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Workshop on Fluid Mechanics

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Physics of convection

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THE HISTORY AND PHYSICS OF BUOYANCY IN FLUIDS

Edward A. Spiegel

The study of convection is a venerable field. Before discussing any detailed work, let us embark on an historical sketch.

1.1 Historical Sketch

c.250 BC	Archimedes discovers the principle of specific gravity and thus quantifies the idea of buoyancy.
1657	Rinaldi demonstrates convection in experiments designed to disprove Aristotles's ideas on the flow of heat (Middleton, 1908).
1749	Ben Franklin gives a geophysical application - the motion of air in a thunderstorm (Middleton, 1968).
1798	Rumford (in trying to discover why soup cools faster than apple pie)does experiments with convection in which a trace (yellow resin) is introduced for the first time (Brown, 1979).
1834	Pruitt coined the word "convection" (Brown, 1979).
1840	Rayleigh, Espy and others looked at the instability of an unstably stratified fluid and derived the "Schwartzchild" interior.
1861	Lord Kelvin introduced convective equilibrium (Lamb).
1885	Jevons discussed double diffusive convection.
1900	Benard observed hexagonal convection cells upon heating a thin layer of fluid above the critical Rayleigh number. Unfortunately, he probably did not realize surface tension had a very important influence on his results.
1903	Boussinesq's approximate equations for convection in a thin, almost adiabatic layer were published (1903, in his Theorie Analytique de la Chaleus").
1916	Lord Rayleigh discussed marginal stability of Boussinesq convection and introduced the bey stability parameter.
1926	Jeffreys (See Saltzman, 1962) looked at the case of insulating top and bottom and found that the most unstable modes were horizontally infinite in extent.
Up to 1940	Better B.C.'s and better calculations of the critical Rayleigh number and wave length at convective instability were done culminating in a paper of Pellev and Southwell (1940) (See Saltzman, 1962). Their results are summarized in Figure 1.

1952ff Overstability was found in convection (See Chandrasekhar, 1961.) These were of two kinds. Rotation could couple modes with vertical vorticity to horizontal motion and produce overstability (later done by nonlinear terms with no rotation by Busse, Busse and Clever). Computing instabilities like magnetic and thermal effects could produce overstability in a generic way. This opened the quantitative study of computing instabilities continued by Townsend (JFM, 1959), Stern (1961), and onward.

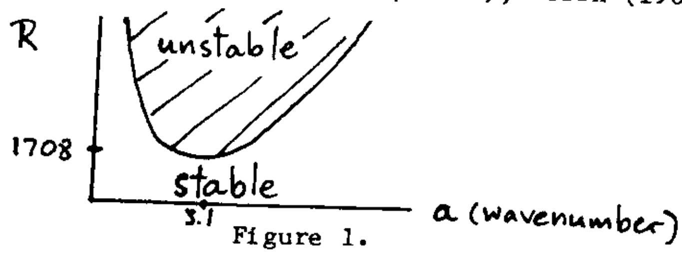

Figure 1.

1956 Malkus and Veronis found for R slightly greater than R_c, that steady small amplitude convection motions took place and found their form by perturbation theory.

1965 Busse took this analysis to higher order to show that the only stable steady convective solutions were two-dimensional rolls. Above a second critical Rayleigh number R_b these become unstable. Unlike the initial instability this bifurcation depends on the Prandtl number $\sigma = \frac{\nu}{\kappa}$. For $\sigma > \sigma_c$, the steady solution bifurcates into two steady solutions. For $\sigma < \sigma_c$, it bifurcates into oscillatory modes. The picture is as in Figure 2, in which we are looking at the amplitude $\|\underset{\sim}{u}\| = A$ of the motion vs R.

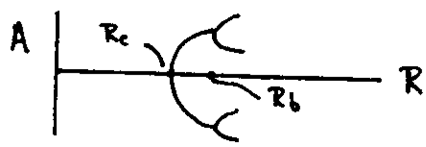

Figure 2.

1.2 A Motivation in Astrophysics

For the most part, the study of convection has been an outcast of GFD, since most geophysical processes involve high Rayleigh number, turbulent convection, which has generally been modelled as a small-length scale mixing process whose effects on larger scales has been treated as an eddy viscosity. However, we will try to keep in mind a real problem which clearly involves convection and several interacting length scales, namely the transport of heat from the solar interior.

Photographs of the apparent surface of the sun in visible light reveal "granulation" in the form of cellular patterns of about 1000 km diameter. These cells clump into globs of about 5000 km, and the surface velocity shows "super-granularity" at a 30,000 km scale. Superimposed on these convective patterns are sunspots, magnetic flux tubes usually in pairs of positive and negative polarity. In the sunspots, the strong magnetic field inhibits convection and so they are cooler at the surface, and therefore a prominent feature of pictures of the sun.

Sunspots obey an 11-year cycle of abundance; between cycles the maximum number of sunspots varies dramatically. At some times, such as the Maunder Minimum of 1550-1600, rediscussed by Jack Eddy, almost no sunspots were seen. The goal of this course will be to try to provide a model which possesses similar behavior.

We now go back and try to capture some of the features of buoyancy forces in fluids by looking at a simple model. It will be the scope of future lectures to show how the set of equations describing such a model is relevant to the general problem of convection.

We adopt a Lagrangian viewpoint and focus our attention on the motion of an idealized fluid particle through a surrounding fluid. The fluid particle has mass m, volume V, density ρ and is uniquely identified by its position z. The forces on such particles are due to gravity, buoyancy and drag and can be written in the equation of motion as:

$$m\ddot{z} = -g(\rho-\rho_0)V - \mu\dot{z}$$

We have chosen g to act in the negative z direction. We can write $m = \rho V$ and if the fluid is almost incompressible we can approximate m by writing $m \simeq \rho_0 V$. Dividing through we get:

$$\ddot{z} = -g\left(\frac{\rho-\rho_0}{\rho_0}\right) - \nu\dot{z} \tag{1}$$

Where, in the spirit of the Boussinesq approximation, we choose ν to be a constant. This is the first of our equations. If the fluid is completely incompressible ρ , ρ_0 = const and (1) has solution:

$$z(t) = \kappa e^{-\nu t} - \frac{g}{\nu}\left(\frac{\rho-\rho_0}{\rho_0}\right)t \tag{2}$$

After the transient decays (2) describes the motion of a particle moving at constant velocity. The effective buoyancy force is equal to the viscous drag and the direction of motion depends on the sign of ($\rho-\rho_0$).

More generally we expect the density of the fluid particle to depend on the thermodynamic state and to obey some equation of state like $\rho = \rho(p, T)$. If we assume that the temperature and pressure of the particle deviate only slightly from the ambient temperature and pressure we can approximate the equation of state by:

$$\rho = \rho_0\left[1 + \kappa_T(p-p_0) - \alpha(T-T_0)\right]$$

Where we have defined the isothermal compressibility $\kappa_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T$ and the thermal expansivity $\alpha = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$.

Again, for a Boussinesq fluid the variations induced by pressure differences can be shown to be negligible and the equation of state reduces to

$$\rho = \rho_0\left[1 - \alpha(T-T_0)\right]$$

We now consider the heat exchanged between the fluid particle and its surroundings; assuming this obeys Newton's Law of cooling we write

$$\dot{T} = -\gamma(T-T_0) + \partial_t T_0$$

for some constant γ . The background temperature T_0 depends on z, and possibly t. If we introduce a temperature excess $\theta = (T - T_0)$ and $\beta = -\partial T_0/\partial z$ the last equation becomes

$$\dot{\theta} = \beta \dot{z} - \gamma \theta \tag{3}$$

This is the second of our equations. Equations (1) and (3) cannot, as yet, be solved as we made no assumptions regarding the evolution of β. A possibility is to assume T_0 to be linear in z and $\beta = \beta_0$ = const and then the equations are linear. For such a set of equations a solution of the form $z \propto e^{\eta t}$ exist provided η satisfied:

$$\eta \left[(\eta + \gamma)(\eta + \nu) - g \alpha \beta_0 \right] = 0 \tag{4}$$

We notice that the translational invariance of the system shows up with η as one of the factors.

If $\nu = 0$ we have no viscous dissipation, and for $\gamma^2 \gg g\alpha\beta_0$ (i.e., the rate of exchange of heat is much larger than the rate at which buoyancy does work) then the solution has

$$\eta \simeq \frac{g \alpha \beta_0}{\gamma}$$

which shows that heat conduction slows down but does not prevent the runaway of the fluid particle.

If $\nu \neq 0$ there exists a marginal mode provided

$$R = \frac{g \alpha \beta_0}{\gamma \nu} = 1$$

and solutions become unstable for $R > 1$. If β is not constant we can get our third and last equation by, for example, writing the evolution of β as an expansion in powers of the heat transport. To first order:

$$\dot{\beta} = c \left[\dot{z} \theta - \kappa (\beta - \beta_0) \right] \tag{5}$$

for some constant C and K. The term $\dot{z}\theta$ in (5) is the advection of heat and the term $K\beta$ is the conduction.

Equations (1), (3) and (5) for variables $\dot{z}$, β, θ are known as the Lorenz equations. Their analysis is an interesting topic in itself, which will be dealt with by other lecturers.

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NOTES SUBMITTED BY
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SMALL NONDISSIPATIVE MOTIONS AND THE ANELASTIC APPROXIMATION

Edward A. Spiegel

Equations

We first recall a few technical points in the study of fluid (and other) dynamics. The fluid is often pictured as a continuum of particles obeying Newton's Laws. A fluid particle or element has an orbit in space that is parameterized by time. The distinction among orbits is made by other (Lagrangian) parameters. Thus x(t), a particular orbit, contains in its description parameters (not written explicitly here) which distinguish it from the others. By $\dot{x}$ we mean we $\partial x/\partial t$ evaluated on an orbit, and to emphasize this we write

$$\dot{x} = \frac{Dx}{Dt}$$

For any function defined on an orbit

$$\partial_t f\,\big|_{x \text{ fixed}} + \dot{x}\;\partial_x f\,\big|_{t \text{ fixed}} = \frac{Df}{Dt}\,.$$

Finally, if we can express the parameters which characterize the orbit in terms of x and t, we can write

$$x(t,\ \text{parameters}) = (x,t)$$

So $\quad \dfrac{Dx}{Dt} = \underset{\sim}{u}(x,t).$

$$\frac{Df}{Dt} = f_t + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})f,$$

we shall also write $\dfrac{\partial f}{\partial t} = \partial_t f = f_t$.

We now introduce the three conservation equations that are the basis of this course:

$$\rho_t + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{u}) = 0$$

$$(\rho\underset{\sim}{u})_t + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{u}\underset{\sim}{u}) = \underset{\sim}{g}\rho + \underset{\sim}{\nabla}\cdot\underset{\sim}{T} + \underset{\sim}{f}_{ext}$$

$$(\rho S)_t + \nabla\cdot(\rho\underset{\sim}{u}S) = \dot{Q}/T\rho$$

where ρ is density, $\underset{\sim}{u}$ is velocity, $\underset{\sim}{T}$ is the fluid stress tensor, $f_{e\,t}$ is any externally applied force field in addition to gravity (whose acceleration is g), S is the specific entropy and $\dot{Q}/T\rho$ is a vague notation for thermal dissipative processes. I use Malkus-Veronis notation, which is based on the idea that a reader can distinguish among T, T , $\mathbf{T}$ and the like.

The simple model used in what follows is

$$T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})$$

and we shall stay always in Cartesian coordinates. The thermodynamic equation will be written using $dH = Tds + \frac{1}{\rho}\,dp$ where T is temperature and H is specific enthalpy. Also we write

$$dH = C_p\, dT$$

and get (with some further conventional assumptions)

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \dot{Q} = \nabla \cdot \underset{\sim}{F} + \Phi$$

where $\underset{\sim}{F}$ be heat flux and Φ be viscous dissipation into heat. We note that

$$\Phi = \frac{\partial u_i}{\partial x_j}\left(T_{ij} + p\,\delta_{ij}\right)$$

We take $\underset{\sim}{F} = -K \underset{\sim}{\nabla} T$. The equations of state that arise in convection problems are varied but, in this course, we shall not deal with Messrs. Saha and Gruneisen. We consider the simplest gas as an example:

$$p = R \rho T$$

with R constant. The entropy for a perfect gas is

$$S = C_v \ln\left[p/\rho^{1/\gamma}\right]$$

where γ is the ratio of specific heats. For small perturbations, $(p, \underset{\sim}{u}, P)$ about the static state $(\rho_0, \underset{\sim}{U}, P_0)$,

$$\rho_t + \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{u}) = 0$$

$$(\rho_0 \underset{\sim}{u})_t = -\nabla p + \rho \underset{\sim}{g}$$

$$P_t + \underset{\sim}{u} \cdot \nabla P_0 = c^2(\rho_t + \underset{\sim}{u} \cdot \nabla \rho)$$

where $c^2 = \gamma P_0/\rho_0$

These equations are combined to obtain a wave equation

$$\underset{\sim}{u}_{tt} = \underset{\sim}{\nabla}\left(c^2 \underset{\sim}{\nabla} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{g}\right) + c^2 \underset{\sim}{\nabla}\left\{\ln \frac{P_0}{\rho_0^{1/\gamma}}\right\} \nabla \cdot \underset{\sim}{u}$$

For an isothermal atmosphere

$$p_0 \propto \rho_0 \propto e^{-z/H}$$

where $H = RT_0/g$ is the scale height.

Note that

$$c^2 = \gamma RT_0$$

and

$$\underset{\sim}{\nabla} \ln \frac{P_0}{\rho_0^{1/\gamma}} = \left(\frac{1-\gamma}{\gamma H}\right)\hat{z}$$

are both independent of (x, y, z). We seek wave solutions,

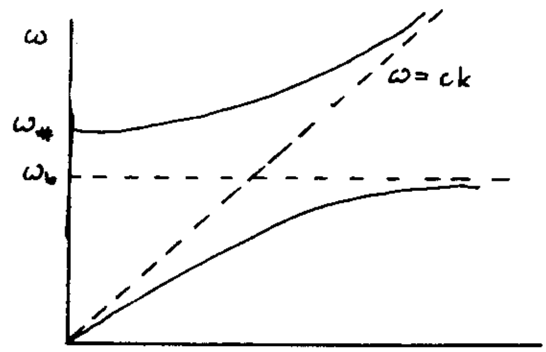
$$\underset{\sim}{u} = \underset{\sim}{u}(z)\, e^{i(\ell x + my + nz - \omega t)}$$

and find the approximate dispersion relations

$$\omega_{ac}^2 = c^2\left(k^2 + n^2 + \frac{1}{4H^2}\right)$$

$$\sigma_g^2 = \left(\frac{\gamma-1}{\gamma H}\right)\left(\frac{gk^2}{k^2+n^2+\frac{1}{4H^2}}\right)$$

$$k^2 = \ell^2 + m^2$$



where

$$\omega_* = \frac{\gamma g}{4H} = \frac{c^2}{4H^2}$$

$$\omega_b = \frac{(\gamma-1)g}{\gamma H}$$

Convective instabilities are associated with gravity modes where $\omega_g^2 < 1$. Choosing a long time scale, relative to ω_* is one way to separate out gravity waves. This is roughly equivalent to letting

$$\gamma \to \infty, \quad \omega_* \to \infty, \quad \omega_b \to \frac{g}{H}$$

For the polytropic atmosphere

$$\frac{dT}{dz} = -\frac{T_0}{z_0}$$

the dispersion curves look something like

The time scales for the two modes in general do not separate, hence it is not easy to filter out one kind of motion and study the other. If the temperature profile is

$$T(z) = T_*\left(1 - \frac{z}{z_*}\right)$$

Scale heights can be defined

$$H(z) = H_*\left(1 - \frac{z}{z_*}\right)$$

The static state is

$$p(z) = P_*\left(1 - \frac{z}{z_*}\right)^{m-1}$$

$$\rho(z) = \rho_*\left(1 - \frac{z}{z_*}\right)^{m}$$

Where the polytropic index

$$m = 1 + \frac{z_*}{H_*}$$

In the limit $z/z_* \to 0$ $(m \to \infty)$

$$p(z) \propto \rho(z) \propto e^{-z/H_*}$$

Hence is the limit of large z_*, the atmosphere can be treated locally as isothermal. The atmosphere characterized by

$$p \propto e^{\Gamma}$$

with entropy profile given by

$$\frac{ds}{dz} = (\Gamma - \gamma)\,\frac{d\ln p}{dz}$$

where $\Gamma = \frac{m-1}{m}$ is the polytropic constant. The atmosphere is said to be in convective equilibrium where $\Gamma = \gamma$, i.e., $m = \frac{1}{\gamma - 1}$.

Wave solutions $\underset{\sim}{u} = \underset{\sim}{u}(z)\, e^{i(\omega t - \ell x - my)}$ admit gravity modes (see Lamb, 1931; Spiegel and Unno, 1962) with the dispersion relation

$$\frac{2\mu\omega^2}{(m+2)g} M(\mu+1; m+2; 2k) + \left[\frac{(m+1)\omega^2}{gk} - \frac{(\omega^2 - 2g)}{gk}\right] M(\mu; m+2; 2k) = 0$$

where

$$2\mu = \frac{m+1}{\gamma}\left(\frac{gk}{\omega^2} - \frac{\omega^2}{gk}\right) - \frac{2gm}{\omega^2} + m + 2$$

and M(a;b;c) is the confluent hypergeometric function (see Magnus and Oberhettinger, 1954). The relation is obtained by assuming rigid boundaries at z = (0,1). This law has the property

$$\text{sign}\ (\omega^2) = \text{sign}\ \left(\frac{\gamma}{\Gamma} - 1\right)$$

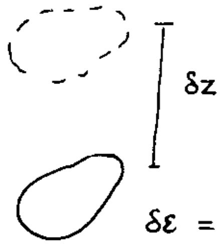
from which it follows $\omega^2 = 0$ for $\Gamma = \gamma$. Hence if the atmosphere is in approximate convective equilibrium, the nongravity modes are suppressed by setting

$$\rho_t = P_t = 0$$

This is the idea behind the anelastic approximation of Charney (see Ogura S. Phillips, 1962):

$$\nabla\cdot(\rho_0 \underset{\sim}{u}) = 0$$

In order to understand how the gravity modes become unstable consider the energetics of a displaced fluid element



$$\delta\varepsilon = g\,\delta z + c_p dT$$

The Schwartzchild criterion for instability is $\delta\varepsilon < 0$ or

$$\frac{dT}{dz} + \frac{g}{c_p} < 0$$

from which using $P = R\rho T$ we obtain

$$\frac{dS}{dz} < 0$$

as a necessary condition for instability. Hence, if the atmosphere is marginally unstable, $dS/dz = 0$, i.e., $\Gamma = \gamma$, and the anelastic approximation can be invoked to filter out the acoustic modes while retaining the convective instability.

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CONVECTIVE EQUILIBRIUM AND THE WARM-UP PROBLEM

Edward A. Spiegel

Convective Equilibrium

Consider a hydrostatic atmosphere of layer thickness z_0 with a linear vertical temperature profile of gradient $\beta = T_*/z_0$. The vertical coordinate z is defined positive downwards, with z = 0 at the top of the layer (Fig. 1).

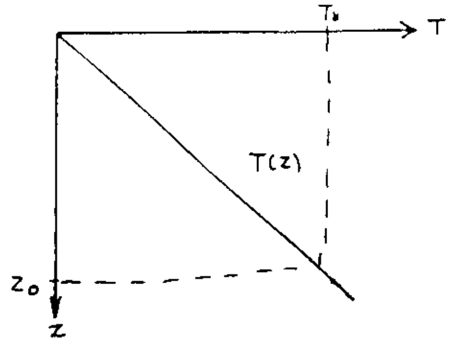

FIGURE 1.

In the static state, the pressure, density, and temperature satisfy

$$\frac{\partial p}{\partial z} = g\rho \qquad \text{hydrostatic equation}$$

$$p = \mathcal{R}\rho T \qquad \text{equation of state for ideal gas}$$

$$T = \beta z \qquad \text{linear temperature profile.}$$

where $\mathcal{R}$ and g are constant.

We can solve for the pressure and density of this <u>polytropic</u> atmosphere as a function of z

$$p = \Pi \left(\frac{z}{z_0}\right)^{m+1}$$

$$\rho = \frac{\Pi}{\mathcal{R}T_*} \left(\frac{z}{z_0}\right)^{m}$$

where Π is the pressure at $z = z_0$ and m, the <u>polytropic index</u>, is given by

$$m = \frac{g}{\mathcal{R}\beta} - 1 = \frac{z_0}{H_*} - 1$$

H_* being the pressure scale height ($H_* = \frac{\mathcal{R}T_*}{g}$). In terms of the <u>polytropic exponent</u>, defined as

$$\Gamma = \frac{m+1}{m}$$

p and ρ are related by an equation of the form

$$p = \text{const} \cdot \rho^{\Gamma}.$$

In a state of convective equilibrium, the temperature gradient is equal to the so-called adiabatic lapse rate

$$\beta = \beta_{\text{adiabatic}} = g/c_p$$

where c_p is the specific heat at constant pressure. In this case

$$m = \frac{1}{\gamma - 1}$$

and

$$\Gamma = \gamma$$

where γ is the ratio of specific heats at constant pressure and volume ($\gamma = c_p/c_v$). Using the Schwartzchild discriminant, we see that for convective equilibrium

$$\frac{1}{T}\left(\frac{dT}{dz} - \frac{g}{c_p}\right) = \frac{1}{c_p}\frac{dS}{dz} \quad \left(\equiv \frac{d\ln\theta}{dz}\right) = 0$$

i.e., the entropy S and the potential temperature θ are constant with height.

