid, we expect the cross-stream concentration to equilibrate s.t. the along stream advection balances cross stream diffusion

$$\frac{1}{\phi}\frac{\partial c}{\partial t} + \underline{\left(\frac{u}{\phi}\right)\frac{\partial c}{\partial x}} = D\left(\frac{\partial^2 c}{\partial x^2} + \underline{\frac{\partial^2 c}{\partial z^2}}\right)$$

Write $c = \bar{c}(x) + \hat{c}(x,z)$



$\bar{c}$ = mean concentration at location x $\int_0^w dz\, c(x,z) / w$

$\hat{c}$ = fluctuations about mean, $\hat{c} = c - \bar{c}$

$$\left[\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} + \underline{\int \hat{u}\frac{\partial \hat{c}}{\partial x} dz / w} = D\frac{\partial^2 \bar{c}}{\partial x^2} \right]$$ undeslined term is dispersive flux

Expect $\frac{\partial^2 \hat{c}}{\partial z^2} \sim \left(\frac{\hat{u}}{\phi D}\right)\left(\frac{\partial \bar{c}}{\partial x}\right)$

i.e. shear induces cross stream conc'n gradients and these are diffused across stre

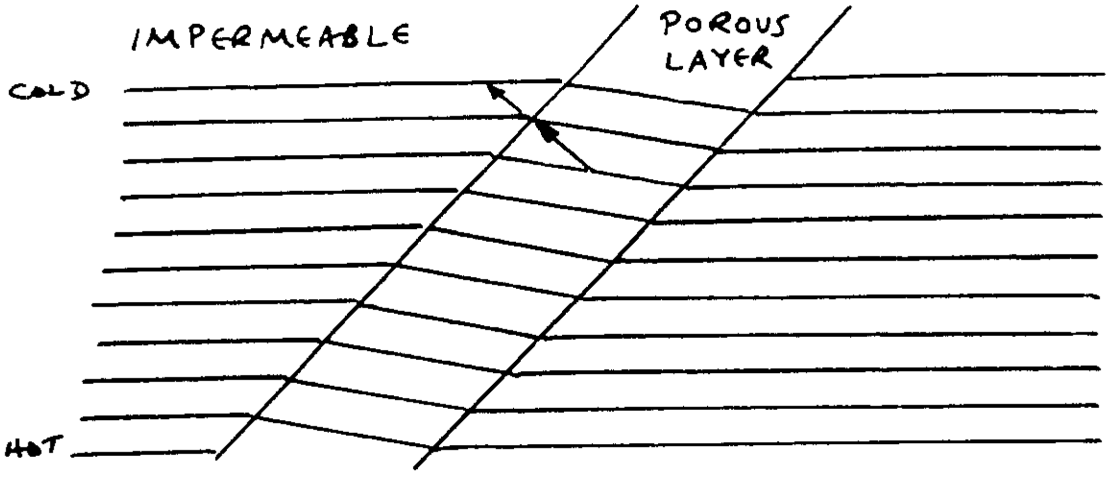
$\hat{u}$ is magnitude of perturbation of flow from mean.

So $\hat{c} \sim \int^{z}\int^{z'} \frac{\hat{u}}{\phi D}\frac{\partial \bar{c}}{\partial x} dz' dz''$

$\hat{c} \sim \frac{|\hat{u}|}{\phi D}\frac{\partial \bar{c}}{\partial x} W^2$

$\Rightarrow$ Dispersive flux along the aquifer $= \int_0^w \hat{u}\hat{c}\, dz \sim \frac{|\hat{u}|^2 W^3}{\phi D}\left(\frac{\partial \bar{c}}{\partial x}\right.$

Hence in stationary frame $\frac{\partial \bar{c}}{\partial t} \sim -\bar{u}\frac{\partial c}{\partial x} + \left(D + \frac{|\hat{u}|^2 w^2}{\phi D}\right)\phi\frac{\partial^2 \bar{c}}{\partial x^2}$



Suppose there is no flow

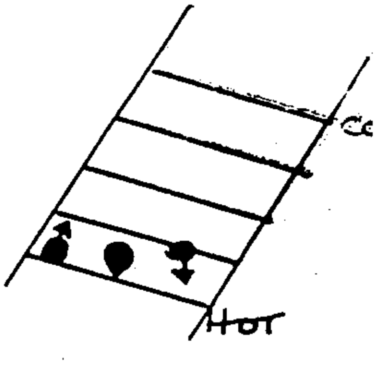
If thermal conductivity in porous layer exceeds that in the rock, then the temperature gradient ⊥'r to the wall is larger in the rock (as sho

(otherwise it is smaller)

In the rock, the isotherms are horizontal

→ In the porous layer they are inclined

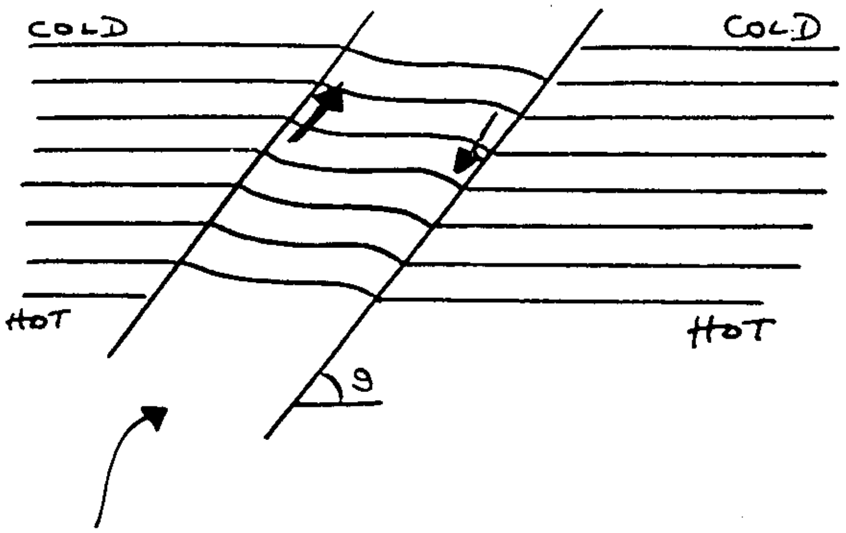
But $\rho(T) = \rho(0)(1-\alpha T)$



So fluid near upper wall is

hot → less dense → rises

Fluid near lower wall cold → more dense → sinks



$[k\, T_\eta] = 0$

on each wall

$\int u\, d\eta = 0$

no mean flow

Flow Induced in the porous layer

$$\underline{u} = -\frac{k}{\mu}\left(\underline{\nabla} p - \rho \underline{g}\right)$$ ← Darcy with gravity

$$(\underline{u}\cdot\underline{\nabla})\, T = \bar{\kappa}\, \nabla^2 T$$ ← Steady flow HEAT CONS'

So $$\underline{\nabla}\wedge\underline{u} = -\left(\frac{k}{\mu}\right) g\, \rho(0)\, \alpha \left(\underline{\nabla}\wedge T\,\hat{\underline{z}}\right)$$ Vorticity Eqn.

