INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR. /55/21

Workshop on Fluid Mechanics

(7 - 25 March 1994)

The chaotic solar cycle

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These are preliminary lecture notes, intended only for distribution to participants

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THE CHAOTIC SOLAR CYCLE

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1. Preface

Almost 400 hundred years ago Galileo noticed that the period of a pendulum is the same for all small amplitudes. Not long afterwards, Galileo and his contemporaries (see Figure 1) proved that sunspots really were on the sun. So the same person was involved in discovering the paradigm of periodicity and establishing an exemplar of irregularity. But just how irregularly do sunspots behave? In modern terms, this question comes down to asking how many degrees of freedom are involved in the phenomenon. If the mechanism I am going to describe here, *on/off intermittency*, is operative, this question cannot be answered soon (Platt *et al.*, 1993a).

That I should begin this discussion by mentioning aperiodicity is a sign of where we are in the long saga of sunspot studies. Shortly after Galileo's discoveries, serious work on sunspots got under way. This was somewhat disappointing for a time because sunspots had become quite scarce, with only a few per year being detected. This intermission in solar activity lasted approximately throughout the life of Newton, being most extreme when he was in his prime and ending about a decade before his death (Eddy, 1978). So the question of the changing level of solar activity must have been much on astronomers' minds at that time. By the time this puzzle was fading from memory, a new issue was raised in the middle of the nineteenth century, when it was noticed that the level of solar activity (as judged mainly by sunspots) was found to vary with some regularity. The variation was taken to be periodic with a ten-year period on (at first) insubstantial evidence, perhaps because the assumption of periodicity came naturally to those indoctrinated with the behaviour of the pendulum. This variation was supported by ancient Chinese observations (Fritz, 1882) and it may have been one motivation for the careful recording of sunspot (or Wolf) numbers in Zurich for the last hundred years. In any case, it quickly must have become clear that the sunspot number was not varying periodically but, as someone wisely put it, cyclically, with a time scale of eleven years.

The variation of the annual sunspot number with time over the past two centuries is shown in Figure 2. It is natural to look for an oscillator driving this phenomenon and to ask how many degrees of freedom are represented in the mechanism. I shall argue here that we can model the process as a relatively simple dynamical system that has both the desired cyclic character and the strong intermittency revealed in the so-called Maunder minimum that occurred in Newton's time. Perhaps, from such a mathematical model, we can attempt to read something of the physical nature of the process itself. This is not the usual direction of astrophysical research, which begins by trying to isolate the physical mechanisms behind observed processes. In trying to proceed in terms of a generic mathematical description, I am illustrating the approach of what I call *astromathematics.* However, both approaches have been used in getting to the model described here so that a certain amount of physical background is given. The end product is a set of equations

To appear in Proceedings of Dynamo Workshop, I. Newton Inst. {M.R.E. Proctor / A.D. Gilbert} eds

whose output looks like the observed variation of the sunspot number. Nevertheless, I have had to forgo calling this "an astromathematician's apology" since I have not strictly followed the rules of the game, as well as for more obvious reasons.

For those who like to read only introductions, let me say here that the proposed mathematical model has two essential ingredients. First, it contains a simple oscillator. And secondly, the model exhibits extended periods with little activity. This behavior is like the intermittency seen in turbulent fluids aand has been called on/off intermittency. It is built into the mathematics by arranging for the equations to admit an invariant manifold within which the system does not exhibit the behavior that will be called activity. The manifold has both stable directions along which the system is occasionally drawn into its neighborhood for extended periods and unstable directions in which it flies out again to resume the large oscillations that here represent the solar cycle. One can make several versions of this process, differing in detail, but what I am after here is the isolation of specific mathematical mechanisms that may be incorporated in such models so as to capture the main temporal features of the global solar cycle (Platt *et al.*, 1993b). Such models can also be made spatio-temporal, and this is being done.

2. The Solar Tachocline

At any stage of the solar cycle, sunspots are concentrated in a particular band of latitude whose location drifts toward the equator as the cycle progresses, beginning at $\pm 40°$ and decreasing to $\pm 5°$ in the course of eleven years. By the time a given cycle is ending at $\pm 5°$, the next one has already begun to appear. Of course, it is not the individual spots themselves that move toward the equator, for spots rarely last more than a month or so. This progression in latitude gives the impression that there are *solitary* waves of solar activity whose propagation time is eleven years. The nature of such waves and their fate when they meet at the equator raise questions that I will address presently. Why should the spots not appear all over the place, given that they are appearing at all? The confinement in latitude is a hint that the activity might originate in a physically distinct layer and the wave-like motion of the locale is suggestive of the influence of a wave guide whose thickness reflects the width of the activity zone. One such layer might be the convection zone itself, whose thickness, $\frac{1}{3}R_\odot$, is not all that much greater than the width of the band of sunspot activity. However, the strong spots have fields of several kilogauss. At fields well below this, magnetic tubes will have lowered density inside them and be buoyed up to the surface. Unless they are in the deep convection zone, or below this, they will emarge before they have time to develop the field strength seen in many spots. But in the deep convection zone ten kilogauuss ~~likely that~~ counts for less, ~~and~~ Brandenburg and Tuominen (1991) report that there is sufficiently strong downwelling in compressible convection, to overcome the effects of magnetic buoyancy. Hence the lower convection zone might serve as the seat of solar activity (DeLuca, 1986). Other possible sites for the origin of solar activity have been considered as well (for example, Layzer *et al.*, 1979).

Another distinct layer with the right properties is the one that mediates the transition between the differential rotation of the solar convection zone and the rotation of the bulk of the sun, or radiative interior. Such a layer was discussed twenty years ago (Spiegel, 1972) but its existence became a reality when helioseismologists were able to infer the distribution of the solar differential rotation well into the sun (Brown *et al.*, 1989, Goode *et al.*, 1991). According to them, the variation of rotational velocity with latitude that is seen on the solar surface continues with little change through the solar convection zone. Throughout the convection zone, the equator turns faster than the poles with a velocity that is constant on cones. I will take this motion in the convection zone as specified, much as oceanographers take the wind stress on the surface of sea as prescribed, though I am sure we are both somewhat in error. Of course, oceanographers allow for time dependence of the wind stress, but helioseismology is not old enough to give us accurately the corresponding variability for the large scale flow of the solar convection zone. Even the picture of constancy on cones remains tentative (Gough *et al.*, 1992).

The inner sun turns rigidly, at least down to depths at which acoustic sounding works. And between these two regimes there is an unresolved transition that is reminiscent of the thermal transition layer between the earth's atmosphere and the deep ocean. In fact, it is perhaps even more like the layers in planetary atmospheres that produce lively activity and are called weather layers. At least, I am claiming that this transition layer produces the magnetic weather in the sun called solar activity. This analogy to geophysical layers like the oceanic thermocline led to the name *tachocline* for the solar rotational transition layer (Spiegel and Zahn, 1992).

