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Workshop on Fluid Mechanics

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The flow at the top of the core

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These are preliminary lecture notes, intended only for distribution to participants

Computation of the flow at the CMB

1) Hypothesis :

1) CMB spherical $r = b$

2) mantle insulating

not critical

(but critical when studying core-mantle coupling).

$\vec{B}(b,t)$ known at the bottom of the mantle : $b+0$

2) The induction equation.

In the conducting liquid core:

From

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

$$\nabla \times \vec{B} = \mu \vec{J}$$

$$\vec{J} = \sigma (\vec{E} + \vec{u} \wedge \vec{B})$$

comes the induction equation:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \wedge \vec{B}) + \eta \nabla^2 \vec{B}} \quad (1)$$

η : magnetic diffusivity

$\eta = (\mu \sigma)^{-1}$ $\quad \mu \sim \mu_0$

$\sigma \sim 10^6 (\Omega m)^{-1}$

$\eta \sim 1 \ m^2 s^{-1}$

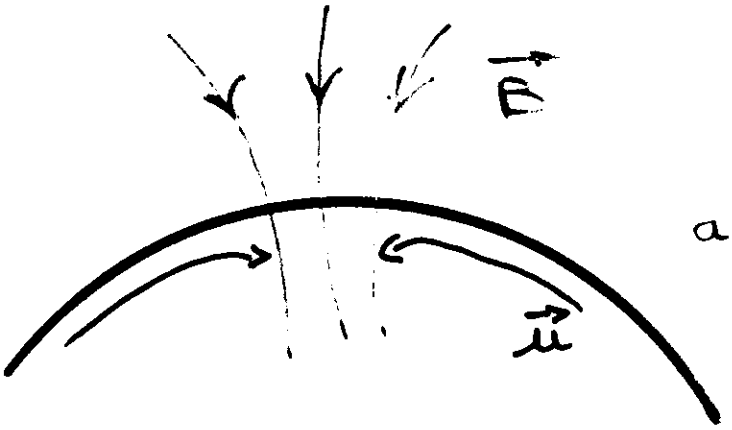
at the top of the core $r = b - 0$ (beneath a viscous - Ekman - boundary layer of thickness $\sim (\nu / \Omega)^{1/2}$)

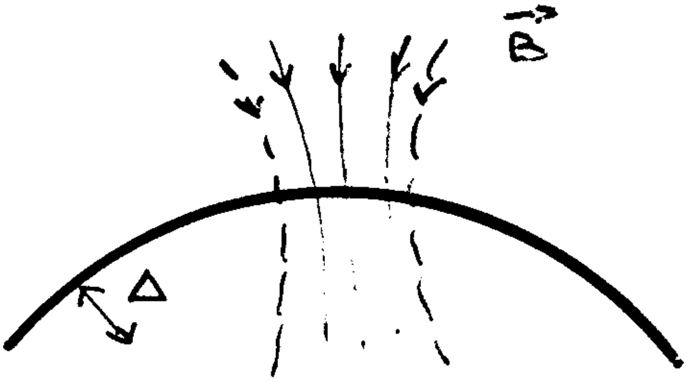
$u_r = 0$. The radial component

$\nu = \frac{\eta}{\rho}$

of (1) is :

$$\boxed{\frac{\partial B_r}{\partial t} = - \vec{\nabla}_H \cdot (\vec{u} B_r) + \frac{\eta}{r} \nabla^2 (r B_r)} \quad (2)$$





Variation temporelle
(séculaire)

$$\frac{\partial \vec{B}}{\partial t} = \text{terme d'advection} + \text{terme de diffusion}$$

3). Boundary conditions at the CMB

B_r continuous

($\hat{r} \wedge \vec{B}$ continuous if σ_c finite)

$$B_r(b-0) = B_r(b+0)$$

⟶ $B_r(b-0)$ known from observation

$\frac{\partial B_r}{\partial t}$ known.

