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Workshop on Fluid Mechanics

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Nonlinear instability theory of convection

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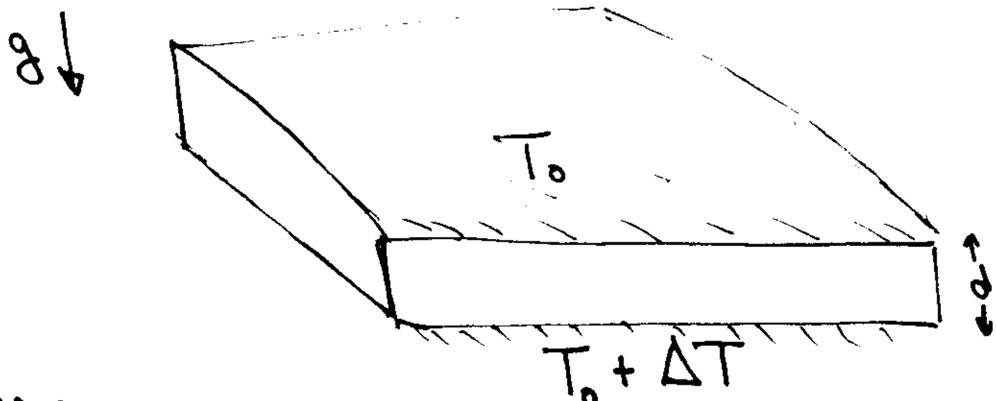
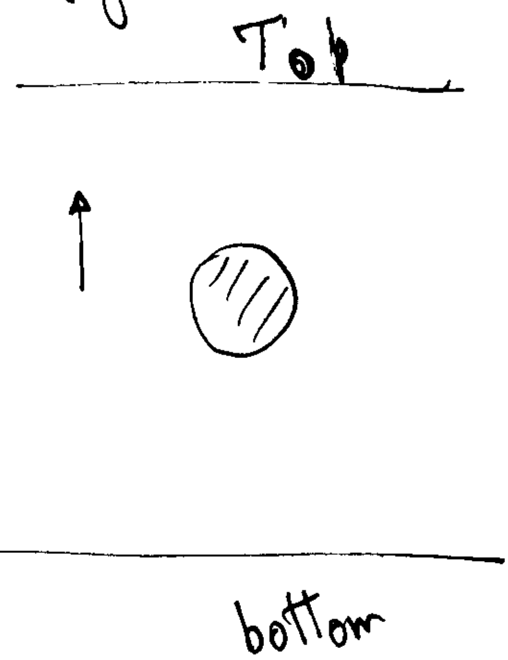
These are preliminary lecture notes, intended only for distribution to participants

Nonlinear Instability Theory of Convection



No dissipation

Instability when a perturbation can lower the energy.



$$\underbrace{\underbrace{c_p\, dT}_{\text{perturbation of internal energy}} + \underbrace{g\, dz}_{\text{pert. of gravitational energy}}}_{\text{per unit mass}} < 0$$

$$\frac{dT}{dz} + \frac{g}{c_p} \leq 0$$

For perfect gas,

sp. entropy: $s = c_v \ln \dfrac{p}{\rho^\gamma}$

$\gamma = c_p/c_v$ $\quad p = R\rho T$

$$\frac{ds}{dz} = \frac{c_v}{p}\frac{dp}{dz} - \frac{c_p}{\rho}\frac{d\rho}{dz}$$

$$\frac{ds}{dz} = \frac{c_v}{p}\frac{dp}{dz} - c_p\left[\frac{1}{p}\frac{dp}{dz} - \frac{1}{T}\frac{dT}{dz}\right] = -\frac{R}{p}\frac{dp}{dz} + \frac{c_p}{T}\frac{dT}{dz}$$

$$\frac{dS}{dz} = + \frac{R}{p} g\rho + \frac{C_p}{T}\frac{dT}{dz}$$

$$= \frac{g}{T} + \frac{C_p}{T}\frac{dT}{dz}$$

$$\frac{1}{C_p} T \frac{dS}{dz} = \frac{dT}{dz} + \frac{g}{C_p}$$

This is a qualitative derivation.
If the layer is thin, many properties are nearly constant. $\frac{dT}{dz} = -\frac{\Delta T}{d}$ {note sign!

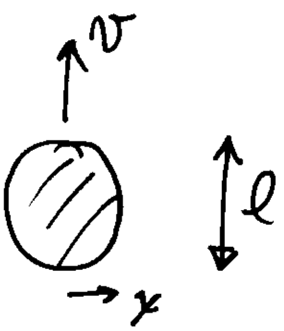
Criterion becomes $\Delta T > \frac{gd}{C_p}$.

For laboratory convection, and some geophysical cases $\frac{gd}{C_p}$ is taken to be ≈ 0.

Or define Θ so that
$\Delta\Theta = \Delta T - \frac{gd}{C_p}$ and think of Θ as "temperature"

When there is viscosity and conductivity, we need a stronger condition: For sustained motion, the buoyancy force must exceed the viscous drag.

$$|g(m-m_0)| \gtrsim \mu \left|\frac{\partial v}{\partial x}\right| \ell^2$$



$$m - m_0 \approx (\rho - \rho_0)\,\ell^3$$
$$= -\alpha(T - T_0)\,\ell^3$$

$$-\rho_0 \alpha = \left(\frac{\partial \rho}{\partial T}\right)_p \qquad \left|\frac{\partial v}{\partial x}\right| \sim \frac{v}{\ell}, \qquad T - T_0 = \theta$$

$$\nu = \frac{\mu}{\rho_0}$$

$$\frac{g\alpha\theta}{\nu}\,\frac{\ell^2}{v} \gtrsim 1$$

But, to keep this up, need to retain the buoyancy. Thermal lifetime $\sim \frac{\ell^2}{\kappa}$; $\kappa = \rho C_p k$

Require buoyancy retention: characteristic dynamic time, $\frac{\ell}{v}$, should be less than ℓ^2/κ. We already require

$$\frac{g\alpha\theta\ell}{\nu} \gtrsim \frac{v}{\ell} \quad \text{and we want} \quad \frac{v}{\ell} \gtrsim \frac{\kappa}{\ell^2}.$$

So for instability

$$\frac{g\alpha\Theta l^3}{\kappa\nu} \gtrsim 1$$

$l \lesssim d, \quad \Theta \lesssim \Delta T$

So to make l.h.s. as large as possible Θ is replaced by ΔT and l by d.

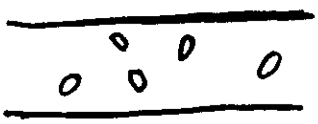
$$\text{Rayleigh number} = R = \frac{g\alpha\Delta T d^3}{\kappa\nu} \geq 1. \qquad \left(\begin{array}{c}\text{Really} \\ \Delta T \text{ is } \Delta\Theta\end{array}\right)$$

In reality l is π/wave number ~~and~~ and so forth, so we should replace the r.h.s. by "1" or really by R_c, where R_c has to computed honorably. The point of this qualitative discussion is show how R comes into the story. We see that neither viscosity nor conductivity can stabilize a situation with ΔT $(\Delta\Theta)$ positive. They are both needed. The actual value of R_c depends on boundary conditions and is ~ "few" $\pi^4 \sim 10^3$.

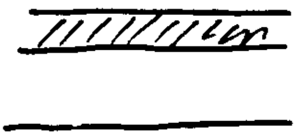
What happens when $R > R_c$?

The answer is complicated. We know very little about the case $R \gg R_c$ or even when $\frac{R-R_c}{R_c}$ is not small. So let us talk about the case $\epsilon = \frac{R-R_c}{R_c} \ll 1$.

For thin layers of fluid, the Boussinesq approximation may usually be adopted. (Some laboratory examples may fool us, such as bioconvection.