Even now that the tachocline's existence has been confirmed, it is not clear why it is there. We may reasonably assume that the stresses exerted by the differential rotation of the convection zone on the interior will produce effects in the stable layers. But the implied turbulent spindown process will tend to spread the effects well into the interior and not leave a well-defined layer. However, strongly anisotropic turbulent stresses that produced by horizontal shear in a vertically stabilized layer (Zahn, 1975) could short-circuit this spreading, as could strong horizontal magnetic stresses. To show how this works, in the case of tuurbuulent viscosity, let me give an equationless summary of our estimate of the tachocline thickness on the assumption of a steady tachocline.

If we take the rotational flow in the convection zone as given, its mismatch to the interior rotation will cause a large scale convective pumping process that drives a vertical velocity, w, just below the convection zone (Bretherton and Spiegel, 1968). This will generate a meridional current with north-south component

$$u \sim \frac{R_\odot}{\ell} w \tag{2.1}$$

with vertical extent ℓ. Strictly, a density gradient term is needed, but this is not important as long as ℓ is less than the density scale height, which is rather large in the tachocline.

We need to balance the Coriolis force caused by the north-south motion. If we do this with eddy viscosity operating on the azimuthal flow, v, we have

$$\Omega u \sim \left(\frac{\nu_H}{R_\odot^2}\right) v \,, \tag{2.2}$$

where ν_H is the eddy viscosity of the horizontal turbulence. We have included only horizontal turbulent stresses since they may be expected to dominate in a medium with strongly stable vertical stratification (Zahn, 1975).

The azimuthal flow will also produce a Coriolis force and we need a north-south pressure head to balance it:

$$\rho v \Omega \sim \frac{\Delta p}{R_\odot} . \tag{2.3}$$

We are assuming that the scale of variation in latitude is of the order of the solar radius. The pressure perturbation has a vertical derivative which is hydrostatically balanced:

$$\frac{\Delta p}{\ell} \sim g \Delta \rho \sim \frac{g \rho \Delta T}{T} , \tag{2.4}$$

where signs are ignored.

Once we bring in the temperature perturbation, we need to worry about maintaining it and that requires advection of heat to balance radiative diffusion:

$$w \frac{d\Theta}{dz} \sim \frac{\kappa}{\ell^2} \Delta T , \tag{2.5}$$

where κ is the thermal diffusivity and $\frac{d\Theta}{dz}$ is the vertical, unperturbed potential temperature gradient (that is, the entropy gradient in some units). Since we are in the radiative zone, κ is the radiative diffusivity, and the contribution by turbulence is small.

The condition that these balances should be mutually compatible is

$$\ell \sim R_\odot \left(\frac{\tau_H}{t_{ES}} \right)^{\frac{1}{4}} , \tag{2.6}$$

where the horizontal eddy time, τ_H, is $R_\odot^2/\nu_H$ and $\tau_{ES} = (N R_\odot)^2/(\kappa \Omega^2)$ is the Eddington-Sweet time. If the theory is carried out with an isotropic turbulent stress tensor, spin down spreads the effects vertically and the tachocline thickens inexorably. But as long as the stable vertical stratification favors a strong horizontal turbulence, we can maintain a thin tachocline, though there will be some vertical spreading from the initial mismatch, or from any time dependent forcing.

A thin tachocline with horizontal turbulence will engender the coherent structures — vortices and flux tubes — that are between the lines of this discussion. On the other hand, (2.6) does not stand by itself as we know neither ℓ nor ν_H, but the observations ought to tell us the former before long. For now, we may note that the value of ℓ does not depend sensitively on the details of the flow in the convection layer and requires only that there be a mismatch between that flow and that of the deep interior. Then $\ell \sim 20{,}000(\kappa/\nu_H)^{\frac{1}{4}}$ km. Although a similar story might be made with magnetic stresses, the eddy viscosity approach leads to qualitative agreement between the empirical isorotation curves (Morrow, 1988) and the theoretical ones (Spiegel and Zahn, 1992) as shown in Figure 3.

3. The Solar Oscillator

In the analogy between the solar tachocline and the oceanic thermocline, the solar convection zone is like the earth's atmosphere and the solar interior is the abyssal ocean.

Instead of rain we have plumes twisting downward through the convection zone, dragging down (and perhaps enhancing) magnetic fields. Such thermals are known in experimental convection and in the earth's atmosphere. Simulations of highly stratified convection shows that the descending plumes are frequent, but there are no comparable rising plumes. In the simulations reported by Brandenburg and Tuominen (1990) downwelling brings magnetic fields to the depths of the convection zone with a vigour that may overcome the opposing tendencies of magnetic buoyancy.

The descending matter, with its trapped magnetic field, will be entrained by the turbulent motions in the tachocline where it is sheared out to build up a toroidal component over long times. How extensive this reservoir is, or how it is structured, are questions that have troubled me for a long time. Other issues like the structure at high latitudes and the effects of the meridional circulations are also worrisome, since they may bear on observational details. To get to the mathematical model we do not need to answer these questions, but they must be faced some day. For now I will simply assume that the toroidal field is there in the tachocline in describing the scenario that S. Meacham and I have been trying to develop over the past few summers in Woods Hole for feeding this field into the convection zone to maintain some kind of balance and, incidentally, to produce spots in the process.

If the tachocline is like an atmospheric weather layer, such as the oceanic thermocline, we must expect it to develop vortices, as does every such layer we can observe well (Dowling and Spiegel, 1990). These vortices will have more or less vertical axes and, when a toroidal magnetic filament impinges on one, it will wind the field up. If the process were confined to the tachocline, we might expect flux expulsion from the vortex (Parker, 1979, Chap. 16). But the local strengthening of the field produces magnetic buoyancy that will lift the field-containing region up into the convection zone. In this way, a rising magnetic tube will be extruded from the tachocline like the output of a cotton-candy machine. Such rising helical tubes return the field to the convection zone in a process that is the surrogate of evaporation in the magnetic weather cycle. A buoyant tube will ultimately protrude through the solar surface to form a single spot or a strong tube may go beyond the surface before falling back to produce a second, more diffuse region of magnetic disturbance.

Whatever the details of such a cycle, the general picture is that the tachocline has a source of field from above to which it may return the field by this and other processes (Spiegel and Weiss, 1980). If there is field stretching, much of it occurs as the helix is twisted out of the tachocline. One form of such a process is in Cattaneo *et al.*, 1990. (For another vision of the role of vortices, see Parker, 1992.)

I mention these images to motivate the construction of (what engineers call) a lumped model of the solar cycle. At that coarse level, we ignore all the spatial detail implied by the magnetic meteorology and simply introduce a parameter, β say, that measures the degree of instability of the magnetic field in the tachocline. When $\beta > 0$, the convection zone is feeding the process abundantly and the magnetic buoyancy is able to extrude strong, ordered fields. This could work in several ways.

There could simply be overstable magnetoconvection giving rise to oscillatory instability and β would measure something like the difference between a magnetic Rayleigh number and its critical value (Childress and Spiegel, 1981). Or there could be a dynamo process, such as an $\alpha - \omega$ dynamo and β could be related to the dynamo number. In the lumped model, we simply need a potentially unstable oscillator and may assume that its operation is described by the normal form for the appropriate bifurcation, either a Hopf bifurcation or a BLT bifurcation (Bogdanov-Lyapunov-Takens).