$\frac{\eta}{b} \nabla^2 (r B_r)$ implies vertical derivatives of B_r not continuous.

If the diffusive term $\frac{\eta}{b} \nabla^2 (r B_r)$ is small compared to the secular variation $\frac{\partial B_r}{\partial t}$ the induction equation (2) reduces to:

$$\boxed{\frac{\partial B_r}{\partial t} = - \vec{\nabla}_H \cdot (\vec{u} B_r)} \qquad (3)$$

4) The frozen-flux approximation.

a) Justification.

$$\frac{\eta}{b} \nabla^2 (r B_r) \sim \frac{\eta B_r}{L^2}$$

$$\partial B_r / \partial t \sim B_r / T$$

$$\longrightarrow \quad 1 \ll \frac{L}{?}$$

si $L \sim 10^{6}$ m

$$T \ll 10^{12} \sim 10^{4} \text{ years}$$

The frozen-flux approximation holds for most components of the S.V if the hypothesis $L \sim 10^{6}$ m is legitimate. In fact, if the flow $\vec{u}$ has a typical length of 10^{6} m. If it is not 2-scale

(Braginsky and Le Mouël, 199).

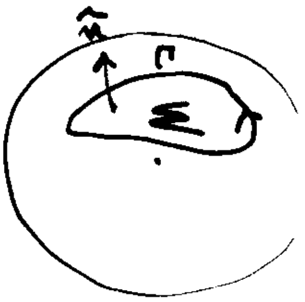
We will make this hypothesis.

b) Consequences. Integral constraints

Let Γ be a material curve on the sphere $(r=b)$

enclosing the patch

Σ



$$\frac{d}{dt}\iint_{\Sigma} B_r\, dS = \iint_{\Sigma} \frac{\partial B_r}{\partial t}\, dS + \int_{\Gamma} B_r (\vec{u}\wedge \vec{dl})\cdot$$

$$= \iint_{\Sigma} -\nabla_H\cdot(\vec{u}\, B_r) + \int_{\Gamma} B_r (\vec{dl}\wedge\vec{u})\cdot$$

$$= 0$$

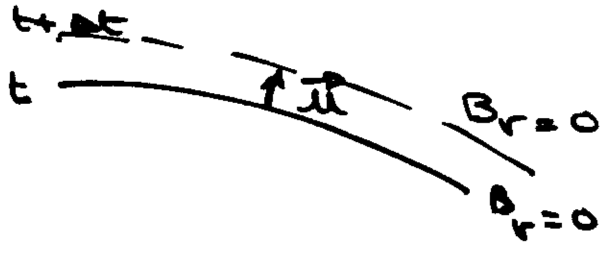
(cas particulier de $\frac{d}{dt}\iint_{\Sigma} \vec{B}\cdot\vec{dS} = 0$
in the frozen flux approximation)

<u>In particular the null-flux curves ($B_r=0$) are material curves:</u>

$$\frac{dB_r}{dt} = \vec{u}\cdot\vec{\nabla}_H B_r + \frac{\partial B_r}{\partial t}$$

$$= \cancel{\vec{\nabla}_H\cdot(\vec{u}\, B_r)} - B_r\, \vec{\nabla}_H\cdot\vec{u} + \cancel{\frac{\partial B_r}{\partial t}}$$

$$\uparrow \quad \frac{dB_r}{dt} = -B_r\, \vec{\nabla}_H\cdot\vec{u}$$



<u>Numerical test</u>:

(Jackson, Bloxham, Gubbins)

cartes de

5). Computation of the flow $\vec{u}$.

$$\vec{u} = \nabla_H \phi(\theta,\varphi) - \hat{n} \wedge \nabla_H \psi(\theta,\varphi) \qquad (4)$$

poloidal $\langle \phi \rangle = 0$ toroidal $\langle \psi \rangle = 0$

Is it possible to determine $\vec{u}$ from

$$\frac{\partial B_r}{\partial t} = - \vec{\nabla}_H \cdot (\vec{u} B_r) \qquad (3) \quad ?$$

<u>dubious</u> : One scalar equation

two unknown scalars ϕ, ψ

In fact, letting :

$$\vec{v} = B_r \vec{u} = \nabla_H S - \hat{n} \wedge \nabla_H T$$

(3) ⟶

$$\frac{\partial B_r}{\partial t} = - \nabla_H^2 S \qquad (5)$$

The operator

$$\nabla_H^2 = -L^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

is inversible. Then S can be determined from (5), uniquely.