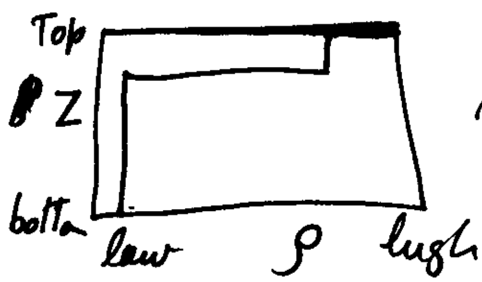


swimming organisms in a pleasant soup.



arrive in a sublayer so density is non-Boussinesq)

This is often unstable to convection



We shall not be concerned with mathematical details but will stress the ideas involved

Equations have this structure

$$\frac{\partial \vec{v}}{\partial t} = \mathcal{V}\left(\vec{v}, T, \text{other variables like } \vec{B}, \text{salinity}, \dots\right)$$

$$\frac{\partial T}{\partial t} = \mathcal{J}\left(\vec{v}, T, \dots\right)$$

and so on.

We know that in the Boussinesq approximation we have the equilibrium solution

$$T_{eq} = T_0 + \Delta T\left(1 - \frac{z}{d}\right)$$

$$\vec{v} = 0 \qquad \text{--} \quad (0 \le z \le d)$$

In linear stability theory we treat <u>small</u> perturbations on the equilibrium neglecting quadratic (and worse) terms in $\Theta = T - T_{eq}$ and $\vec{v}$. That produces <u>linear</u> equations which are (in the easiest cases) separable. The linear theory can be expressed in terms of

vertical velocity $w = \hat{z} \cdot \vec{v}$
Temperature fluctuation $\theta = T - T_{eq}$
vertical vorticity: $\zeta = \hat{z} \cdot (\nabla \times \vec{v})$

Since you can reconstruct $\vec{v}$ from $\nabla \cdot \vec{v}$ and $\nabla \times \vec{v}$

and if $\nabla \cdot \vec{v} = 0$, w and ζ can be seen to give us $\vec{v}$ (up to a gauge).

(Hidden traps lurk, but we press on.)

The modes that first give instability correspond to solutions of the form

$$\begin{pmatrix} w \\ \theta \\ \zeta \end{pmatrix} = \begin{pmatrix} W(z) \\ \Theta(z) \\ 0 \end{pmatrix} f(x,y)\, e^{st}$$

[Θ is not the same as before (sorry)] $f(x,y)$ is the planform function.

This gives (in linear theory) an equation for $W(z)$ (or $\Theta(z)$). The parameters in this equation are R, σ and two <u>separation constants</u>: s and k where we find that

$$\Delta f = -k^2 f$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. (2-d Laplacian)

and $\sigma = \nu/\kappa$ (Prandtl number)

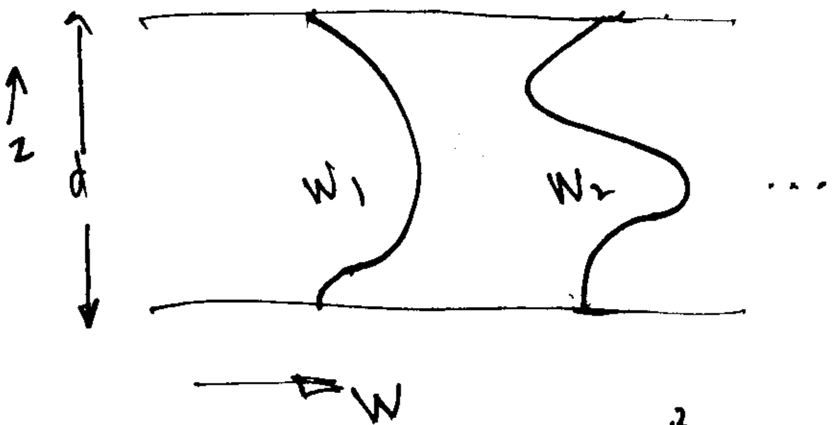
The case $s=0$ is the marginal case of instability. That is, solutions of the linear problem lie on surfaces in the space with coordinates $R, \sigma, kd, sd^2/\kappa$ (I have used kd and sd^2/κ instead of k and s since that means I want to use <u>natural</u> units: d for length and d^2/κ for time.) Let $a = kd$

$$\eta = sd^2/\kappa.$$

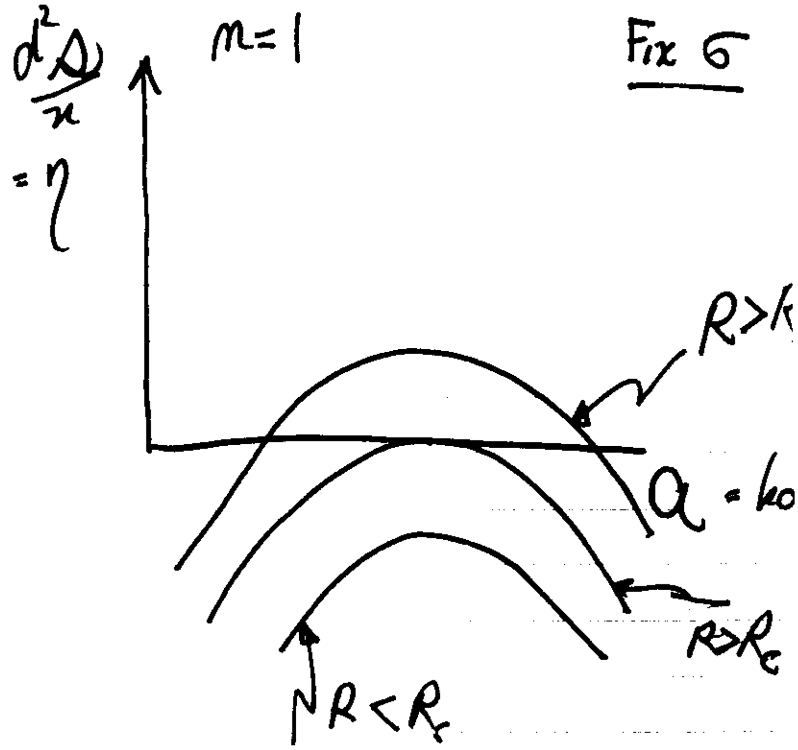
There is a solution surface for any allowed $f(x,y)$ (affected by container shape:

but we shall study the case of simple shapes and the case of infinite horizontal extent),

every a and every vertical "mode number"

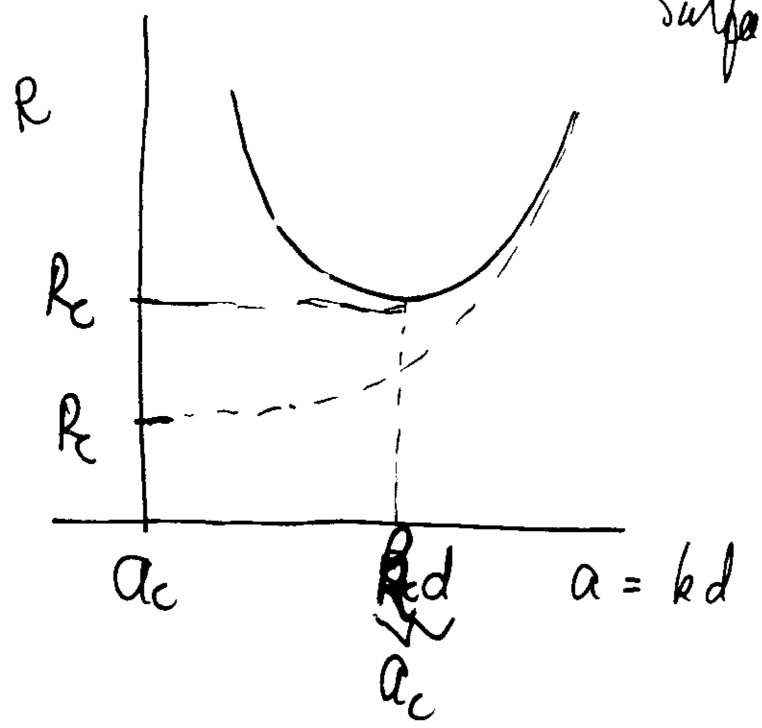


Called gravest mode.