If flow is purely along slot ; η = distance across slot

then $$u_{\eta\eta} = g\,\frac{k}{\mu}\,\rho(0)\,\alpha\; T_{\eta\eta} \sin\theta$$ ← from vorticity eqn.

$$\sin\theta\;\; u\, T_{0z} = \bar{\kappa}\; T_{\eta\eta}$$ ← from diffusion eqn.

$$\Rightarrow \boxed{u = \left(\frac{\bar{\kappa}\, \mu\, \text{[redacted]}}{\rho(0)\, g\, \alpha\, T_{0z}}\right)(\sin\theta)^{-2}\, u_{\eta\eta}}$$

So $$u_{\eta\eta\eta} = \left(\frac{\sin^2\theta \; T_{0z} \; g k \; \rho(0) \alpha}{\mu \bar{k}} \right) u_\eta$$

$$\Rightarrow \quad \underline{Ra = \left(\frac{\sin^2\theta \; T_{0z} \; g k \; \rho(0) \alpha}{\mu \bar{k}} \right) d^2}$$

Length of flow in cross slot direction

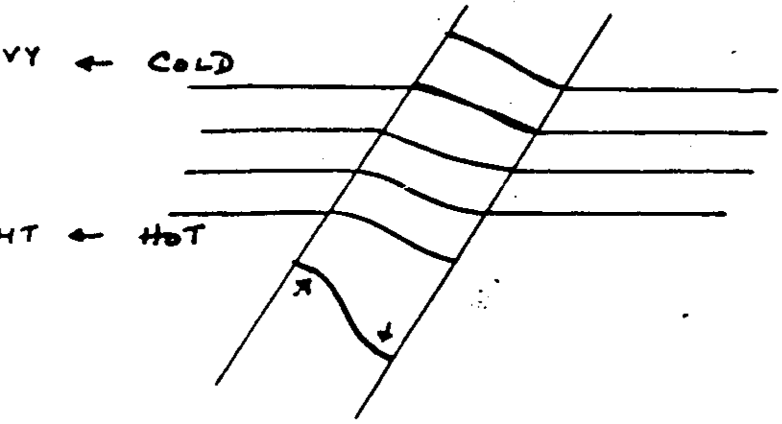
is $\underline{\ell \sim (|Ra|)^{-\frac{1}{2}} d}$

Stable $= A \sinh\left(\frac{\eta |Ra|^{1/2}}{d} \right)$ $\left\{ \begin{array}{l} \Rightarrow \text{low } Ra \text{, then flow extends across layer} \\ \quad \text{(diffusion cross slot faster than advection alo} \\ \quad \text{sl} \\ \Rightarrow \text{high } Ra \text{, flow confined to b'dy layer} \end{array} \right.$

$A = \frac{(\Delta \bar{k}) |Ra^{-1/2}| \cot\theta}{\cosh(Ra^{1/2})}$

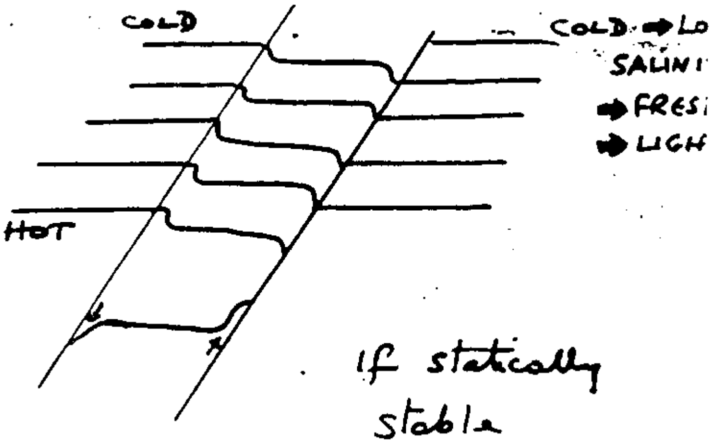
Unstable → Flow always extends across layer

$= A \sin\left(\frac{\eta |Ra|^{1/2}}{d} \right)$



If statically unstable

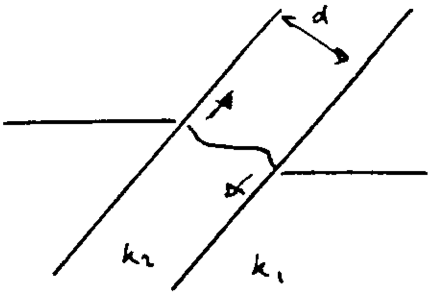
α > 0



If statically stable

α < 0 .

Process may be important in a number of situations, even with zero mean flow –



Consider flow induced by geothermal temperature gradient and differences in the thermal conductivity (end of lecture notes; set 1).

Flow scales as $u \sim \left((\varepsilon - 1)\frac{\kappa_1}{d}\right)\cot\phi \; g(Ra)$ $\quad \varepsilon = \frac{k_1}{k_2}$

(Dimensional argument flow : $f(Ra)\left(\frac{\kappa}{d}\right)$)

Velocity scale of diffusion. If $Ra >> 1$ then flow becomes confined to b'dry layer.

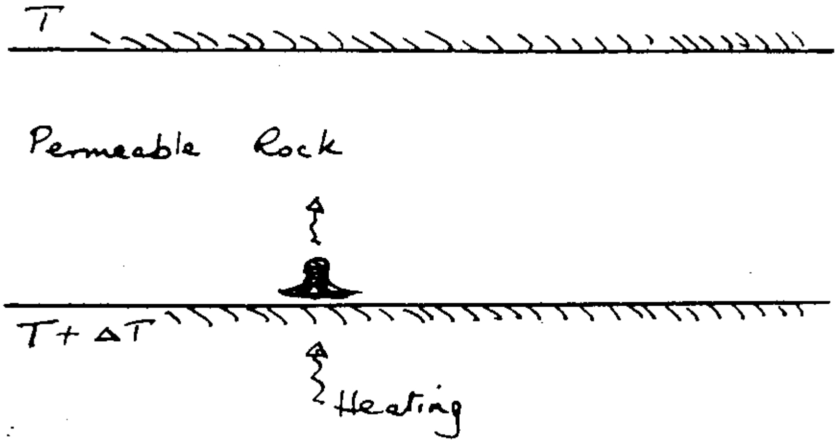
Then dispersion $\sim \left\{\frac{|\hat{u}|^2 d^2}{\phi D}\right\} \sim \left(\frac{\kappa^2}{\phi D}\right)\cot\theta$

So $\frac{\partial \bar{c}}{\partial t} \sim \left(\phi D + \alpha\frac{\kappa^2}{D}\right)\frac{\partial^2 \bar{c}}{\partial z^2}$ $\quad (\alpha : \cot\theta(\varepsilon - 1))$ $\quad (\alpha \sim 1)$

if $\kappa >> D$, as is typical $\frac{\kappa}{D} \sim 10^2$, then material is dispers 10^2 faster than by diffusion $\Rightarrow l_D \sim 10\, l_m$

Instabilities in Porous Layers.

1. Rayleigh - Darcy Instability



As a result of heating the floor, thermal instability and hence motion may ensue

However, conduction of heat + friction tend to suppress this motion

Velocity scale $\frac{k}{\mu}(\alpha \Delta T) g \rho(0)$ (from Darcy's Law)

Effective conduction velocity scale $\frac{\kappa}{d}$ ← velocity scale for rate of heat transfer by conductio

if $\frac{\frac{k}{\mu}(\alpha \Delta T) g \rho(0)}{(\kappa/d)} > O(1)$ then

we expect motion

In fact $\left(\frac{K \alpha \Delta T g}{\nu \kappa}\right) d > 4\pi^2$ for motion

Formally use stability theory (c.f. $\mathbf{H}^2$)

① $$\frac{\partial p}{\partial z} = -\frac{\mu}{k_v} w - \rho g$$ Vertical component of Darcy

② $$\underline{\nabla}_h p = -\frac{\mu}{k_H} \underline{u}_h$$ horizontal part of Darcy $\nabla_h = (\partial_x, \partial_y)$

③ $$\underline{\nabla}.\underline{u} = 0$$ incompressibility ⇒ Boussines

④ $$\Gamma \frac{\partial T}{\partial t} + (\underline{u}.\underline{\nabla})T = \kappa \nabla^2 T$$ thermal equ. ⇒ cons'n of thermal ener