In the former, for fixed β, the complex amplitude of the oscillation is given by the normal form for a Hopf bifurcation

$$\dot{A} = (\beta + i\omega)A - |A|^2 A. \tag{3.1}$$

I have presumed for definiteness that the bifurcation is supercritical and have scaled the coefficient in the nonlinear term equal to unity. If we were starting from first principles, we should be able to relate the parameters to the physical properties of the model. For now, I simply assume that $\pi/\omega \approx 11$yrs and leave β free. If this is the oscillator that describes the solar cycle at some level, some property of A should be the measure of the toroidal field that is somehow forced to poke out of the sun and produce spots.

Alternatively, we might favor the more subtle BLT bifurcation (as, for some years, I did). In a simplified version with linear friction, the real amplitude of the oscillation is governed by

$$\ddot{\mathcal{A}} = \beta\mathcal{A} - \gamma\dot{\mathcal{A}} - \mathcal{A}^3. \tag{3.2}$$

As they stand, neither of these oscillators will by itself adequately describe the complications of the solar cycle. To make the oscillations aperiodic and intermittent — in a word, chaotic — we allow β to vary slowly.

4. On/Off Intermittency

An oscillator like (3.2) becomes chaotic when its parameters are made to vary suitably in time. We may impose this time dependence, or it may come about through a feedback of the oscillation on the mechanism that determines the value of the parameter. For example, suppose that instead of having β constant in (3.2), we let it vary according to

$$\dot{\beta} = -c[\beta + a(\mathcal{A}^2 - 1)] , \tag{4.1}$$

where a and c are specified parameters. A simple transformation turns (3.2) and (4.1) into the Lorenz equations, originally devised in the study of thermal convection. So there is little doubt that this is a system capable of producing aperiodic behavior for appropriate values of the parameters.

This way of producing chaotic systems, by letting simple oscillators feed back on their parameters (Marzec and Spiegel, 1980) may be used to generate equations for excitable media, so perhaps in a case like this, we ought to refer to hysterical media. But I would prefer to reserve this usage for the case of intermittency, for an example of which, suppose that in (3.2) $\beta = Z - 2Y$ and that for Y and Z we have the equations

$$\ddot{Y} = -Y^3 + ZY - \gamma\dot{Y} - \mathcal{A}^2 \tag{4.2}$$

$$\dot{Z} = -\epsilon[Z + a(Y^2 + \mathcal{A}^2 - 1)]. \tag{4.3}$$

When $\mathcal{A} = 0$, equations (4.2) and (4.3) constitute the form of the Lorenz equations that I just mentioned. So $\mathcal{A} = 0$ is an invariant manifold of the fifth-order system that combines these two equations with (3.2). Figure 4 (from Spiegel, 1981) shows $\mathcal{A}(t)$ for $\epsilon = 0.1$, $a = 6.5$ and $\gamma = 0.4125$. This example of intermittent behavior with episodes of inactivity in $\mathcal{A}$ recalls the inactive sun of Newton's time.

The term intermittency has been used in dynamical systems theory to describe alternation between two modes of activity, as in the Pomeau-Manneville (1980) theory. To restore the meaning of the word as used by fluid dynamicists, the term *on/off intermittency* has been proposed (Platt *et al.*, 1993a) to connote alternation between activity of a certain kind and inactivity, as in Figure 4. The present interest of the model is that there is continuous chaos in the invariant manifold, but the behavior of $\mathcal{A}$ alone shows on/off intermittency. In this metaphor for the solar cycle, chaos in the Lorenz system represents convection and $\mathcal{A}$ the solar activity. The merit of the model is that it captures the kind of intermittency that the cycle manifests, but otherwise Figure 4 does not look very much like Figure 2. One reason is that the effect of the solar activity ($\mathcal{A}$) on the convection (Y, Z), is pronounced and this makes for great irregularity. There must really be such coupling, but it is likely to be weaker than in this model. We turn to a model which better captures the nature of the solar cycle. In this one there is no feedback of the oscillator on the chaotic driver.

In on/off intermittency, the intermittent behaviour is organized by an unstable invariant manifold with stable and unstable manifolds coming into and out of it (Platt *et al.*, 1993a). When the system moves away from the manifold, it bursts into activity until it is brought back very close to the manifold along a stable manifold to hover inactively before being sent out again. This may be seen as a chaotic relaxation oscillation , or a higher dimensional version of homoclinic chaos, or as what is called bursting in neurophysiology (Hindmarsh and Rose, 1984). The general idea is to make a potentially unstable oscillator whose stability parameter is the variable of an associated chaotic system. There are many ways to set this up, so what we are isolating is not a particular model but a particular mechanism, on/off intermittency. Whether the oscillation really is generated by an instability of the tachocline is a separate issue that is not central to the mathematical description. We do not even need the tachocline for the mathematical model to work, though it is useful to think in such explicit terms. An interesting analysis of on/off intermittency has recently been given by Heagy *et al.* (1993) and there are by now several discussions of this kind of process (Yamada and Fujisaka, 1986-87; Hughes and Proctor, 1990; Pikovsky and Grassberger, 1991). One key result is that, if this process is going on in the solar cycle, we have no real hope of determining the dimension of the solar attractor by any of the presently known means. It is not just that the data are inadequate for the purpose, as has already been objected (Spiegel and Wolf, 1987), but that the on/off process imposes a sort of indeterminism on dimension determination (Platt *et al.*, 1993a).

Here is a mathematical model for the solar cycle (Platt *et al.*, 1993b) that has the features I have outlined. We take the standard form (3.1) for the oscillator, which we couple to a chaotic system by letting $\beta = \beta_0(\mathcal{U} - \mathcal{U}_0)$ where β_0 and $\mathcal{U}_0$ are fixed parameters. So (3.1) becomes

$$\dot{A} = [\beta_0(\mathcal{U} - \mathcal{U}_0) + i\omega]A - |A|^2 A \,. \tag{4.4}$$

This says that the instability is strongly affected by $\mathcal{U}$, which is determined by something else in the system. In particular we generate $\mathcal{U}$ with this third order system:

$$\ddot{\mathcal{U}} = r\mathcal{U} - \mathcal{U}^3 - q\dot{\mathcal{U}} - \mathcal{V} \tag{4.5}$$

$$\dot{\mathcal{V}} = \delta[\mathcal{V} - p\mathcal{U}(\mathcal{U}^2 - 1)] \,, \tag{4.6}$$

where (r, q, p) are more parameters. Like (4.2), (4.5) is a modification of (3.2).

This time, the chaotic driver is a particular case of a model that was constructed to clarify the physics of doubly diffusive convection (Moore and Spiegel, 1966). The system (4.4)-(4.6) makes a fair model of the solar cycle, at least in the coarse-grained sense. We do have to make some decision about what to compare to the sunspot number, though this appears not to be crucial. In Figure 5 we see a plot of the square of ReA vs. time showing several intermissions in activity. Within a long period of activity, the cycle will be chaotic as we see clearly in Figure 6, a portion of Figure 5 with an enlarged time scale. These results are robust and we do not need a lot of fine tuning of all these parameters to get this behaviour.