These curves do not depend on f

Put $\eta = 0$ look at intersection of $n = 1$ surface with R–a plane

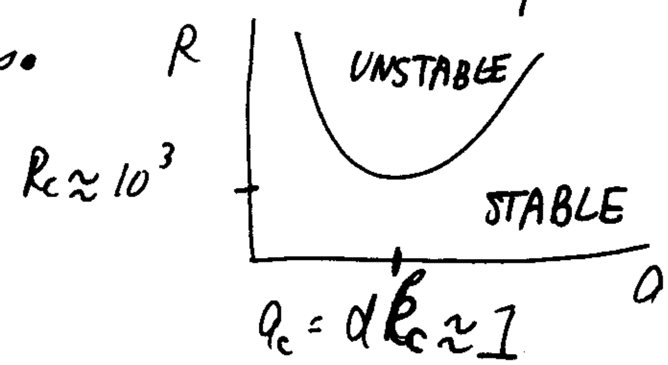


Independent of σ and f.

— $\theta = 0$ on top and bottom (fixed temperature)

--- $\frac{\partial \theta}{\partial z} = 0$ on top and bottom (fixed heat flux)

The value of a_c (the wave number of the "most dangerous" modes is very dependent on thermal boundary conditions. We shall stick with fixed temperatures on the boundaries.



For $\epsilon^2 = \frac{R - R_c}{R_c}$ small the width of the band of unstable a is $O(\epsilon)$ and $\eta = O(\epsilon^2)$.

We have near marginality $\left\{ \begin{matrix} k = k_c + O(\epsilon) \\ R = R_c\, O(\epsilon^2) \end{matrix} \right\}$

$$\eta = \epsilon^2 + \xi\,(a^2 - a_c^2)^2 + O(\epsilon^4)$$ where ξ is a constant of order unity (depends on $\sigma, \ldots$)

might have a coefficient here for some B.C.s

To master this material, work through the first few chapters of H.&H.S. by S. Chandrasekhar (or see attached notes.).

Note: $$(a^2 - a_c^2)^2 = (a - a_c)^2 (a + a_c)^2 = O(\epsilon^2).$$

Now we look at the nonlinear development that occurs when amplitudes grow and nonlinear terms may not be neglected.

First We consider a fixed $f(x,y)$. It generally not easy to "clamp" the planform in reality. This part is to give an idea of the procedures and to suggest nature of some solutions, albeit unstable ones.

To think about specifics, it is best to look at the simplest case with no vertical vorticity: $\zeta = 0$.

To go even further, let the problem be two-dimensional with $\vec{v} = \nabla \times (\hat{y}\psi)$.
The stream function is $\psi(x,z)$.
The equations are

$$(\partial_t - \sigma\nabla^2)\nabla^2\psi = -R\sigma\frac{\partial\theta}{\partial x} + J(\psi, \nabla^2\psi)$$

$$(\partial_t - \nabla^2)\theta = -\frac{\partial\psi}{\partial x} + J(\psi,\theta)$$

where
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

and
$$J(\psi,\theta) = \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x}.$$

The planform function is $f(x)$ and it is trigonometric. For finite extent in x we get only discrete values of a in linear theory.

We get from the linear theory a set of convective modes:

$$\begin{pmatrix} W_1(z) \\ \Theta_1(z) \\ 0 \end{pmatrix} f_{a_1}(x,y) \,, \quad \ldots \quad \begin{pmatrix} W_m \\ \Theta_m \\ 0 \end{pmatrix} f_{a_m}(x,y) \quad \ldots$$

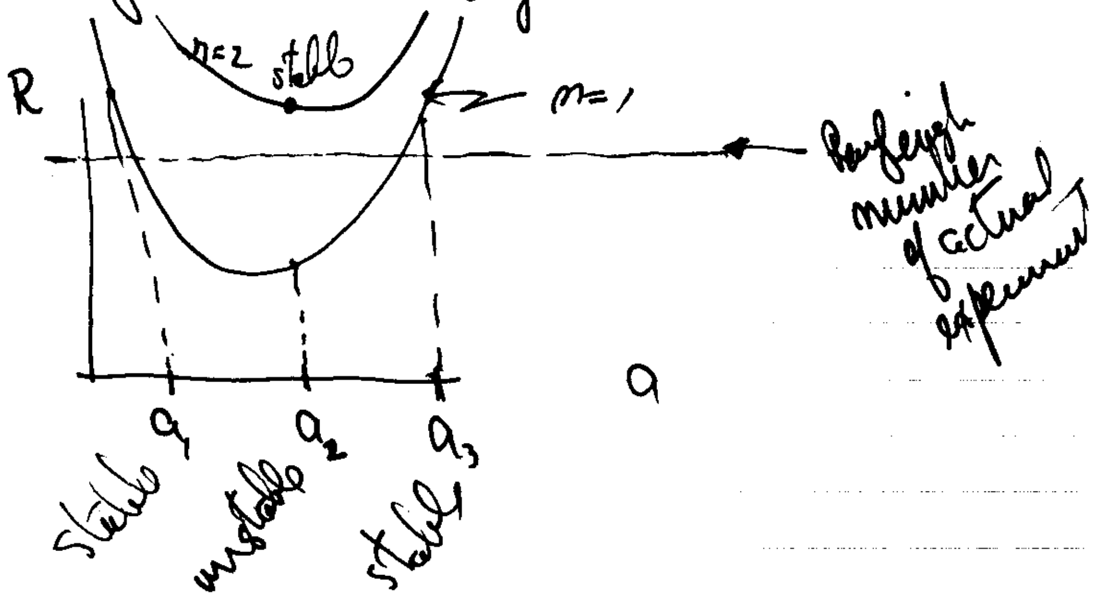
Just the spatial part here.

I am assuming that the f_{a_i} have a common shape and differ only in scale

$$\left(\partial_x^2 + \partial_z^2\right) f_{a_m} = - k_m^2 f_{a_m} \,.$$

$$(a_m = k_m d)$$

This is what we get for a layer that is finite in horizontal



We use the modes as a basis. They are eigenfunctions of a certain linear operator so they might not form a complete set. If so, the set would need to be completed by some additional modes (more on this later if time permits). In any case, there are other kinds of modes such as vertical vorticity modes:

$$\begin{pmatrix} w \\ \theta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Z(z) \end{pmatrix} f(x,y)\, e^{+\sigma t}$$

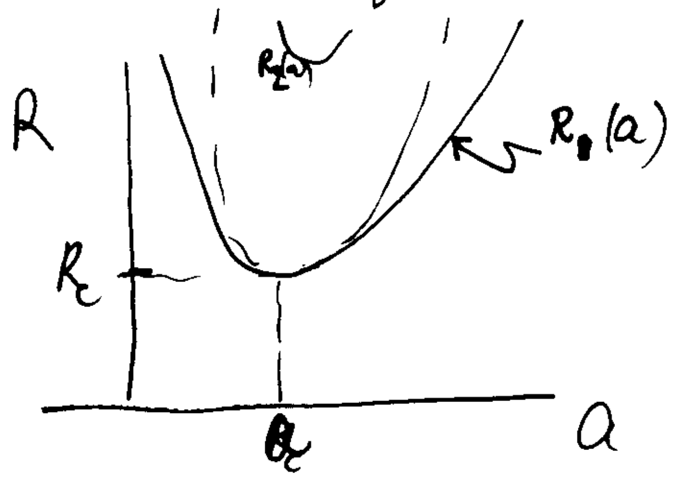
For these modes $\sigma \leq 0$.

We take the spatial parts of the modes and call them $\Xi^{\bar{\ell}}_{m,n}$ where m is the label on a_m
n is the vertical mode #
$\bar{\ell}$ is the mode type.
"ℓ bar"

As to mode type, there are three principal types.

1) Vertical vorticity modes.

2) Two kinds of convective mode



The marginal curve is fit by a parabola at the bottom. To good approximation there and qualitatively elsewhere,

$$\eta^2 + (\text{factor}_1)\,\eta + (\text{factor}_2)\big(R_m(a) - R\big) = 0$$

Both factor$_1$ and factor$_2$ are +.

Two roots! What are the two modes?

For one set of modes,

$$\eta \approx \frac{R - R_m(a)}{\text{factor}_1}\,\text{factor}_2$$

Any of these modes can go unstable; just raise R above $R_m(a)$

For the other $\eta \approx -\,\text{factor}_2$.

Always stable.