⑤ $$\rho = \rho_0 (1 - \alpha T)$$

State of Rest is a Solution $\quad \underline{u}_h = w = 0, \quad p = p_0(z)$

$$\frac{\partial p_0}{\partial z} = -\rho g$$

{ Steady Conduction across the layer } $$T = T_0 + \left(1 - \frac{z}{h}\right)\Delta T$$

<u>Perturb about state of Rest</u>

$$\underline{u} \rightarrow \underline{u}'(x, y, z, t) \qquad p \rightarrow p_0(z) + p'(x, y, z, t)$$

$$T \rightarrow T_0 + \Delta T\left(1 - \frac{z}{h}\right) + T'(x, y, z, t)$$

assume ' is a small perturbation ⇒ linearise

① $$\frac{\partial p'}{\partial z} = -\frac{\mu}{k_v} w' + \rho_0 g \alpha T'$$

② $$\underline{\nabla}_h p' = -\frac{\mu}{k_h} \underline{u}'_h$$

$$\underline{\nabla}.\underline{u}' = 0$$

③ $$\Gamma \frac{\partial T'}{\partial t} + \cancel{\underline{u}'.\underline{\nabla} T'} - w' \frac{\Delta T}{h} = \kappa \nabla^2 T'$$

and seek 2-Din'l, temporal perturbations of the form

$$\psi \sim e^{nt} e^{i(kx + mz)}$$

Combining ① + ② from last page, we have

$$0 = -\frac{\mu}{K_v} \psi_{xx} - \frac{\mu}{K_H} \psi_{zz} - \rho_0 g \alpha T_x' \qquad ④$$

$$③ \Rightarrow \qquad \Gamma \frac{\partial}{\partial t} T' + \psi_x \frac{\Delta T}{h} = \kappa \nabla^2 T' \qquad ⑤$$

On the boundaries, upper + lower,

$$T' = 0 \qquad \text{and} \qquad w = -\psi_x = 0$$

hence we set

$$\begin{cases} T' = \hat{T} e^{nt} \sin\left(\frac{\pi m z}{h}\right) e^{ikx/h} & ⑥ \\ \psi' = \hat{\psi} e^{nt} \sin\left(\frac{\pi m z}{h}\right) e^{ikx/h} & ⑦ \end{cases}$$

$$\left\{ \begin{array}{l} m \text{ is vertical wave no.} \\ l \text{ is horizontal wave no.} \end{array} \right\}$$

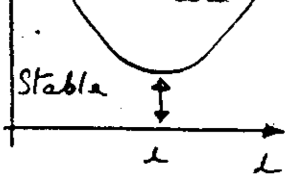
n is growth rate

Combining ④ + ⑤ and substituting ⑥ + ⑦ we have

$$n \frac{h^2 \Gamma}{\kappa} = \frac{l^2}{r l^2 + m^2 \pi^2} Ra - (l^2 + m^2 \pi^2) \qquad r = \frac{k_H}{k_v} ; \; Ra = \frac{g\alpha\Delta T}{\nu l}$$

$\underline{n < 0}$ we have stability $\Rightarrow$ $Ra \leq \dfrac{(l^2 + m^2\pi^2)(r l^2 + m^2 \pi^2)}{l^2}$

$$Ra \leqslant \frac{(\ell^2 + m^2\pi^2)(r\ell^2 + m^2\pi^2)}{\ell^2}$$



m is integer → vertical wave number

fix m, then Ra has a minimum as ℓ varies

Minimum occurs at $\ell^2 = \frac{1}{\sqrt{r}} m^2\pi^2$ and so we require $\underline{Ra \leqslant (1+\sqrt{r})^2 m^2 \pi^2}$.

This must hold for all m → $m = 1$ is most unstable
(i.e. 1 cell in vertical).

For total stability, we therefore need

$$Ra \leqslant (1+\sqrt{r})^2 \pi^2$$

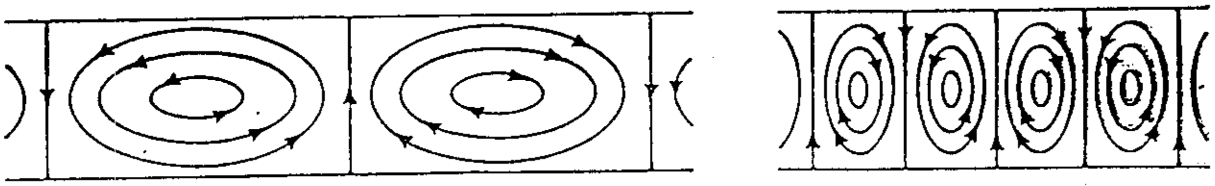
if $r = 1$ then $\left(\frac{k_H}{k_V}\right)$

$$\underline{Ra = \frac{k \alpha \Delta T g}{\kappa \nu} \leqslant 4\pi^2}$$

for no motion

More permeable in horizontal
→ cells asymmetric (long, squat)

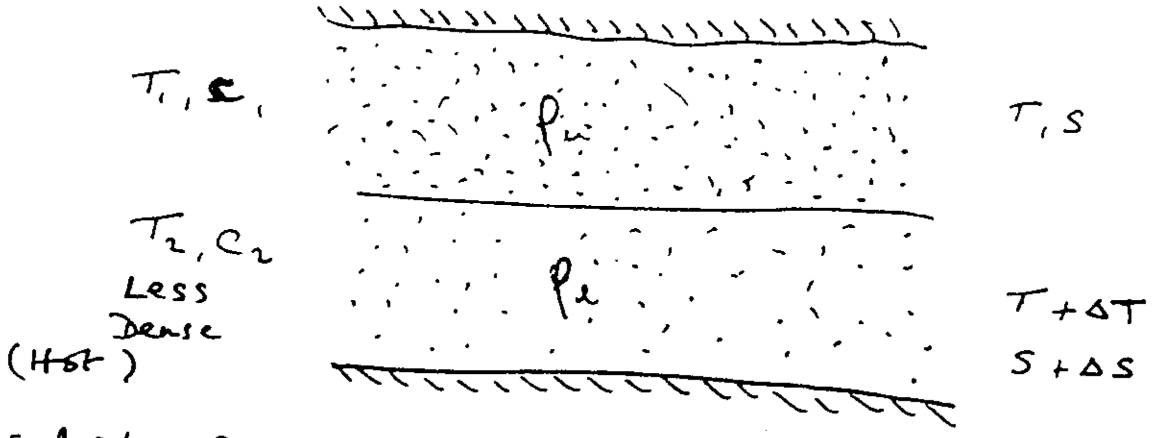
More permeable in vertical
→ cells narrow, tall



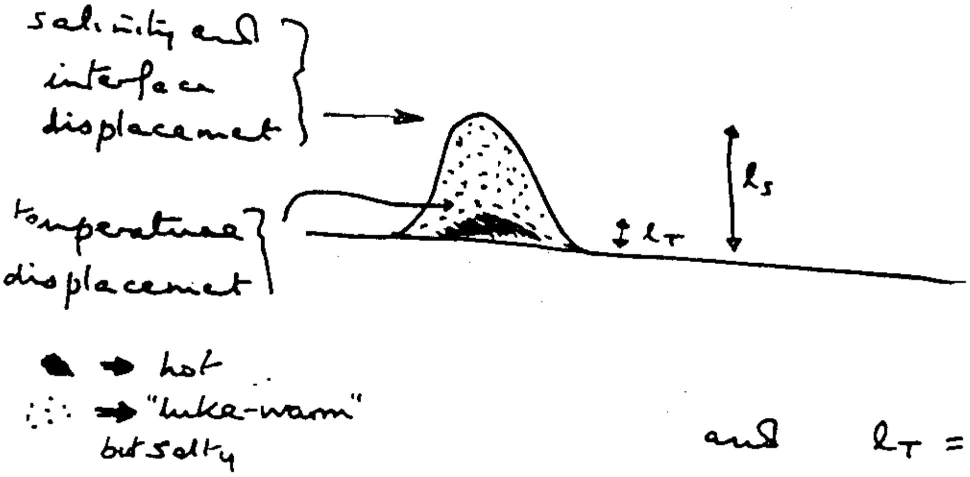
$k_H = 10k_V$ $k_H = 0.1k_V$

Figure 5.3. Cell shapes for the most unstable disturbances for k_H/k_v = 10 and 0.1. When $k_H = k_v$, the cells are square.