In fact, the observed sunspot number variation is much more ragged than this model predicts, as we see from in Figure 2. So there is evidence that more is happening than just an intermittent oscillation such as is shown here. If the cycle does come from a deep layer, we are seeing it through the convection zone, which will add its own direct input while distorting the "true" signal. That can be modeled too (Platt *et al.*, 1993b) and, when such effects are included, the qualitative agreement seems (to us) very good. But here I omit such fine points of the cycle in the belief that they are incidental.

5. Solar Activity Waves

A plot showing the latitudes of vigorous sunspot activity vs. time looks like a row of butterflies (see Weiss' discussion in Chapter 2). This Maunder butterfly diagram is a space-time plot showing the propagation of solar activity. Lines along the activity maxima are world lines of motion toward the equator. But what is moving? The most likely prospect is that we are seeing some kind of wave motion and, in one version, these are dynamo waves (Parker, 1979). The idea I wish to describe next is that the the butterfly diagram represents the propagation of solitary waves (Proctor and Spiegel, 1991).

If an oscillation arises in a thin layer like the thermocline, we might expect to see simple waves produced. Since the layer is thin, there should be a dense spectrum of allowed wavenumbers. If the wave numbers are closely spaced, there is effectively a continuum of them. A packet of such waves could have a solitary wave as envelope that would make a nice descriptor of the activity band in latitude. The generic form of the propagation equation should be the same for any simple overstability. The astromathematical approach allows us to discuss the butterfly diagram in a general way, even though the precise instability mechanism has not yet been isolated. There are in fact several possible instabilities, including magneto-convective overstability, instability caused by the vertical shear of the tachocline or instability of a dynamo in the tachocline. Since they would have a common mathematical description in the present coarse-grained discussion, we are not hampered on that account.

The amplitude equation for the Hopf bifurcation is based on a model in which one mode has, in linear theory, a time dependence like $\exp(\beta t + i\omega t)$ with $|\beta|$ small, and all the other modes are rapidly damped. If there is just one mode with small $|\beta|$, its complex amplitude, A, evolves according to (3.1). If the seat of the instability is a thin layer like the tachocline, there can be a band of modes with small β. But now β is a function of the wavenumber along the channel, k, and such modes can propagate.

To describe the nonlinear development of the instability, we construct a packet of waves, in which the amplitude of each component, A, depends on k. If the system is axisymmetric in the large, we need consider only a one-dimensional case. We factor out the carrier frequency and wavenumber defined as those of the most unstable mode, and we characterize the packet by $\Psi(x,t)$, the Fourier transform of A, where x is latitude. The amplitudes measure the size of all disturbance quantities but further details will differ from case to case depending on things like the relation of packet width to the wavelength of the carrier wave. That in turn depends on subtleties like interface conditions which we are here blurring over.

On general grounds, we expect the equation for Ψ to be the complex Ginzburg-Landau equation, which is like (3.1), but with spatial derivatives as well. Strictly speaking, the governing equations are two coupled Ginzburg-Landau equations, one for each direction. Though we know how to write these down (Bretherton and Spiegel, 1983), we do not as yet have solutions relevant to the solar case, so I shall discuss only the single G-L equation here (Manneville, 1990). The reason for the limited progress is that there is a more serious complication that has to be dealt with first, one that Proctor and I (1991) have so far treated in a phenomenological way. This is the variation of underlying conditions, such as local stability, with latitude in the sun.

In the phenomenological view, the magnetic rain probably varies with latitude, and certainly the shear in the tachocline does. This inhomogeneity should induce a drift mode into the problem in addition to the one we are already omitting. However, we have so far left out this extra mode and have attempted to make amends by putting a positional dependence into the coefficients in the G-L equation. As the correct positional dependences are as yet unknowable, we have used simple forms for it. This parameterization will have to serve until we have a better understanding of the underlying variations in tachocline structure.

In the wave packet, frequencies and growth rates depend on the wave number in linear theory. We treat only situations where the width of the packet is small, as measured by some small parameter, ϵ. Linear theory provides a group velocity c_0 that we use to provide a basic reference frame. The peak of the packet is nearly stationary in the frame with coordinate $\xi = x - c_0 t$. The form of the equation, when we choose units to minimize the number of parameters, is

$$\partial_t \Psi - c(\xi)\partial_x \Psi - (\epsilon + i)\partial_x^2 \Psi + (\nu - i)|\Psi|^2\Psi = [\beta(\xi) + i\omega(\xi)]\Psi. \tag{5.1}$$

Here we have allowed for a dependence of the stability parameter, β, and of the linear frequency, ω, on the location of the solitary wave. The parameter $c(\xi)$ is a local drift speed with respect to the preferred frame.

We assume that the instability is weak and write $\beta(\xi) = \epsilon\mu(\xi)$. When $\epsilon \to 0$, (5.1) reduces to the cubic Schrœdinger equation, which admits a soliton solution that we may write as

$$\Psi(x,t) = \mathcal{R}\, e^{i\Theta(x,t)} \;, \tag{5.2}$$

where

$$\mathcal{R}(x,t) = \sqrt{2}R sech[R(x-x_0)] \;, \tag{5.3}$$

and

$$\Theta(x,t) = U(x-x_0) + \int (U^2+R^2)dt \;. \tag{5.4}$$

This soliton contains two arbitrary parameters, R and U, with $x_0 = 2Ut$. The presence of arbitrary parameters is related to symmetry groups of the nonlinear Schrœdinger equation.

The soliton, for all its remarkable stability, is a rather dull object when left to itself. When we introduce dissipation and instability into the system, a richer behavior arises. The arbitrariness of the parameters permits us to accommodate the dissipation and instability terms that come in when $\epsilon \neq 0$. For small ϵ, we let both R and U be functions of ϵt. Then the methods of singular perturbation theory lead to equations of motion for the parameters. These equations form a dynamical system that control the behaviour of the solitary wave, much as a mind does for a person. In this way, the otherwise mindless soliton is provided with a rather simple mind in the case of the standard complex G-L equation that goes right to a fixed point. However, that situation is enriched when the domain is large enough to allow instabilities that produce other solitary waves (Bretherton and Spiegel, 1983).

In the solar case, when the parameters depend on position, even a single activity wave shows a certain amount of interesting behaviour. The theory for $\epsilon \neq 0$ shows that when the amplitude and position of the solitary waves depend on ϵt, (5.2)-(5.4) represent a solution of (5.1) provided that these equations are satisfied:

$$\dot{R} = 2R[\mu(\xi) - \kappa(\xi) - U^2] - \frac{2}{3}(1+4\nu)R^2 \tag{5.5}$$

$$\dot{U} = U[2\kappa(\xi) - \frac{4}{3}R^2] + \lambda(\xi) \tag{5.6}$$

$$\dot{\xi} = 2U - c_0 \;, \tag{5.7}$$

where $\kappa = dc/d\xi$ and $\lambda = d\omega/d\xi$ (Proctor and Spiegel, 1991).

In modeling the dependences of the given quantities on latitude, we need to look at the structure of the tachocline. The helioseismological studies suggest that the rotation in the solar interior is the same as the surface rotation at somewhere around 35° latitude. The model (Spiegel and Zahn, 1992) agrees with this and predicts that the vertical shear has a minimum at this latitude. Since we expect the shear to drive instability in a dynamo, either directly or indirectly, we represent this either as a quadratic dependence over a whole hemisphere or, more crudely, as a linear dependence over the zone of sunspot activity. In either case the qualitative results are similar and, for the linear case, with μ proportional to ξ, we get results like those in Figure 7 (Proctor and Spiegel, 1991). The results are for a single hemisphere and thus represent a series of one-winged butterflies. To this extent, the model is satisfactory. It suggests that at the end of a cycle the activity wave survives and returns rapidly to midlatitudes maintaining very small amplitude, there to begin another trip to the equator.