The sign of dS/dz gives us information about the stability of a layer. We expect convective instability if $dS/dz > 0$. For an ideal gas with constant specific heats

$$S = c_v \ln(p/\rho^\gamma) \qquad + \text{const}$$

$$= c_v \left(\frac{\Gamma - \gamma}{\Gamma}\right) \ln p \qquad + \text{const} .$$

Thus

$$\frac{dS}{dz} = c_v \left(\frac{\Gamma - \gamma}{\Gamma}\right) \frac{d\ln p}{dz}$$

and for instability Γ must be greater than γ, since $d\ln p/dz > 0$. Just as the conductive heat flux can be written in terms of a thermal conductivity as

$$\underset{\sim}{\mathcal{F}}_{\text{conductive}} = -K \, \nabla T$$

we may conjecture in this case that

$$\mathcal{F}_{\text{convective}} = -K_+ \nabla\theta \propto -K_+ \nabla S$$

where K_+ is a sort of eddy conductivity.

As an example, consider the sun, where the profile of S with distance from the center looks something like:

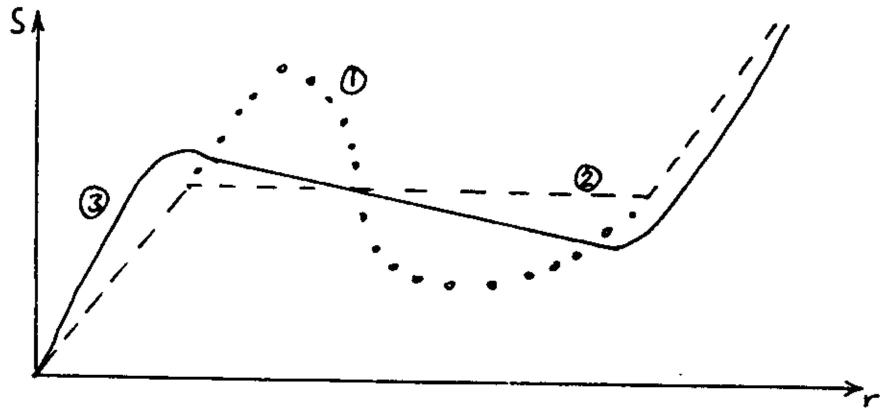

FIGURE 2.

Here we assume that Γ , which is evaluated locally, has a variation with height. Curve 1 is the profile in the absence of convection, computed assuming that radiative diffusion is the only mechanism for heat transport from the interior to the surface. There is a convectively unstable region where $ds/dr < 0$. If convection tends to bring the profile to convective equilibrium we would expect a profile like curve 2, where $ds/dr = 0$ in the former convective zone. The actual profile must be slightly super-adiabatic in order for convection to be maintained (curve 3).

Response Time for Perturbation from Convective Equilibrium - the Warm-up Problem.

Suppose that we change the temperature profile of an atmosphere initially in convective equilibrium by heating it from "the side", that is, we introduce a small perturbation to the temperature profile at time $t = 0$ by uniformly heating the atmosphere from a vertical wall such that

$$T(Z) \rightarrow (1 + \varepsilon)T(Z) = (1 + \varepsilon)\,\beta Z$$

where ε is assumed small. What is the characteristic "warm-up" time τ_w for the atmosphere to reach a new state of equilibrium with this new temperature profile? This warm-up problem is analogous to the spin-up problem in rotating fluids and to the heat-up problem in stably stratified incompressible fluids (Veronis, 1970). We introduce:

a) The dynamical time τ_{dyn} for this problem as the time needed for a sound wave (gravity waves are excluded in convective equilibrium) to propagate vertically across the atmosphere. The scale height is given by

$H_* = z_0/(m + 1)$, and for m of order one, $H_* \sim z_0$. Thus

$$\tau_{dyn} = z_0/c \sim H_*/c$$

where c is the speed of sound ($c^2 = \gamma g H_*$). We anticipate that an atmosphere subject to thermal disturbances alters its hydrostatic structure on a time scale τ_{dyn}.

b) The thermal time of the atmosphere $\tau_{th} \sim z_0^2/\kappa$, where κ is roughly the mean of thermal diffusivity over z, weighted in favor of smaller values.

In the warm-up problem, the region that is at first affected by thermal perturbation is that adjacent to the side wall on which the temperature perturbation is made. Nothing very significant occurs until a time τ_{dyn} passes. In that interval the side wall disturbance has had time to diffuse horizontally and establish a thermal boundary layer whose thickness, δ_{th} is given by

$$\frac{H_*}{c} \sim \frac{\delta_{th}^2}{\kappa}$$

where κ is the thermal diffusivity. We obtain for the boundary layer thickness

$$\delta_{th} \sim \left(\frac{\kappa H_*}{c}\right)^{1/2}$$

This is the analog of the Ekman layer thickness in the spin-up problem.

The physical response of the system to the perturbation may be summarized as follows:

Initially sound waves are excited from the wall. These sound waves are presumably of little dynamical importance, much like the inertial waves which are the initial response in the spin-up problem. In the thermal boundary layer near the wall the vertical extent of the atmosphere locally changes by an amount

$$\delta H_* \sim \delta z_0 = \varepsilon z_0 .$$

This change in thickness is caused by a vertical velocity in the boundary layer which is

$$w \sim \frac{\delta z_0}{H_*/c} \sim \frac{\varepsilon z_0 c}{H_*}$$

From continuity this vertical motion must also engender a horizontal suction velocity u in the direction of the wall

$$\frac{u}{\delta_{th}} \sim \frac{w}{H_*} + \frac{w}{z_0} = \frac{w}{H_*}\left(1 + \frac{H_*}{z_0}\right)$$

It is this induced horizontal circulation which ultimately determines the warm-up time. Consider a ring of material of radius a with its axis perpendicular to the wall (Fig. 3). The ring feels the effect of the suction long before it feels the effect of the direct diffusion of heat from the wall. As the ring is drawn towards the wall it expands isentropically

$$\delta S = 0 = \delta\left(\ln \frac{p}{\rho^{\gamma}}\right)$$

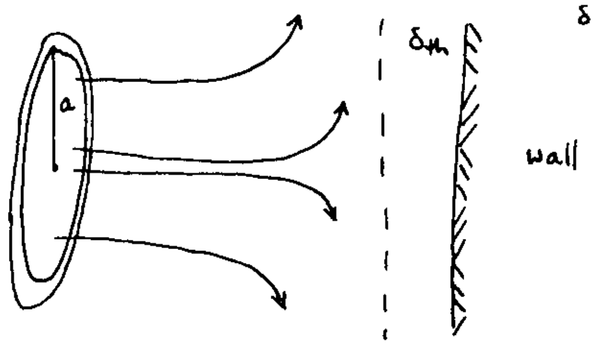

FIGURE 3.

Using the ideal gas law and the relationship $\delta T/T = \varepsilon$, we have for the relative change in the ring's density

$$\frac{\delta\rho}{\rho} \sim \frac{\varepsilon}{\gamma-1}$$

The relative change in the radius of the ring is given by

$$\frac{\delta a}{a} \sim \frac{\delta\rho}{\rho}$$

Assuming a is of the same order as z_0

$$\delta a \sim \frac{\varepsilon z_0}{\gamma - 1}$$

The characteristic warm-up time for this problem is given by the time it takes for the ring to come to the new equilibrium dictated by the altered sidewall condition. Since the distance traversed by the ring is of the same order as δa

$$\tau_w \sim \frac{\delta a}{u}$$

Substituting in the previously derived expressions for δa and u

$$\tau_w \approx \left(\frac{1}{\gamma-1}\right)\left(\frac{z_0}{z_0 + H_*}\right)\left(\frac{H_*^3}{c\kappa}\right)^{1/2}$$

For $z_0 \sim H_*$, τ_w is approximately given by the geometric mean between the dynamical time $\tau_{dyn} = H_*/c$ and the thermal time for the whole atmosphere $\tau_{therm} = H_*^2/\kappa$. Generally

$$\tau_{dyn} < \tau_w < \tau_{therm}$$

Just as in the spin-up problem, a characteristic response time calculated assuming only simple diffusion without consideration of the induced circulation is much too long -- the induced circulation brings the fluid to equilibrium much more quickly than diffusion alone could.

As an example of where this distinction between response times is important, consider a two-layer model of the sun consisting of the base of the convective zone and the top of the inner radiative zone. The two layers adjust to small perturbations on different time scales. In the upper turbulent layer τ_{therm} is about one month and τ_{dyn} is about half an hour, yielding a warm-up time t_w of a few hours. In the lower diffusive layer the thermal time is very long -- about 10^6 years, while τ_{dyn} is again about one-half hour. Thus τ_w is on the order of a few years in the subconvective layer. There is now a strong suspicion that the luminosity of the sun is not constant -- it appears to vary in consort with the solar cycle of 11 years. Why should we observe a phenomenon with a period of a few years? Both the dynamical and thermal times of the diffusive layer are of completely different orders of magnitude, but the warm-up time fits the bill, and we should expect the sun to be very responsive on this time scale. A model has been proposed where "ropes" of hot material are pulled up from below by magnetic effects, engendering thermal changes in solar magnetocline which manifest themselves in a variable solar luminosity with a period of about τ_w.

Finally, we note that several other characteristic times may be defined. Just as the thermal time z_0^2/κ represents the time for heat to diffuse across a layer of thickness z_0, the viscous time z_0^2/ν represents the time needed for momentum diffusion in a fluid with kinematic viscosity ν. The convective time is the time it takes for a parcel of fluid to "fall" across the layer

$$\tau_{\text{convective}} \sim (z_0/g\varepsilon)^{1/2}$$

where $g\varepsilon$ is the reduced gravity due to the buoyancy force. In the next lecture we will study the anelastic mode when we move slightly off convective equilibrium. This "quasi-anelastic" approximation is analogous to the quasi-geostrophic approximation. Following Ogura and Phillips (1962) (see references in ":Small Nondissipative Motions and the Anelastic Approximation") , this relevant time scale for the perturbation equations is

$\tau_{\text{convective}}$. As long as $\tau_{\text{convective}} \gg \tau_\omega$ we are near convective equilibrium, and sound waves are effectively filtered from the system.

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NOTES SUBMITTED BY
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QUASIANELASTICITY

Edward A. Spiegel

In many geophysical situations, we are trying to model a system in which rapid vertical mixing is taking place. We would like to scale the equations of fluid motion so as to use the nearness of the system to convective equilibrium. The result of such a scaling is the anelastic approximation.

We will, for simplicity, restrict ourselves to an ideal gas whose thermodynamic properties -- K, the thermal conductivity; μ , the viscosity; Cp, the specific heat at fixed pressure; R, the gas constant normalized with the molecular weight; γ , the ratio of specific heats, and local gravity g -- are constant. Our first task is to identify static equilibrium states of the gas, which must obey:

$$\frac{dp}{dz} = \rho g$$
$$\frac{d^2T}{dz^2} = 0$$
$$p = \rho \mathcal{R} T$$

With the geometry of Figure 1, we can see that the solutions to the above must have a linear temperature gradient and thus must be polytropic:

$$p = \Pi\, p_1(z)$$
$$\rho = P\, \rho_1(z)$$
$$T = \Theta\, T_1(z)$$

where Π , P , and Θ are the pressure, density, and temperature at the bottom of the gas, and

$$
\begin{aligned}
&\Pi = \mathcal{R}\, P\, \Theta \\
&\Theta = g z_0 / m \mathcal{R} \\
&p_1(z) = (z/z_0)^{m+1}, \quad \rho_1(z) = (z/z_0)^{m}, \quad T_1(z) = z/z_0
\end{aligned}
\tag{1}
$$

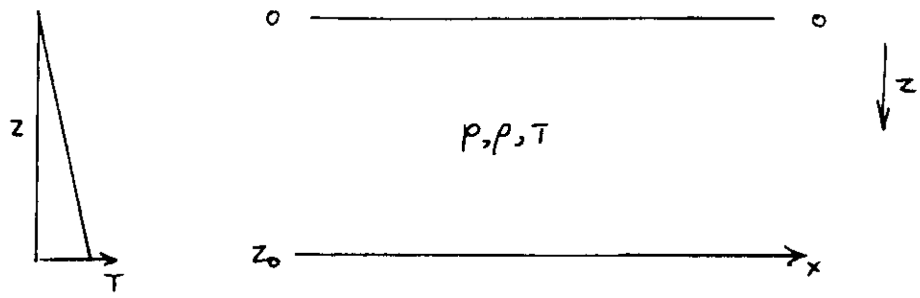


FIGURE 1. THE Geometry of the Gas

To isolate the gravitationally driven motions we assume small vertical entropy gradients, that is, that the atmosphere is nearly in convective equilibrium. This condition,

$$\frac{d}{dz} \ln \frac{p}{\rho^{\gamma}} \approx 0$$

implies

$$m \approx \frac{1}{\gamma - 1}$$

and thus that vertical temperature gradient is near its adiabatic value

$$\beta_c = \left.\frac{dT}{dz}\right|_{m = \frac{1}{\gamma-1}} = \frac{g}{\mathcal{R}} \left(\frac{\gamma - 1}{\gamma}\right)$$

Imagine a real gas in motion. We would like to consider the motion as a perturbation away from an adiabatic $(m = 1/\gamma - 1)$ static state described by (1). Which state do we pick? Out of the two parameter family, we can pick, for instance, the static state which preserves the average mass and average basal pressure of a fluid column:

$$\Pi = \overline{p(x, z_*, t)}$$

$$P \int_{*}^{z_0} \rho_1(z)\, dz = \overline{\int_0^{z_*} \rho(x, z, t)\, dz}$$

where p and ρ are the pressure and density in the real gas, z_* its depth, and —— is a horizontal and temporal average. If

$$\mathcal{E} = \left| \frac{p(x, z_r, t) - \Pi\, p_1(z)}{\Pi\, p_1(z)} \right| \ll 1$$

the above atmosphere is near convective equilibrium.

The time scale of convective motions then becomes $\tau_{\text{conv}} = \left(\frac{z_0}{g\varepsilon}\right)^{1/2}$ which is much longer than the other time scales in the problem:

$$\tau_{\text{conv}} \sim \varepsilon^{-1/2} \gg \tau_{\text{warm}} \sim \varepsilon^{-1/4} \gg \tau_{\text{sound}} \sim \varepsilon^{0}$$

Thus, on the convective time scale other adjustments are effectively instantaneous, sound waves just being an ignorable background sea of noise.

Consider the small, order ε, perturbations to the basic state as given by (1)

$$\begin{aligned} p &= \Pi\,(p_1 + \varepsilon \tilde{p}) \\ \rho &= \mathcal{P}\,(\rho_1 + \varepsilon \tilde{\rho}) \\ T &= \Theta\,(T_1 + \varepsilon \tilde{T}) \end{aligned} \tag{2}$$

and nondimensionalize the kinematic variables with respect to the reference length z_0 and the reference time $\tau_{\text{conv}} = (z_0/g\varepsilon)^{1/2}$.

Upon the neglect of relatively small terms, the momentum equation becomes

$$\rho_1 \frac{D\underset{\sim}{u}}{Dt} = -\left(\frac{\gamma-1}{\gamma}\right) \underset{\sim}{\nabla} \tilde{p} - \tilde{\rho}\,\hat{z} + \left(\frac{\nu_r^2}{z_0^3 g\varepsilon}\right)^{1/2} \nabla \cdot \underset{\approx}{T} \tag{3}$$

here the nondimensional viscous stress tensor is given by

$$\underset{\approx}{T}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

and the effective kinematic viscosity $\nu_* = \frac{\mu}{\mathcal{P}}$. Similarly, the entropy equation is simplified to

$$\rho_1 \frac{D\tilde{T}}{Dt} - \left(\frac{\gamma-1}{\gamma}\right) \alpha T_1 \frac{D\tilde{p}}{Dt} = \left(\frac{\kappa}{\nu_*}\right)\left(\frac{\nu_r^2}{z_0^3 g\varepsilon}\right)^{1/2} \frac{1}{\rho c_p} \nabla^2 \tilde{T} + \left(\frac{\nu_r^2}{z_0^3 g\varepsilon}\right)^{1/2} \Phi \tag{4}$$

where Φ is the nondimensionalized mechanical dissipation. The substitution of (2) and the rescaling of the mass conservation equation and the equation of state is straightforward. In equations (3) and (4) some nondimensional constants appear. The factor $\frac{\gamma-1}{\gamma}$ is just a number of $O(1)$. The constant $Gr = z_0^3 g\varepsilon/\nu_r^2$, called the Grashof number, also appears and must be assumed to be $O(1)$ so that viscous dissipation can be included. For thermal diffusion to be retained we must also have the Prandtl number $\sigma = \nu_*/\kappa$ of $O(1)$. Only when these two numbers are $O(1)$ are the convective, thermal and viscous time-scales of the same order of magnitude.

Having established which terms are neglected in the quasi-anelastic approximation we can now return to the dimensional form of the equations. In their general form, allowing for the variation of some of the thermodynamic properties of the gas, they read

$$\nabla \cdot (\rho_c \underset{\sim}{u}) = 0$$

$$\rho = \rho_c \left[1 - \alpha_c (T - T_c) + K(T_c)(p - p_c)\right]$$

$$\rho_c \frac{D\underset{\sim}{u}}{Dt} = -\nabla p - \rho g \hat{z} + \nabla \cdot \underset{\approx}{T}$$

$$\rho_c C_{pc} \frac{DT}{Dt} - \alpha_c T_c \frac{Dp}{Dt} = \underset{\sim}{\nabla} \cdot \left[K_c \underset{\sim}{\nabla} (T - T_c) + (K - K_c) \underset{\sim}{\nabla} T_c\right]$$
$$+ \Phi + (\alpha - \alpha_c) T_c \frac{\partial p_c}{\partial z} - \alpha_c (T - T_c) \frac{\partial p_c}{\partial z} + \left\{ -C_{pc}(\rho - \rho_c) w \frac{dT_c}{dz} \right.$$
$$\left. - \rho_c (C_p - C_{pc}) w \frac{\partial T_c}{\partial z} \right]$$

The original equations have essentially been modified by ignoring the effect of the small density deviations from convective equilibrium in the inertia and in mass conservation. The equations have been linearized in T, p and ρ, but the problem is still nonlinear due to advection of momentum, temperature, and pressure.

The Boussinesq Approximation

In many applications the layer of convecting fluid covers only a small fraction of a scale height. This is used in the shallow layer approximation which will be dealt with later. If additionally the horizontal scales are assumed to be of the same scale as the vertical motions, then the Boussinesq approximation follows.

Long horizontal scales can be generated in the Boussinesq approximation*. In such cases the approximation needs to be reexamined. It is important to remember that the Boussinesq approximation may not be a consistent scaling of some convection problems, e.g., convection at large Prandtl number, and even as a model may not have the correct bifurcation structure.

The Boussinesq approximation follows from

$$\delta = \frac{d}{H_0} \ll 1$$

where d is the convecting layer depth and H_0 is the minimum scale height in the layer (note that $\hat{\underset{\sim}{z}}$ is now vertically upwards). All the previously defined timescales are replaced by changing the atmospheric height z_0 by the layer depth d, the amplitude of the motion being determined by a balance between the new convective and dissipative timescales. Using this assumption we find that

$$\rho_c = \rho_c(0) + O(\delta)$$

and thus ρ_c may be taken to be a constant ρ_0. The pressure perturbations in the layer are also small and can be ignored in the equation of state.

*For example, by constant flux boundary conditions, large scale modulations of small scale structures or in cells much longer in one direction than another.

In the heat equation we find that the dissipation Φ is small compared with the diffusive term (at least for O(1) Prandtl number) and that to leading order

$$\frac{Dp}{Dt} = \frac{Dp_c}{Dt} = -\rho_o g w$$

Thus the Boussinesq version of equation (5) is just

$$\begin{aligned} \rho_o \frac{D\underset{\sim}{u}}{Dt} &= -\nabla p + \rho g \hat{z} + \mu \nabla^2 \underset{\sim}{u} \\ \rho_o c_p \frac{DT}{Dt} &= -\rho_o g w + \underset{\sim}{\nabla}\cdot(K \nabla T) \\ \rho &= \rho_o [1 - \alpha (T - T_*)] \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u} &= 0 \end{aligned} \qquad (6)$$

where T_* is some reference temperature at which $\rho = \rho_o$.

The boundary conditions to be used at z = 0,d are somewhat arbitrary. No simple and physical choice is obvious. Usually used are rigid boundaries:

$$\underset{\sim}{u} = 0 \text{ and } T = \text{given};$$

or stress free boundaries

$$\underset{\sim}{u}\cdot\hat{n} = \underset{\sim}{u}_z \times \hat{n} = 0; \; T = \text{given}.$$

Note that at a rigid boundary the above could be changed to $\underset{\sim}{u} + C \frac{\partial \underset{\sim}{u}}{\partial z} \times \hat{n} = 0$ where c measures the mean free path of the fluid's constituent particles. If an eddy viscosity is used, then perhaps c should be related to the eddy size.

Nondissipative Boussinesq Linear Dynamics.

To simplify the problem even further we consider the inviscid non-diffusive Boussinesq equation (μ = K = 0) for small velocities. This allows the nonlinearity in the velocity to be neglected. The analysis is relevant to the stability of the purely conducting trivial solution.