Note <u>Hot</u> may be salty + dense – (liquid saturated + in ≡'m wt rock
or simply light, owing to thermal expansion



Stable configuration possible even if $\rho_u > \rho$



extra salinity in fluid displaced $= l_s \Delta S$

extra temperature $= l_T \Delta T$

and $l_T = l_s \frac{\phi (\rho C)_{fluid}}{(\rho C)_{total}} \sim \phi \, l_s$

Hence density excess in displaced fluid $= l_s(\beta \Delta S - \alpha \phi$

Thus lower layer is stable if $\Delta S > (\phi \Delta T)($

However, density of lower layer fluid (relative) is $\beta \Delta S - \alpha \Delta T$

and so it is possible that

temperature
causes net
difference
n density
but
ermal inertia
llows salt to
stabilize

$\beta \Delta S - \alpha \Delta T < 0$ (apparently unstable, i.e. lower layer less dense)

but $\beta \Delta S - \phi \alpha \Delta T > 0$ (stable to displacement due to thermal inertia

So that the medium is stable.

(top diagram inverted)

However, in some situations, the medium can become unstable to double-advective (diffusive) instabilities $\Rightarrow$ fluid migrates throughout rock leading to variety of reactions

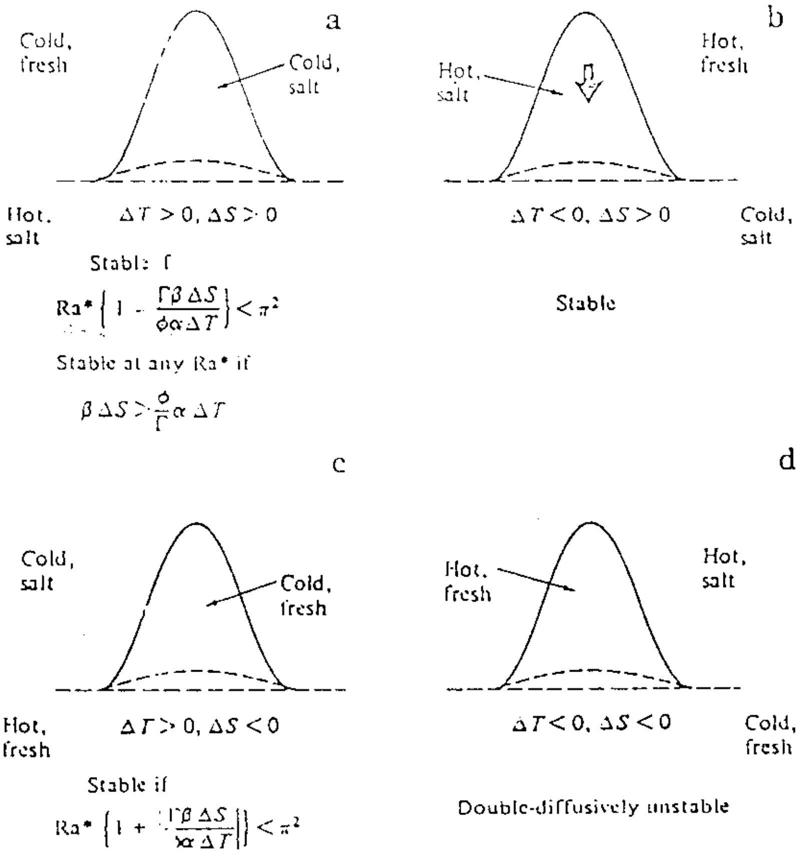


Figure 5.5. Isohaline (solid line) and isotherm (dashed line) displacements in diffusive saturated permeable media, with the associated stability characteristics when $\kappa \gg \kappa_s$. In case (c), when $\Delta T \to 0$, the stability condition given is replaced by one analogous to equation (5.1.15) but with $-\beta\Delta S/\phi\kappa_s$ replacing $\alpha\,\Delta T/\kappa$.

Salt acts
to stabilize
the thermal
estabilization
imilar to
R-D instability

Always
Stable
T, S both
stabilize

T, S
estabilize
—
stion if
ffective
'a large
enough to
overcome
diffusion —

Buoyancy
decreases)

Salt
fingers
←

Buoyancy Increases

For example: Dolomites { Permian Basin of West Tex

Time Scale for formation Rapid. Magnesium req'd to convert limestone to dolomite → for double-diffusive / advective motion

Velocity $w \sim \left(\frac{k_v}{\nu}\right)(g\beta\Delta s)$

c.f. ($u = -\frac{k}{\nu}\nabla$ D'arcy La

⇒ Time for flow to migrate through a depth h is

$$T \sim \frac{h}{w} \sim \left(\frac{h\nu}{k_v g\beta\Delta s}\right)$$

if $h \sim 500\,m$; $\nu \sim 10^{-2}\,cm^2/s$
$k_v \sim 10^{-14}$ and $\beta\Delta s \sim 0.05$

then $\left\{\begin{array}{l} T \sim 3000\ yr \\ w \sim 15\ cm/yr. \end{array}\right\}$ Geologicall Short

c.f. 10^6
for diffus
$D \sim 10^{-8}$
$t \sim \frac{h^2}{D}$
$\sim 10^{5}\cdot 10^{8}$