The observations reveal that a new cycle begins in midlatitudes before the previous cycle ends near the equator. This is not seen in Figure 7. On the other hand, that picture is based on the solitary wave being a rigid object regarded as a small particle. In fact, the real reflection process is a more complicated affair lasting about the time it takes the wave to travel its own width. This seems about right for the overlap period of the two cycles. Moreover, we ought to see some of the spots associated with the return trip of the activity wave to the midlatitudes, so part of the overlap may be on that account.

Another feature of Figure 7 that is not in good agreement with the facts is that the maximum of activity occurs virtually at the beginning of the cycle. This may be a result of the form adopted for the latitude dependence of the parameters. If this model turns out to be on the right track, the phase of maximum activity may ultimately permit us to study the latitude variation of the tachocline structure.

The cycle shown in Figure 7 is periodic, but this is not surprising at this stage of the story. In the next level of development, when we include two solitary waves in the description, one for each hemisphere, we obtain a coupled pair of sets of equations like (5.6)-(5.7). This leads to chaos and north-south asymmetry, more in accord with observation. However, Proctor and I are not yet sure about coupling terms in this description of both hemispheres, so I do not give details here. In fact the major cause of aperiodicity is likely to come from input variations from the convection zone, expressed once again by a chaotic origin of β in (5.1). This will produce spatio-temporal on/off intermittency of the kind we see in the sun and the next step should be to include this mechanism in the theory.

6. Final Remarks

Since we do not have a theory of turbulence, it is not possible to make a fully deductive theory of the solar cycle on account of the involvement of the solar convection zone. Nevertheless, we can hope to make phenomenological models of increasing precision. In the work described here, there are two parallel developments along those lines, one physical and one mathematical. Both are frankly qualitative, but in the mathematical case, this may be a desirable feature.

The mathematical models discussed here are aimed at showing how the apparently complicated spatio-temporal behavior of the solar cycle can be reasonably well reproduced with relatively simple equations. This encourages us to attack the physical model in a more detailed way, despite our inability to cope with the turbulence problem. The equations describe a simple oscillatory instability fed by an aperiodic process. The sun provides the necessary ingredients for all the processes that can be read from the model equations.

The solar tachocline, the rotational transition layer between the convection zone and the deep interior, offers a natural site in which to unfold our scenario. Fed from above by plunging plumes it can entrain fluid carrying tangled magnetic field and stretch the field out into some more orderly configuration only to expel it in discrete structures. We have several promising mechanisms to choose from before setting out to follow one through to a quantitative model. But before embarking on such daunting calculations, we need to see a way through some of the unsolved problems. The main ones seem to me to be concerned with time dependence.

The solar differential rotation appears to vary on the time scale of the activity cycle (Howard and LaBonte, 1980). We do not know whether this is incidental or fundamental. In the picture I am describing, the dynamical coupling between the tachocline and the convection zone is strengthened when the spot fields link them. This time-dependent interaction could modify the structure of the tachocline and produce large-scale motions like azimuthal rolls. Whether such effects are fundamental or just secondary is not yet clear. Similarly, I do not know whether the polarity reversals that occur with the solar cycle point to some deep process or represent some superficial feature of the cycle. The true physical nature of this behavior is not reliably known and the corresponding mathematical in generic instability models has not been isolated.

Another question that has to be faced at some stage is the quantitative determination of properties of the cycle, such as the eleven-year time scale. Eleven years is very long compared to the travel time of any of the obvious waves across the tachocline, which is about a quarter of an hour for sound waves. On the other hand, eleven years is quite short compared to the conventional Kelvin-Helmholtz, or thermal, time of the tachocline of a million years, or so. The changes that the solar cycle must work in the tachocline would seem to encounter a sort of fluid-dynamical impedance mismatch between the driving frequency and these response times of the tachocline. However, the hydrostatic adjustment time of the tachocline is approximately the geometric mean of the acoustic travel time and the thermal time (Spiegel, 1987), which is of the order of years. So it may be that the period of the cycle is less of a clue to the actual process than it is to the structure of the tachocline. If this is true, we have another means of estimating its thickness. I offer this as an example of a feature of the cycle that might be fundamental but might just as well be secondary.

There are many places to seek further clues to the processes discussed such as other solar type stars (Belvedere, 1991) and turbulent disks that might show solar type processes. Indeed there are hot stars that seem to show activity resembling that of the sun (Casinelli, 1985). It difficult to know which phenomena are central to the sunspot cycle and the decision is usually subjective. The models I have described are rooted in elementary mathematical processes that seem robust. They suggest a vision of the solar activity process that differs from the conventional solar dynamo and avoid some of the difficulties solar dynamo theory faces. I am sure that the present models will also face numerous problems as they are elaborated and I look forward to learning what these will be.

Acknowledgments

The work described here was supported by the Air Force Office of Scientific Research under contract number F49620-92-J-0061 to Columbia University. Nigel Weiss kindly acted as a "referee" and I am grateful for his comments; his own views appear Chapter 2.

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FIGURE CAPTIONS

1. Sunspots in the seventeenth century from Heß(1911), courtesy of E.L. Schucking.

2. The yearly mean sunspot number as a function of time with the period of the Maunder minimum (*ca.* 1650-1720) not shown.

3. The structure of the solar tachocline from the observed (Morrow 1988) and theoretical viewpoints (Spiegel and Zahn, 1992). The tachocline thickness in the lower (theoretical) figure is arbitrary.

4. On/off intermittency from eqns. (3.2), (4.2) and (4.3) (after Spiegel, 1981). Even when the oscillator is inactive, there is chaos in the invariant manifold.

5. The activity predicted by (4.4)-(4.6) for $r = 0.7$, $q = 0$, $p = -0.5$, $\beta_0 = 1$, $U_0 = -0.15$, $\delta = 0.03$.