T_{ψ} ~~ψ~~ is undetermined.

Then if u_0 is a solution of (3)

$$\vec{u} = \vec{u}_0 - \frac{1}{B_r}\,\hat{n}\wedge\vec{\nabla}_H\,\psi^T \qquad (6)$$

is another solution. ψ^T arbitrary (except that ψ^T must be constant on the curves $B_r = 0$).

(6) illustrates the underdetermination shown by Backus (1965).

Condition of compatibility.

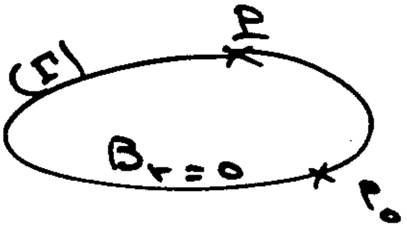
B_r, $\partial B_r/\partial t$ given. Is there any solution of (3)?

As $$\vec{v} = B_r\,\vec{u}$$

→ Necessary condition:

$\vec{v} = 0$ on null-flux curves $B_r = 0$

$$\longrightarrow \quad \vec{n}\wedge\vec{\nabla}_H T = \vec{\nabla}_H S$$



$$T(P) = T(P_0) - \int_{P_0}^{P}(\vec{n}\wedge\vec{\nabla}_H S)\cdot\vec{dl}$$

$$\oint_{\Gamma} (\vec{n} \wedge \vec{\nabla}_H S) \cdot d\vec{\ell} = 0$$

$$\iint_{\Sigma} \vec{rot}\, (\vec{n} \wedge \vec{\nabla}_H S) \cdot \vec{dS}_{\varepsilon} = 0$$

(Th. Stokes. Ampère)

$$\rightarrow \quad \iint_{\Sigma} \vec{\nabla} \times (\vec{\nabla} \times S \vec{n}) \cdot \vec{dS}_{\varepsilon} = 0$$

Finally:

$$\iint_{\Sigma} \nabla_H^2 S \, d\Sigma = \iint_{\Sigma} \frac{\partial B_r}{dt} \, d\Sigma = 0$$

already met. Compatible with observations.

5.1. How to reduce the ambiguity?

$$\vec{u} = \vec{u}_0 - \frac{1}{B_r} (\vec{n} \wedge \vec{\nabla} \overset{T}{\psi}) \qquad (6)$$

needs assumptions on $\vec{u}$.

$\vec{u}$ ~~stationnary~~ . Fig.

$\vec{u}$ purely toroidal $= \vec{n} \wedge \vec{\nabla} \psi$

The flow is tangentially geostrophic.

Dynamical considerations.

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla\vec{u}\right) = -\vec{\nabla}p + \vec{J}\wedge\vec{B} - 2\rho\,\vec{\Omega}\wedge\vec{u}$$
$$+ \rho\nu\nabla^2\vec{u} - \rho_1 g\,\vec{r} \quad (7)$$

ρ : density of the core fluid $\rho(r)$

$-\rho_1$: density heterogeneity

$\vec{J}$: electric current density

$\vec{\Omega}$: Earth's rotation

ν : kinematic viscosity

g : gravity

ν (molecular) $\sim 5\,10^{-6}\,m^2 s^{-1}$

$\Omega \sim 7\,10^{-5}\;s^{-1}$

Let us compare the different terms of (7) to the Coriolis force (acceleration).