I call the two kinds of stable modes
+ modes and − minus modes.

In the + modes $\quad W \textcircled{H} > 0 \rightarrow$ can go unstable

In the − modes $\quad W \textcircled{H} < 0 \rightarrow$ can never go unstable.

For vertical vorticity modes $\quad W \textcircled{H} = 0$

For a complete description of convection need all the modes. But near onset the gravest + convective modes is the most important.

We have $t = +, 0, -$.

Now fix t. At some instant t, the solution

$$\begin{pmatrix} w \\ \theta \\ \xi \end{pmatrix}$$

has some <u>spatial</u> distribution. We describe it by an expansion in modes:

$$\begin{pmatrix} w \\ \theta \\ \xi \end{pmatrix} = \sum_{m,n,t} A_{m,n,t} \, \Xi_{m,n,t}(x,y,z)$$

Can do this for any time. We just get a different set of amplitude values $A_{m,n,t}$.

So really for all t,

$$\begin{pmatrix} w \\ \theta \\ \xi \end{pmatrix} = \sum_{m,n,\ell} A_{m,n,\ell}(t)\, \Xi_{m,n,\ell}(x,y,z) .$$

Now for some hard work — that we do not do — put this expansion into the equations and derive a set of equations for the $A_{m,n,\ell}(t)$. [Not too big a chore for the two-d case with $\xi=0$.] We get (typically):

$$\dot{A}_{1,1,+} = \eta_{1,1,+} A_{1,1,+} + \text{nonlinear terms.}$$

$$\vdots$$

$$\dot{A}_{m,n,\ell} = \eta_{m,n,\ell} A_{m,n,\ell} + \text{nonlinear terms}$$

$$\vdots$$

Nonlinear terms are (for Boussinesq convection) ~~[illegible]~~ quadratic: $\sum C_{m'n'\ell' m''n''\ell''} A_{m'n'\ell'}(t)\, A_{m''n''\ell''}(t)$.

This description of convection does not look nice, but it will reveal how we may simplify matters when $0 < \frac{R - R_c}{R_c} = \epsilon^2 \ll 1$.

In the new notation,

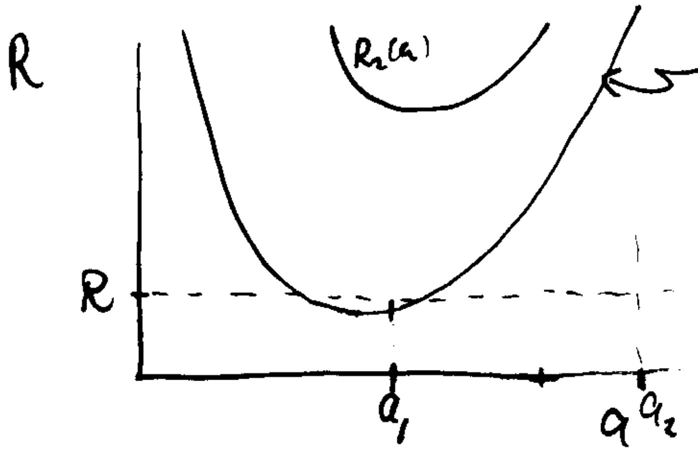
$$R_c = \min_{a_m} R_1(a)$$

We can expect that

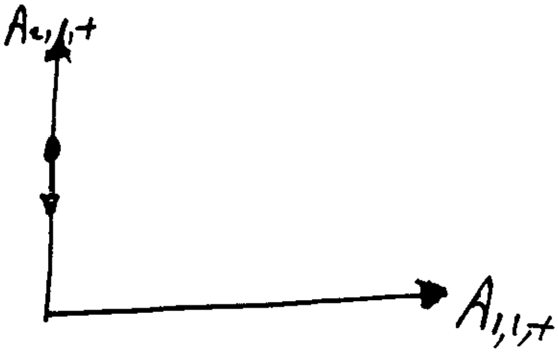
$$\eta_{1,1,+} > 0 \qquad \text{and, in fact,} \quad \eta_{1,1,+} = O(\epsilon^2)$$

and $\eta_{m,n,\pm} < 0$ unless $(m,n,\pm) = (1,1,+)$.

For a box that is not large in the horizontal dimensions we can arrange that $\eta_{1,1,+} = O(\epsilon^2)$ and all other $\eta < 0$

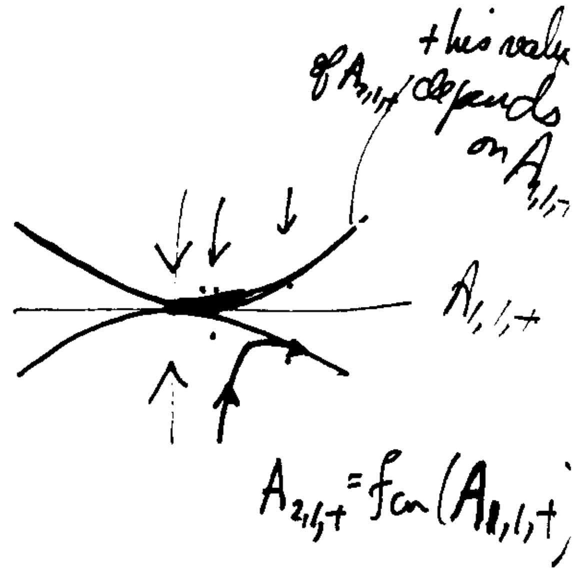


The $A_{m,n,t}$ can be used as coordinates in a space of convective states. $A_{m,n,t}(t)$ defines a path in this space.

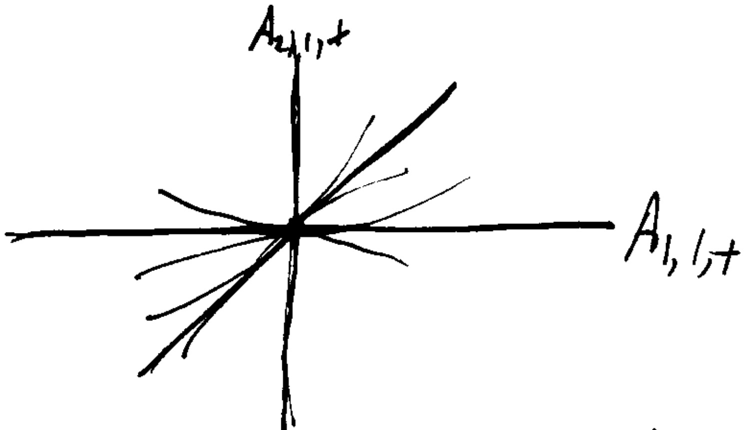


For $A_{1,1,+}(0) = 0$
and
$A_{m,n,t}(0)$ $O(\epsilon)$

get initial <u>decay</u> of all modes



For $A^{(0)}_{1,1,+} \neq 0$, $A_{1,1,+}$ grows



Call $A_{11+} = A$

So the motion of the system in phase space, after a transient time goes to a subspace where

$$A_{mnt} = a_{mnt}(A)$$

in Boussinesq Approximation,
For convective modes, if

is a solution, so is

Invariant under $A \to -A$

This is not an exact symmetry in general.