6. A blowup of a portion of Figure 5.

7. The dynamics of a single-winged butterfly according to (5.5)-(5.7) (after Proctor and Spiegel, 1991).

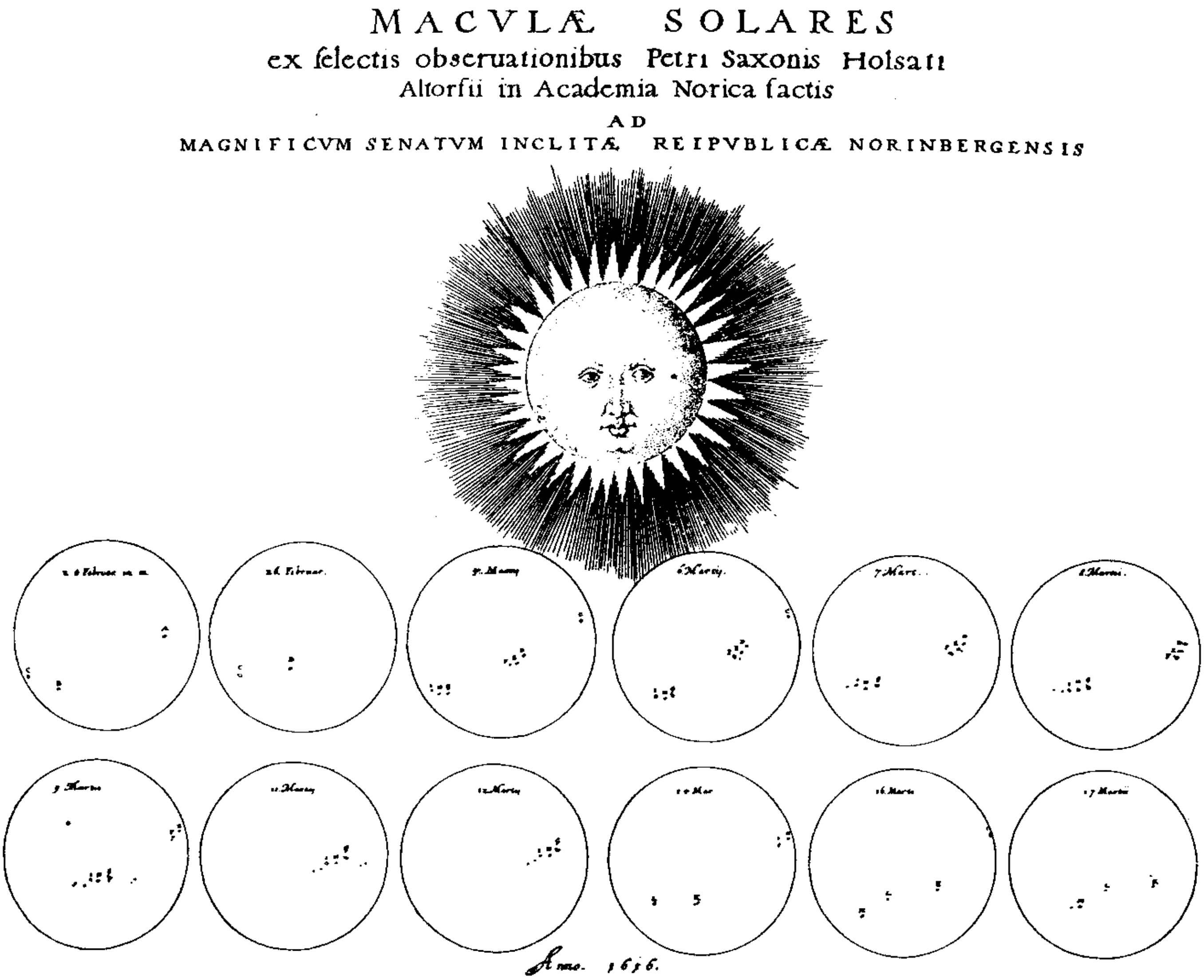

Abb. 23. Sonnenflecken, aufgenommen 1616 auf der Nürnbergischen Akademie zu Altdorf.