Viscous term $\dfrac{\rho\nu\nabla^2\vec{u}}{U\,\Omega\wedge\vec{U}} \sim \dfrac{\nu}{L_u^2\,\Omega} \ll 1$ ~~if $L_u \gg$~~

if $L_u \gg \delta = \sqrt{\dfrac{\nu}{\Omega}} < 50\,m$

~~CMB to δ~~ : Ekman layer

$$\frac{\rho \frac{\partial \vec{u}}{\partial t}}{2\rho \vec{\Omega}\wedge\vec{u}} \sim \frac{1}{\Omega T_u} \ll 1$$

if $T_u \gg 1$ day

$T_u \geqslant T_B \sim$ qq 10^2 years

(S-V data).

advective acceleration

$$\frac{\rho \vec{u}\cdot\nabla\vec{u}}{2\rho\vec{\Omega}\wedge\vec{u}} \sim \frac{U}{L\Omega} \ll 1 \quad \text{if } \frac{L_u}{U}$$

$\gg 1$ day

$\uparrow$

1 number

In fact $\frac{L_u}{U} \gtrsim 10^2$ years.

We are left with:

$$\underline{\vec{\nabla} p} \; \underline{+ 2}\rho\, \underline{\vec{\Omega}\wedge\vec{u}} = \underline{\vec{J}\wedge\vec{B}} - \rho_1 g\, \underline{\vec{r}} \qquad (8)$$

a) if $B \sim B_r$ continued from the Earth's surface

$$\frac{\vec{J}\wedge\vec{B}}{2\rho\vec{\Omega}\wedge\vec{u}} \sim \frac{B_r^2}{2\mu_0\rho\Omega U L_B} \sim 5\,10^{-4} \left(\vec{J} = \mu_0^{-1}\vec{\nabla}\times\vec{B}\right)$$

with the values

° $B_r \sim 10^6$ nT, $\mu_0 = 4\pi 10^{-7}$, $\rho = 10^4$, $L \sim 10^6$ m, $u = 10^{-3}$ ms^{-1}

3) there is a strong toroidal field in the core.

Many theories of the dynamo (not all) give :

$$B_T \sim \underset{\substack{\uparrow \\ \text{magnetic Reynolds number} \\ \sim 100}}{(UL\mu\sigma)}\, B_p$$

$$\frac{\vec{J}\wedge\vec{B}}{2\rho\vec{\Omega}\wedge\vec{u}} \sim \frac{B_T B_p}{\rho\Omega U \mu_0 L_B} \sim$$

But, close to the CMB

$$B_T \ll B_T \text{ in the core}$$

$$B_T = 0 \quad \text{at the CMB if } \sigma_m = 0$$

Then the approximation

$$|\vec{J}\wedge\vec{B}| \ll \text{Coriolis} \qquad \text{is legitimate}$$

$$\rightarrow \boxed{\vec{\nabla}p + 2\rho\,\vec{\Omega}\wedge\vec{u} = -\rho_1 g\,\vec{r}} \qquad (9)$$

horizontal component :

$$2\rho(\vec{\Omega}\wedge\vec{u})_H + \vec{\nabla}_H p = 0 \qquad (10)$$

$$\boxed{\vec{u} = \frac{1}{2\rho\Omega\cos\theta}\,\vec{r}\wedge\vec{\nabla}_H p} \qquad (11)$$

$\vec{u}$ tgt geostrophic

$\vec{u}$ large scale : $L_u \sim 10^6 m$

5.2. Remaining ambiguity.

$$\boxed{\frac{\partial B_r}{\partial t} = - \mathrm{div}_H(\vec{u} B_r)} \quad (3) \qquad \left.\begin{array}{l}\frac{\partial B_r}{\partial t} \\ B_r\end{array}\right\} \text{Known}$$

imposing, from (11)

$$\boxed{\mathrm{div}_H(\vec{u}\cos\theta) = 0} \qquad (12)$$

(12) says that a geostrophic flow creates no S.V from an axial dipolar field.