So for small $|A|$, expected for small ϵ^2;

$$A_{ment}(A) = A_{ment}(-A)$$

$$= \text{constant } A^2 + \cdots$$

More in

$$\dot{A} = \eta_{1+} A + \text{nonlinear}$$

we have nonlinear terms $A A_{mnt}$ and $A_{mnt} A_{mn}$

Cannot have A^2 since $A \to -A$ is not respected left.

$$\underset{O(\epsilon^3)}{\dot{A}} = \underset{O(\epsilon^2)}{\eta_{1+}} \underset{O(\epsilon)}{A} + \text{const. } \underset{O(\epsilon^3)}{A^3} + \cdots$$

The idea is that after rapid transient,
A controls dynamics with $\eta = \eta_{11+} = O(\epsilon^2)$

So in $\dot{A}_{m,n,\ell} = \eta_{mn\ell} A_{mn\ell} +$ nonlinear

after transient $\dot{A}_{mn\ell} \sim \eta_{11+} A_{mn\ell} = \eta A_{mn\ell}$

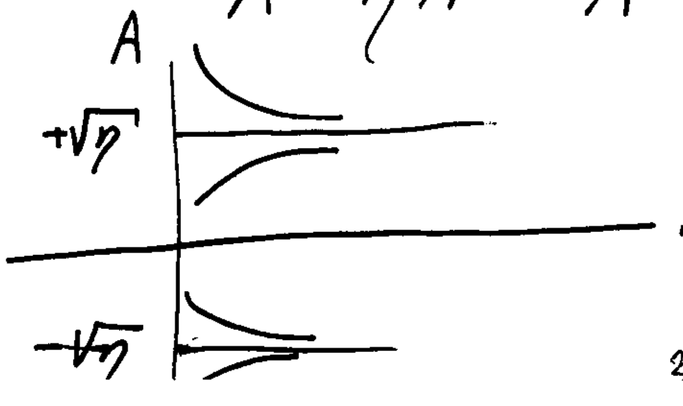
$$\eta \ll |\eta_{mn\ell}|$$

So $$A_{m,n,\ell} \approx \frac{-\text{Nonlinear}}{\eta_{mn\ell}} \approx \frac{1}{\eta_{mn\ell}}\left[A^2 + \cdots \right]$$

This is how you get $A_{mn\ell}$

A is gotten from solution of the <u>Landau equation</u>

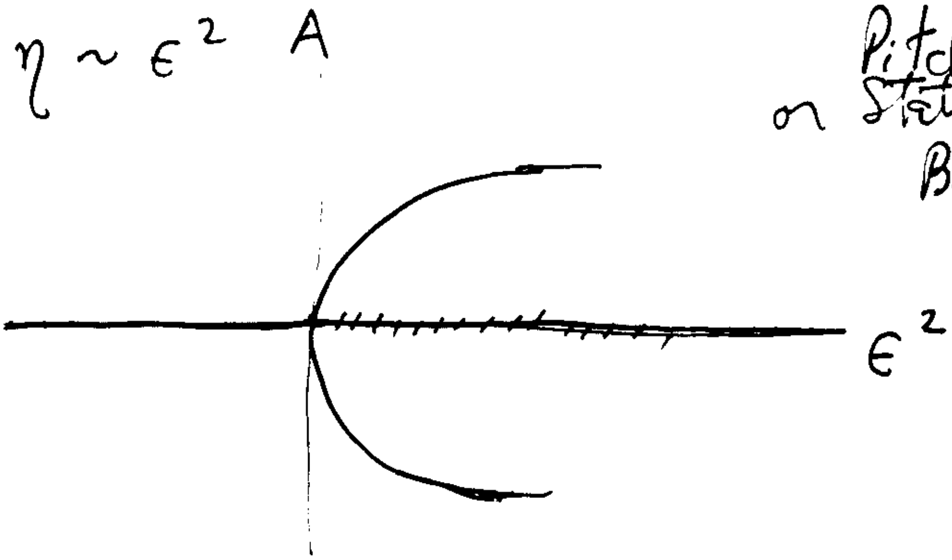
$$\dot{A} = \eta A - A^3 \quad (+\cdots)$$



We will see something of how such an equation is derived in a bit.

$\eta \sim \epsilon^2$

Pitchfork or Stationary Bifurcation



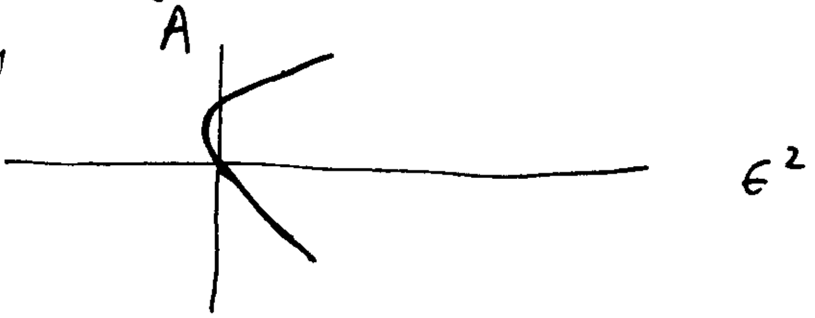
In non-Boussinesq convection, the situation is that

$$\dot{A} = \eta A + \alpha A^2 \left(+ \beta A^3 + \cdots \right)$$

For *slightly* non-B convection, $\alpha = O(\epsilon)$

$$\dot{A} = \eta A + \alpha A^2 + \beta A^3 \; (+ \cdots) \quad (\beta = O(1))$$

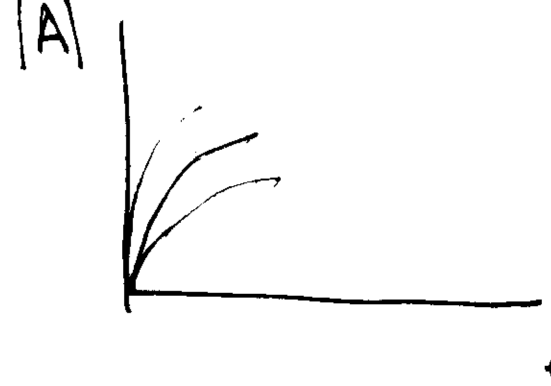
Typically get



Transcritical bifurcation

Working out the coefficients is a lengthy calculation.

This all looks very simple. But the real problem is a lot worse. The troubles come from the complications involving f. It is really not so easy to set connection with a single f. People have tried for a long time, starting with Avsec (Thèse, 1939, Paris). What really goes on is qualitatively like this



There are several bifurcation pictures shown each "calculated" for a different f, as if that were the only f. But you really should put them all in (at least all those allowed by the b.c.'s). Or you can do what Busse may have described last week and take a solution for one f and see what instabilities it has. I will now describe the situation where you allow for many f s. Again I will think of the "simplest" version of this problem; ~~but~~ this is the case of infinite horizontal extent. In this case, the linear planform function that we use is quite general. Before, we had

$$(\Delta + k^2) f = 0,$$

$\Delta = \partial_x^2 + \partial_y^2$ for the planform function. Now we want to find the generalization of this equation for the nonlinear, time-dependent situation. We will do this for $k \approx k_c$.

We will call $\begin{pmatrix} w \\ \theta \\ \xi \end{pmatrix}$: U

Our equations can be thought of as having this form

$$\partial_t U = \mathcal{F}(U, \partial_x).$$

(The truth is close to this but a bit messier.)

Now we are dealing with a continuous spectrum of horizonal wave numbers but a discrete spectrum of vertical wavenumbers so we write

$$U(x,y,z,t) = \int U_{\vec{k}}(z,t)\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$$