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Figure 1

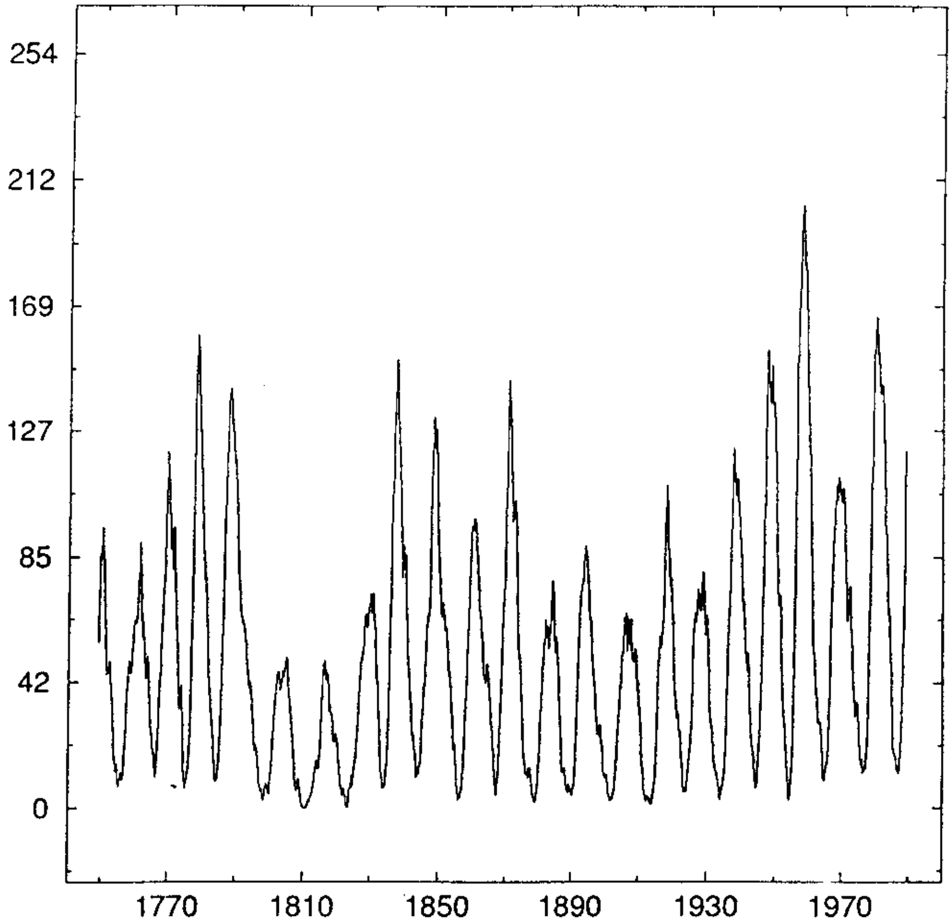

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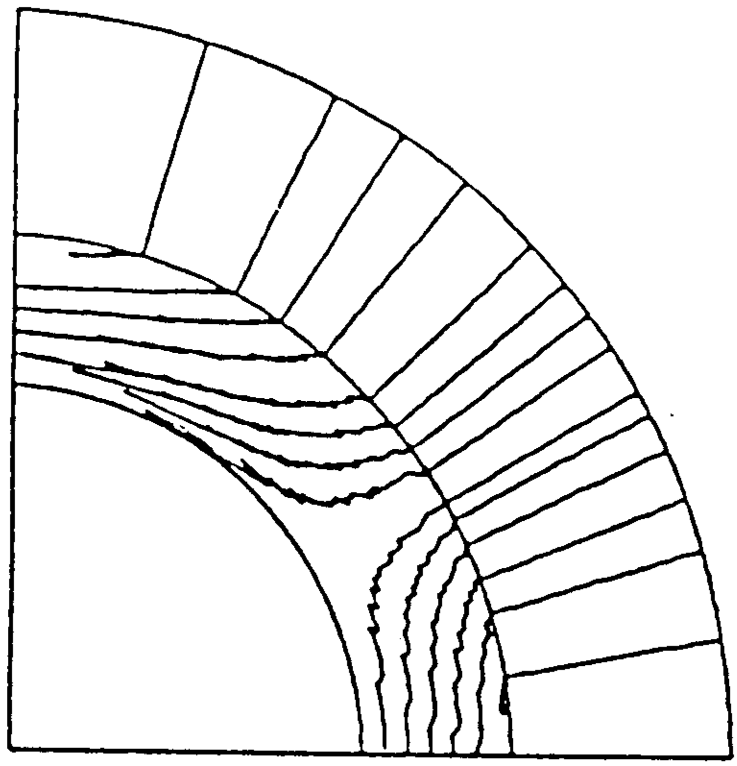
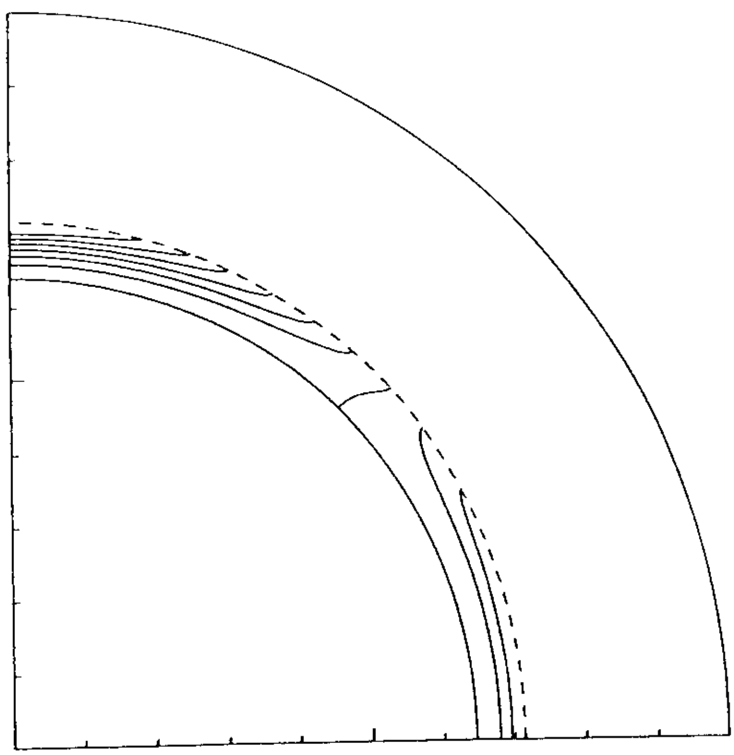
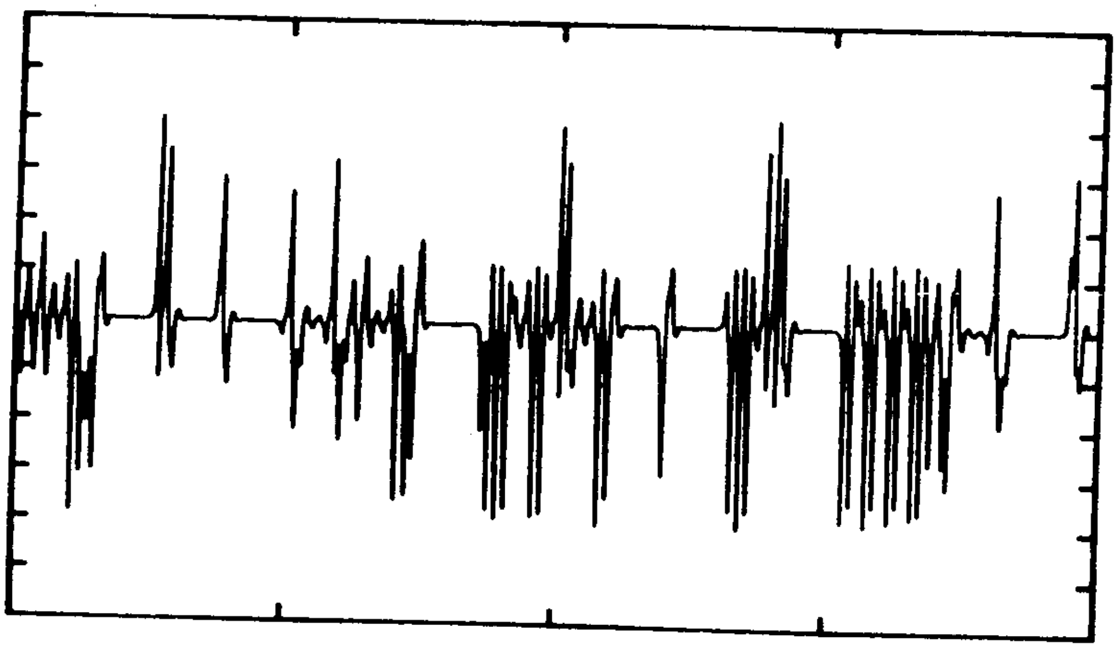
Fig 2