Let $T_O(z)$ be some initial given temperature field and introduce the dependent variable $\theta = T - T_o$ to replace T. The set of equations (6) reduce to

$$\begin{aligned} \rho_o \frac{\partial \underset{\sim}{u}}{\partial t} &= -\nabla p + g \alpha \theta \hat{z} \\ \frac{\partial \theta}{\partial t} &= \beta w \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u} &= 0 \end{aligned} \qquad (7)$$

where

$$\beta = -\left(\frac{dT_o}{dz} + \frac{g}{c_p}\right)$$

Taking the z-component of the $\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times$ of the first equation in (7) gives the following simple equation

$$\nabla^2 \frac{\partial w}{\partial t} = \nabla_1^2 (g \alpha \theta)$$

Try for a solution of the form

$$\begin{pmatrix} w \\ \theta \end{pmatrix} = e^{\gamma t} f(x,y) \begin{pmatrix} W(z) \\ \Theta(z) \end{pmatrix}$$

and the above equations give the following equation for f :

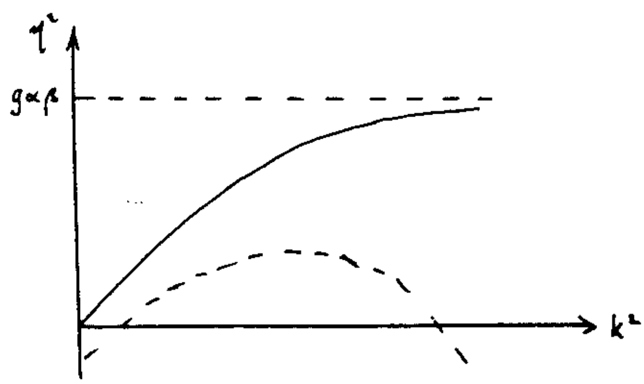
$$\nabla_1^2 f = -k^2 f$$

where k is a measure of the horizontal wave number. This equation guides the choice of the horizontal geometry of convection, the question being: how to tessellate the plane? Usually a planform of regular polygons is preferred.

The expression for the growth rate of the modes is

$$\gamma^2 = \frac{g\alpha\beta k^2}{k^2 + n^2 a^2}$$

where n is the vertical wave number. The solid line in the following figure is a graph of n^2 as a function of k^2:



Notice that in the limit of convective equilibrium ($\beta \to 0$) there is no growth. The upper limit of the growth rate is given by $g\alpha\beta$ instead of O(gH), and is due to geometric constraints rather than the fluid's compressibility. The inclusion of the dissipative terms, which extract energy preferentially from short wave lengths, causes the growth rate γ to be maximized at some finite wave number k, and gives rise to the dotted line of the figure.

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NOTES SUBMITTED BY
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DISSIPATIVE BOUSSINESQ DYNAMICS

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We want to investigate how the inclusion of dissipative terms alters the behavior of the fluid at the onset of convective instability.

We know that for a given horizontal wave number a, there is a value of the Rayleigh number R_0, above which motion on the corresponding horizontal scale becomes possible. Our first task is to determine the relationship between R and a.

The full Boussinesq equations are:

$$\partial_t \underset{\sim}{u} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\rho_0^{-1} \nabla p + g\alpha\theta \hat{z} + \nu \nabla^2 \underset{\sim}{u} \tag{1}$$

$$\underset{\sim}{\nabla} \underset{\sim}{u} = 0 \tag{2}$$

$$\partial_t \theta + \underset{\sim}{u} \cdot \nabla \theta = \beta w + \kappa \nabla^2 \theta \tag{3}$$

where $\underline{u} = (u, v, w)$, the temperature $T = T_{static} + \theta$ and $\beta = \left(\frac{dT_s}{dz} + \frac{g}{c_p}\right)$

At the onset of convective instability we expect the velocity and temperature fluctuations to be small. We may therefore determine the initial time dependence from the linearized equations with the understanding that if we find instability the linearization quickly loses validity. If we omit $\underline{u} \cdot \underset{\sim}{\nabla} \underline{u}$ from (1) and apply $\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times$ we find

$$(\partial_t - \nu\nabla^2)\underset{\sim}{u} = g\alpha \nabla^2 (\theta \hat{\underset{\sim}{z}}) - g\alpha \nabla \left(\frac{\partial\theta}{\partial z}\right) \tag{4}$$

where $\hat{\underline{z}}$ is a unit vector. Equation (4) has the components

$$(\partial_t - \nu\nabla^2)\nabla^2 u = -g\alpha \, \partial_{xz}^2 \theta \tag{5a}$$

$$(\partial_t - \nu\nabla^2)\nabla^2 v = -g\alpha \, \partial_{yz}^2 \theta \tag{5b}$$

$$(\partial_t - \nu\nabla^2)\nabla^2 w = g\alpha \, \nabla_1^2 \theta \tag{5c}$$

while the linearized version of (3) is

$$(\partial_t - \kappa\nabla^2)\theta = \beta w \tag{5d}$$

Before we proceed further we discuss the kinematic boundary conditions:

A number of possibilities are available but it is usual to restrict the choice to:

$$\left.\begin{array}{ll} \text{rigid boundary} & w = \underline{u} \cdot \hat{\underline{z}} = 0 \\ \text{free boundary} & w,\ \partial_z u,\ \partial_z v = 0. \end{array}\right\} \text{ top and bottom}$$

We postpone the choice of the thermal boundary conditions. If we take $\partial_y(5a) - \partial_x(5b)$ and define the vertical vorticity $\zeta = \partial_x v - \partial_y u$ we get:

$$(\partial_t - \nu\nabla^2)\nabla^2\zeta = 0 \tag{6}$$

We can seek solutions of the form:

$$\zeta \propto e^{st} f_\zeta(x,y)\, Z(z)$$

Then we get $\nabla_1^2 f_\zeta = -k^2 f_\zeta$ and

$$[s - \nu(D^2-k^2)](D^2-k^2)\, Z(z) = 0 \tag{7}$$

where

$$\nabla^2 = \nabla_1^2 + D^2, \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad D = \frac{\partial}{\partial z}$$

If $z = 0$ and $z = d$ are the top and bottom, the free B. C.'s give $Z(0)$, $Z(d) = 0$ and we get

$$s = -\nu\left(\frac{n^2\pi^2}{d^2} + k^2\right) \tag{8}$$

which implies that in the linear regime the vertical vorticity decays. Clearly we expect it to couple to other modes at finite amplitude, but for the moment we leave it at that.

In a similar manner we may separate the horizontal structure from (5a) and 5(d) to obtain

$$[\partial_t - \nu(D^2-k^2)][D^2-k^2]\,\hat{w} = -g\alpha k^2 \hat{\Theta} \tag{9a}$$

$$[\partial_t - \kappa(D^2-k^2)]\,\hat{\Theta} = \beta\hat{w} \tag{9b}$$

$$\nabla_1^2 f = -k^2 f \tag{9c}$$

where $\hat{w}$ and $\hat{\Theta}$ are functions of Z and t only.

We choose d as the unit of length, d^2/k as the unit of time and $\Delta T = \beta d$ as the unit of temperature, and let

$$R = \frac{g\alpha\Delta T d^3}{\nu\kappa} \qquad \sigma = \frac{\nu}{\kappa}$$

(the Rayleigh and Prandtl numbers respectively).

The nondimensional equations are

$$(\sigma^{-1}\partial_t - \Delta)\Delta\hat{w} = -R a^2 \hat{\Theta} \tag{10a}$$

$$(\partial_t - \Delta)\hat{\Theta} = \hat{w} \tag{10b}$$

where $\Delta = D^2 - a^2$ and $a = kd$ is the dimensionless horizontal wave number. We still have the choice of thermal B. C. Again we consider two idealizations that are used in the literature for the top and bottom boundaries:

$$\Theta(0) = \Theta(1) = 0 \quad \text{fixed temp.}$$

$$\partial_z \Theta(0) = \partial_z \Theta(1) = 0 \quad \text{fixed flux}$$

corresponding to perfectly or poorly conductive surfaces at $Z = 0,1$. Clearly, for physical applications a combination of the two would be appropriate but for illustration we choose $\Theta = 0$ on top and bottom. Then if we try solutions of the form:

$$\begin{pmatrix} \hat{w} \\ \hat{\Theta} \end{pmatrix} = e^{\eta t} \begin{pmatrix} w \\ \Theta \end{pmatrix}$$

we obtain trigometric functions as the eigenvectors of the operators in (10). This was, in fact, the motivation behind our choice of B. C. Equations (10) give

$$\det \begin{bmatrix} (\frac{\eta}{\sigma} + q^2) & -\frac{Ra^2}{q^2} \\ 1 & -(\eta + q^4) \end{bmatrix} = 0$$

as the dispersion relation for η, where $q_n^2 = n^2\pi^2 + a^2$. This gives

$$\eta = \frac{1}{2} q^2 (1+\sigma) \left[-1 \pm \sqrt{1 + \frac{4\sigma}{(\sigma+1)^2}\left(\frac{R}{R_0} - 1\right)} - 1 \right] \tag{11}$$

and if $\left|\frac{R}{R_0} - 1\right| \ll 1$ the two roots are

$$\eta \simeq q^2 \left(\frac{\sigma}{\sigma+1}\right)\left(\frac{R}{R_0} - 1\right) \tag{12a}$$

$$\eta \simeq -q^2(1+\sigma) \tag{12b}$$

The condition for marginal stability is obtained by setting $\eta = 0$; for $n = 1$ this gives

$$R = R_0 = q^6/a^2 \tag{13}$$

where $q^2 = q_1^2$.

In the R_0 - a plane we get a curve like Figure 1.

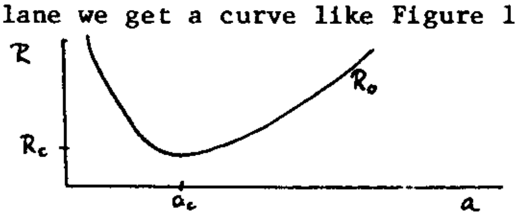


FIGURE 1.
The marginal value of R as a function of horizontal wave number. The minimum (or critical) value, R_c occurs at a_c.

We now want to find how the curve in Figure 1 is modified by a change in the thermal B. C. This time we fix the flux and consider the equations governing the steady state:

$$\Delta^2 w = R a^2 \Theta \qquad (14a)$$

$$\Delta \Theta = -w \qquad (14b)$$

The boundary conditions are $D\Theta = 0$ top and bottom. In effect, this problem was partly solved by Jeffrey, though its meaning was not appreciated until the 1960's by Sani and Hurle, Jakeman and Pike (1967). The essential point is that R has its minimum at a = 0. Since a is small, we rescale W; let $\Omega = W/a^2$. Integrating (14b) from Z = 0 to Z = 1 we find that

$$a^2 \int_0^1 \Theta \, dz = \int_0^1 w \, dz \qquad (15)$$

To zeroth order we find

$$D^2 \Theta_0 = 0$$

$$D^4 \Omega_0 = R_0 \Theta_0$$

This gives Θ_0 = const $\Omega_0 = R_0 \Theta_0 P(z)$ where P(Z) is a 4th order polynomial such that $P'''(Z) = 1$. The next order gives

$$D^2 \Theta_2 = \Theta_0 \left(1 - R_0 P(z)\right)$$

which if integrated from 0 to 1, becomes, on use of (15)

$$R_0 = \left[\int_0^1 P(z) \, dz \right] \Rightarrow R_0 = 5\,!$$

It can be shown that the linear stability curve becomes Figure 2.

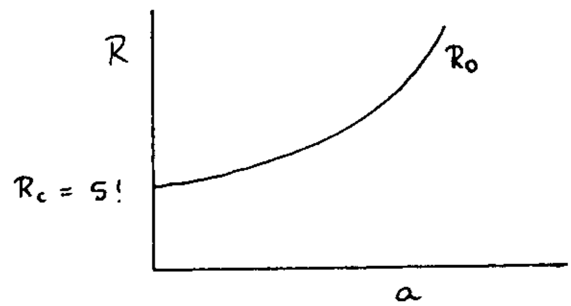


FIGURE 2.

Linear stability calculations are a useful tool to determine the value of R_0 and the initial structure of the motion. They are clearly unsatisfactory when evolved in time as they predict that the amplitude of the motion will grow exponentially without limit. To get a more reasonable picture we must include the nonlinear properties of the equations. These describe the reaction of the stable modes to the exponential growth of an unstable one. We expect these modes to provide, eventually, a way to prevent the unlimited growth of the unstable modes.

If we describe the amplitude of the motion by A we expect that for R just supercritical the evolution of A is governed by the Landau equation

$$\partial_t A = \underset{\text{exp growth}}{\eta A} - \underset{\text{nonlinear int. with virtual modes.}}{\xi A^3}$$

(Terms of order A^2 are omitted as the equations are invariant under $A \to -A$).

We now seek to derive the coefficients of the Landau equation for the case of 2D convection. The velocity is solenoidal and can be written in terms of a stream function:

$$u = (-\psi_x, 0, \psi_z)$$

The equations for ψ and θ become

$$(\partial_t - \sigma\nabla^2)\nabla^2\psi = -\sigma R\theta_x + J(\psi, \nabla^2\psi) \tag{16a}$$

$$(\partial_t - \nabla^2)\,\theta = -\psi_x + J(\psi, \theta) \tag{16b}$$

where $J(f,g) = f_x g_z - f_z g_x$. We fix the temperature concentrate on top and bottom and assume that on the sidewalls of the cells ($x = 0, 2\pi/a$) that there is no heat flux ($\theta_x = 0$) and tangential viscous stress ($\psi_{xx} = 0$). On top and bottom, θ, ψ, $\nabla^2\psi$ all vanish. Let R be slightly above marginal:

$$R = R_0(1 + \varepsilon^2).$$

The natural time variable would be ηt but suppose that $\sigma \sim 1$ and use $\varepsilon^2 t$. We rescale the variables:

$$t \to t/\varepsilon^2, \qquad \psi \to \varepsilon\psi, \qquad \theta \to \varepsilon\theta/\lambda$$

$$R = \lambda^2, \qquad \lambda = \lambda_0 + \varepsilon^2\lambda_2$$

and get

$$\mathbb{L} V = \mathbb{I} \tag{17}$$

where

$$\mathbb{L} = \begin{pmatrix} \nabla^4 & -\lambda_0\partial_x \\ -\lambda_0\partial_x & \nabla^2 \end{pmatrix} \qquad \mathbb{L}^+ = \begin{pmatrix} \nabla^4 & \lambda_0\partial_x \\ \lambda_0\partial_x & \nabla^2 \end{pmatrix}$$

$$\mathbb{I} = \varepsilon\begin{bmatrix} \sigma^{-1} J(\psi, \nabla^2\psi) \\ J(\psi, \theta) \end{bmatrix} + \varepsilon^2 \begin{bmatrix} \sigma^{-1}\partial_t\nabla^2 & \lambda_2\partial_x \\ \lambda_2\partial_x & \partial_t \end{bmatrix} V$$

and

$$V = \begin{pmatrix}\psi \\ \theta\end{pmatrix}$$

We than expand V in powers of ε : $V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2$.
and find

$$\mathbb{I} = \mathbb{I}_0 + \varepsilon \mathbb{I}_1 + \varepsilon^2 \mathbb{I}_2$$

At each order n in powers of ε in order to solve for V_n we must impose

$$(\tilde{V}, \mathbb{I}_n) = 0 \qquad (18)$$

where $\mathbb{L}^{+}\tilde{V} = 0$. This is known as the solvability condition in classical perturbation theory, the removal of resonant terms in the suppression or the suppression of secularities.

By inspection we see that $\mathbb{I}_0 = 0$; also it is easy to deduce

$$\mathbb{I}_1 = \begin{bmatrix}\sigma^{-1} J(\psi_0, \nabla^2\psi_0) \\ J(\psi_0, \theta_0)\end{bmatrix} = \begin{bmatrix} 0 \\ J(\psi_0, \theta_0)\end{bmatrix}$$

$$\mathbb{I}_2 = \begin{bmatrix}\sigma^{-1}\{J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0)\} \\ J(\psi_0, \theta_1) + J(\psi_1, \theta_0)\end{bmatrix} + \begin{bmatrix}\sigma^{-1}\nabla^2\partial_t & \lambda_2\partial_x \\ \lambda_2\partial_x & \partial_t\end{bmatrix} V_0$$

To zeroth order we recover the linear theory result $\mathbb{L}V_0 = 0$.

$$V_0 = \begin{pmatrix}A \sin ax \sin \pi z \\ B \cos ax \sin \pi z\end{pmatrix} \qquad \mathbb{L}V_0 = \begin{pmatrix}\gamma^4 & \lambda_0 a \\ -\lambda_0 a & -q^2\end{pmatrix} V_0 = 0$$

$\Rightarrow \lambda_0 = \gamma^3/a$ choosing the $+ve$ root and B = $-$9A. A = A(t) and is arbitrary thus far. Clearly we also have $\tilde{V} \propto \begin{pmatrix}\sin ax \sin \pi z \\ \cos ax \sin \pi z\end{pmatrix}$. We plug V_0 into our expression for $\mathbb{I}_1$, and get

$$\mathbb{I}_1 = -\frac{A^2}{2}\gamma a\pi \begin{bmatrix} 0 \\ \sin 2\pi z\end{bmatrix}$$

Condition (18) is satisfied for all A and we find

$$V_1 = \begin{pmatrix} 0 \\ c \sin 2\pi z\end{pmatrix} + V_0 \qquad c = \frac{A^2 a \gamma}{8\pi}$$

We now go to the next order and calculate

$$\mathbb{I}_2 = \begin{bmatrix} -\sin ax \sin \pi z \left(\sigma^{-1}\dot{A}\gamma^2 - \lambda_2 A \gamma a\right) \\ \\ -\cos ax \sin \pi z \left(\gamma \dot{A} - \lambda_2 a A + \pi a A C\right) + \pi a\, AC \cos ax \sin 3\pi x \end{bmatrix} \qquad (19)$$

Condition (18) now gives an equation for A, namely

$$\dot{A} - \frac{2a\sigma}{\gamma(1+\sigma)} A + \frac{a^2\sigma}{\gamma(\sigma+1)} A^3 = 0 \qquad (20)$$

This describes the bifurcation of steady solutions from the static one at $R = R_c$. Schematically the result is displaced in Figure 3.

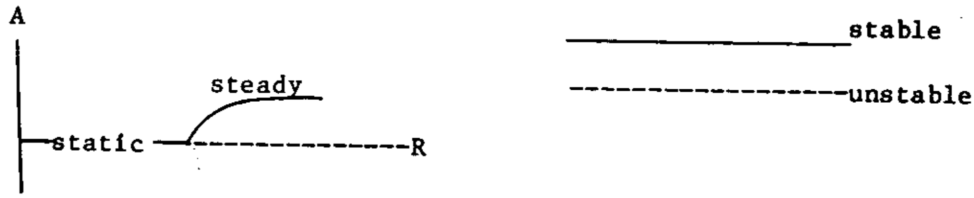
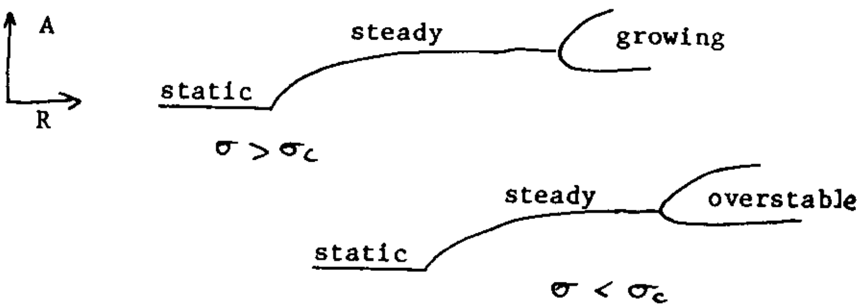


FIGURE 3.

One of the characteristics of this result is that the value at which steady solutions bifurcate is independent of σ . By following the evolution of the steady solution Busse has shown that at $\sigma \to \infty$ two more growing branches bifurcate from the stable steady ones and for σ finite two overstable branches appear. Near σ_c we may expect codimension two behavior (see below).



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CONVECTION ON VERY LARGE HORIZONTAL SCALES

Edward A. Spiegel

1) What are the realistic boundary conditions?

The determination of boundary conditions is not a trivial matter, especially thermodynamic ones. Usually a constant temperature is imposed at the boundaries, thus assuming perfectly conducting walls and inhibiting large scale horizontal motions. Consider the more realistic situation (see Fig. 1) of a liquid layer bordered by two thick conducting plates

k_s plate

k_f liquid

k_s plate

FIGURE 1.