$$(\vec{x}) = (x,y)$$

In z we use the vertical functions W_m, Θ_m, and so

to expand $U_{\vec{k}}(z,t)$ as

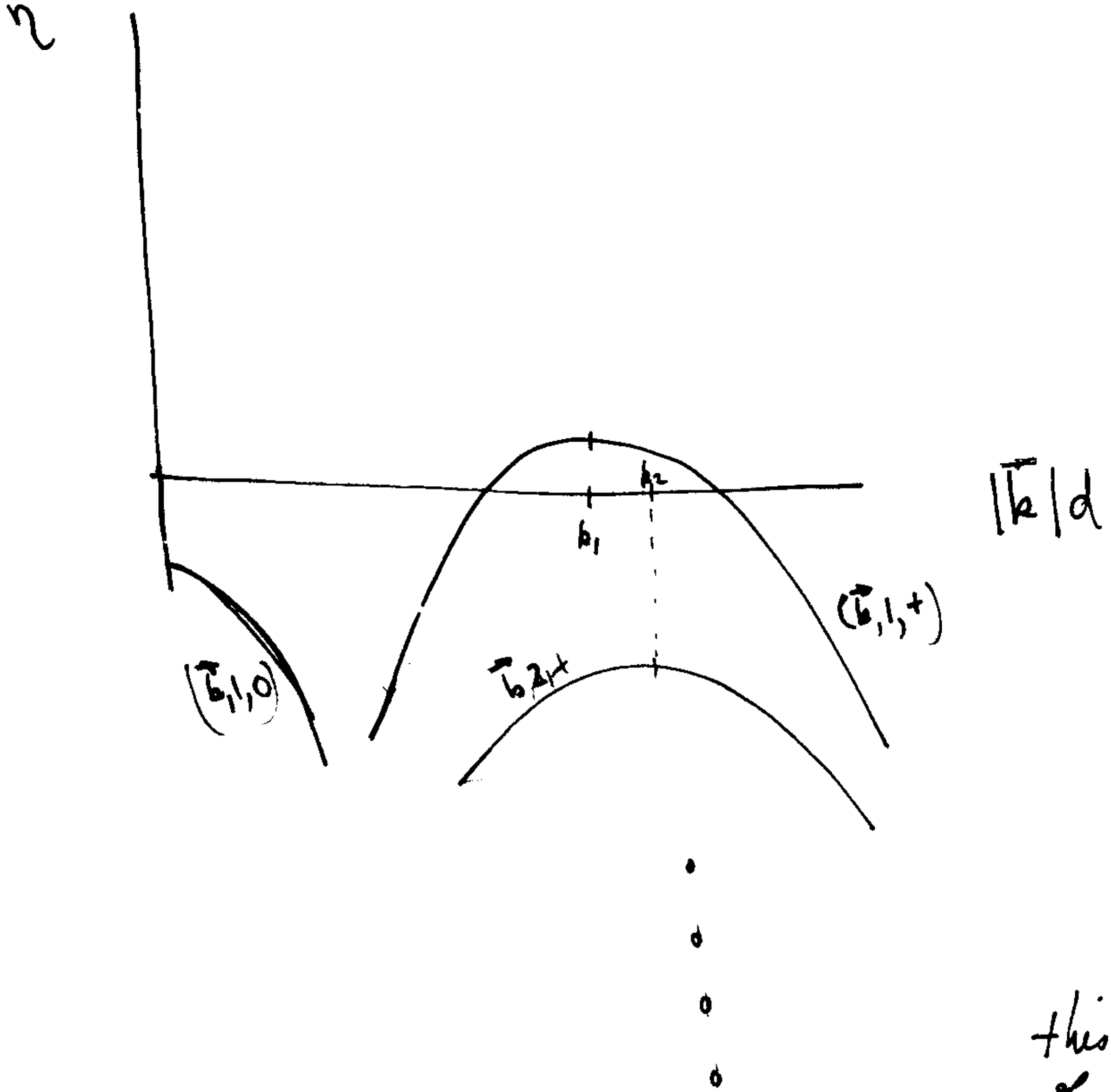
$$U_{\vec{k}}(z,t) = \sum_{\ell} A^{(m)}_{\vec{k}}(t)\, H_{m,\ell}(z)$$

where $H_{m\ell}$ consists of modes $\begin{pmatrix} W_m(z) \\ \Theta_m(z) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ Z_n \end{pmatrix}$

In linear theory

$${}_{\ell}\dot{A}^{(m)}_{\vec{k}}(t) = \eta_{\vec{k}n\ell}\, {}_{\ell}A^{(n)}_{\vec{k}}$$

For a given values of R and σ we have



this is a "sum of terms of a different t → really a convolution

So now, when we expand we get

$$\partial_t \, {}_tA^{(1)}_{\vec{k}} = \eta_{\vec{k},1,+} \, {}_tA^{(1)}_{\vec{k}} + \text{nonlinear terms}$$

As before, we expect to have

${}_tA^{(n)}_{\vec{k}} = f[{}_tA^{(1)}_{\vec{k}}]_{\vec{k}}$ where $[\cdots]$ means that there could be a functional dependence

This a complex issue worked on by quite a few people — in connection theory Swift and Hohenberg were first ~~in the~~ general style — and I will just sketch the procedure.

Instead of the complication of the full problem with infitely many modes in $\vec{k}$ we proceed with just two vertical modes to give the idea: there are

$$A_{\vec{k}} = {}_{+}A^{(1)}_{\vec{k}} \quad , \quad B_{\vec{k}} = {}_{+}A^{(2)}_{\vec{k}}$$

We will have something like

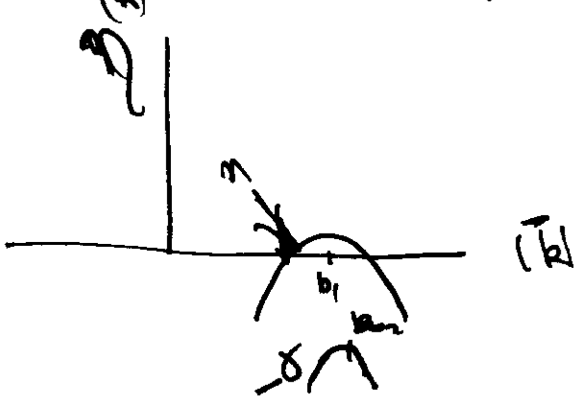
$$\partial_t A_{\vec{k}} = \eta A_{\vec{k}} + \text{nonlinear}$$

$$\partial_t B_{\vec{k}} = -\gamma B_{\vec{k}} + \text{nonlinear}$$

where

$$\eta = \eta_{k,1,+} \cong \epsilon^2 - \xi_1(\vec{k}^2 - k_1^2)^2$$

$$\gamma = -\eta_{k,2,+} = \lambda - \xi_2(\vec{k}^2 - k_2^2)^2$$



I have been reluctant to write the nonlinear terms explicitly because they are not very pleasant. When you expand in vertical modes the quadratic terms make a bit of a mess: ~~body~~ $u^2 \sim (AW_1+BW_2)^2$

$$\sim A^2W_1^2 + 2AB\,W_1W_2 + B^2W_2^2$$

If W_1 is something like $\sin \pi z$

and W_2 " " " $\sin 2\pi z$

and we project onto $\sin \pi z$ and $\sin 2\pi z$

we get a term like AB in the equation for A

and " " " A^2 " " " " B.

But we also Fourier transform in $\vec{k}$. The expanded fluid equation look like

$$\partial_t A_{\vec k} = \eta A_{\vec k} + \int B_{\vec q}\, A_{\vec k - \vec q}\, d\vec q$$

$$\partial_t B_{\vec k} = -\gamma B_{\vec k} + \int A_{\vec q}\, A_{\vec k - \vec q}\, d\vec q$$

Of course, there are more terms in these equations and more such equations.

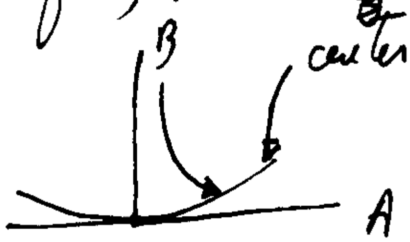
Let us simplify further to 2-d convection. All this reduces to

$$\partial_t A_k = \eta A_k + \int_{-\infty}^{\infty} B_q A_{k-q}\, dq$$

$$\partial_t B_k = -\gamma B_k + \int_{-\infty}^{\infty} A_q A_{k-q}\, dq .$$

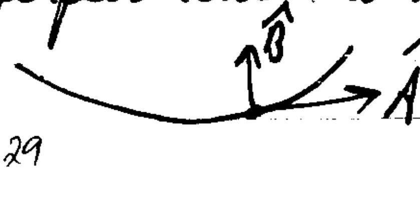
So this is a 'metaphor' for the full problem which has an infinitude of such equations, each of which contains many more nonlinear terms (used with Louis Tao)

We now surmise that the B_k go rapidly to equilibrium — and that form of the equilibrium is determined by A_k. Before, we saw that, in the discrete case,



$$B = B(A).$$

We can use coordinates $\tilde{A}$ and $\tilde{B}$ which are respectively in and perpendicular to the invariant manifold.