Non uniqueness.

$$\vec{u} = \frac{1}{2\Omega\cos\theta}(\vec{n}\wedge\vec{\nabla}p) \quad : \quad \text{one scalar, } p$$

(3) is <u>one</u> scalar equation.

But ambiguity remains in certain domains

$$(3) \longrightarrow \quad \partial_t B_r + \vec{\nabla}_H \cdot \left(\psi\left(\hat{n} \wedge \vec{\nabla}_H q\right)\right) = 0$$

$$\psi = \frac{B_r}{\cos\theta} \qquad q = \frac{p}{2\rho\Omega}$$

or:

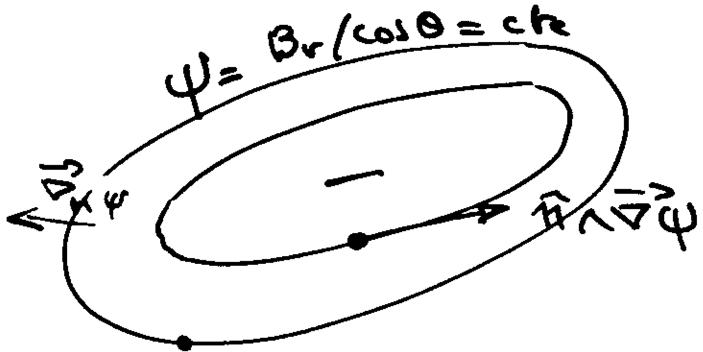
$$\vec{\nabla}_H \psi \cdot \left(\hat{n} \wedge \vec{\nabla}_H q\right) = -\partial_t B_r$$

but

$$\vec{\nabla}_H \psi \cdot \left(\hat{n} \wedge \vec{\nabla}_H q\right) = -\left(\vec{n} \wedge \vec{\nabla}_H \psi\right) \cdot \vec{\nabla}_1 q$$

$$\longrightarrow \quad \boxed{\vec{\nabla}_H q \cdot \left(\vec{n} \wedge \vec{\nabla}_H \psi\right) = -\partial_t B_r} \qquad 12$$

(Backus and Le Mouël 1986)

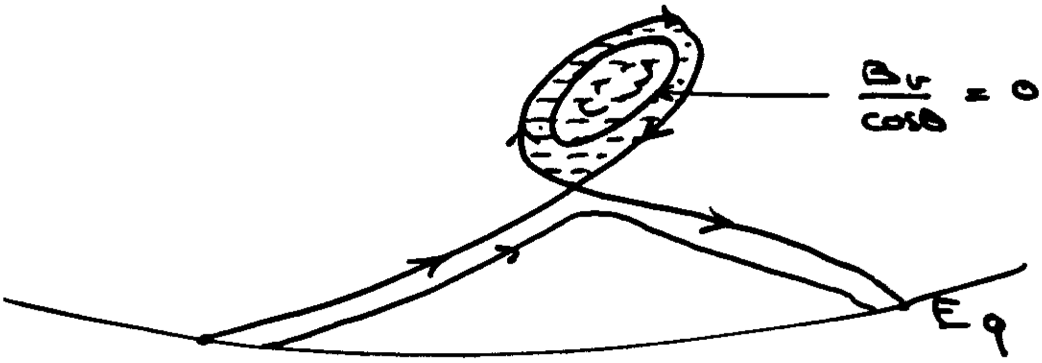


→ q known on every level curve iso_value of $\psi = B_r/\cos\theta$ if known on one point of the curve.

$p = p_0$ $(= 0)$ on the equator.

Visible set : set of all points of $(r=b)$ which are connected to the geographical equator by level curves of ψ.

q, then $\vec{u}$ are uniquely determined in the visible set.