(constant conductivity k_f, k_s respectively). In the plates heat is transported only by conduction, hence, in the static state $\nabla^2\theta_s = 0$, where θ_s is the temperature profile. (We consider only the case of the upper plate; the arguments can be applied to the lower plate as well.) The horizontal dependence can be decomposed in periodic solutions of horizontal wave number a. The vertical axis Z is positive upward, the origin is the lower surface. The solution is

$$\theta_s = A(x,y)\cosh(az) + B(x,y)\sinh(az)$$

where A and B are combinations of sinusoid all with wave length a. The upper boundary (not in contact with the fluid) is a constant temperature, hence

$$A(x,y) = -B(x,y)\tanh(ah)$$

At the interface of the fluid and plate $\theta_s = A(x,y)$. It is natural to assume that flux and temperature are continuous. Thus, we have

$$\theta_s(0) = \theta_f(0)$$

$$k_s \frac{d\theta_s}{dz}(0) = k_f \frac{d\theta_f}{dz}(0)$$

where θ_F represents the temperature in the fluid. Then,

$$A = \theta_F(0)$$

$$k_s a B = k_F \frac{\partial \theta_F(0)}{\partial z}$$

Eliminating A by using the former equation and the boundary condition, we get

$$\theta_F(0) = -\frac{k_F}{k_s}\frac{\tanh(ah)}{a}\frac{d\theta_F(0)}{dz}$$

or

$$\theta_F(0) + B_k \frac{d\theta_F(0)}{dz} = 0 \qquad \text{where} \qquad B_k = \frac{k_F}{k_s}\frac{\tanh(ah)}{a}$$

A similar calculation shows that, at the upper surface of the lower plate, we get

$$\theta_F(0) - B_k \frac{d\theta}{dz} = 0$$

The change of sign results from the presumed antisymmetry of the geometry.

As a result, we are able to generalize the boundary condition at the interface to

$$B\theta^+ + C\theta_z^+ = 0 \quad \text{at the upper boundary}$$

and

$$B\theta^- - C\theta_z^- = 0 \quad \text{at the lower boundary}$$

where B and C are appropriately determined functions of x, y, a and the geometry of the plates. It is interesting to consider two limiting cases of the above equations. That is, when C $\rightarrow$ 0, we obtain the constant temperature condition, and then, when B $\rightarrow$ 0, we obtain constant heat flux case which was already discussed by Hurle et. al. (1967) in the case of the Boussinesq problem (see "Dissipative Boussinesq Dynamics", this volume). The change of boundary condition greatly affects the stability curve (Hurle et. al., 1967). Such a change is schematically illustrated in Figure 2.

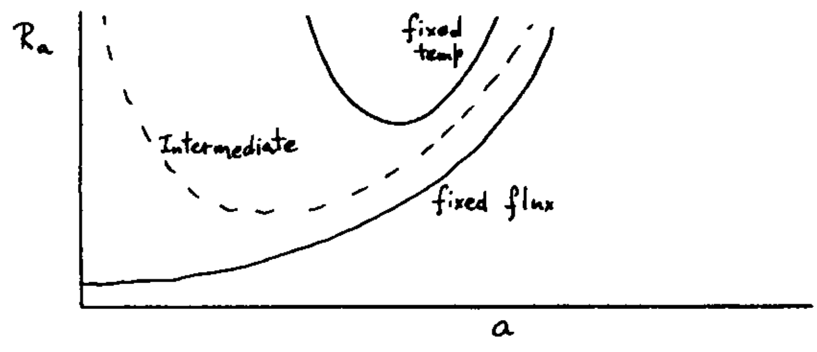

FIGURE 2.

As the boundary becomes a poorer heat conductor the critical Rayleigh number becomes smaller and the wave number of the most unstable mode approaches zero (in the case of fixed heat flux condition it becomes zero). These phenomena can be understood as the thermal penetration of the convection layer into its boundaries. If the boundary is perfectly conducting, the thermal penetration does not occur. However, on the contrary, if the boundary becomes a poorer conductor, it will penetrate deep into the boundary and in case of fixed heat flux, it will do so indefinitely (see Fig. 3).

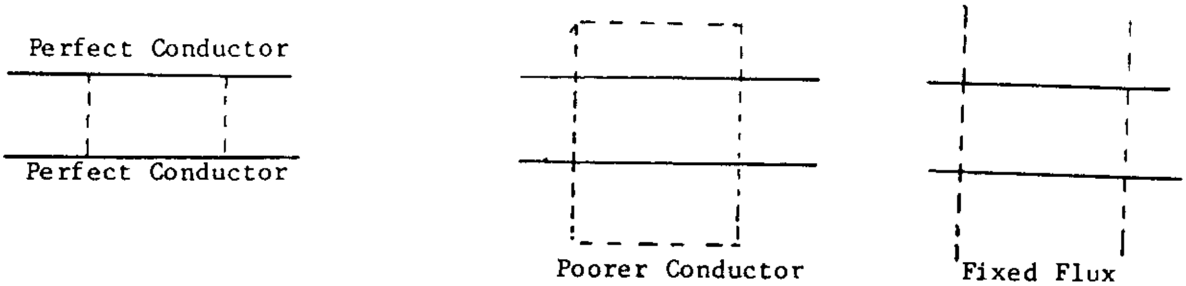

FIGURE 3.

This is the reason why the large horizontal scale motion may be preferred in linear theory with fixed-flux boundary conditions.

2) Some Examples of NonBoussinesq Convection

As an example of nonBoussinesq convection, we consider a fluid containing microorganisms, (such as tetrahymena pyroformis) which are negatively geotactic, i.e., like to swim upward. The density is expressed as,

$$\rho = \rho_0 \left[1 + \alpha [c] \right]$$

where C is the concentration, satisfying the equation of motion,

$$C_t + \nabla \cdot \Big[\underbrace{cU\hat{z}}_{\text{upward swimming}} - \underbrace{\kappa \nabla c}_{\text{dispersion}} + \underbrace{c\,\mathbf{u}}_{\text{advection}} \Big]$$

where $-U$ is the geotactic velocity, $K(c,Z)$ is dispersion coefficient that models random swimming, and $\mathbf{u}$ is the fluid velocity. If $k/U = f(c) \propto c^k$ you get a polytropic solution, $C \propto z^k$. The convection arises as high concentration fluid descends and organisms swim up to maintain the concentration gradient.

Another example of nonBoussinesq convection very similar to bio-convection are: "chromium plated" stars which have an excess of (Cr, Mg, ---) in their spectra. These elements sense the radiation force, and are levitated to the surface, giving rise to a positive concentration gradient, in an analogous fashion to the negative-geotactic microorganisms. A similar instability occurs as shown by Lin (1980, Columbia dissertation).

3) NonBoussinesq Convection on a Very Large Horizontal Scale

When we consider convection having large horizontal scale, we can find that nonBoussinesq terms may become significant. In other words, to guarantee the validity of the Boussinesq approximation, the condition that horizontal scale be much smaller than a critical value is also necessary when the heat flux is fixed on the boundaries (Depassier and Spiegel, 1981).

Consider convection having large horizontal scale. We suppose that the thermal boundary conditions at the upper and lower surfaces are those of fixed flux. We find, from the first order perturbation analysis, that the most unstable mode has the wave length of infinite length (Hurle et. al., 1967).

Assume the equation of state is slightly nonBoussinesq,

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) + \alpha \delta q (T - T_0) \right] \tag{1}$$

q is an arbitrary function with q(0) = 0, and δ is a measure of the deviation from the Boussinesq condition, assumed small. The basic equations are

$$\rho_0 (\underset{\sim}{u}_t + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}) = - \nabla p - g \rho \hat{z} + \mu \nabla^2 \underset{\sim}{u} \tag{2}$$

$$\nabla \cdot \underset{\sim}{u} = 0 \tag{3}$$

$$\rho_0 c_p (T_t + \underset{\sim}{u} \cdot \nabla T) = \underset{\sim}{\nabla} \cdot (k \nabla T) \tag{4}$$

We restrict ourselves to two-dimensional flow, and introduce a stream function ψ. After normalization d (depth of convective layer) as a unit length, $\rho_0 c_p d^2 / k_0$ as a unit of time, ρ_0 as a unit of density, Fd/κ_0 as a unit of temperature (we use conventional symbols of fluid mechanics). We obtain

$$\nabla^4 \psi = R \left[1 - \delta q'(\theta - z) \right] \theta_x + \sigma^{-1} \left[\nabla^2 \psi_t + \frac{\partial(\nabla^2 \psi, \psi)}{\partial(x, z)} \right] \tag{5}$$

$$\nabla^2 \theta = \theta_t - \psi_x + \frac{\partial(\theta, \psi)}{\partial(x, z)} \tag{6}$$

where the primed means $\partial/\partial z$, $R = g\alpha d^4 \rho_0^2 c_p F / k^2 \mu$, $\sigma = \frac{c_p \mu}{k}$ and θ is a perturbed temperature defined by $T = T_0 - z + \theta$. We take the coordinate as indicated in Figure 4.

fixed flux = F

Z = 1/2d ————————————————

Z = 0 -

Z = -1/2d ————————————————

fixed flux = F

FIGURE 4.

The boundary conditions are

$$\begin{aligned} \psi &= 0 \\ \psi_z &= 0 \qquad \text{at} \quad z = \pm \tfrac{1}{2} d \\ \theta_z &= 0 \end{aligned} \tag{7}$$

We rescale the above three equations as follows:

$$\xi = \varepsilon x \quad , \quad s = \varepsilon^4 t \, , \quad \psi = \varepsilon \Psi \quad , \quad \delta = \varepsilon^2 \tag{8}$$

and obtain

$$\theta_{zz} = \varepsilon^2 \left[\theta_{\xi\xi} - \Psi_\xi - \Psi_z \theta_\xi + \Psi_\xi \theta_z + \varepsilon^4 \theta_s \right] \tag{9}$$

$$\Psi_{zzzz} = R \left[1 - \varepsilon^2 g'(\theta - z) \right] \theta_\xi - 2 \varepsilon^2 \Psi_{\xi\xi zz} + \frac{\varepsilon^2}{\sigma} \frac{\partial(\Psi, \Psi_{zz})}{\partial(\xi, z)} \tag{10}$$

Integrating Equation (9) over Z from -1/2 to +1/2 we find

$$\varepsilon^4 \langle \theta_s \rangle = \langle \theta \rangle_{\xi\xi} + \langle \Psi \rangle_\xi - \langle \Psi \theta_z \rangle_\xi \tag{11}$$

where $\langle \cdots \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \, dz$

In the process of integration, we used the temperature boundary conditions at $Z = \pm 1/2$ of (7). Equation (11) is an evolution equation for $\langle \theta \rangle$.
We expand

$$\begin{aligned} \Psi &= \Psi_0 + \varepsilon^2 \Psi_1 + \varepsilon^4 \Psi_2 + \cdots \\ \theta &= \theta_0 + \varepsilon^2 \theta_1 + \varepsilon^4 \theta_2 + \cdots \\ R &= R_0 + \varepsilon^2 R_1 + \varepsilon^4 R_2 + \cdots \end{aligned} \tag{12}$$

and substitute these into Equations (9) and (10). We obtain the lowest order linear equations

$$\begin{aligned} \theta_{0zz} &= 0 \\ \Psi_{0zzzz} &= R_0 \theta_{0\xi} \end{aligned} \tag{13}$$

Considering the boundary conditions, we have

$$\theta_0 = f(\xi, s) \tag{14}$$

Then we can write as

$$\Psi_0 = R_0 f_\xi \mathcal{P}(z) \tag{15}$$

where

$$\mathcal{P}^{(iv)} = 1 \tag{16}$$

At this point we use the kinematic boundary condition of (17), that is,

$$\mathcal{P} = 0 \ , \quad \mathcal{P}' = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2} \tag{17}$$

We have

$$\mathcal{P} = \frac{1}{4!}\left(z^4 - \frac{1}{2}z^2 + \frac{1}{16}\right) \tag{18}$$

Now, the problem is to derive the equation of f by using (11). Putting Equation (15) and (18) into (11) we get fxx = 0 or $R_0 = 6!$ in the leading term. We need the computation of the next order. Selecting the case of $R_0 = 6!$ which is coincident with the critical value of R derived from the Boussinesq approximation, we find

$$f_s + r f_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} - \mu (f_\xi^3)_\xi - F_{\xi\xi} = 0 \tag{19}$$

where $r = R_1/R_0 = R_1/6!$, $F(f) = R_0\left(Q_6 - \frac{1}{2}Q_4 + \frac{1}{12}Q_3\right)$, $\mu = \frac{10}{7}$ and

$K = \frac{17}{462}$. Q is defined as

$$Q_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi_1 \int_{-\frac{1}{2}}^{\xi_1} \cdots \int_{-\frac{1}{2}}^{\xi_{n-1}} g(f - \xi_n)\, d\xi_n \tag{20}$$

If we assume that $(T-T_0) \propto (T-T_0)^2$, that is, the expression which we can expand by the Taylor series, we find that $F = f^2 + \text{const}$. We have the evolution equation as follows:

$$f_s + r f_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} - \mu (f_\xi^3)_\xi - \lambda (f^2)_{\xi\xi} = 0 \tag{21}$$

(note $F = \lambda f^2$).

In this equation, λ represents the nonBoussinesq contribution. In solving equation (21) we must require suitable boundary conditions. They may be $\Theta_x = 0$ and $\Psi_{xx} = 0$ at the end points describing no heat flux from the neighbor and shear stress free. These conditions are converted to

$$\begin{aligned} f_\xi &= 0 \\ f_{\xi\xi\xi} &= 0 \end{aligned} \qquad \text{on} \quad \xi = \pm \frac{\pi}{\alpha} \tag{22}$$

Another important condition derived from (22) is

$$\overline{f_s} = 0 \tag{23}$$

where

$$\bar{f} = \frac{\alpha}{2\pi} \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} f \, d\xi$$

To the order of our present approximation conservation of mass gives

$$\bar{f} = 0 \tag{24}$$

To solve (21) with conditions (22) and (24) we expand f by a new small parameter $\hat{\delta}$, that is,

$$\begin{aligned} f(\xi,s) &= f_0 + \hat{\delta} f_1(\xi, \hat{\delta}^2 s) + \hat{\delta}^2 f_2(\xi, \hat{\delta}^2 s) \\ r &= r_0 + \hat{\delta} r_1 + \hat{\delta}^2 r_2 + \cdots \end{aligned} \tag{25}$$

In leading term we have

$$f_0 = A(\hat{\delta}^2 s) \cos(\alpha \xi) \tag{26}$$

with $r_0 = \kappa \alpha^2$

$$(\text{i.e.}, R = R_m = 6!(1 + \varepsilon^2 \kappa \alpha^2) = R_c(1 + \kappa \alpha^2)) \tag{27}$$

In higher order we find a Landau equation,

$$\dot{A} = \alpha^2 r_2 A - \frac{3}{4}\left(\mu \alpha^4 - \frac{2}{9}\frac{\lambda^2}{\kappa}\right) A^3 \tag{28}$$

where r_2 measures $R-R_m$. Equation (28) states that bifurcation from R = R_m to the subcritical area exist, if the wave number α is less than a certain transition wave number α_0 .

$$\alpha_0^4 = \frac{2\lambda^2}{9\mu\kappa} \quad \text{or} \quad \alpha_0 = 1.43\sqrt{|\lambda|}$$

The rough sketch of this situation is shown in Figure 5.

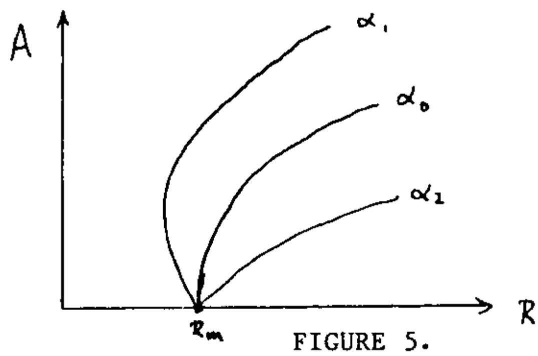

FIGURE 5.

It is clear that, if we do not take the nonBoussinesq term into account (λ = 0), we cannot find such a solution. This may be interpreted as follows:

The small parameter δ may be considered as $\frac{d}{H}$ where H is a vertical characteristic length associated to a nonBoussinesq effect. To guarantee the Boussinesq approximation we require the condition

$$\frac{d}{\alpha_0} \sim \frac{d}{\sqrt{\delta}} \sim \sqrt{Hd}$$

must be small.

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MILDLY NONBOUSSINESQ CONVECTION WITH FIXED FLUX

Edward A. Spiegel

Using the scaling

$$\xi = \varepsilon x \ , \qquad s = \varepsilon^4 t \ , \qquad \psi = \varepsilon \Psi$$

we have arrived at the evolution equation of two-dimensional fixed-flux convection for the leading order temperature perturbation $\theta = f(\ \xi\ , s) + O(\ \varepsilon^2\ )$. It is

$$f_s + \kappa f_{\xi\xi\xi\xi} + r f_{\xi\xi} = \nu\left(f_\xi^3\right)_\xi + \lambda\left(f^2\right)_{\xi\xi} \qquad (1)$$

where the Rayleigh number of the flow is determined by r in the relation

$$R = R_0(1 + \varepsilon^2 r)$$

and K and ν are constants depending upon the boundary conditions (K = 17/462, ν = 10/7 for rigid boundaries, and λ is a parameter measuring the non-

Boussinesq effects. Equation (1) is solved on an interval in ξ of length $2\pi/k$ subject to zero velocity ($\Rightarrow$ no neat heat flux) and no viscous stress boundary conditions at $\xi = 0,\ 2\pi/k$.

Now suppose that we are interested in that part of the parameter regime where the behavior of f is weakly nonlinear. Hence f can be represented as an asymptotic sequence in some small parameter h that measures the nonlinearity in f. Thus substitute

$$\tau = h^2 s$$

$$f = h F_0(\xi,\tau) + h^2 F_1(\xi,\tau) + \cdots$$

$$r = r_0 + h r_1 + h^2 r_2 + \ldots$$

into equation (1), group like powers of h and solve the resultant recursive set of equations.

The first order equation is

$$r_0 F_{0\xi\xi} + \kappa F_{0\xi\xi\xi\xi} = 0$$

with solution

$$F_0 = A(\tau)\cos(k\xi) \quad , \quad r_0 = \kappa k^2$$

where the condition $r_0 = \kappa k^2$ implies that near linear behavior of f is only found for values of r near this particular r_0. The equation at second order is

$$r_0 F_{1\xi\xi} + \kappa F_{1\xi\xi\xi\xi} = r_1 A k^2 \cos(k\xi) - 2\lambda A^2 k^2 \cos(2k\xi)$$

To eliminate the secular forcing term in $\cos(k\xi)$ we must choose $r_1 = 0$. Thus the second order solution is

$$F_1 = B(\tau)\cos(k\xi) - \frac{\lambda A^2}{6\kappa k^2}\cos(2k\xi)$$

Substituting the solutions of the first and second order problems into the third order equation we find that it has the form

$$r_0 F_{2\xi\xi} + \kappa F_{2\xi\xi\xi\xi} = \left[-\dot{A} + r_2 k^2 A + \left(\frac{\lambda^2}{6\kappa} - \frac{3\nu}{4}k^4\right)A^3\right]\cos k\xi$$
$$+ [\cdots]\cos(2k\xi) + \{\cdots\}\cos(3k\xi)$$

where $\dot{A} = A_t$. To eliminate the secular term in the solution for F_3 we must again have that if the coefficient of $\cos(k\xi)$ in the righthand side is zero, hence defining

$$k_0^4 = \left(\frac{2}{9\kappa\nu}\right)\lambda^2 \qquad (2)$$

we end up with the following Landau equation for A(t)

$$\dot{A} = r_2 k^2 A + \left(\frac{15}{14}\right)(k_o^4 - k^4)A^3.$$

From this equation we see that the amplitude of the steady solution is

$$A_{st}^2 = \frac{r_2 k^2}{k^4 - k_o^4}$$

It is then also easy to see that for $k < k_o$ we must have $r_2 < 0$ and the steady solution is unstable; while for $k > k_o$ we must have $r_2 > 0$ and the steady solution is stable, see Figure 1.

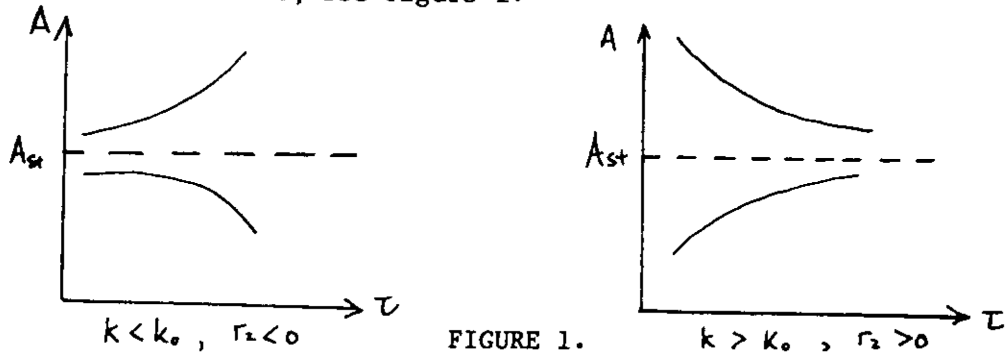


FIGURE 1.

We now turn our attention to the steady nonlinear solutions of the evolution, equation (1). Firstly, note that since $\bar{f}_s = 0$, $\bar{f}$ is constant in time and mass conservation requires that this constant be zero to the accuracy of the Boussinesq approximation. Define new variables ϕ and y and new parameters ρ and α by

$$\phi(y) = \left(\frac{2\nu}{\kappa}\right)^{1/2} \mathrm{sgn}(\lambda)\, f(\xi) \quad , \quad y = k_o \xi$$

$$\rho = \frac{r}{\kappa k_o^2} \quad , \quad \alpha = k/k_o$$

where k_o is defined by equation (2) as before. Substituting into equation (1) and integrating once we find the following nonlinear ordinary differential equation for ϕ

$$\dddot{\phi} - \frac{1}{2}\dot{\phi}^2 - 3\phi\dot{\phi} + \rho\dot{\phi} = 0 \tag{3}$$

where $\cdot$ denotes d/dy . Note that there is now no λ dependence appearing explicitly in the problem, it does, however, occur implicitly through the definition of k_o.