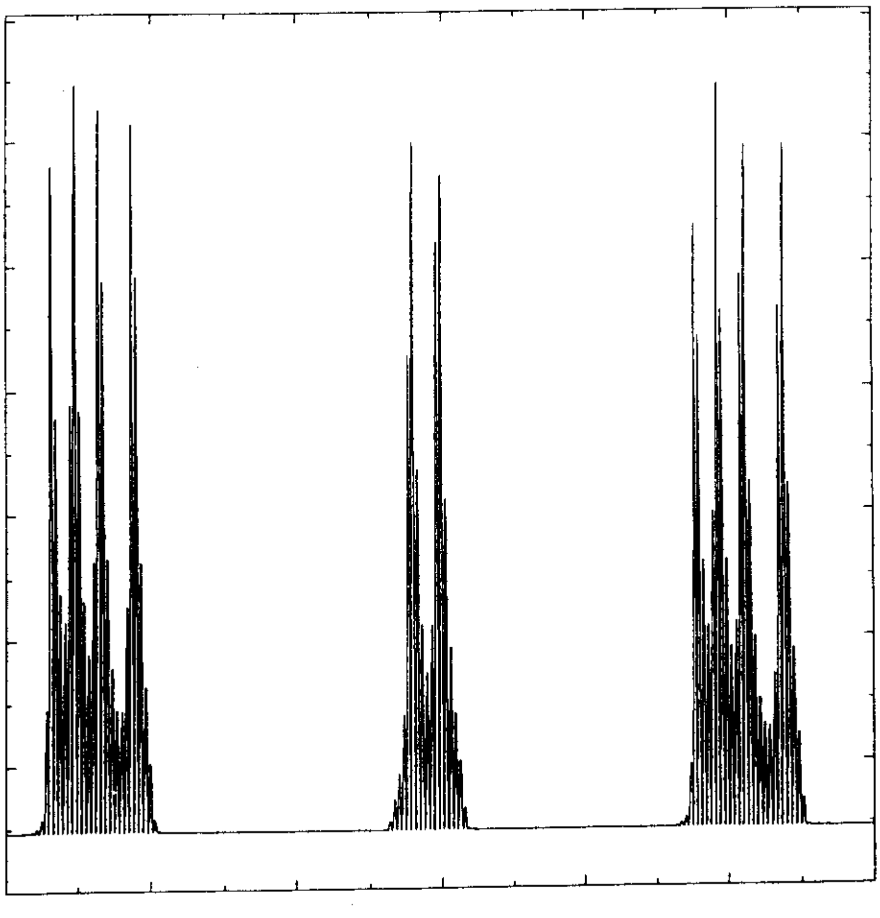


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Fig. 5

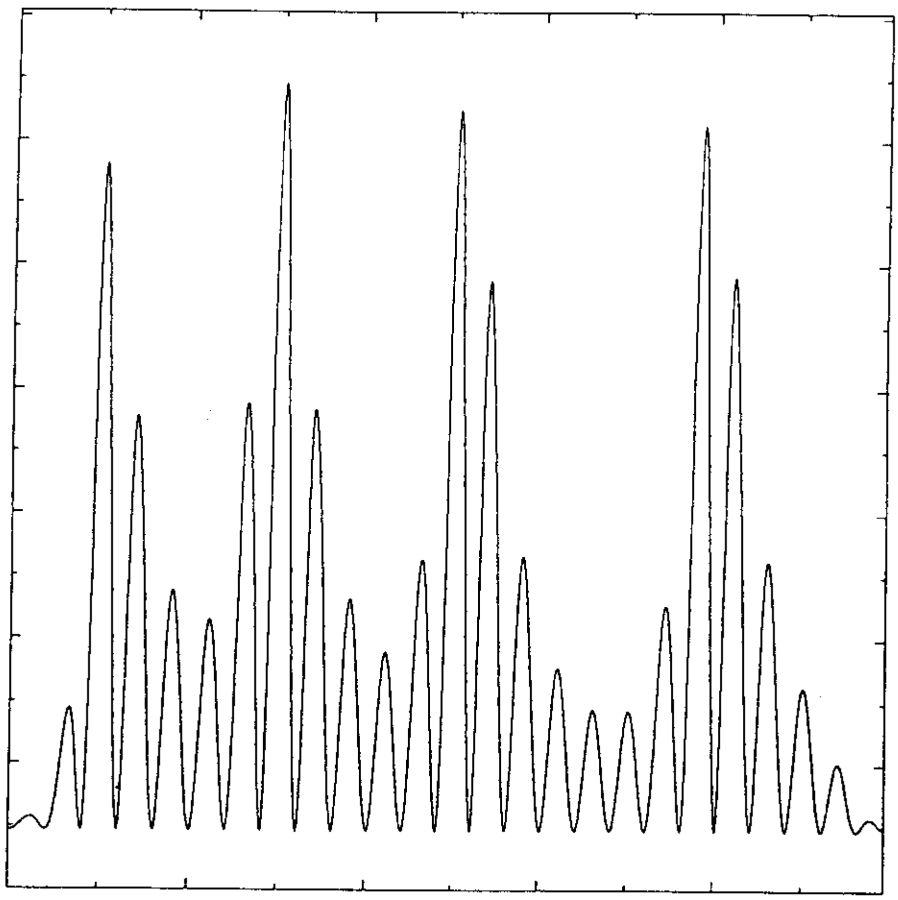

Fig. 6 21

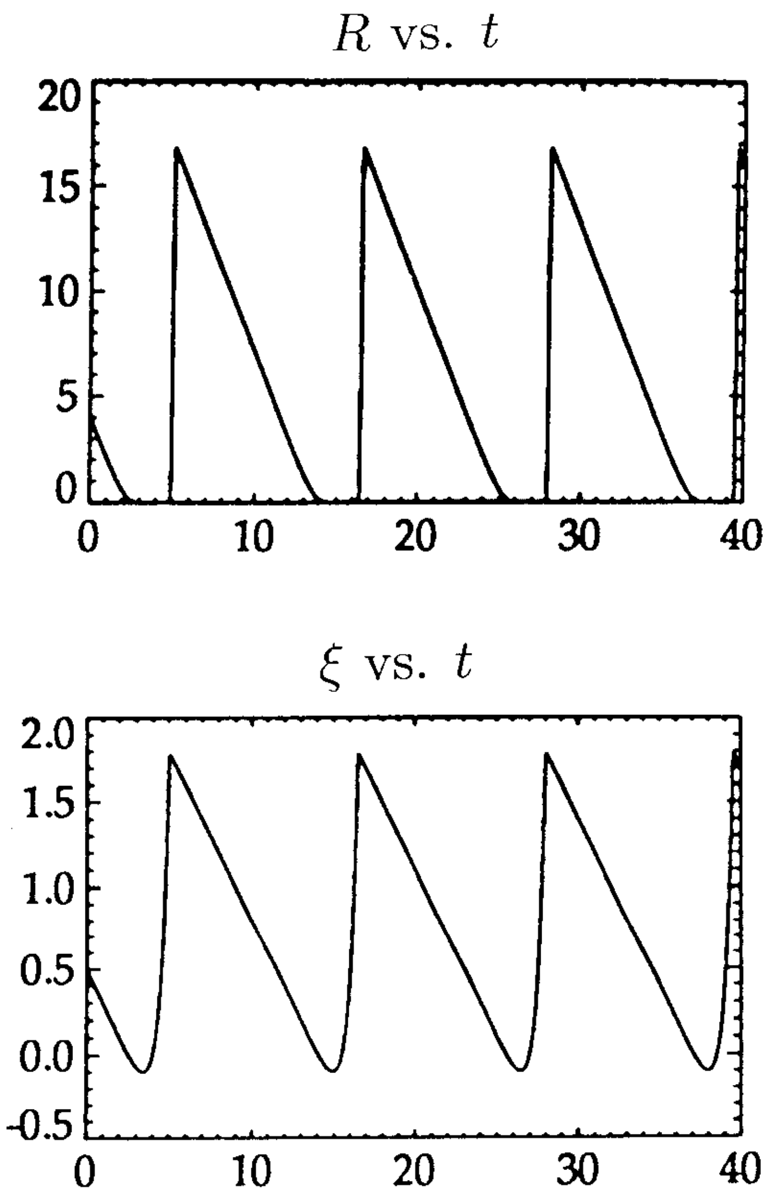

Figure 7

Modeling a Maunder Minimum

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December 26, 1993

Abstract

The effects of stochastic fluctuations in a two-dimensional solar mean-field dynamo model are investigated. It is found that fluctuations with typical times scales of the order of 3 years can produce phenomena similar to Maunder minima.

subject headings: Solar dynamo – MHD – turbulence

1 Introduction

The sun shows variability on a broad range of time scales, from miliseconds to millenia. Here we are interested in the longer time scales associated with the solar cycle and its intermissions. A salient manifestation of this cyclic behavior is seen in the sunspot number, whose annual mean shows a cyclic variation on a scale of eleven years (or twenty-two years, if one goes by magnetic polarity variations). The term "cycle" is used in the sense that the sunspot number performs the same kind of oscillation approximately every eleven years, but the magnitude of the oscillation varies strongly in a way that is reminiscent of a chaotic ocillator. Although we do not have sufficient data to conclude that the solar oscillation is chaotic, in the sense that it is describable by a low order deterministic dynamical system (Spiegel & Wolf, 1987), this possibility has been conjectured by Tavakol (1978), Ruzmaikin (1981) and others. Like the sun, simple chaotic systems repeat themselves over and over again, but never the same