Inside the ambiguous patches

$$q = q_0 + f(\psi) \qquad f \text{ arbitrary function of } \psi$$

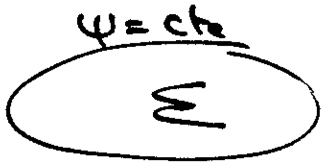
Remark.

Equation (12) $\quad \mathrm{div}_H(\vec{u}\cos\theta) = 0$

implies that the curves

$$\psi = (B_r/\cos\theta) = \text{cte}$$

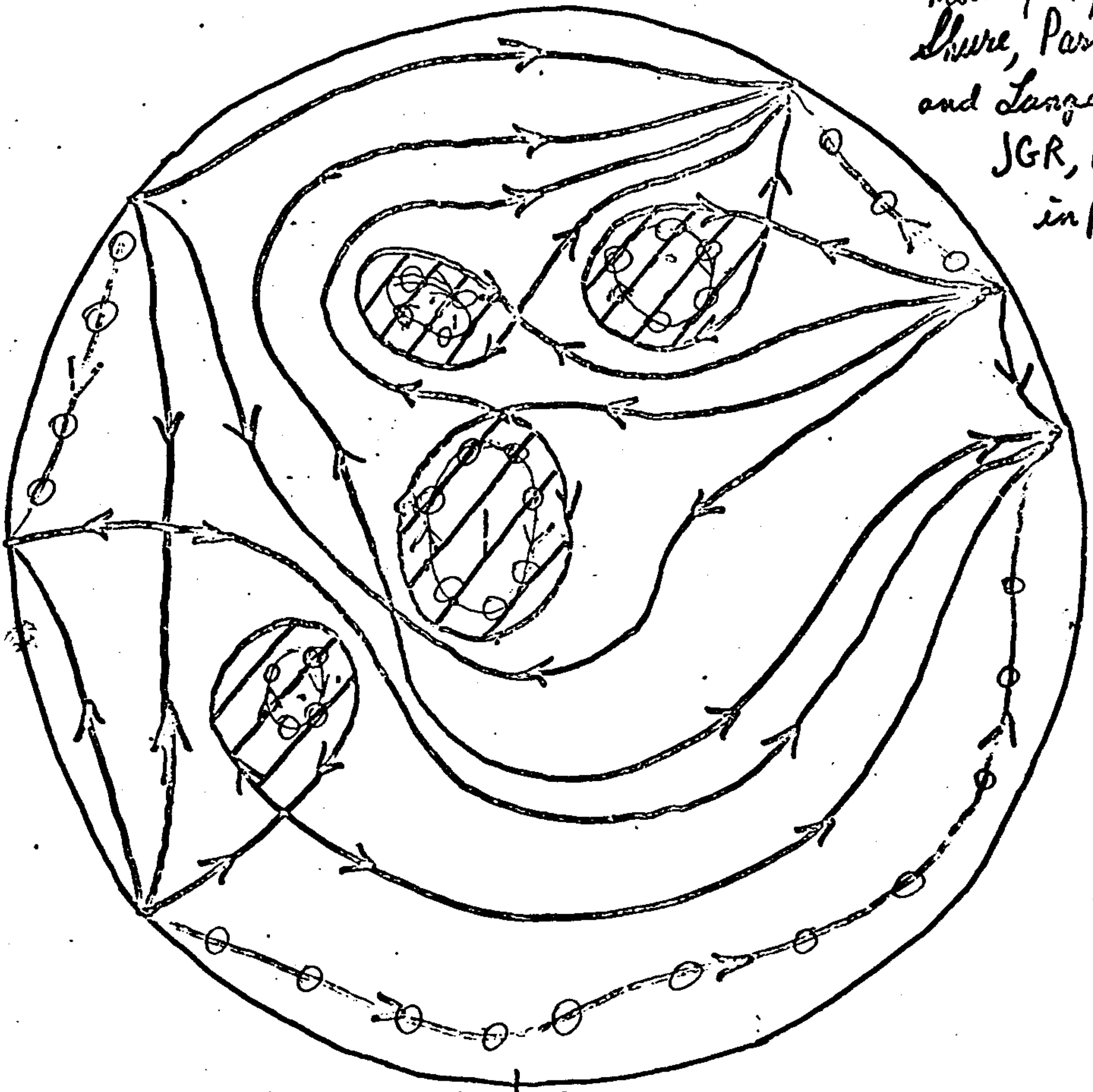
are material curves.



$$\longrightarrow \iint_{\Sigma} B_r \, dS = 0$$

allows some checking.