Replace ρ by the new parameter P where

$$\rho = \alpha^2 + P$$

then for P near zero equation (3) has solutions that are only weakly nonlinear. In terms of this new parameter the Rayleigh number is given by

$$R = R_o \left[1 + \kappa a^2 + \kappa a_o^2 \, P \right]$$

where $a = \varepsilon k$ and $a_o = \varepsilon k_o$ are wave numbers in the x coordinate. Thus we are looking at solutions for Rayleigh numbers near the marginal $R_m = R_o(1 + \kappa a^2)$ for the given wave number, the difference from R_m is measured by P.

Equation (3) can be integrated twice by introducing the function

$$G(\phi) = \dot{\phi}$$

which transforms equation (3) to the equation

$$\frac{d^2}{d\phi^2} G^2 = G^2 + 6\phi - 2\rho$$

which can be solved to give

$$\dot{\phi}^2 = 2A \cosh\phi + 2B \sinh\phi - 6\phi + 2\rho \qquad (4)$$

where A and B are integration constants. This equation can be written in the form of a particle in a potential well. Let

$$\phi_0 = -\min_y \phi(y) \quad , \qquad \phi_1 = \max_y \phi(y)$$

$$\Phi = \phi_0 + \phi_1$$

$$X(y) = \phi(y) + \phi_0$$

then Φ measures the amplitude of the motion and X represents the "particle's height". Equation (4) can then be written as

$$\tfrac{1}{2} \dot{X}^2 + V(X) = 0$$

where

$$V = 3X + E(\cosh X - 1) + C \sinh X$$

$$C = \left[E(1 - \sinh\Phi) - 3\Phi \right] / \sinh\Phi$$

$$E = \rho + 3\phi_0$$

$$\bar{X} = \phi_0$$

Observe that the shape of the potential well varies as the "energy" E is changed. The outcome is that there are no solutions unless

$$\rho = P + \alpha^2 > -3 \cdot \frac{(\sinh\Phi - \Phi)}{(\cosh\Phi - 1)}$$

This behavior is summed up in Figure 2 where we see that for $\alpha > 1$ there is a supercritical bifurcation to a nonlinear steady solution; for $\alpha < 1$ there is a subcritical bifurcation and the subcriticality is limited by the $\alpha = 0$ asymptote at P = -3.

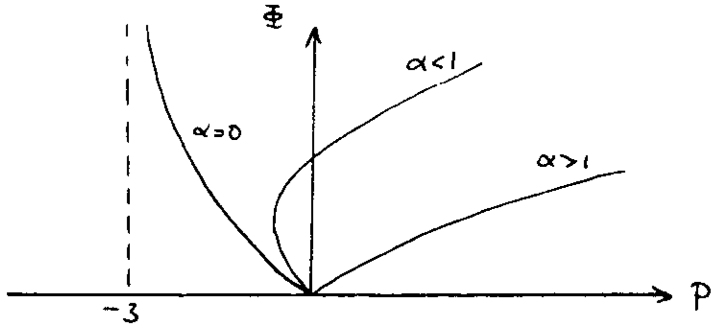
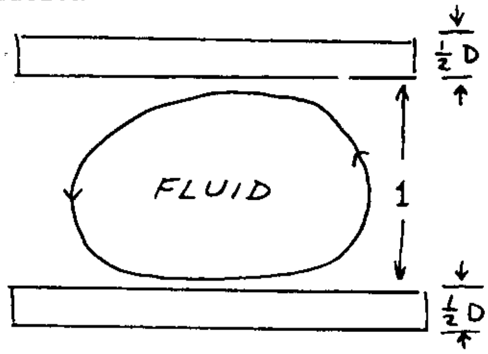


FIGURE 2.
Amplitude of steady nonlinear solutions for various wave number parameters $\alpha = k/k_0$.

Transition to Finite Critical Wave Number

Consider the situation



where the convecting fluid is sandwiched between two conducting plates of thickness $\frac{1}{2}$ D. The fixed flux temperature boundary condition is now applied on the outside of the plates. In the fluid we use the previously derived equations of motion while in the plates there is the purely diffusive problem

$$\frac{\partial \theta^{\pm}}{\partial t} = \kappa_s \nabla^2 \theta^{\pm}$$

We consider the case where the temperature boundary conditions of the fluid and the plates are just

$$\theta^{\pm}_{z} = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2}(1 + D)$$

$$\theta, \kappa \frac{\partial \theta}{\partial z} \text{ continuous on } \quad z = \pm \tfrac{1}{2}$$

As before we can obtain (Poyet, 1979) an evolution equation for the temperature perturbation which is

$$(1+D) f_s + r f_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} = \nu (f_\xi^3)_\xi - \lambda (f^2)_{\xi\xi}$$

But now κ depends on D and $\tau = \kappa_s/\kappa$ (the ratio of the plates to the fluids diffusivity). The relationship is given by

$$\kappa = \left[68 + 213\, D\tau - 154\,(D\tau)^2/\tau - 262\,(D\tau)^3 \right] / 1848$$

and the regions of different behavior in the (D,t) plane are shown in Figure 3.

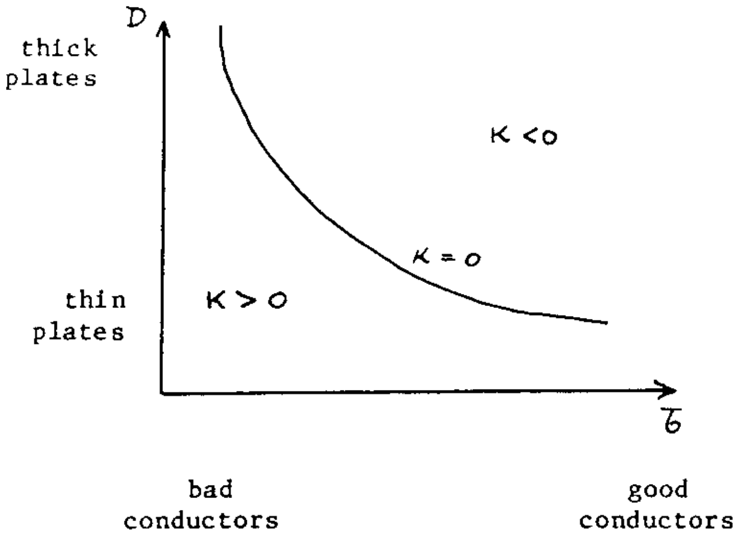
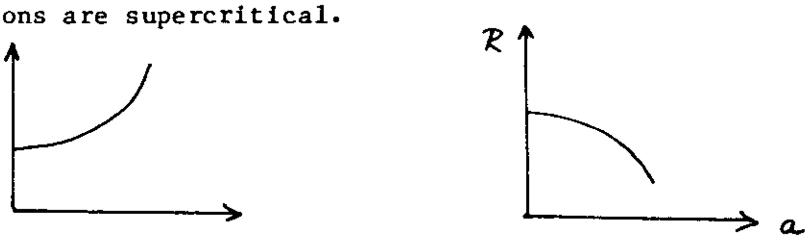


FIGURE 3.

We conclude that in the regime of plate configurations where K > 0 the Rayleigh number versus wave number marginal stability curve looks like Figure 4a, the situation is similar to the previous discussion and so there is a subcritical bifurcation for wave numbers smaller than some critical value. For K > 0 the Rayleigh number, wave number curve looks like Figure 4b and bifurcations are supercritical.



(a) $\kappa > 0$ (b) $\kappa < 0$

FIGURE 4.

The transition between one sort of behavior and the other, when K is small and of order ε^2, is also interesting. We have to go right back to the beginning, set

$$R = R_0 + \varepsilon^4 R_4$$

and scale the time with ε^6 and θ with ε. Then the analogous evolution equation to equation (1), involving a term in $f_{\xi\xi\xi\xi\xi\xi}$, is sixth order instead of fourth order. It turns out that R_4 is positive and so Figure 4b can be redrawn to give Figure 5 for values of K just less than 0.

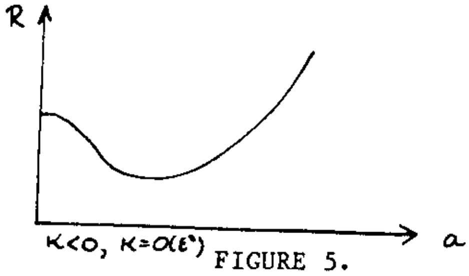


FIGURE 5.

In conclusion we remark that we know that the Boussinesq approximation breaks down on a horizontal length scale $(dH)^{1/2}$ corresponding to some wave number a_B, say. Then the Boussinesq approximation may still be reasonable for situations covered by Figure 5 whenever the wave number of the minimum of the R(a) curve occurs at a wave number bigger than a_B.

Equations of Shallow Convection - One Version

We now derive a variation on the traditional two-dimensional shallow-water equations. The usual shallow-water model assumes a homogeneous fluid (not necessarily water) with constant density ρ_0 , but here we relax this assumption and include a Boussinesq contribution

$$\rho = \rho_0 (1 - \alpha\theta)$$

where α is again the thermal expansivity and θ is the temperature perturbation. Bulk parameters such as α , κ and ν are assumed constant throughout the fluid. Although both friction and baroclinicity will be included in our model, we will assume that to first order both θ and the horizontal velocity component u and independent of Z. The height of the free surface above the bottom is denoted by h(x,t) (see Figure 6).

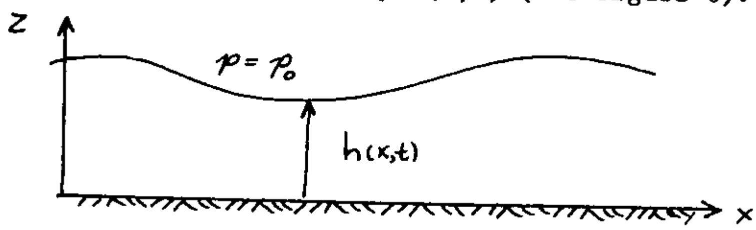


FIGURE 6.

The boundary conditions are that w = 0 at z = 0, $p = p_0$ at z = h(x,t), and that a particle on the surface remains at the surface, i.e., z = h is a material surface. The boundary conditions for and u remain as yet unspecified.

Assuming hydrostatic balance

$$p_z = -g\rho \quad .$$

So, upon integration

$$p = \int_z^h \rho \, dz + p_0$$

where we have used the fact that $p = p_0$ at $z = h$. Substituting for ρ

$$p = p_0 + g\rho_0 \left[h - z - \alpha \int_z^h \theta \, dz \right] .$$

Now using the fact that $z = h$ is a material surface, we have for $z = h$

$$\frac{D}{Dt}(h-z) = h_t + uh_x - w = 0$$

or

$$w(z = h) = h_t + uh_x .$$

Integrating the incompressibility condition $\nabla \cdot \underset{\sim}{u} = 0$ vertically from $z = 0$ to $z = h$

$$\int_0^h (u_x + w_z) dx = 0 ,$$

which implies

$$hu_x + w(z = h) = 0$$

where we have employed our presumption of $O(1)$ absence of vertical shear in u. Using the result for $w(z = h)$,

$$\boxed{h_t + (hu)_x = 0}$$

which is the expression for conservation of mass in the context of shallow-water theory.

In the Boussinesq approximation the x-component momentum equation, with the previously derived expression for the pressure inserted, is

$$\rho_0 \left(u_t + uu_x + wu_z \right) = - g\rho_0 \left[h - z - \alpha \int_z^h \theta \, dz \right]_x + \mu \nabla^2 u$$

This can be rewritten in flux form

$$u_t + (uu)_x + (wu)_z = - g \left[h_x - \alpha \int_z^h \theta_x \, dz \right]$$

$$+ \nu u_{xx} \qquad \text{+ higher order terms.}$$

The higher order terms might include a boundary term in θ if we stray from a constant temperature B. C. on the free surface or a νu_{zz} term if we have a no-slip bottom boundary layer.

Again integrating vertically and ignoring any z-dependence on u and θ

$$hu_t + h(uu_x) + u(h_t + uh_x)$$
$$= - g \left[hh_x - \tfrac{1}{2} \alpha \theta_x h^2 \right] + \nu hu_{xx}$$

or, collecting terms

$$\boxed{(hu)_t + (hu^2)_x = -\left(\tfrac{1}{2} g h^2\right)_x + \nu h u_{xx} + \tfrac{1}{2} g \alpha h^2 \theta_x}$$

The thermodynamic equation can be written

$$\theta_t + \nabla\cdot(\underset{\sim}{u}\theta) = \kappa \nabla^2\theta + \beta w$$

Here

$$\beta = -\left(\frac{dT_0}{dz} + \frac{dT}{dz}\bigg|_{\text{adiabatic}}\right)$$

where the temperature $T = T_0 + \theta$, and $T_0(Z)$ is the static temperature profile. Performing another vertical integration

$$h\theta_t + h(u\theta)_x + \theta(h_t + uh_x)$$
$$= h\kappa\theta_{xx} + \left(\Delta T - \frac{gh}{c_p}\right) w(z=h)$$

Regrouping, we have our third governing equation for h, u and θ

$$\boxed{\begin{aligned}&(h\theta)_t + (hu\theta)_x \\ &\quad = \kappa h\theta_{xx} + \left(\Delta T - \frac{gh}{c_p}\right)(h_t + uh_x)\end{aligned}}$$

The existence of a free surface now allows the propagation of gravity waves - in particular a thermally-induced bore or shock wave might be possible if the fluid is heated from below. Another approach is to use amplitude expansions as in the derivation of Boussinesq or Kortwig-DeVries equations. That has been begun by Depassier in order to study the possible existence of convective solitary waves. The corresponding double-diffusive problem with a free surface does seem to show coherent wave-like solutions at marginal stability.

REFERENCE

Poyet, J.-P., 1980. Dissertation, Astronomy Department, Columbia University.

NOTES SUBMITTED BY
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DOUBLE CONVECTION

Edward A. Spiegel

"If I have seen less far than other men, it is because I have stood behind giants". - Et. al.

In this lecture we will explore the codimension two bifurcation of thermohaline convection, which occurs when the heat and salt gradients are so adjusted as to make a direct and an oscillating instability both close to marginally stable. The idea is to study the mechanism by which competing instabilities bring in complicated dynamical behavior in a situation which permits the use of current analytical techniques.

1. A Mean Field Model

First, let us imagine a qualitative generic physical model of double convection such as semi-convection (Moore and Spiegel, 1966). Imagine a blob of fluid, volume $V(t)$, which has average density $\rho(t)$, temperature $T(t)$ and salinity $\Sigma(t)$, and which is moving vertically in a medium with density $\rho_o(z)$, temperature $T_o(z)$, and salinity $\Sigma_o(z)$ (Figure 1). We make the following assumptions to strip the physics to its bare bones:

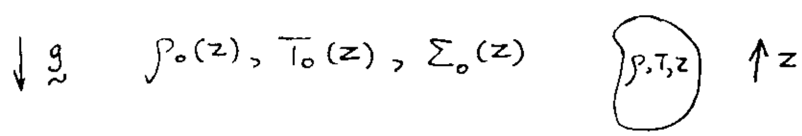

FIGURE 1.

a) Variations of ρ_o, T_o, Σ_o are small enough so that they can be neglected except when their vertical derivatives appear explicitly. So, ρ_o, T_o and Σ_o are "constant with a nonzero derivative".

b) ρ_o, T_o, Σ_o are not influenced by the parcel. We concentrate on the kinematics of the parcel, but at the expense of two more equations for changes in dT_o/dz and $d\Sigma_o/dz$ caused by excess heat or salt transport by the parcel (see "The History and Physics of Bouyancy in Fluids", in this volume), we could get a fully coupled fifth order system analogous to the modal truncation of Veronis (see Weiss in these proceedings), coupling the mean field back to the representative blob.

c) The drag on the parcel is negligible.

d) The rate at which salt diffuses from the blob is much slower than the heat diffusion rate, and so can be neglected (at least for a few thermal diffusion times). This is a good approximation, since κ_s/κ_T 10^{-2} in brine.

The momentum is:

$$\rho V \ddot{z} = -g(\rho-\rho_o)V \tag{1}$$

Now, by assumption (a),

$$\frac{\rho-\rho_o}{\rho} \simeq \frac{\rho-\rho_o}{\rho_o} \simeq -\alpha_T(T-T_o) + \alpha_\Sigma(\Sigma-\Sigma_o)$$

The parcel loses heat by Newtonian cooling:

$$\dot{T} = -q(T-T_o(z)) \tag{2}$$

and keeps the initial salinity:

$$\Sigma = \Sigma(t = 0) = \text{constant}.$$

It is convenient to work with a temperature perturbation $\theta(t, z(t)) = T(t) - T_0(z)$ and a salinity perturbation $S(z(t)) = \Sigma - \Sigma_o(z)$.
From equation (2)

$$\dot{\theta} = -\gamma\theta - \dot{z}\frac{dT_o}{dz}$$

Rewriting the momentum equation (1) in terms of θ and S, we get

$$\ddot{z} = g\left\{-\alpha_T\theta + \alpha_\Sigma S\right\}$$

Elimination of θ gives a third order equation in time,

$$\dddot{z} + \gamma\ddot{z} + g\left\{\alpha_T\frac{dT_o}{dz} - \alpha_\Sigma\frac{d\Sigma_o}{dz}\right\}\dot{z} + g\alpha_\Sigma\gamma S(z) = 0 \qquad (3)$$

Imagine a parcel which, due to strong convection or nonBoussinesq effects has vertically varying gradients of T_0 and Σ_o, as in Figure 2. Then (5) can, because the terms in z and $\dot{z}$ are nonlinear in Z, can produce complicated dynamical behavior since, depending on the parcel position, both, either, or neither direct or oscillating instability can be important, and the blob is kicked between different dynamical regimes.

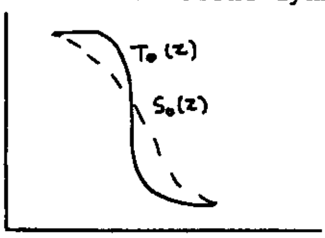

Fig. 2. Typical Mean Temperature and salinity fields.

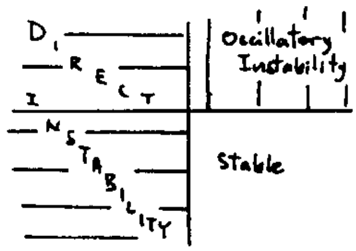

Fig. 3. Stability diagram for ideal thermohaline convection

However, to look at the nature of the instabilities, we restrict ourselves to a constant environment in which dT_o/dz and dS_o/dz are independent of z. Measure z from the level at which the salinity of our blob is equal to the ambient salinity Σ (z). Then

$$S(z) = -z\frac{d\Sigma_o}{dz}$$

and (3) is a homogeneous linear equation for z with eigenmodes $z(t) \propto e^{\lambda t}$, for which

$$\lambda^3 + \gamma\lambda^2 + g\left\{\alpha_T\frac{dT_o}{dz} - \alpha_\Sigma\frac{d\Sigma_o}{dz}\right\} - g\alpha_\Sigma\gamma\frac{d\Sigma_o}{dz} = 0 \qquad (4)$$

A direct instability ($\lambda > 0$, real) is found if

$$\gamma\alpha_\Sigma\frac{d\Sigma_o}{dz} > 0 \quad \Rightarrow \quad \frac{d\Sigma_o}{dz} > 0$$

An oscillating instability (Re $\lambda >$ 0, λ complex) is found for

$$g\alpha_\Sigma g \frac{d\Sigma_0}{dz} < 0 \quad , \quad g\left\{\alpha_T \frac{dT_0}{dz} - \alpha_\Sigma \frac{d\Sigma_0}{dz}\right\} < -g\alpha_\Sigma \frac{d\Sigma_0}{dz} \Rightarrow \frac{d\Sigma_0}{dz}, \frac{dT_0}{dz} < 0$$

In this case, in any situation in which the potential energy can be lowered there is an instability which does it.

Physically, if $\frac{d\Sigma_0}{dz} > 0$, a parcel displaced slowly downward radiates off its excess heat and falls due to its high salinity, causing "salt fingers". If $d\Sigma_0/dz < 0$ and $dT_0/dz < 0$ a blob displaced down feels a strong upward buoyancy force due to the stable density gradient, augmented by the buoyancy produced by the heat diffusing into the parcel, shooting it up faster than it came down to produce an overstable oscillation.

Clearly, when there are very small gradients of T_0 and Σ_0, both instabilities are nearly marginal, and small inhomogeneities in the mean field (perhaps produced by the convection itself) can bounce the parcel between regimes of oscillating and direct instability. z(t) may at different times reflect both of these behaviors. The influence of nonzero viscosity and salt diffusion changes the particular gradients for which the instabilities compete, but the qualitative behavior near the point of competition is much the same.

2. Reconstitution

We will now aim to describe a co-dimensional two bifurcation in the realistic thermohaline case, by suitable recombination of the equations found by an amplitude expansion. The "reconstituted" equation, which gives a complete description of the dynamics near the bifurcation, is a Van der Pol-Duffing equation for the roll amplitude. It can be derived by a variety of means, which do not expand all variables in powers of ε and thus reduce manipulation. One such technique is described in Knobloch and Proctor (preprint).