This change of coordinates is designed to simplify the nonlinear terms; the linear terms are already as simple as we can expect. So we transform to new coordinates:

$$A_k = \tilde{A}_k + \mathcal{F}_k[\tilde{A}_k, \tilde{B}_k]$$

$$B_k = \tilde{B}_k + \mathcal{G}_k[\tilde{A}_k, \tilde{B}_k]$$

where $\mathcal{F}$ and $\mathcal{G}$ are strictly nonlinear functionals. (This is sometimes called a near-identity transformation.) We now replace $\mathcal{F}_k$ and $\mathcal{G}_k$ by their functional Taylor series:

$$\mathcal{F}_k[\hat{A}_k, \tilde{B}_k] = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \; \tilde{A}_p \tilde{A}_q \, \mathcal{I}_k(p,q) + \cdots$$

$$\mathcal{G}_k[\tilde{A}_k, \hat{B}_k] = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \; \tilde{B}_p \tilde{A}_q \, \mathcal{J}_k(p,q) + \cdots$$

(I write this with the idea in mind that the A's are generally larger than the B's.)

Our hope is that this transformation can transform the equations into

$$\partial_t \tilde{A}_k = \eta \tilde{A}_k + \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \; \Phi_k(p,q) \tilde{A}_p \tilde{A}_q + \cdots$$

$$\partial_t \tilde{B}_k = -\gamma \tilde{B}_k + \tilde{B}_k \left\{ \int_{-\infty}^{\infty} dp \; \Psi_k(p) \tilde{A}_p + \cdots \right\}$$

The leading terms in this form give us an equation in $\tilde{A}_k$ alone for the first equation (a functional Landau equation) and an equation for B_k that says that once $\tilde{B}_k = 0$, it stays that way. Hence $\tilde{B}_k = 0$ defines the functional center manifold whose form is

$$B_k \cong \mathcal{G}_k[A_k, 0] \cong \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \; A_p A_q \; J_k(p,q)$$

This is no more that statement that

$$|\partial_t B_k| \ll |\gamma B_k|$$

So, in leading order, the equation for A_k is of this form

$$\partial_t A_k = \eta A_k + \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dr \; A_p A_q A_r \mathcal{K}_k(p,q,r) + \cdots$$

The kernel $\mathcal{K}_k$ is worked out by following the substitutions. Now define

$$A(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} A_k(t)\, e^{-ikx}\, dk$$

Transformation of the equation for A_k gives, when we recall that $\eta = \epsilon^2 - \xi(k^2-k_1^2)^2$,

$$\partial_t A = [\epsilon^2 - \xi(\Delta + k_1^2)^2]A + \text{cubic and higher terms.}$$

These cubic terms are generally complicated. In the ~~one-d~~ two-d case, they may be simplified. Swift & Hohenberg suggested that a good model would be (for the general case)

$$\partial_t A = [\epsilon^2 - \xi(\Delta + k_1^2)^2]A - A^3 .$$

As I say, this is essentially right for 2-d cases and works quite well for 3-d convection. It is a generalization of the Landau equation to the situation where the amplitude of the convection depends on position. This is an equation governing the horizontal pattern of convection. We set $k_1 = k_c$ to make the notation standardand see that this equation in the steady, linear problem is just the planform equation for linear theory when $\epsilon^2 = 0$, where linear theory is relevant.

Write $L_\epsilon = \epsilon^2 - \xi(\Delta + k_c^2)^2$. Then we see that

$$\partial_t A = \frac{\delta}{\delta A} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\tfrac{1}{2} A L_\epsilon A - \tfrac{1}{4} A^4\right]$$

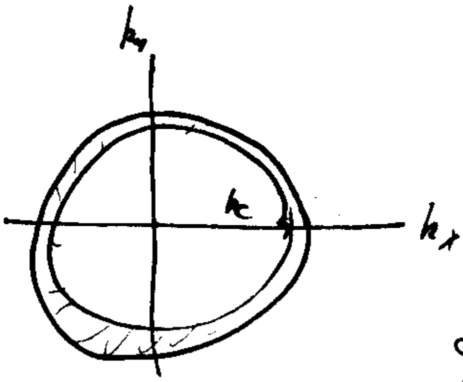
Let $$\mathbf{F}[A] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\tfrac{1}{4} A^4 - \tfrac{1}{2} A L_\epsilon A\right]$$

Then we can show that

$$\frac{d\mathbf{F}}{dt} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \,(A_t)^2 \leq 0$$

However, I have not worried about boundary conditions in this calculation. The interesting convective patterns one sees are probably much affected by boundary conditions. Also there does not seem to be a lower bound unless the derivatives are small in this sense:

In Fourier space $\Delta \sim -k^2$. $\Delta + k_c^2 \sim -k^2 + k_c^2$



If only wave numbers in the annulus arise

$$(\Delta + k_c^2)^2 \sim \epsilon^2$$

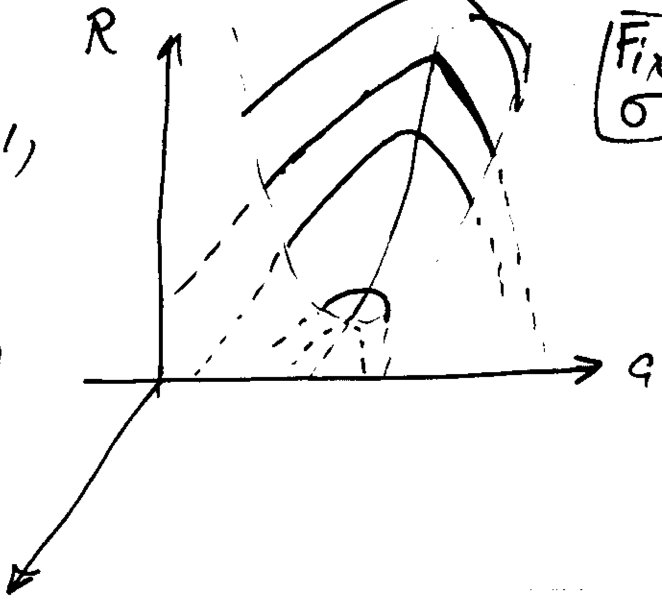
So F could be bounded from below. But defects (irregularities) in the patterns form, and this ~~[illegible]~~ makes for real complications although they strictly are not allowed since we assume η is small.

The Swift-Hohenberg model contains some of the dynamics of the all instabilities discovered by Busse. Usually, people try to simplify the equation before turning to that.

The aim is to go back to A_k eqn. and notice that $(k^2 - k_c^2)^2 \cong 4k_c^2(k-k_c)^2 = 2k_c^2\epsilon^2K^2$; $\underline{k-k_c = \epsilon K}$
(one-d only!)

Now instead an amplitude depending x, we get one depending on $\epsilon x = X$. Standard reductions give a simplified equation called G-L (Ginzburg-Landau) equation. I leave this out here under the assumption that someone else will talk about this rather standard topic.

We saw that when $\vec{k}$ is not in the critical annulus, η may not be that small. Let us back up and look at this.



$\boxed{\text{Fix } \sigma}$

In η-R-a space, for $n=1$, $\eta(R,a)$ is a <u>Ridge</u> which is above sea-level ($\eta=0$) for $R > R_1(a)$ and submerged ~~for~~ ($\eta<0$) for $R < R_1(a)$.

$$R_1(a) = \frac{(a^2+\pi^2)^3}{a^2}$$ for B.C. <u>cooked</u> to make $W_1(z) = \sin \pi z$

But R_1 always looks like this for general B.C. <u>provided</u> $\theta = 0$ on $z=0$, And

$$\eta^2 + \alpha\eta + \beta\left(R_1(a) - R\right) = 0$$

where α and β are parameters that are positive and depend on σ. (This exact for the cooked B.C.s and qualitative otherwise.)