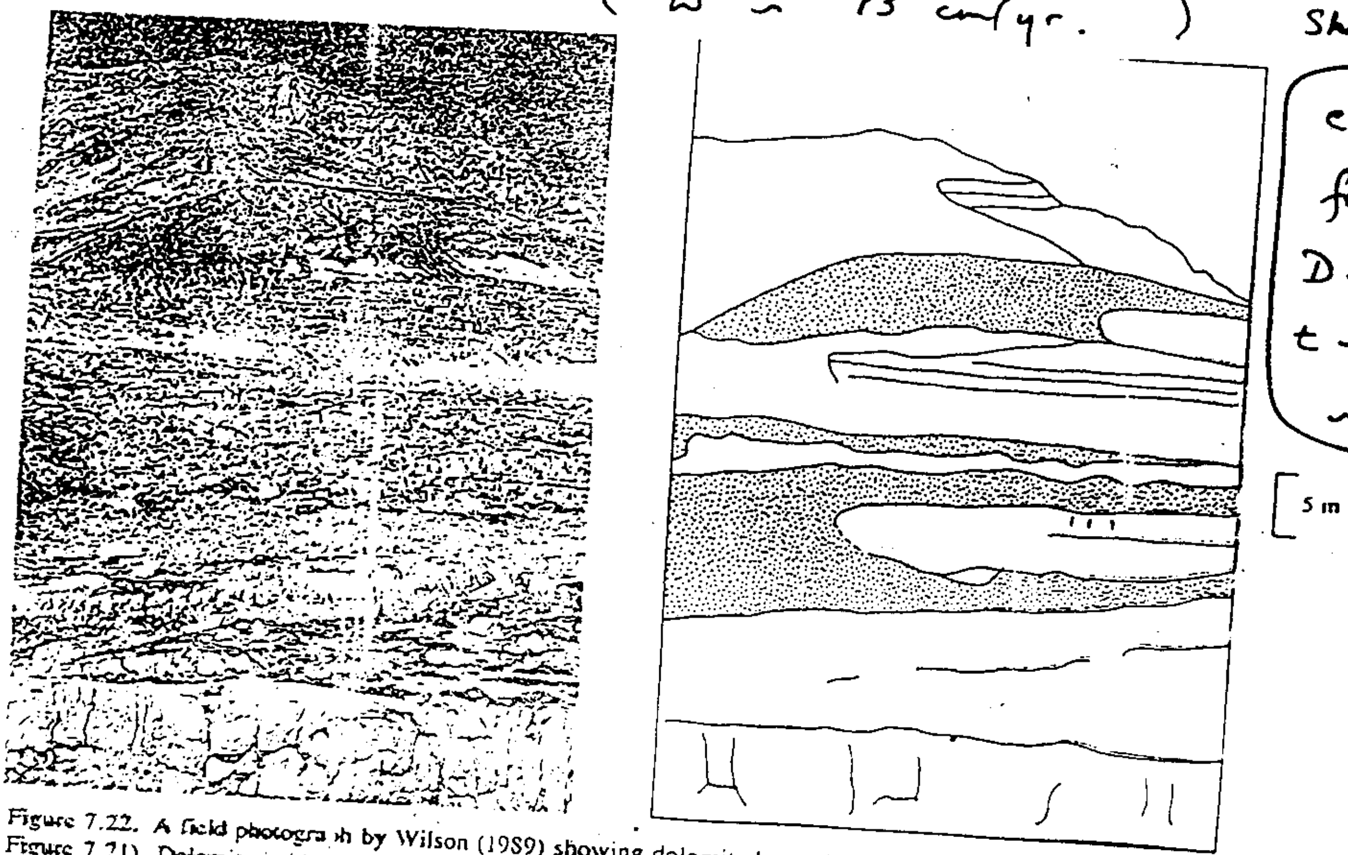
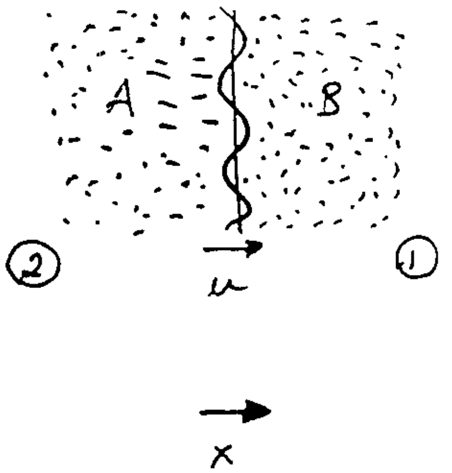


Figure 7.22. A field photograph by Wilson (1989) showing dolomite lenses in the upper, sparsely dolomitized zone (3 in Figure 7.21). Dolomite is identified in the line drawing by stippling.

Saffman Taylor Instability



if B more viscous than A then unstable

finger of A is less viscous than surrounding fluid B

⇒ finger grows.

Suppose Interface moves with speed u

$\rightarrow$ Transport (Darcy) velocity is ϕu.

$$\nabla p_1 = -\mu_1 \frac{\phi u}{k} \rightarrow P_1 = -\mu_1 \frac{\phi u}{k} x + P_0$$

$$\nabla p_2 = -\mu_2 \frac{\phi u}{k} \rightarrow P_2 = -\mu_2 \frac{\phi u}{k} x + P_0$$

Add a perturbation to interface, $u \rightarrow u + \sigma a e^{inx+\sigma t}$

$$\left\{ \begin{array}{l} P_1 \rightarrow P_1 + \mu_1 \frac{\phi}{k} \left(\frac{\sigma a}{n}\right) e^{inx - nx + \sigma t} \\ P_2 \rightarrow P_2 + (-)\mu_2 \frac{\phi}{k} \left(\frac{\sigma a}{n}\right) e^{inx + nx + \sigma t} \end{array} \right\}$$

from $\nabla^2 p =$
and $p \rightarrow 0$
as $x \rightarrow \pm$

$$\boxed{\begin{array}{l} \underline{u} = -\frac{k}{\mu} \nabla p ; \underline{\nabla} \cdot \underline{u} = \\ \Rightarrow 0 = -\frac{k}{\mu} \nabla^2 p \end{array}}$$

But on the moving interface $P_1 = P_2$

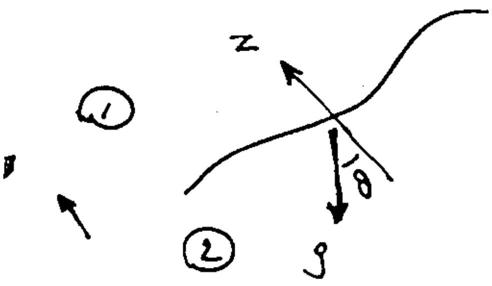
hence $$-\mu_1 \frac{\phi u}{k} + \mu_1 \frac{\phi \sigma a}{nk} = -\mu_2 \frac{\phi u}{k} - \mu_2 \frac{\phi \sigma a}{nk}$$

$$\Rightarrow (\mu_1 - \mu_2) \frac{\phi u}{k} = \sigma \left(\frac{a \phi}{nk}\right)(\mu_1 + \mu_2)$$

More viscous displaced by less visc

If the two fluids are of different density, then a gravitational force exists and may act to stabilize the front

If interface has angle θ to the vertical



then $p_1 \to p_1 - \rho_1 g z \cos\theta$

$p_2 \to p_2 - \rho_2 g z \cos\theta$

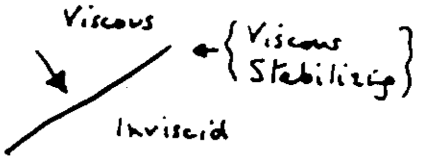
hence dispersion relation becomes

$$(\mu_1-\mu_2)\frac{\phi u}{k} + \underline{(\rho_1-\rho_2) g \cos\theta} = \sigma\left(\frac{a\phi}{nk}\right)(\mu_1+\mu_2$$

if $\sigma \geq 0$, $u(\mu_1-\mu_2) > -\left(\frac{k g \cos\theta}{\phi}\right)(\rho_1-\rho_2)$

Instability requires $u(\mu_1-\mu_2)$ sufficiently large if $\rho_1 < \rho_2$ $\left(\frac{\text{light}}{\text{dense}}\right)$

→ density stabilizes if flow is upward, and heavy fluid displaces light fluid



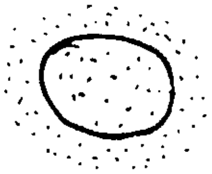
if $\rho_2 < \rho_1$ → always unstable for upflow

Also note downflow of heavy, viscous fluid into light, inviscid fluid may be stabilised by viscous forces

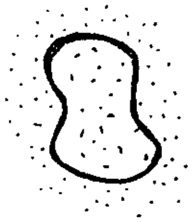
Case of circular geometry :

$u = -\frac{k}{\mu} \nabla P$ seek modes of the form

$$\eta = \eta_0(t) \exp(in\theta)$$



$n = 1$



$n = 2$

$n = 3$ etc

Now $\nabla^2 P = 0 \Rightarrow P_{rr} + \frac{1}{r} P_r - \frac{n^2}{r^2} P = 0$

for $P = P(r) \exp(in\theta)$

Matching velocity and stress at boundary gives

$$[\Delta p] = \sigma \left(\frac{1}{R} - \frac{\eta + \frac{d^2\eta}{d\theta^2}}{R^2} \right)$$

where the surface tension

radius of interface $= R + \eta$

hence

$$\left(\frac{1}{\eta}\right)\frac{d\eta}{dt} = \frac{Q n}{2\pi R^2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right) - \frac{Q}{2\pi R^2} - \frac{\sigma n(n^2-1)}{R^3}\left(\frac{M_1 M_2}{M_1 + M_2}\right)$$