We examine Boussinesq thermohaline convection in a box with stress-free boundaries (Figure 4). The temperature and salinity are fixed on the top and bottom, while their fluxes through the horizontal boundaries are zero. Define a streamfunction ψ with $u = \psi_z$, $w = -\psi_x$. Nondimensionalize distances with d, times with d^2/κ_T and temperatures and salinities by their difference across the layer. Work with perturbations T and S from the conduction state. Then

$$\begin{aligned}
\nabla^4\psi - R_T T_x + R_S S_x &= \sigma^{-1}\left\{\nabla^2\psi_t + J(\nabla^2\psi, \psi)\right\} \\
\nabla^2 T - \psi_x &= T_t + J(T, \psi) \\
\tau\nabla^2 S + \psi_x &= S_t + J(S, \psi)
\end{aligned} \tag{5}$$

The nondimensional parameters are

$\sigma = \nu/\kappa_T$ (the Prandtl number)

$\tau = \kappa_S/\kappa_T$ (the Schmidt number)

$\mathcal{R}_T = \dfrac{g\alpha_T \Delta T d^3}{\nu \kappa_T}$; $\mathcal{R}_S = -\dfrac{g\alpha_S \Delta S d^3}{\nu \kappa_T}$ (the Rayleigh numbers, with $\alpha_T, \alpha_S > 0$)

Our interest centers on values of R_T and R_S within a small distance, call it $O(\varepsilon^2)$, the codimension bifurcation, that is, of the joint occurrence of the two instabilities. There are two times in the problem an $O(\varepsilon)$ frequency periodic orbit due to the oscillating instability and a slow $O(\varepsilon^2)$ frequency on which dissipation and forcing act. Thus, define

$$t_* = \varepsilon^2 t$$
$$s = \varepsilon t$$

so $F_t = \varepsilon F_s + \varepsilon^2 F_t$

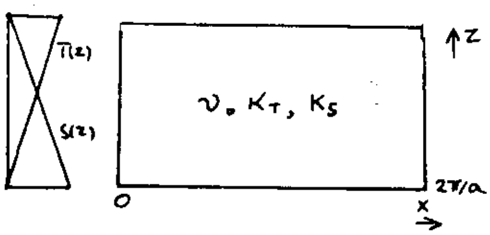

Fig. 4 The physical situation

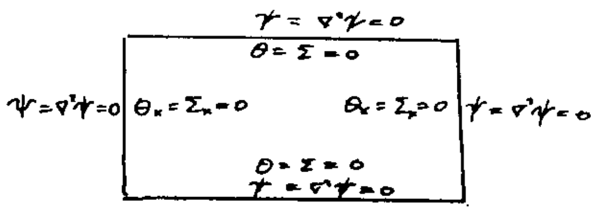

Fig. 5 Boundary condition on the scaled equation

Rescale the variables to symmetrize the linear operator and take into account the weak supercriticality

$$\mathcal{R}_T = M^2 \quad , \qquad \mathcal{R}_S = -\Gamma^2$$

$$T = \frac{R}{M}\theta \quad , \qquad S = \frac{\theta}{\Gamma}\Sigma \quad , \qquad \psi = \varepsilon \Psi$$

and define a state vector

$$\Phi = \begin{pmatrix} \Psi \\ \theta \\ \Sigma \end{pmatrix}$$

The scaled equations are

$$\begin{aligned}
\nabla^4 \Psi - M\theta_x - \Gamma \Sigma_x &= \frac{\varepsilon}{\sigma}\left\{\nabla^2 \Psi_s + \varepsilon \nabla^2 \Psi_{t_*} + J(\nabla^2 \Psi, \Psi)\right\} \\
\nabla^2 \theta - M \Psi_x &= \varepsilon \left\{\theta_s + J(\Psi, \theta) + \varepsilon^2 \theta_{t_*}\right\} \\
\nabla^4 \Sigma + \Gamma \Psi_x &= \varepsilon \left\{\Sigma_s + J(\Sigma, \Psi) + \varepsilon^2 \Sigma_{t_*}\right\}
\end{aligned} \qquad (6)$$

We expand

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \cdots$$
$$\Gamma = \Gamma_0 + \varepsilon \Gamma_1 + \cdots$$
$$M = M_0 + \varepsilon M_1 + \cdots$$

to get a sequence of linear problems, which determine the Φ_n's . The boundary conditions are (Figure 5)

$$\psi = \nabla^2\psi = \theta = \Sigma = 0 \qquad \text{at} \quad z = 0, 1$$

$$\psi = \nabla^2\psi = \Theta_x = \Sigma_x = 0 \qquad \text{at} \quad x = 0, \frac{2\pi}{a}$$

<u>0(1)</u>

The analysis is pivoted on the linear operator:

$$L = \begin{bmatrix} \nabla^4 & -M_0\partial_x & -\Gamma_0\partial_x \\ -M_0\partial_x & \nabla^2 & 0 \\ \Gamma_0\partial_x & 0 & \tau\nabla^2 \end{bmatrix} \tag{7}$$

At 0(1) or system (6) reduces to

$$L\Phi_0 = 0 \tag{8}$$

With the given boundary conditions, we can have solutions with $\Phi_0 \propto \sin n\pi z$. However, when $n \gg 1$, the system will be violently unstable to a $\sin n\pi z$ mode when the $n > 1$ mode is marginally stable, so we restrict ourselves to the ansatz.

$$\Phi_0 = \begin{bmatrix} A_0(s,t_*) & \sin ax & \sin\pi z \\ B_0(s,t_*) & \cos ax & \sin\pi z \\ C_0(s,t_*) & \cos ax & \sin\pi z \end{bmatrix}$$

whence (8) yields a homogeneous linear system for A_0, B_0, C_0 which has a solution only if

$$M_0^2 - \frac{\Gamma_0^2}{\tau} = \frac{\gamma^6}{a^2} \quad , \quad \gamma^2 = a^2 + \pi^2 \tag{9}$$

Then the eigenvector has

$$\Phi_0 = A_0(s,t_*)\begin{bmatrix} 1 & \sin(ax)\sin(\pi z) \\ -\frac{M_0 a}{\gamma^2} & \cos(ax)\sin(\pi z) \\ \frac{\Gamma_0 a}{\tau\gamma^2} & \cos ax \sin(\pi z) \end{bmatrix}$$

Adjoint

For higher order in ε, we require a condition on the righthand side of such that a solution Φ_n exists. Therefore, we inspect the adjoint operator

$$L^{\dagger} = \begin{bmatrix} \nabla^4 & M_0 \partial_x & -\Gamma_0 \partial_x \\ M_0 \partial_x & \nabla^2 & 0 \\ \Gamma_0 \partial_x & 6 & \tau \nabla^2 \end{bmatrix}$$

and require R_n be orthogonal to all solutions φ of $L^{\dagger}\varphi = 0$. The form of Φ_0 hints that any solution φ with vertical wave number unequal to u or horizontal wave number unequal to a will automatically be orthogonal to Φ_n. Thus only one solution φ is important:

$$\underset{\sim}{R}_n \perp \varphi = \begin{pmatrix} \sin ax \sin \pi z \\ \cos ax \sin \pi z \\ \cos ax \sin \pi z \end{pmatrix}$$

with the inner product defined in the natural way. Explicitly

$$0 = \langle \Xi_n \sin ax \sin \pi z \rangle + \frac{M_0 a}{g^2} \langle H_n \cos ax \sin \pi z \rangle + \frac{\Gamma_0 a}{\tau g^2} \langle z_n \cos ax \sin \pi z \rangle \quad (10)$$

where $\langle\ \rangle$ is the spatial integral:

$$\langle f \rangle = \int_0^1 dz \int_0^{\pi/a} dx\, f(x,z)$$

$O(\varepsilon)$

At this order the Jacobian terms and the time dependence of the slow periodic orbit come in to produce an inhomogeneity on the righthand side, and the system (6) can be written

$$L\, \Phi_1 = \begin{pmatrix} \Xi_1 \\ H_1 \\ Z_1 \end{pmatrix}$$

where

$$\begin{pmatrix} \Xi_1 \\ H_1 \\ Z_1 \end{pmatrix} = \begin{bmatrix} \left\{ -\frac{g}{\sigma} A_{0s} + \frac{a^2}{g^2}\left(M_0 M_1 - \frac{\Gamma_0\Gamma_1}{\tau}\right) A_0 \right\} \sin ax \sin \pi z \\ \left\{ M_1 a A_0 - \frac{M_0 a}{g^2} A_{0s} \right\} \cos ax \sin \pi z \\ \left\{ -\Gamma_1 a A_0 + \frac{\Gamma_0 a}{\tau g^2} A_{0s} \right\} \cos ax \sin \pi z \end{bmatrix} + \frac{\varepsilon}{2}\frac{a^2}{g^2} A_0^2 \begin{bmatrix} 0 \\ M_0 \sin 2\pi z \\ -\frac{M_0}{\tau} \sin 2\pi z \end{bmatrix}$$

The resultant solvability condition implies

$$M_0^2 - \tau^{-1}\Gamma_0^2 = -g^6/\sigma a^2 \quad , \quad M_0 M_1 - \tau^{-1}\Gamma_0\Gamma_1 = 0$$

which, combined with the zeroth order condition (9) implies

$$\Gamma_0^2 = \tau^2\left(\frac{\gamma^6}{a^2}\right)\left(\frac{\sigma+1}{1-\tau}\right) + M_0^2 = \left(\frac{\gamma^6}{\sigma a^2}\right)\left(\frac{\sigma+\tau}{1-\tau}\right) \tag{11}$$

The second condition merely tells us that the only allowed O(ε) changes in the Rayleigh numbers are perpendicular to the marginal stability boundary for oscillations. This is because an O(ε) movement away from the doubly degenerate parameters along that boundary caused an oscillation with a frequency O($\varepsilon^{1/2}$), which is not allowed by our scaling. In fact, since we want growth and decay on a time scale O(ε^2), and only an oscillation on time of O(ε), we must choose the supercritically to be O(ε^2), not O(ε), so

$$M_1 = {}_1 = 0$$

We can now solve for Φ_1 . The most general solution is

$$\Phi_1 = \begin{bmatrix} A_1 \qquad\qquad \sin ax \sin\pi z \\ -\frac{aM_0}{\gamma^2}A_1 + \frac{M_0 a}{\gamma^4}A_{0s} \cos ax \sin\pi z \\ \frac{a\Gamma_0}{\tau\gamma^2}A_1 - \frac{\Gamma_0 a}{\tau^2\gamma^4}A_{0s} \cos ax \sin\pi z \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{a^2}{8\pi\gamma^2}M_0 A_0^2 \sin 2\pi z \\ \frac{a^2}{8\pi\tau^2\gamma^2}\Gamma_0 A_0^2 \sin 2\pi z \end{bmatrix} \tag{13}$$

where A_1x is the coefficient of the O(ε) contribution to the homogeneous linear problem

O(ε^2)

One can compute from (6) the O(ε^2) inhomogeneity. Thus

$$L\,\Phi_2 = \begin{bmatrix} \Xi_2 \\ H_2 \\ Z_2 \end{bmatrix}$$

where we can calculate with tedious algebra

$$\Xi_2 = \left(-\frac{\gamma^2}{\sigma}A_{1s} - \frac{\gamma}{\sigma}A_{0t_*} + \frac{a^2}{\gamma^2}Q_{21}A_0\right)\sin ax \sin\pi z$$

$$\begin{aligned} H_2 = {} & \left(\frac{M_0 a}{\gamma^4}A_{0ss} + M_2 a A_0 - \frac{M_0 a}{\gamma^2}A_{0t_*}\right)\cos ax \sin\pi z \\ & + \frac{\pi M_0 a^2}{2\gamma^4}\left(-\left(1+\frac{\gamma^2}{2\pi^2}\right)A_{0s}A_0 + \gamma^2 A_0 A_1\right)\sin 2\pi z \\ & + \frac{M_0 a^3}{4\gamma^2}A^3 \cos ax \sin\pi z \cos 2\pi z \end{aligned}$$

$$\begin{aligned} Z_2 = {} & \left(-\frac{\Gamma_0 a}{\tau^2\gamma^4}A_{0ss} - \Gamma_2 a A_0 + \frac{\Gamma_0 a}{\tau\gamma^2}A_{0t_*}\right)\cos ax \sin\pi z + \frac{\pi\Gamma_0 a^2}{2\tau^2\gamma^4}\left(\left(1+\frac{\gamma^2}{2\pi^2}\right)A_{0s}A_0\right. \\ & \left. - \frac{\gamma^2}{\tau}A_0 A_1\right)\sin 2\pi z - \frac{\Gamma_0 a}{4\tau^2\gamma^2}A_0^3 \cos ax \sin\pi z \cos 2\pi z \end{aligned}$$

We have defined the symbol:

$$Q_{mn} = M_0 M_n - \frac{\Gamma_0 \Gamma_m}{\tau^n}$$

The solvability condition for the existence of Φ_2 is

$$Q_{03} A_{0ss} + 2 Q_{21} \gamma^4 A_0 - \frac{1}{8} a^2 \gamma^2 Q_{03} A_0^3 = 0 \tag{14}$$

Note this is a <u>conservative</u> equation. To leading order, energy exchanging effects on the slow time t_* do not enter the equation, because A_0 moves around the orbits so much faster than it crosses them that the latter effect is subjugated to the next order. This system is not structurally stable, since the addition of infinitesimal dissipation destroys most of the periodic orbits of (14), so we must go to next order to recover the dissipation and get a true picture of the behavior near the bifurcation.

So, we compute Φ_2 and use it back in (6) for the O(ε^3) behavior. To save space, its precise form will be omitted.

<u>O(ε^3)</u>

The equation to be solved is

$$L \Phi_3 = \begin{pmatrix} \Xi_3 \\ H_3 \\ Z_3 \end{pmatrix}$$

We will only use the solvability condition, so we expand Ξ_3, H_3, Z_3 in Fourier components. The only resonant term is $\sin \pi z \begin{pmatrix} \sin ax \\ \cos ax \\ \cos ax \end{pmatrix}$, so we isolate the coefficient of this term.

$$\begin{pmatrix} \Xi_3 \\ H_3 \\ Z_3 \end{pmatrix} = \begin{pmatrix} \Xi_3^{(11)} \sin ax \sin \pi z \\ H_3^{(11)} \cos ax \sin \pi z \\ Z_3^{(11)} \cos ax \sin \pi z \end{pmatrix} + \text{nonresonant terms}$$

Knowing Φ_2 , we may calculate the inhomogeneous term. We find

$$\Xi_3^{(11)} = \frac{a^2}{\gamma^2} Q_{21} A_1 - \frac{a^2}{\gamma^4} Q_{22} A_{0s} - \frac{\gamma^2}{\sigma} A_{2s} - \frac{\gamma^2}{\sigma} A_{1t_*}$$

$$H_3^{(11)} = M_2 a A_1 - \frac{M_0 a}{\gamma^2} A_{2s} - \frac{1}{\gamma^2} H_{2s}^{(11)} + \frac{M_0 a}{\gamma^2} A_{1t_*} + \frac{M_0 a}{\gamma^2} A_{0st_*} + \pi a \left[H_2^{(02)} A_0 + H_1^{(02)} A_1 \right]$$

$$Z_3^{(11)} = -\Gamma_2 a A_1 + \left(\frac{\Gamma_0 a}{\tau\gamma^2}\right) A_{2s} - \frac{1}{\tau\gamma^2} Z_{2s}^{(11)} + \frac{\Gamma_0 a}{\tau^2\gamma^2} A + \frac{\Gamma_0 a}{\tau^2\gamma^2} A_{0st_*}$$
$$+ \bar{u} a \left[Z_2^{(01)} A_0 + Z_1^{(02)} A_1 \right]$$

$H_m^{(kl)}$, $Z_m^{(kl)}$ are the coskaxsinl π y components of H_m and Z_m, which can be deduced from our earlier expressions for H_m and Z_m, m = 1,2. After gruesome agony we deduce the solvability condition

$$Q_{03} A_{1ss} + 2\gamma^4 Q_{21} A_1 - \frac{3a^2\gamma^2}{8} Q_{03} A_0^2 A_1 = \frac{1}{\gamma^2} Q_{04} A_{0sss}$$
$$+ 2\gamma^2 Q_{22} A_{0s} - \frac{Q_{04}}{8} a^2 \left(4 + \frac{\gamma^2}{2\pi^2}\right) A_0^2 A_{0s} - 2 Q_{03} A_{0st_*} \tag{15}$$

Now, here we can take two tacks. First, we can find the condition on the slow time dependence of A_0 that results from insisting that A_1 remain O(1). To do this, we multiply (15) by A_{0s} and average over a large number of periods of A_0. Let P(t*) be the period of the fast oscillation of A_0. Then

$$\frac{1}{n}\int_0^{nP} \left[Q_{03} A_{1ss} + 2\gamma^4 Q_{21} A_1 - \frac{3a^2\gamma^2}{8} Q_{03} A_0^2 A_1 \right] A_{0s}\, ds$$

$$= \frac{1}{n}\int_0^{nP} \left[\frac{1}{\gamma^2} Q_{04} A_{0sss} A_{0s}\, ds + 2\gamma^2 Q_{22} A_{0s}^2 - Q_{04} \frac{a^2}{8}\left(4 + \frac{a^2}{2\pi^2}\right) A_0^2 A_{0s}^2 - 2 Q_{03} A_{0st_*} \right] ds \tag{16}$$

Integrate the left-hand side by parts to get some boundary terms, which if A_1 remains bounded are $O(1/n)$ and an integral of the $O(\varepsilon^2)$ solvability condition. Thus, the left-hand side is $O(1/n)$. The right-hand side is simplified by substituting the $O(\varepsilon^2)$ solvability condition to get

$$A_{0sss} = -\left(\frac{2Q_{21}}{Q_{03}}\right)\gamma^4 A_{0s} + \frac{1}{8} a^2 \gamma^4 (A_0^3)_s \tag{17}$$

The equation (16) now reads

$$O\left(\frac{1}{n}\right) = \frac{1}{n}\cdot n \cdot \left\{ \int_0^P \left[2\gamma^2\left(Q_{22} - \frac{Q_{21}}{Q_{03}} Q_{04}\right) A_{0s}^2 - Q_{04}\frac{a^2}{8}\left(1 + \frac{\gamma^2}{2\pi^2}\right) A_0^2 A_{0s}^2 \right] ds \right.$$
$$\left. - \int_0^P Q_{03} A_{0s}^2\, ds \right.$$

or, taking the limit $n \to \infty$,

$$Q_{03}\frac{\partial}{\partial t_*}\int_0^P A_{0s}^2\, ds = 2\gamma^2\left\{Q_{22} - \frac{Q_{21}}{Q_{03}} Q_{04}\right\}\int_0^P A_{0s}^2\, ds - Q_{04}\frac{a^2}{8}\left(1 + \frac{a^2}{2\pi^2}\right)\int_0^P A_0^2 A_{0s}^2\, ds \tag{18}$$

This equation gives a recipe for the slow effect of dissipation and forcing on the orbit (and can be used to find the orbit which is asymptotically approached for large t_*).

An alternate way of using (15) is to use it in conjunction with the solvability condition for A_0, (14), to get an improved amplitude equation for the total amplitude $A = A_0 + \varepsilon A_1$. While it is not essential for small ε, the use of (18) to eliminate A_{osss} from (14) puts the equation in a Van der Pol-Duffing form and reduces the order of the equation to the minimum order necessary to describe the small ε physics. The time derivative is also resummed by defining the total time derivative, in term of a time $\tau = \varepsilon t$

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial s} + \varepsilon \frac{\partial}{\partial t_*}$$

We add (14) to ε (15), using (17), to derive the "reconstituted" equation:

$$A_{\tau\tau} + \frac{2\gamma^4 Q_{21}}{Q_{03}} A - \frac{1}{8} a^2 \gamma^4 \left[A_0^3 + 3\varepsilon A_0^2 A_1\right] = \varepsilon \left[\frac{2\gamma^{?}}{Q_{03}}\left(Q_{22} - \frac{Q_{21}}{Q_{03}} Q_{04}\right) - \frac{a^2}{8} \frac{Q_{04}}{Q_{08}}\left(1 + \frac{\gamma^2}{2\pi^2}\right) A_0^2\right] A_{0\tau}$$

But

$$A_0^3 + 3\varepsilon A_0^2 A_1 = (A_0 + \varepsilon A_1)^3 + O(\varepsilon^2)$$

and

$$A = A_0 + O(\varepsilon)$$

so

$$A_{\tau\tau} + \varepsilon \left[\frac{2a^4}{Q_{03}}\left[Q_{04}\frac{Q_{21}}{Q_{03}} - Q_{22}\right] + \frac{a^2}{8}\frac{Q_{04}}{Q_{03}}\left(1 + \frac{\gamma^2}{2\pi^2}\right) A^2\right] A_\tau + 2 Q_{21} \gamma^4 A - \frac{1}{8} a^2 \gamma^2 Q_{03} A^3 = O(\varepsilon^2) \tag{19}$$

All the dependence of A on the slow time can be incorporated in $\frac{\partial}{\partial \tau}$. (19) is a "post-asymptotic equation" insofar as it combines two orders of the equation, and in fact contains both time scales of the problem. It is proposed that for ε not small, that (19) is the most robust description of the dynamics, even though it now neglects corrections of $O(\varepsilon^2) \sim O(1)$. The reason is that it represents the normal form for the co-dimension two bifurcation of thermohaline convection, and is structurally stable to the addition of higher order terms, in the sense that small perturbations make no qualitative changes in the phase plane dynamics.