Near <u>onset</u> $R - R_1$ is small, so we concentrated on the small root,

$$\eta \cong \frac{\beta}{\alpha}\left[R - R_1(a)\right]$$

We replaced η by $\frac{\partial}{\partial t}$ (by a longish more formal approach)

and approximated $\frac{\beta}{\alpha}\left[R - R_1(a)\right]$

by $\epsilon^2 - \xi(a^2 - a_c^2)^2$. So we had $a \to k$ by conventional usage

~~But~~ $\partial_t A_k = \left[\epsilon^2 - \xi(k^2 - k_c^2)^2\right] A_k + \cdots$

However there is also a larger root

$$\eta \cong -\alpha.$$

So we might look an equation for the mode $(k, 1, -)$ of the form

$$\partial_t C_k = -\alpha C_k + \cdots$$

Or we could be more daring and go straight into

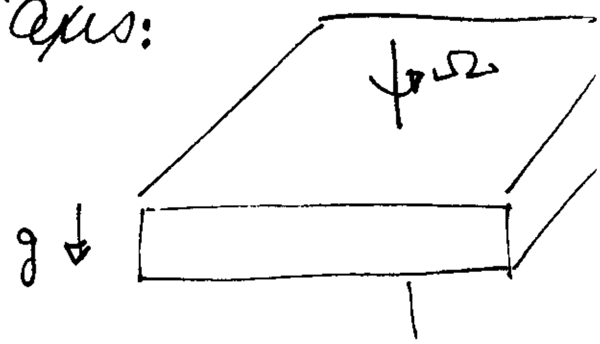
$$\left\{ \partial_t^2 + \alpha \partial_t - \left[\alpha \epsilon^2 - \xi (k^2 - k_c^2)^2 \right] \right\} a_k = \text{nonlinear} = a_k^3 \text{ (say)}$$

In other words, when our limited discussion breaks down, there must be more appropriate simplifications short of doing the full problem. There are two ways (or more) to get such "improved" equations: (1) is by brute force guessing and aggressive approximation such as you have seen just now or (2) is based on a careful tuning of the parameters that permits an asymptotically sound derivation of such an equation.

(1) is not very reliable and (2) is of very limited applicability.

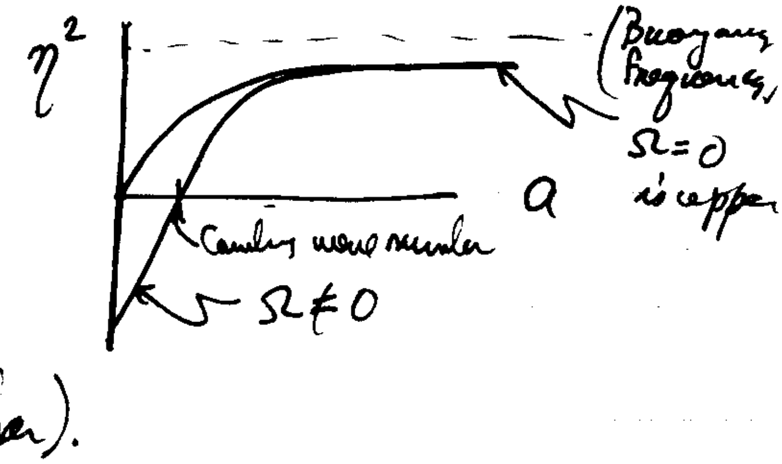
I like them both because they serve their purpose: to give us an idea of the kind of solutions the equations possess and of the kind of behavior we can expect to observe. Also, some of the equations derived in this way are of interest in themselves.

Let me now indicate the way method (2) goes. We complicate the problem by rotating the layer about a vertical axis:



This problem was studied by Cowling over 40 years ago.

With no viscosity or conductivity convective modes have this growth rate



Now put in conductivity and the stable modes go unstable with growing oscillations (Chandrasekhar).

For the onset of growing oscillations in the <u>discrete</u> case, we get a simple generalization of the Landau equation, for <u>complex</u> amplitude, A:

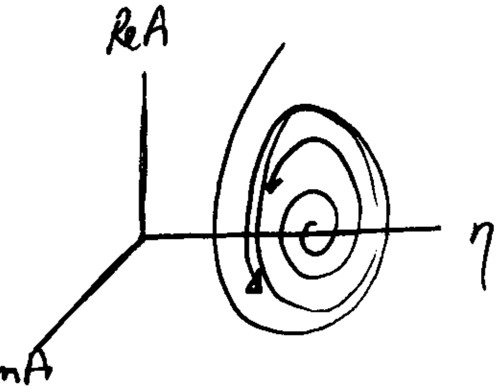
$$\dot{A} = [i\omega + \eta] A - \gamma |A|^2 A$$

where η is again growth rate involving $R - R_1(a;\Omega)$ and ω is the frequency in Cowling's work. γ is a parameter

Let

$$A = R e^{i\theta}$$

$$\left[\begin{array}{l} \dot{R} = \eta R - \gamma_R R^3 \\ \dot{\theta} = \omega + \gamma_i R^2 \end{array}\right.$$

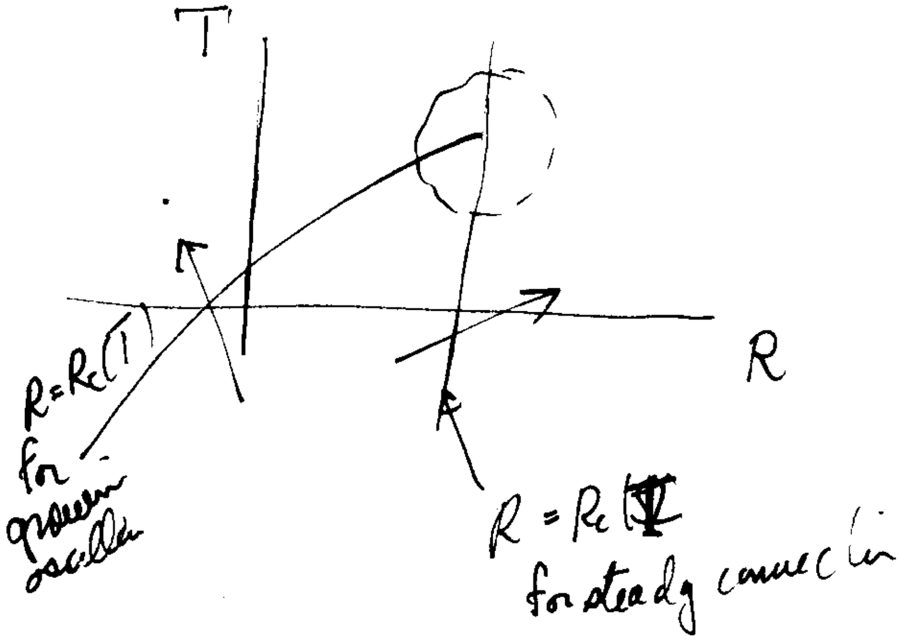


R is independent of θ

$\left\{\begin{array}{l}\text{The } R \text{ equation in the continuous case} \rightarrow \text{Swift-Hohenberg} \\ \text{But now amplitude and phase vary in space.}\end{array}\right.$

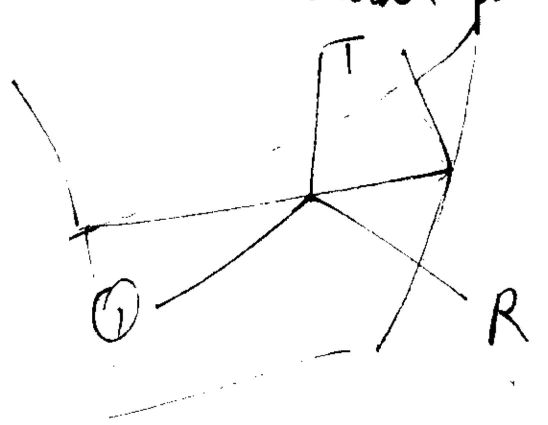
But there are more complications since ξ does not decouple in linear theory and the equation for η is a cubic. (It also without rotation, but that factors in an evident manner.)

Just as we write the nondimensional ΔT as R we can write the nondimensional Ω as $T = \frac{4\Omega^2 d^4}{\nu^2}$.



The two arrows show the paths to instability that we have discussed.

In the circled region the matter is more delicate since both modes go unstable together. Now put on another dynamical effect such as Magnetic Field — another parameter — Q say



There are now three modes and we have three coupled equations:

$$\begin{cases} \dot{A} = \eta_1 A + f(A,B,C) \\ \dot{B} = \eta_2 B + g(A,B,C) \\ \dot{C} = \eta_3 C + h(A,B,C) \end{cases}$$

The three growth rates η_1, η_2, η_3 are small.
At this point, we get to equations of third order.
We eliminate B and C to get one equation in A:

$$\dddot{A} + \alpha \ddot{A} + \beta \dot{A} + \gamma A = \text{Nonlinear}(A, \dot{A}, \ddot{A})$$