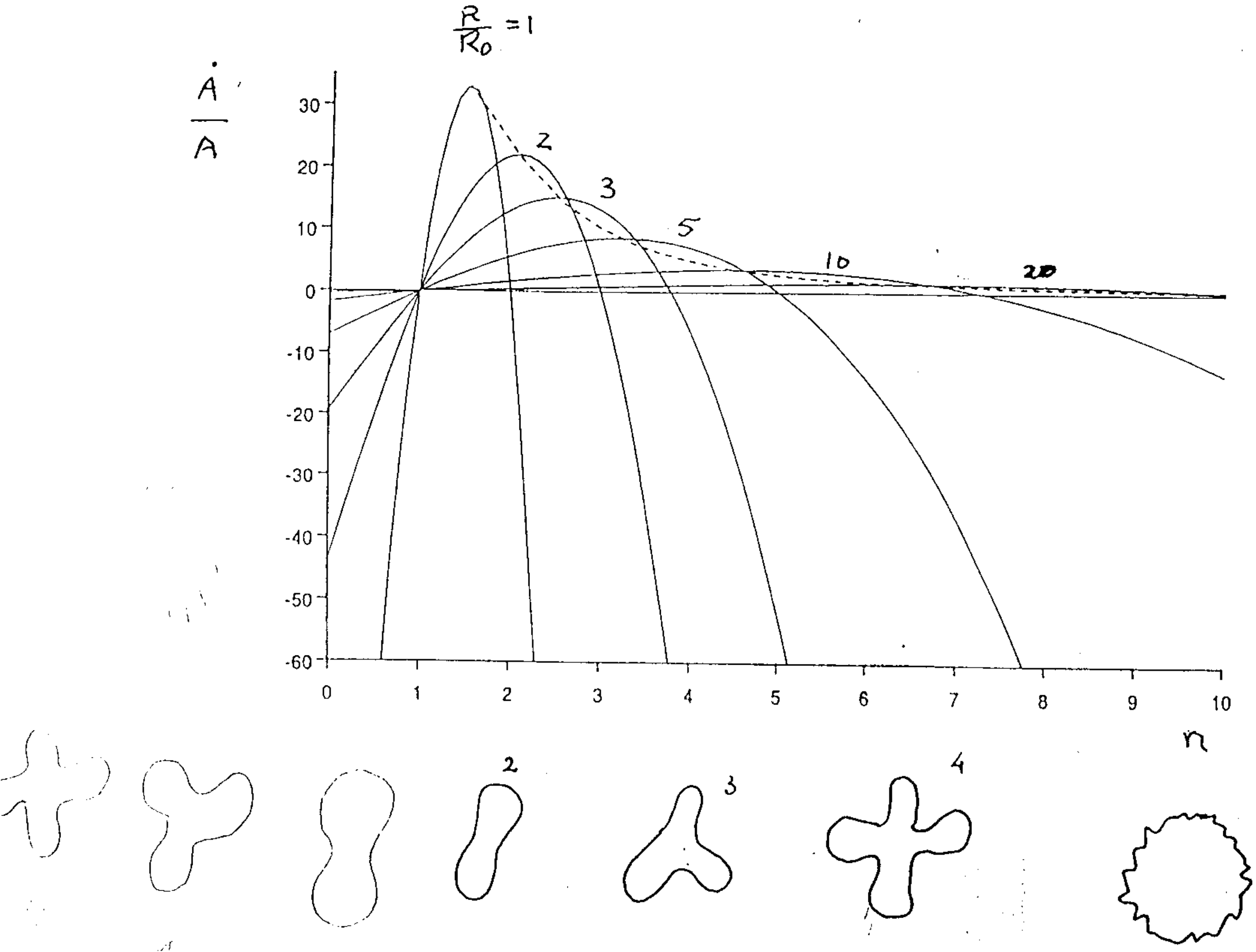
growth rate   viscous inst.   surface tension st.

(with $M_i = \frac{k}{\mu_i}$)

At each R, most instable mode is selected $\Rightarrow n(R)$

for $R < R_c = \frac{4\sigma M_2 \pi}{Q}$ interface absolutely stable

$(M_1 >> M_2)$



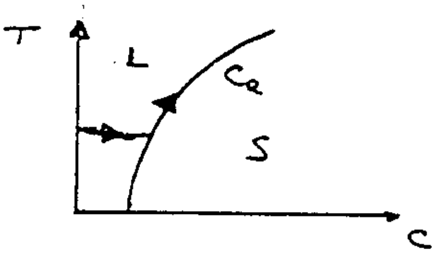
$\frac{R}{R_0} = 1$
$\frac{\dot{A}}{A}$
30
20
10
0
-10
-20
-30
-40
-50
-60
0
1
2
3
4
5
6
7
8
9
10
n
2
3
5
10
20
2
3
4

Reactions : Rock alteration due to fluid flow

Reaction types

(i) Simple Dissolution / Precipitation

$$NaCl \rightleftharpoons Na^{+} + Cl^{-}$$

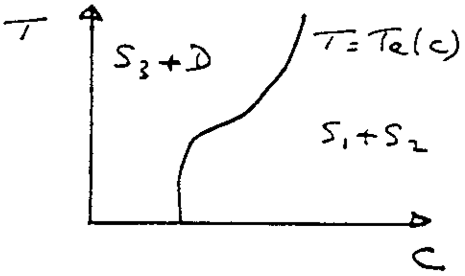


(ii) Dissolution of 2 minerals into a third + a dissolved species

⇒ Wollastonite

$$CaCO_3 + SiO_2 \rightleftharpoons CaSiO_3 + CO_{2\,(aq)}$$

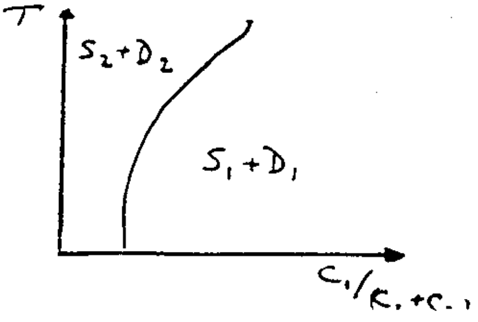
Calcite + Silica ⇌ Wollastonite + dissol. CO_2



