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Workshop on Fluid Mechanics

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Similarity of steady stratified flows

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Similarity of Steady Stratified Flows

Incompressibility:

$$\boxed{u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0}$$

Continuity:

$$\frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0, \quad \text{or} \quad \boxed{\frac{\partial u_\alpha}{\partial x_\alpha} = 0.}$$

Equations of motion:

$$\boxed{\rho u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial p}{\partial x_i} - g\rho\,\delta_{i3}.} \quad (i = 1, 2, 3)$$

$$\delta_{i3} = \begin{cases} 0 & \text{if } i \neq 3 \\ 1 & \text{if } i = 3 \end{cases} \qquad g\downarrow$$

Let $*$ denote "laboratory", $\wedge$ denote "nature".

and let

$$y_i^* = \frac{x_i^*}{L^*}, \qquad \hat{y}_i = \frac{\hat{x}_i}{\hat{L}}.$$

At corresponding points,

$$y_i^* = \hat{y}_i .$$

$$\hat{\rho} = \hat{\rho}_0 + \hat{\rho}_1(\hat{y}_1, \hat{y}_2, \hat{y}_3).$$

$$\hat{u}_\alpha \, \partial\hat{\rho}/\partial\hat{y}_\alpha = 0. \qquad (1)$$

$$\partial\hat{u}_\alpha/\partial\hat{y}_\alpha = 0. \qquad (2)$$

$$\hat{\rho}\,\hat{u}_\alpha \,\partial\hat{u}_i/\partial\hat{y}_\alpha = -\partial\hat{\pi}/\partial\hat{y}_i - \hat{L} g \hat{\rho}_1 \delta_{i3}, \qquad (3)$$

where $\hat{\pi} = p + \hat{\rho}_0 g \hat{x}_3.$

In the laboratory, arrange to have

$$\rho^* = \rho_0^* + \rho_1^*(y_1^*, y_2^*, y_3^*).$$

$$\hat{\rho}_1/\rho_1^* = r.$$

$$m = \hat{L}/L^*.$$

$$\boxed{u_i^* = (\hat{\rho}/rm\rho^*)^{1/2}\,\hat{u}_i.} \qquad (4)$$

and let

$$\pi^* = p^* + \rho_0^* g x_3^*.$$

Then

4 and 1 $\rightarrow$ $\boxed{u_\alpha^* \,\partial\rho^*/\partial y_\alpha^* = 0} \qquad (5)$

[illegible] in [illegible] .

$$\boxed{\frac{\partial u^*_\alpha}{\partial y^*_\alpha} = 0.} \qquad (6)$$

(4) gives

$$\rho^* u^*_\alpha \, \partial u^*_i / \partial y^*_\alpha = \frac{\hat{\rho}}{rm} \hat{u}_\alpha \, \partial \hat{u}_i / \partial \hat{y}_\alpha \qquad \boxed{\begin{matrix} \partial/\partial y^*_\alpha \\ \parallel \\ \partial/\partial \hat{y}_\alpha \end{matrix}}$$

so that (3) becomes

$$rm\rho^* u^*_\alpha \, \partial u^*_i / \partial y^*_\alpha = - \partial \hat{\pi} / \partial y^*_i - r \hat{L} g \rho^*_1 \delta_{i3}$$

or,

$$\boxed{\rho^* u^*_\alpha \, \partial u^*_i / \partial y^*_\alpha = - \partial \pi^* / \partial y^*_i - L^* g \rho^*_1 \delta_{i3}.} \qquad (7)$$

if we take

$$\pi^* = \hat{\pi} / rm$$

and identify π^* with (i.e. $\pi^* \equiv$)

$$p^* + \rho^*_0 g x^*_3 .$$

For compressible fluids,

$$\rho / p^{1/\gamma} = \text{const.}\, e^{-S/c_p}$$

$$(S \equiv \text{entropy}, \quad \gamma = c_p / c_v .)$$

Let

$$\lambda = (\rho/\rho_c)/(p/p_c)^{1/\gamma} .$$

Then

$$\boxed{u_\alpha \, \partial \lambda / \partial x_\alpha = 0}$$

$$\hat{\lambda} = \hat{\lambda}_0 + \hat{\lambda}_1 , \qquad \lambda^* = \lambda^*_0 + \lambda^*_1 .$$