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NOTES SUBMITTED BY
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RECONSTITUTION METHOD FOR A SIMPLE EXAMPLE AND THE LONG BUOYANCY WAVE

Edward A. Spiegel

Reconstitution Method for a Simple Example.

Let us consider the third order differential equation

$$\dddot{z} + \ddot{z} + (\alpha - a z^2)\dot{z} + \beta z - b z^3 = 0 \qquad (1)$$

Linear theory: $z \sim e^{\eta t}$

$$\eta^3 + \eta^2 + \alpha\eta + \beta = 0$$

According to the values of the parameters, two types of instability occur for the solution $z = 0$

- Direct instability or stationary bifurcation (pitchfork due to symmetry $z \Rightarrow -z$)
  $\beta = 0 \quad , \quad \eta^3 + \eta^2 + \alpha\eta = 0 \quad \Rightarrow \quad \eta = 0$
- Overstability or Hopf bifurcation

$$\alpha = \beta = \omega^2 > 0 \qquad \eta^3 + \eta^2 + \omega^2(\eta + 1) = 0 \;\Rightarrow\; \eta = \pm i\omega \;\; (+\, \eta = -1)$$

Now the interesting point is that we can choose the values of the parameters such that the two types of instability occur almost simultaneously.

- Codimension two bifurcation

$$\alpha = \beta = 0 \qquad \eta^3 + \eta^2 = 0 \;\Rightarrow\; \eta^2 = 0 \;\; (+\, \eta = -1)$$

near the degenerate situation $\alpha = \beta = 0$ the characteristic polynomial is written as

$$\left(\eta + 1 + O(\alpha)\right)\left(\eta^2 + (\alpha - \beta)\eta + \beta\right)$$

such that we can expect that the dynamics near the bifurcation is then governed by a second order differential equation whose linear part is written obviously as

$$\ddot{z} + (\beta - \alpha)\dot{z} + \beta z = 0$$

The problem we want to solve is to find the pertinent or asymptotically relevant nonlinear terms of this second order differential equation.

Fortunately, normal form theory tells us what the generic terms are for the interaction between two instabilities described by the characteristic polynomial

$$\eta^2 + \varepsilon_1 \eta + \varepsilon_2 = 0$$

The answer is

$$\ddot{z} + (\varepsilon_1 + \tilde{a} z^2) \dot{z} + (\varepsilon_2 + \tilde{b} z^2) z = 0 \tag{2}$$

Now we try to find the correct amplitude expansion method to derive (2) from (1).

Scaling

$$\alpha = \varepsilon^2 \mu \qquad\qquad \beta = \varepsilon^2 \nu$$

$$z = \varepsilon x \qquad\qquad \tau = \varepsilon \omega t$$

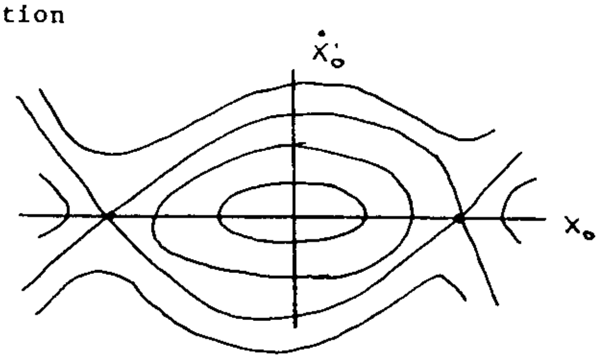
$$x = x_0 + \varepsilon x_1 + \cdots$$

$$\omega = \omega_0 + \varepsilon \omega_1 + \cdots$$

so at the first order we get

$$\omega_0^2 \ddot{x}_0 + \nu x_0 - b x_0^3 = 0 \tag{3}$$

This is the Duffing equation



At this order the absence of dissipative terms does not allow us to determine the solutions. In other words this equation (Hamiltonian system) is not structurally stable in the set of dissipative dynamical systems. In some sense this equation is qualitatively false but for short enough times, it describes qualitatively the behavior of solutions of (1) for α, β small.

At this stage one alternative is to pick solutions of this equation and try to use the next order to determine their stability to higher order effects. In fact, we are not interested in this alternative which allows us only to determine the asymptotic solution. We want to find the correct or generic amplitude equations. At the second order we get

$$\omega_0^2 \ddot{x}_1 + \nu x_1 - 3b x_0^2 x_1 = - \omega_0^3 \dddot{x}_0 - \omega_0 (\mu - a x_0^2) \dot{x}_0 \tag{4}$$

Reconstitution Procedure.

Step 1. Remove the resonances using a solvability condition for any initial condition; X_0 is a given periodic orbit of the Duffing problem of period P.

Now it is easy to verify that if we multiply (4) by X_0 and integrate over the period P the left-hand side vanishes identically. This gives the solvability condition. We get at this order

$$\omega_0^2 = \frac{\int_0^P (\mu - a x_0^2)\, \dot{x}_0^2 \, d\tau}{\int_0^P \ddot{x}_0^2 \, d\tau}$$

Step 2. In general the right-hand side of the second order equation (4) contains higher derivative in X_0. We use the first order equation to compute these derivatives

$$-\omega_0^3 \dddot{X}_0 = \nu \omega_0 \dot{x}_0 - 3 b x_0^2 \dot{x}_0$$

Then (4) becomes

$$\omega_0^2 \ddot{x}_1 + \underbrace{\nu \dot{x}_1}_{\nu x_1} - 3b x_0^2 x_1 = \left[(\nu-\mu) + (a-3b) x_0^2 \right] \omega_0 \dot{x}_0 \tag{5}$$

Step 3. Let

$$x = x_0 + \varepsilon x_1$$

where

$$x \doteq x(\varepsilon t)$$

and form

$$(3) + \varepsilon(5)$$

we get

$$\ddot{x} + \varepsilon \left[(\mu-\nu) - (a-3b) x^2 \right] \dot{x} + \nu x - b x^3 = 0 \tag{6}$$

This is the first order reconstitution of the asymptotic sequence of equation for small amplitudes. This gives the same equation as the equation obtained using the normal form procedure. The reconstitution method in the case of thermohaline convection gives the same kind of amplitude equation

$$\ddot{A} + \varepsilon \left[\kappa + \xi A^2 \right] \dot{A} + \lambda A - \zeta A^3 = 0 \tag{7}$$

λ and κ are given by

$$(R_T)_2 \quad \text{and} \quad (R_S)_2 = \frac{R_{T,S} - R^c_{T,S}}{\varepsilon^2}$$

ξ and ζ depend on the Prandtl numbers. In the phase space the divergence of the flow associated to (7) is given by

$$\nabla \cdot \underline{u} = \frac{\partial \dot{A}}{\partial A} + \frac{\partial \dot{B}}{\partial B} \qquad \text{where} \quad B = \dot{A}$$

$$= -\varepsilon (\kappa + \xi A^2)$$

Solutions for small ε using the averaging nondissipative case (Duffing)

$$\ddot{A}_0 + \lambda A_0 - \xi A_0^3 = 0$$

integral:

$$E(A_0, \dot{A}_0) = \frac{1}{2}\dot{A}_0^2 + \frac{1}{2}\lambda A_0^2 - \frac{1}{4}\xi A_0^4$$

Let $\mathcal{E} = E(A, \dot{A})$

$$\dot{\mathcal{E}} = -\varepsilon(\kappa + \xi A^2)\dot{A}^2$$

Let

$$\bar{\mathcal{E}} = \frac{1}{P}\int_0^P \mathcal{E}(\tau)\, d\tau$$

and $A_0 = A_0(\varepsilon, \bar{\mathcal{E}})$ we get $\dot{\bar{\mathcal{E}}} = -\varepsilon f(\bar{\mathcal{E}})$. This equation gives the selection of a given orbit due to the dissipation term

$$-\varepsilon(\kappa + \xi A^2)\dot{A}$$

Long Buoyancy Waves.

Motivation: Study long buoyancy waves in the two-dimensional thermohaline convection with fixed flux boundary conditions.

Basic Idea: As we have seen in the previous lecture, the thermohaline convection near the doubly degenerate situation with fixed temperature and salinity boundary conditions is described in terms of a second order ordinary differential equation for the most unstable mode (finite wave length): the Van der Pol-Duffing equation obtained using the reconstitution amplitude expansion method. In the fifth lecture we heard that fixed-flux boundary conditions favor large scale motions. The idea here is to derive a second order partial derivative equation using ordinary amplitude expansions, describing a wave packet with small wave number, in the case of the thermohaline convection with fixed-flux boundary conditions. This equation will describe long buoyancy waves. As we shall see, the waves produced by such an equation are in some sense degenerate (lack of dissipation as in the Duffing equation). The next lecture will be a derivation of a "structurally stable" wave equation including dissipation using the reconstitution method.

Equation of the Two-Dimensional Thermaline Convection.

$$(\partial_t - \sigma\nabla^2)\nabla^2\psi + \sigma R\,\partial_x\theta + \sigma\tau S\,\partial_x\Sigma = J(\psi, \nabla^2\psi) \qquad \text{stream funtion equation}$$

$$(\partial_t - \nabla^2)\theta - \partial_x\psi = J(\psi, \theta) \qquad \text{temperature equation}$$

$$(\partial_t - \tau\nabla^2)\Sigma - \partial_x\psi = J(\psi, \Sigma) \qquad \text{salinity equation}$$

where R, S are the Rayleigh numbers
where σ, τ are the Prandtl numbers

Boundary Conditions.

$$z = \pm \tfrac{1}{2} \quad \begin{cases} \partial_z \theta = \partial_z \Sigma = 0 & \text{flux fixed} \\ \psi = \partial_z^2 \psi = 0 & \text{stress free but not deformable boundaries} \end{cases}$$

Scaling and Scaled Equations.

$$R = R_0 + \varepsilon^2 R_2$$

$$S = S_0 + \varepsilon^2 S_2$$

R_0, S_0 are the values of the Rayleigh number which render the fluid neutrally stable and R_2, S_2 are arbitrary

$$\psi = \varepsilon \tilde{\psi} , \quad \tilde{x} = \varepsilon x , \quad \tilde{t} = \varepsilon^3 t$$

We now rewrite the equation dropping the tildes

$$\begin{aligned} \psi_{zzzz} = {} & R\theta_x - \tau S \Sigma_x + \varepsilon^2 \left[-2\psi_{xxzz} + \sigma^{-1}(\psi_z \psi_{zzx} - \psi_x \psi_{zzz}) \right] \\ & + \frac{\varepsilon^3}{\sigma} \psi_{tzz} + \varepsilon^4 \left[-\psi_{xxxx} + \sigma^{-1} (\psi_z \psi_{xxx} - \psi_x \psi_{xxz}) \right] \\ & + \varepsilon^5 \sigma^{-1} \psi_{txx} \end{aligned}$$

$$\theta_{zz} = \varepsilon^2 (\psi_x - \theta_{xx} + \psi_z \theta_x - \psi_x \theta_z) + \varepsilon^3 \theta_t$$

$$\tau \Sigma_{zz} = \varepsilon^2 (\psi_x - \tau \Sigma_{xx} + \psi_z \Sigma_x - \psi_x \Sigma_z) + \varepsilon^3 \Sigma_t$$

Now we expand ψ, θ, Σ in power of ε

$$\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots$$

$$\theta = \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \cdots$$

$$\Sigma = \Sigma_0 + \varepsilon \Sigma_1 + \varepsilon^2 \Sigma_2 + \cdots$$

The boundary conditions are

$$\theta_{nz} = \Sigma_{nz} = \psi_n = \psi_{nzz} = 0 \qquad at \quad z = \pm \tfrac{1}{2}$$

Perturbation Expansion.

- 0th Order.

$\psi_0 = (R_0 f - \tau S_0 g)_x P(z)$, where P(Z) is a given polynomial in Z

$$\Theta_0 = f(x,t)$$
$$\Sigma_0 = g(x,t)$$

f and g are arbitrary funtions to be determined.

- 1st Order.

$$\Psi_1 = (R_0 f_1 - \tau S_0 g_1)_x \, P(z)$$
$$\Theta_1 = f_1(x,t)$$
$$\Sigma_1 = g_1(x,t)$$

f_1, g_1 are arbitrary functions

- 2nd Order.

The solvability condition arises at this order and gives a relation between R_0, S_0

$$(R_0 - S_0) = 5!$$

This determines a critical value for the total Rayleigh number (direct instability). We have at this order a relation between f and g

$$f = \tau g$$

We have for Ψ_2, Θ_2, Σ_2

$$\Psi_2 = \left[R_0 f_2 - \tau S_0 g_2 + (R_2 - S_2) \right]_x P(z) + P_2(z) f_{xxx} + Q_2(z)_x f_{xxx}$$

$$\Theta_2 = f_2(x,t) + H_2(z) f_{xx} + G_2(z) (f_x)^2$$

$$\Sigma_2 = g_2(x,t) + \tau^{-1} H_2(z) f_{xx} + G_2(z) (f_x)^2$$

P_2, Q_2, H_2, G_2 are given polynomial in Z and where f_2 and g_2 are yet two more functions to be found.

- 3rd Order.

The compatability condition gives

$$R_0 - \tau^{-1} S_0 = 0$$

Then we have

$$R_0 = \left(\frac{1}{1-\tau}\right) 5! \quad , \quad S_0 = \left(\frac{\tau}{1-\tau}\right) 5!$$

This gives the condition of the double degeneracy and we have

$$f_{1xx} - \tau g_{1xx} = - \left(\frac{1-\tau}{\tau}\right) f_t$$

We can also compute Ψ_3, Θ_3, Σ_3 but these expressions will be useless to compute the nonlinear wave equation for f obtained in getting the next order.

- 4th Order.

We get as a solvability condition the nonlinear wave equation

$$f_{tt} - \mu\tau f_{xxxx} - \kappa\tau f_{xxxxxx} - \nu\left(f_x\right)^3_{xxx} = 0$$

where

$$\mu = (R_2 - S_2)/5!$$

and κ, ν are given numerical constants. This equation is a nonlinear wave equation whose properties we are at present trying to understand.

Long Buoyancy Waves.

When the amplitude of f is infinitesimal the evolution equation may be linearized and it has a solution of the form

$$f = e^{\eta t} \cos kx$$

This gives us

$$\eta^2 = \tau k^4 (\mu - \kappa k^2)$$

so we have instability whenever

$$\mu \geq \mu_0 = \kappa k^2$$

if the situation is only slightly unstable, we can once again make an amplitude expansion.

Scaling and Unscaled Equations.

$$\mu = \mu_0 + \left(\frac{\lambda}{\tau}\right) \delta^2 \qquad \text{is an arbitrary parameter}$$

$$f = \delta F \quad , \quad s = \delta t$$

We get

$$\kappa\tau \left(F_{xxxxxx} + k^2 F_{xxxx}\right) = \delta^2 \left\{ F_{ss} - \lambda F_{xxxx} - \nu (F_x)^3_{xxx} \right\}$$

We expand again

$$F = F_0 + \delta F_1 + \delta F_2 + \cdots$$

- 0th Order.

$$F_{0\,xxxxxx} + k^2 F_{0\,xxxx} = 0$$

with the solution

$$F_0 = X(s)\cos(kx) + Y(s)\sin kx$$

- <u>1st Order</u>.

The solvability condition (orthogonally to sin and cos of the righthand side) gives two coupled equations for X and Y.
We define

$$X = A\cos\Phi$$

$$Y = B\sin\Phi$$

and get for these variables

$$\ddot{A} - A\dot{\Phi}^2 - \lambda k^4 A - \tfrac{3}{4}\nu k^5 A^3 = 0$$

and

$$A\ddot{\Phi}^2 + 2\dot{A}\dot{\Phi} = 0$$

hence

$$\dot{\Phi} = b/A^2$$

where b is arbitrary. We get the equation for the amplitude

$$\ddot{A} - (b/A^2) - \lambda k^4 A - \tfrac{3}{4}\nu k^5 A^3 = 0$$

This has the integral

$$E = \tfrac{1}{2}\dot{A}^2 + V(A)$$

where

$$V = \tfrac{1}{2}\left(\frac{b^2}{A^2} - \lambda k^4 A^2 - \tfrac{3}{8}\nu k^5 A^4\right)$$

and E is a constant. Solutions may be expressed in elliptic functions, but it is instructive simply to look at plots of the amplitude and phase in the following figure, here for $b = .001$, $\lambda = -2$ and $k = 1$.

Bound solutions of this system exist only for negatives E and those may be expressed in terms of elliptic functions. The waves found are in some sense degenerate. Their amplitudes are arbitrary and determined by initial condition. This is due to the fact that the amplitude equations are nondissipative. The next step will be (next lecture) to use the reconstitution method to get a more generic wave equation.

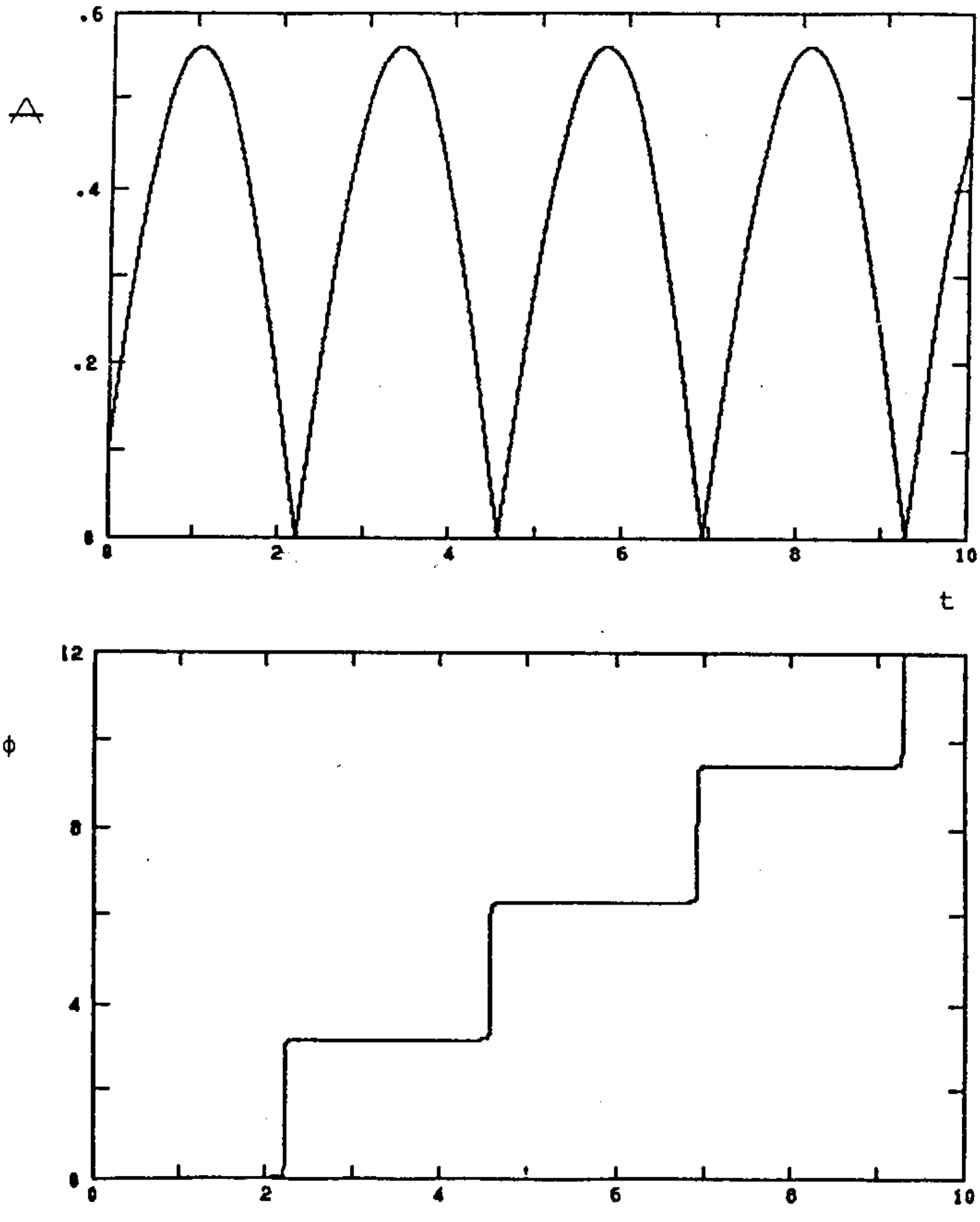


FIGURE 1. Amplitude and Phase as a Function of Time.

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NOTES SUBMITTED BY
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LONG BUOYANCY WAVES AND SOLAR VARIABILITY; A GEOMETRICAL VIEW OF DYNAMICAL SYSTEMS

Edward A. Spiegel

1. A Model for Stellar Variability.

One of the aims of this course has been to indicate a convective explanation of the magnetic activity cycles of the sun and certain other stars. If we look at a plot of mean annual sunspot number, we observe several important features:

1) There is a definite time scale (one crudely would say periodicity) of about 11 years -- the well-known solar cycle.
2) There is also erratic, nonperiodic behavior.
3) There is intermittency where the solar cycle is apparently turned off, as exemplified by the Maunder minimum of the late 16th century when sunspot abundance was extremely low.