(iii) Replacement Reactions Dolomite

$$2\,CaCO_3 + Mg^{2+} \rightleftharpoons CaMg(CO_3)_2 + Ca^{2}$$

Calcite Dolomite



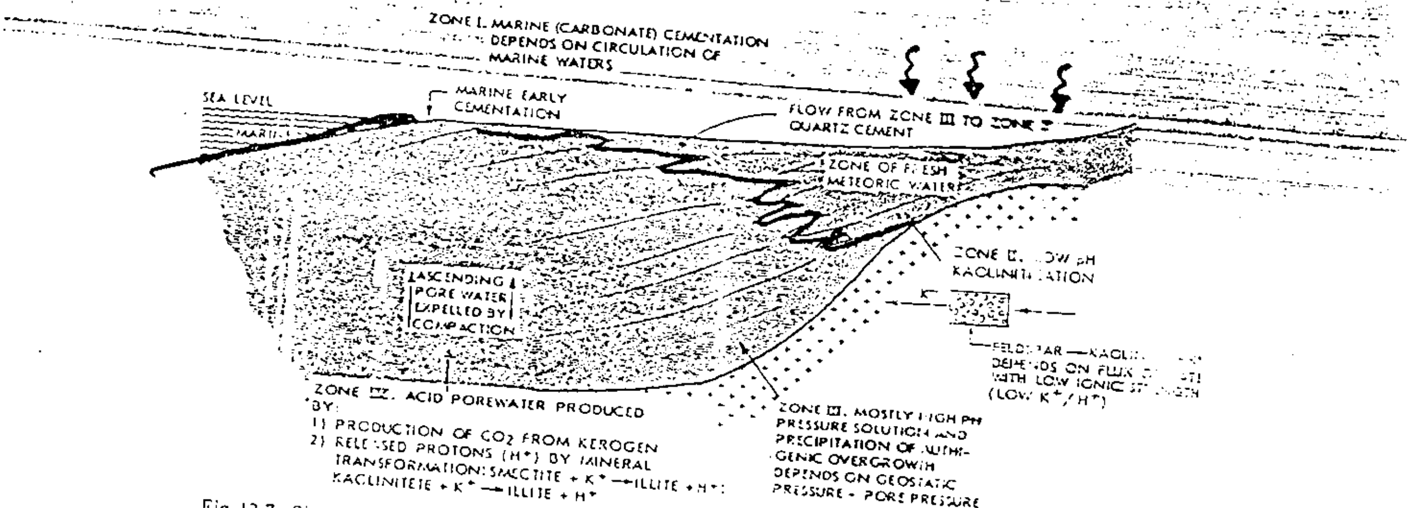


Fig. 12 7. Simplified model of porewater circulation in a sedimentary basin, and the diagenetic reactions which are typical of the various parts of the basin. (Bjørlykke 1983)

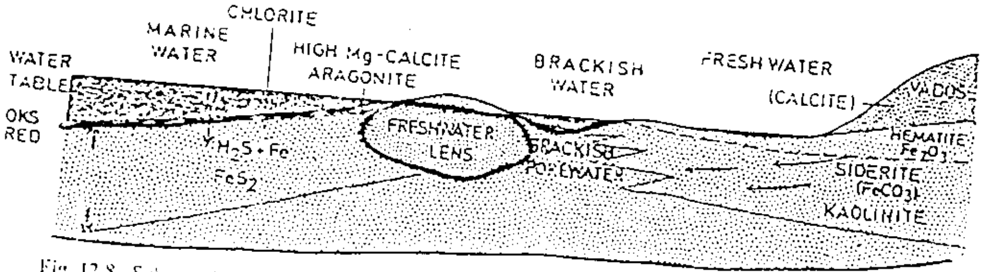


Fig. 12.8. Schematic overview of early diagenetic environment. In the sulphate-reducing zone all iron will precipitate as sulphides. Siderite will only form in the fresh or brackish water environment or below the sulphate-reducing zone

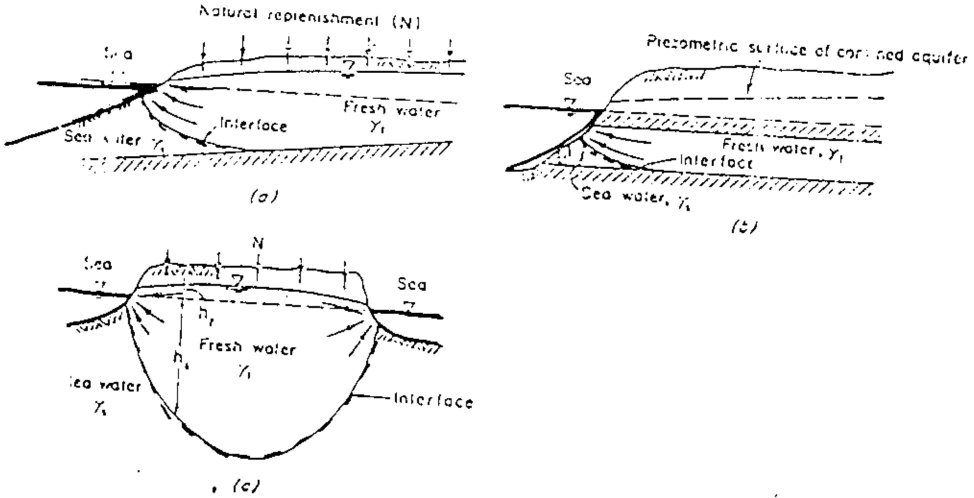
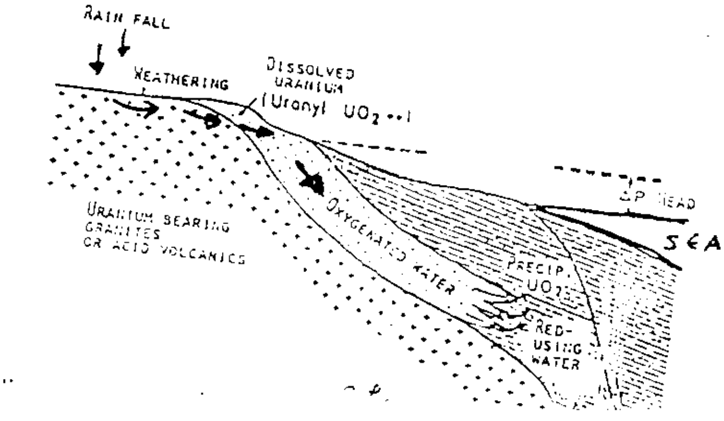
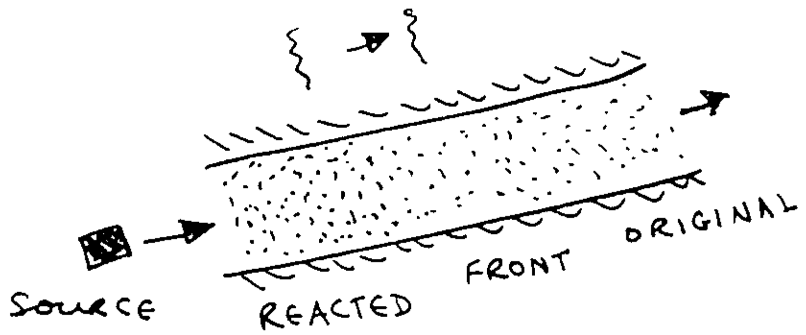


Fig: 9.7.1. Interfaces in coastal aquifers (highly distorted figures).



Types of Reactions driven by Fluid Flow

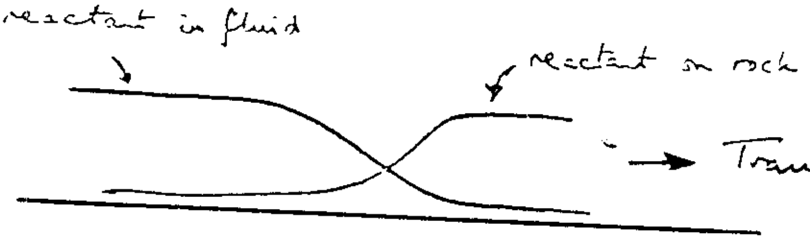


b = reactant on rock
a = reactant in fluid

$$\begin{cases} \lambda a_t + u a_x = -a\,b \\ b_t = -a\,b \end{cases}$$

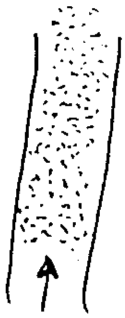
$(1-a), b \sim \left\{ \frac{1}{1-\exp[(x-\right.$