$$r = \hat{\lambda}_1 / \lambda^*_1 .$$

$$u^*_i = (\hat{\lambda} / r m \lambda^*)^{1/2} \, \hat{u}_i , \qquad \pi^* = \hat{\pi} / r m$$

where

$$\pi^* = \int \frac{dp^*}{\rho'^*} + \lambda^*_0 g x^*_3 , \qquad \hat{\pi} = \int \frac{d\hat{p}}{\hat{\rho}'} + \hat{\lambda}_0 g \hat{x}_3 .$$

$$\rho'^* = \frac{\rho^*}{\lambda^*} . \qquad \hat{\rho}' = \frac{\hat{\rho}}{\hat{\lambda}} .$$

milar nclusions as for comp. fluids

$$\boxed{\frac{\rho^{*\prime}}{(p^*)^{1/\gamma}} = \frac{\hat{\rho}'}{(\hat{p})^{1/\gamma}} = \text{constant}}$$

Significance of

$$u_i^* = (\hat{\rho}/rm\rho^*)^{1/2}\hat{u}_i .$$

$$r = \hat{\rho}_1/\rho_1^* ,$$

$$m = \hat{L}/L^* .$$

At corresponding points, then,

$$\frac{\rho^* u_i^{*2}}{\rho_1^* L^*} = \frac{\hat{\rho}\,\hat{u}_i^2}{\hat{\rho}_1 \hat{L}} .$$

If

$$q^{*2} = u_i^* u_i^* , \quad \hat{q}^2 = \hat{u}_i \hat{u}_i ,$$

then, since (flow patterns being the same)

$$r = \frac{\hat{\rho}_1}{\rho_1^*} = \frac{\operatorname{grad}\hat{\rho}_1}{\operatorname{grad}\rho_1^*}\,\frac{\hat{L}}{L^*} . \qquad \operatorname{grad}\hat{\rho}_1 = |\operatorname{grad}\hat{\rho}_1|$$

we have, at corresponding points,

$$\frac{\rho^* q^{*2}}{g(\operatorname{grad}\rho_1^*)L^{*2}} = \frac{\hat{\rho}\,\hat{q}^2}{g(\operatorname{grad}\hat{\rho}_1)\hat{L}^2} , \quad \text{or,}$$

with

$$g^* = g\,\frac{\text{grad}\,\rho_1^*}{\rho_0^*}\,L^* \qquad \hat{g} = g\,\frac{\text{grad}\,\hat{\rho}_1}{\hat{\rho}_0}\,\hat{L}\,,$$

$$u_i'^* = \left(\frac{\rho^*}{\rho_0^*}\right)^{1/2} u_i^* \,, \qquad \hat{u}_i' = \left(\frac{\hat{\rho}}{\hat{\rho}_0}\right)^{1/2} \hat{u}_i \,.$$

$$s^* = |\underline{u}'^*| \,, \qquad \hat{s} = |\hat{\underline{u}}'|\,,$$

we have

$$\frac{s^{*2}}{g^* L^*} = \frac{\hat{s}^2}{\hat{g}\hat{L}} \qquad (8)$$

or

$$F^{*2} = \hat{F}^2. \qquad \text{(Equality of densimetric Froude numbers everywhere.)}$$

Note that (4) implies

$$s^{*2} = \hat{s}^2/rm$$

or

$$(\text{grad}\, s^*)\, L^* = (\text{grad}\,\hat{s})\,\hat{L}\,\left(\frac{1}{rm}\right)^{1/2}$$

or

$$\frac{s^{*2}}{(\text{grad}\, s^*)^2 L^{*2}} = \frac{\hat{s}^2}{(\text{grad}\,\hat{s})^2 \hat{L}^2} \qquad (9)$$

Eqs. (8) & (9) give

$$\frac{g^*}{(\text{grad}\, s^*)^2 L^*} = \frac{\hat{g}}{(\text{grad}\,\hat{s})^2 \hat{L}} \,. \qquad (10)$$

or

$$R_i^* = \hat{R}_i \qquad \text{(Richardson numbers equal everywhere.)}$$