These features have usually been explained as an effect of the solar dynamo and in some sense ultimately are. But we may imagine a model in which the underlying physical mechanism in all of the phenomena of the activity cycle occurs as an offshoot of the main process. We focus on the instability of a thin layer beneath the convective zone. The model permits either monotonic growth or growing oscillations (overstability) of perturbations to a static state, and for low frequencies the nearness of these instabilities in parameter space allows quite complicated dynamics. In the simplified set of third-order equations for magnetoconvection with fixed-flux boundary conditions we will consider, the asymptotic solutions for small but finite perturbations are of the form of nonlinear waves propagating in the meridional direction. The strict Boussinesq equations require an extra term in the heat equation, but we consider a simplified version (Speigel and Weiss, 1981). The period of these waves is determined by the rate at which the toroidal magnetic field is forced down by penetrative convection from the overlying convective zone, and can be matched to the 11-year solar cycle. At this level of simplification we do not yet see any erratic behavior or intermittency, but the extension of the model to triple convection shows the desired chaos.

Since the fixed-flux boundary condition favors large horizontal scales, we will use the 2-D long buoyancy wave model which we developed in the preceding lecture as out starting point.

2. Long Buoyancy Waves.

Picking up where we left off, we recall the evolution equation for the zeroth-order nondimensional temperature perturbation f :

$$f_{tt} - \mu\tau f_{xxxx} - \kappa\tau f_{xxxxxx} - \nu\left[(f_x)^3\right]_{xxx} = 0$$

where τ is the ratio of magnetic to thermal diffusivity, μ is the degree of instability

$$\mu = \frac{(R_2 - S_2)}{5!}$$

and κ and ν are numerical constants. For small amplitude this nonlinear wave equation can be linearized and we obtain a solution of the form

$$f \propto e^{\eta t} \cos kx$$

with the dispersion relation

$$\eta^2 = \tau k^4 (\mu - \kappa k^2)$$

We find that whenever

$$\mu \geq \mu_0 = \kappa k^2$$

we have instability. If we consider only long waves of small amplitude, i.e., μ only slightly greater than μ_0, we can again make an amplitude expansion. Letting

$$\mu = \mu_0 + \left(\frac{\lambda}{\tau}\right)\delta^2$$

where $\delta^2 \ll 1$ and λ is an arbitrary parameter, and scaling the amplitude f and the time t by δ , we expand in powers of δ to obtain, after removing resonances by a set of operations analogous to those used to obtain the original f equation, the zeroth-order solution

$$f = \delta R(\delta t) \cos(kx + \theta(\delta t))$$

Here the amplitude function $R(\delta t)$ and the phase $\theta(\delta t)$ satisfy the equations

$$\ddot{R} - (b^2/R^3) - \lambda k^4 R - \tfrac{3}{4}\nu k^5 R^3 = 0$$

$$\dot{\theta} = b/R^2$$

where b is another arbitrary constant and the dot denotes differentiation with respect to the slow time δt .

The amplitude equation has the first integral

$$\frac{1}{2}\dot{\mathcal{R}}^2 + V(\mathcal{R}) = \mathcal{E}$$

where

$$V = \frac{1}{2}\left[\frac{b}{\mathcal{R}^2} - \lambda k^4 \mathcal{R}^2 - \frac{3}{8}\nu k^5 \mathcal{R}^4\right]$$

and $\mathcal{E}$ is a constant. $\mathcal{E}$ can be thought of as the total energy and V as the potential -- we see a loose analogy to central force motion. Bound solutions to this system exist only for negative $\mathcal{E}$, in terms of elliptic functions. The form of the solutions as a function of time is shown in Fig. 2.

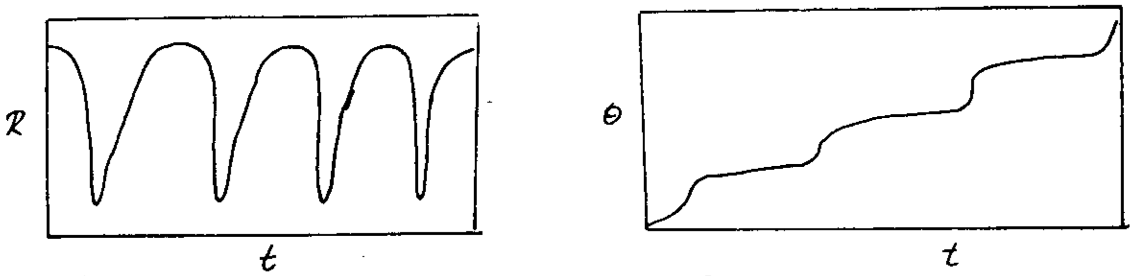

FIGURE 2

We see a sort of "solar cycle" in the periodic behavior shown by the amplitude function, while the phase diagram shows a latitude drift of magnetic activity, a magnetic curtain traveling north-south. There is no indication of any erratic behavior.

To obtain a larger class of phenomena we need to reconstitute the problem to get a more generic co-dimension two equation analogous to the Landau equation in ordinary Rayleigh-Benard convection (Childress and Spiegel, unpublished).

We return to the basic equations as scaled in Lecture #9:

$$\theta_{zz} = \varepsilon^2\left(\psi_x - \theta_{xx} + \psi_z\theta_x - \psi_x\theta_z\right) + \varepsilon^3\theta_T$$

$$\phi_{zz} = \varepsilon^2\left(\frac{1}{\tau}\psi_x - \phi_{xx} + \frac{1}{\tau}\psi_z\phi_x - \frac{1}{\tau}\psi_x\phi_z\right) + \frac{\varepsilon^3}{\tau}\phi_T$$

$$\psi_{zzzz} = \mathcal{R}\theta_x + \tau\delta\phi_x + \varepsilon^2\left(-2\psi_{xxzz} + \sigma^{-1}\psi_z\psi_{zzx} - \sigma^{-1}\psi_x\psi_{zz}\right)$$
$$+ \varepsilon^3\sigma^{-1}\psi_{Tzz} + \varepsilon^4\left(-\psi_{xxxx} + \sigma^{-1}\psi_z\psi_{xxx} - \sigma^{-1}\psi_x\psi_{xxz}\right) + \varepsilon^5\sigma^{-1}\psi_{xxT}$$

Let

$$\theta = f(x,T) + \varepsilon^2 F(x,z,T)$$

$$\phi = g(x,T) + \varepsilon^2 G(x,z,T)$$

$$\psi = (\bar{\mathcal{R}}f + \tau\delta g)_x P(z) + \varepsilon^2\chi$$

$$P^{IV}(z) = 0, \qquad P(\pm\tfrac{1}{2}) = 0, \qquad P'(\pm\tfrac{1}{2}) = 0$$

We find, on introducing

$$\rho = Rf + \tau S g$$

that

$$F_{zz} = \rho_{xx} P - f_{xx} - f_x \rho_x P' + \varepsilon f_T + \varepsilon^2 [\cdots] + \varepsilon^3 F_T + \varepsilon^4 [\cdots]$$

$$\tau G_{zz} = \rho_{xx} P - \tau g_{xx} + g_x \rho_x P' + \varepsilon g_T + \varepsilon^2 [\cdots] + \varepsilon^3 G_T + \varepsilon^4 [\cdots]$$

So

$$F = (\varepsilon f_T - f_{xx})\, g(z) + \rho_{xx}\, p(z) + f_x \rho_x\, p'(z) + O(\varepsilon^2)$$

$$G = (\varepsilon g_T - \tau g_{xx})\, g(z) + \rho_{xx}\, p(z) + g_x \rho_x\, p'(z) + O(\varepsilon^2)$$

where

$$g''(z) = 1 \quad , \quad p'' = P$$

with

$$(\varepsilon f_T - f_{xx})\, g'(\pm \tfrac{1}{2}) + \rho_{xx}\, p'(\pm \tfrac{1}{2}) = O(\varepsilon^2)$$

$$(\varepsilon g_T - \tau g_{xx})\, g'(\pm \tfrac{1}{2}) + \rho_{xx}\, p'(\pm \tfrac{1}{2}) = O(\varepsilon^2)$$

Thus, F and G can be worked out. Then we get

$$\chi_{zzzz} = (RF + \tau S G)_x - 2\rho_{xxx} P'' + \sigma^{-1} \rho_x \rho_{xx} (P' P'' - P P'') + \varepsilon \sigma^{-1} \rho_{Tx} P'' + O(\varepsilon^2)$$

This leads to χ. Putting it all together, we find

$$u_T - \mu \tau v_{xx} - \kappa \tau v_{xxxx} - (\nu \tau^2 / R_c^2)(v_x^3)_x + \varepsilon \left[(1+\tau)\mu u_{xx} + \kappa (1+\tau) u_{xxxx} + \beta \tau \sigma^{-1} v_{Txx} + 2\nu \tau R_c^{-2} (1+\tau) (v_x^2 u_x)_x \right] = 0$$

$$v_T - u_{xx} - \varepsilon \left[\gamma \tau v_{xx} + \tau (1+\tau) R_c^{-2} \nu (v_x^3)_x \right] = 0$$

where

$$u = Rf + Sg \quad , \quad v = Rf + S\tau^{-1} g$$

and

$$\mu = \left(\frac{R}{R_c} - 1 \right) / \varepsilon^2 \quad , \quad \gamma = \left[R + \tau^{-1} S - (R_0 + \tau^{-1} S_0) \right] / \varepsilon^2$$

and the other quantities (κ, ν, β) are numbers that depend only on the choice of boundary conditions.

These are the reconstituted wave equations and the waves they give have well-determined amplitude and velocity that vary slowly.

3. Some Speculation on Turbulence.

What I have been trying to do in these lectures is open up topics for further research. The problems I have presented can be divided into three parts. The first category is the fundamentals. There are still veins to be tapped even at this level, as the warm-up problem attests. At the next level we have the problems occupying the tillers of the field — how to extend the standard calculations we have become familiar with. And next we have the impossible dreams...

In the last category we might place the current interest in strange attractors as a model for chaos. In the usual dynamical problems we are handed a flow field U(X) and asked to compute the Lagrangian variable X (which we can think of as a streakline) from a set of equations of the form

$$\dot{X} = \mathcal{U}(X)$$

If divU is negative, a swarm of points contracts to zero volume and we have an attractor. An example of this sort of behavior is the damped harmonic oscillator, where the dependent variables shrink to a fixed point. Other examples of an attractor are a limit cycle, or in 4-D, a torus.

A worst possible definition of a strange attractor is an attractor that is not a point, limit cycle, or torus. This does not help us much. A distinguishing characteristic of a strange attractor is the existence of erratic, nonperiodic behavior. Consider a 3-D orbit in phase space. If we insert a surface of section we get a pattern of points where the orbit intersects the surface (Fig. 3). We call this a Poincare map.

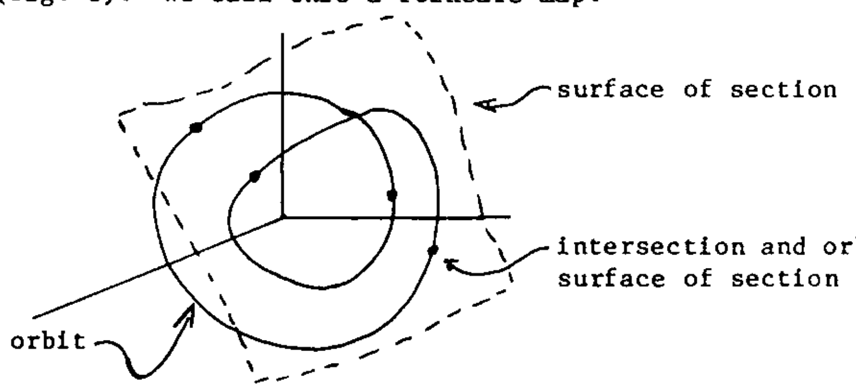

FIGURE 3

For a strange attractor we get an infinite set of points on our Poincare map, with a distribution showing a self-similar structure on all scales -- a Cantor set (Fig. 4).

In the real world a system governed by a strange attractor would thus be highly sensitive to noise. Just as in the hydrogen atom, where ideally there are an infinite number of discrete energy levels, but in real life we observe a finite number of levels due to the inevitable environmental noise, the number of leaves on the Poincare map of a strange attractor is dependent on background noise. A strange attractor could be viewed as a sort of noise amplifier. Nevertheless, we do not need noise to get chaos. Erratic behavior

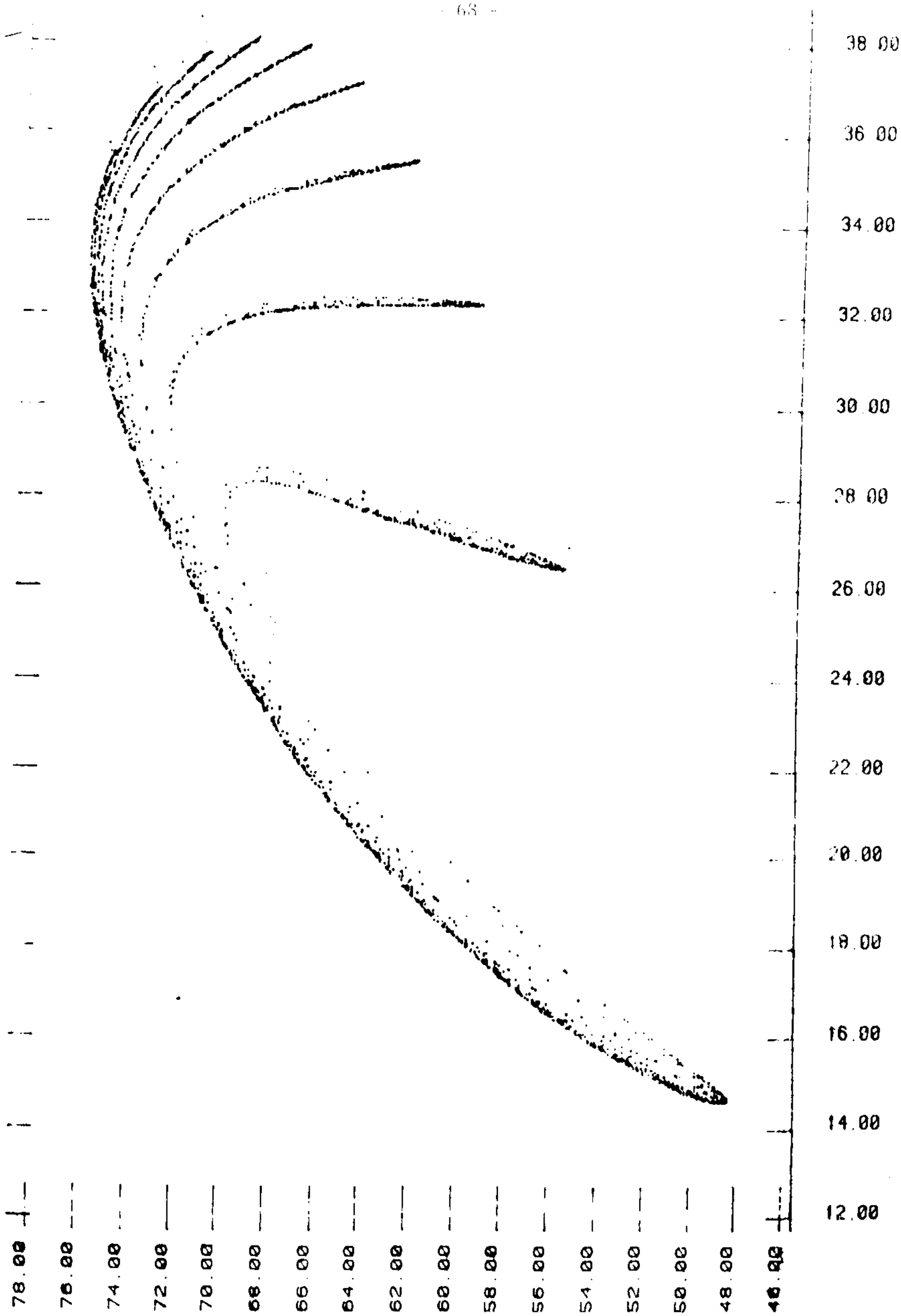

Fig. 4 Plot of Poincaré Map for $\dddot{x} + \ddot{x} + (\alpha - a x^2) + \beta x = 0$.

is part of the very nature of the strange attractor. This inherent chaos and that introduced by the noisy real world may both be important in realizable dynamical systems.

Some people have actually called the chaos exhibited by the strange attractor turbulence. There are others who claim that the sort of truncations that have been made on the governing equations to obtain systems with strange attractors have nothing to do with real fluid mechanics. Then what is turbulence?

The word turbulence has its root in either of the two Latin words turbo-vortex, or turba-mob. Mob is an old abbreviation for mobile. A mob of mobile vortices, if you will. Vorticity lies at the heart of turbulence. I now offer some conjectures on the nature of turbulence and the possibly related class of convection (thermohalence). These run as follows:

1) Solitary waves or objects occur in many real flows, though they are a secondary phenomenon and hard to get our hands on. We found possible solitary wave solutions in double convection; solitons can be shown to run down vortex tubes, and, vorticity being at the heart of turbulence we might expect solitary waves to be important in turbulence, too. In convection thermals are solitary objects, so are oceanic gyres.

2) Solitary objects in an unstable situation which have "metaphorical minds". On a fast time we see a nonlinear wave on coherent structure, but on a slow time the amplitude and phase are governed by dynamical systems (ODE's) of the form

$$\dot{A} = f(A) \qquad ; \quad A \text{ a vector}$$

The attractor of this system is what I mean by the metaphorical mind. The standard KdV and Schroedinger solitons are "mindless", while waves whose attractors are fixed points are "simple-minded". The waves we just studied already show interesting behavior that we might call Zitterbewegung. Certainly, waves whose minds are strange would move chaotically. These solitary waves are the elementary objects of this vague turbulence model. An example arises when we study the reconstituted equation for triple convection with fixed flux.

3) In a fast collision our solitary objects collide more or less like solitons (particles), but on a slow time the collisions are "telepathic" in that the attractors (or metaphorical minds) interact. As a possible example of what I mean by a solitary wave with a mind of its own, the Great Red Spot of Jupiter presents itself. These are my reasonable conjectures: we enter the truly conjectural part of this lecture. If we think of solitary waves as particles, we can write their orbits in the usual way

$$X = v t + X_0$$

where v, on a long time scale, is governed by a system of equations with strange attractor and may behave chaotically. The "old one" does not need to play dice to know what such particles will do, but we have to. If we have a large number of solitary waves we essentially have a problem in statistical mechanics. Fast collisions are assumed elastic while slow collisions are strongly telepathic. Unfortunately we do not know the collision rules, but some clues ought to be calculable from the buoyancy wave theory.

The picture this suggests is an analog of kinetic theory in which the particles can interact simply on a fast time and in which the representative points on each attractor can disturb each other. In fact, one may simply assume that all of the particles have identical minds so that we can visualize all of their representative points on the same attractor. When two or more of them interact, they conceivable distort the attractor itself. So that is the kind of problem I want to study: a large number of representative points moving on an attractor, but with the attractor itself influenced or even shaped by the interactions. How can we write a theory that allows such possibilities? Consider the system

$$\dot{X} = u(X)$$

The effect of the attractor is measured by $\nabla . u$; I see this as similar to a gravitational attractor. Let

$$g_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}$$

and think of g_{ij} as being like a gravitational potential. Similarly, think of $\nabla \times u$, the vorticity, as being like a magnetic field; and since u is the vector potential for vorticity, it is like the electromagnetic potential. So motion around this attractor has a loose analogy to motions in an EM + gravitational field. This should not be taken literally, but the analogy points to ways in which we can model the attractor. One description that has been used to study motion in gravitational and electric fields is to introduce Finsler geometry (Stephenson and Kilmister, 1953). That is what I do for the dissipative dynamical system.

Suppose you consider a space with coordinates x^i and a line element

$$ds = L(x^i, dx^j)$$

such that

$$L(x, k\,dx^i) = k\,L(x^i, dx^i)$$

Let $\dot{x}^i = \frac{dx^i}{ds}$. Then the geodesics given by

$$\delta \int L(x^i, \dot{x}^i)\, ds = 0$$

are the orbits of a Hamiltonian system with L as its Lagrangian.

Let

$$ds = u_i\, dx^i + \sqrt{g_{ij}\, dx^i\, dx^j}$$

where units are chosen suitable to make this nondimensional. The first part of ds, uds, is the usual action. The second part corrects in some way for the effect of the attractor. The second part corrects in some way for the effect of the attractor. The geodesics are given by

$$\ddot{x}^i + \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k + g^{ij} \omega_{jk} \dot{x}^k = 0$$

where

$$\left\{ {i \atop jk} \right\} = \frac{1}{2} g^{il} \left[g_{jl,k} + g_{kl,j} - g_{jk,l} \right]$$

and

$$\omega_{jk} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \right)$$

In other words, this geometry contains all the kinematic information of the original system and it has some close correspondences. These will have to be told in a future summer. I just want to close by noting that the model, too, can be closed by expressing g_{ij} in terms of what the particles on the attractor are doing. If there is a strange attractor for turbulence, I imagine that it will be like that.

I cannot predict this yet, but it appears that the Hamiltonian system develops caustics near where the original system has an attractor. So the particles in the associated system spend a lot of time in the right place. This opens up the possibility of a statistical mechanic for such systems when they are embedded in a heat bath. I am sorry that my time is up and I cannot tell you more about this, but you can perhaps see how it goes.

These are problems for one's dotage, perhaps. I now end the course, and propose to begin my dotage.

NOTES SUBMITTED BY
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