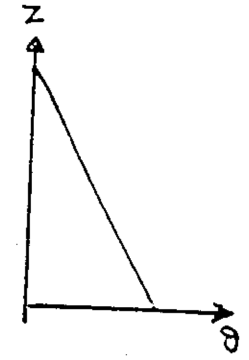
Migrating reaction front



→ Travelling solution

GRADIENT
REACTION
ZONE





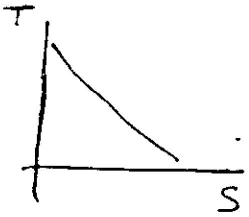
Geothermal
Gradient

Fluid ascends and
remains in local
equilibrium

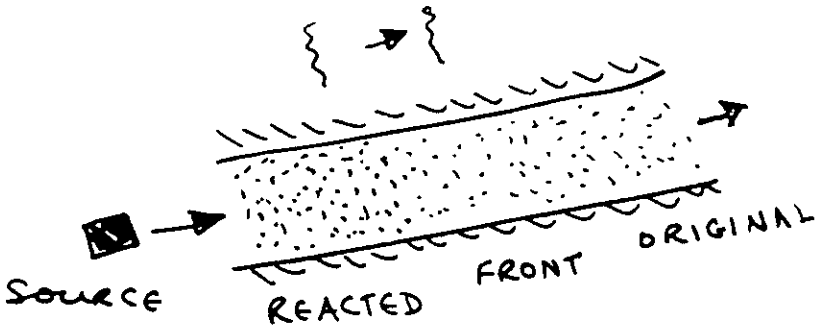
⇒ phase changes occur
as consequence of
background temperature
variation.

→ Flows may be thermally or hydraulically
driven.

→ MIXING ZONES



'ypes of Reactions driven by Fluid Flow

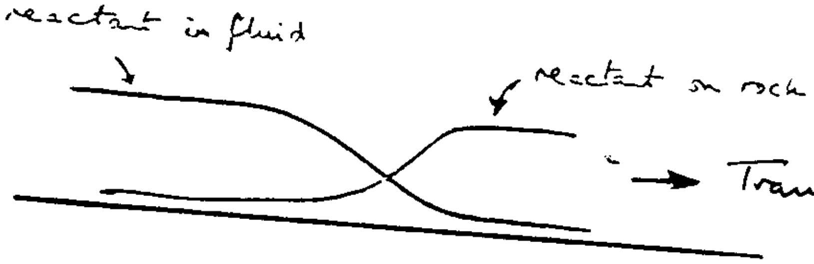


b = reactant on rock
a = reactant in fluid

$$\begin{cases} \lambda a_t + u a_x = -a \\ b_t = -a\,b \end{cases}$$

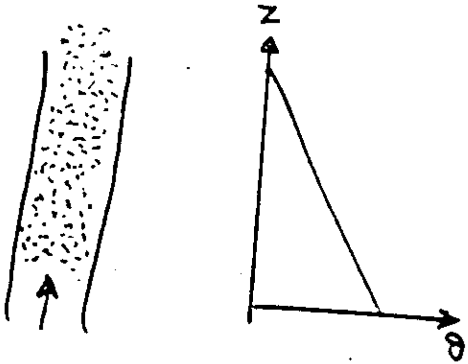
Migrating reaction front

$(1-a), b \sim \left\{ \frac{1}{1-\exp[(x} \right.$



→ Travelling solution

GRADIENT
REACTION
ZONE



Geothermal
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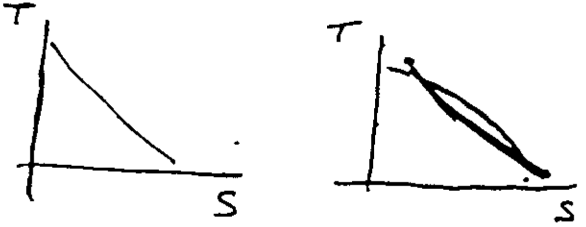


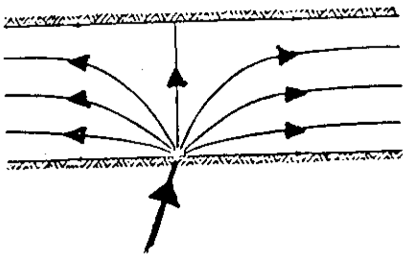


Figure 4.6. Streamlines of flow issuing from a linear fracture into a permeable layer with no superimposed ambient pressure gradient.

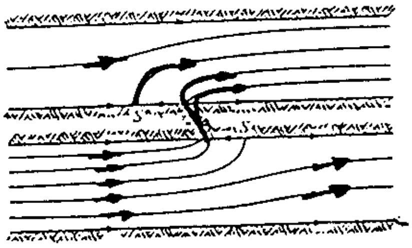


Figure 4.7. Streamlines of flow when fluid moves through a linear fracture from one permeable layer to another.

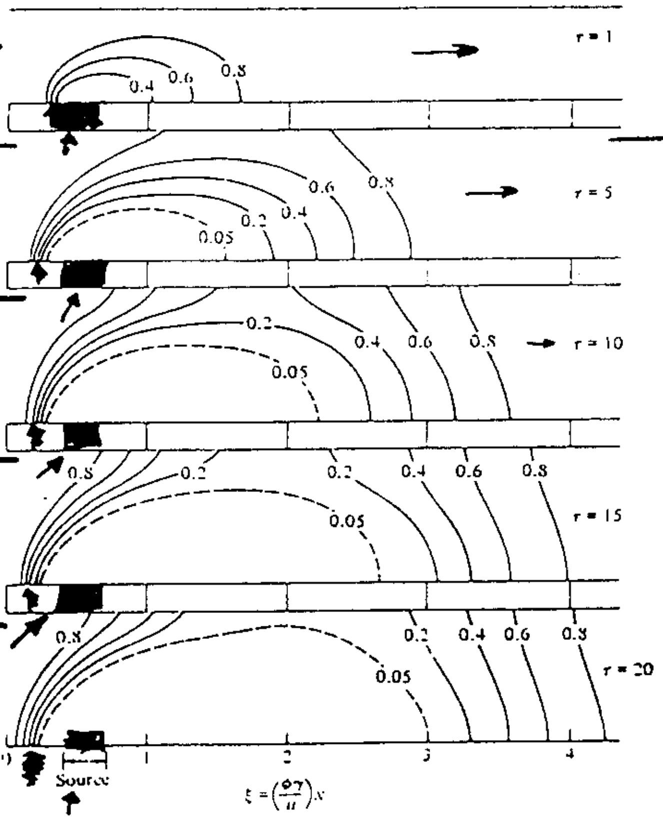


Figure 4.13. Distributions of the unaltered mineral undergoing reaction, $S = s/s_0$. The regions inside the dashed curves indicate zones of virtually complete reaction.

Development of Reaction Zone following fracture thro' impermeable layer

4c

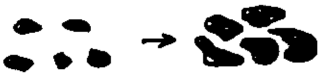
Reaction Front Instabilities

Reaction of interstitial fluid and porous matrix can change the permeability.

Dissolution ⇒ Decrease of matrix blocking flow

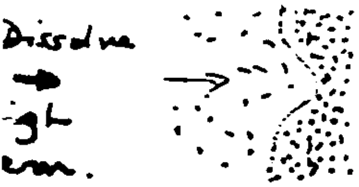
 ⇒ Increased Permeability

Precipitation ⇒ Increase of ~~mat~~ material blocking flow between porous matrix



⇒ Decreased Permeability

As fluid propagates through rock, we expect the dissolution process to generate fingers of high permeability → planar interface unstable



finger offers less resistance to flow
→ fluid tends to flow into finger
→ unstable (c.f. viscous fingering)