NORTHERN HEMISPHERE OF CORE-MANTLE BOUNDARY, SEEN FROM NORTH POLE



Model of B_r from Shure, Parker and Langel, JGR, 1985 in press

—— GEOGRAPHIC EQUATOR
-o-o- NULL FLUX CURVES ($B_r = 0$)
—— LEVEL LINES OF $B_r / \cos\theta$, HIGH ON RIGHT
///// REGION WHERE FLUID VELOCITY IS INCOMPLETELY DETERMINED, EVEN WITH GEOSTROPHY

$$\vec{u} = b \nabla_H S(\theta,\varphi) - b\, \hat{n} \wedge \vec{\nabla}_H T(\theta,\varphi) \qquad (13)$$

poloidal toroidal

$$\vec{\nabla}_H = \vec{\nabla} - \vec{n} \frac{\partial}{\partial r} \qquad \vec{n} = \vec{e}_r$$

$S(\theta,\varphi)$: poloidal scalar

$T(\theta,\varphi)$: toroidal scalar

S and T expanded in surface harmonic

$$\vec{u} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} s_n^{mc}\, \vec{S}_n^{mc} + s_n^{ms}\, \vec{S}_n^{ms} + t_n^{mc}\, \vec{T}_n^{mc} + t_n^{ms}\, \vec{T}_n$$

$$\vec{S}_n^{mc,s} = b\, \vec{\nabla}_H Y_n^{mc,s}$$

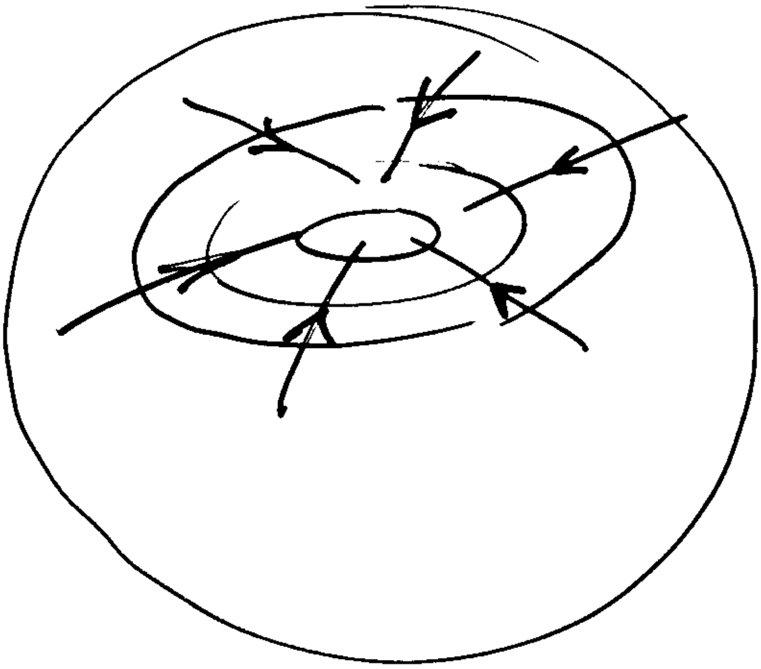
$$\vec{T}_n^{mc,s} = - b\, \vec{n} \wedge \vec{\nabla}_H Y_n^{mc,s}$$

$$Y_n^{mc,s} = P_n^m(\cos\theta)\, \cos m\varphi ,\ \sin m\varphi$$

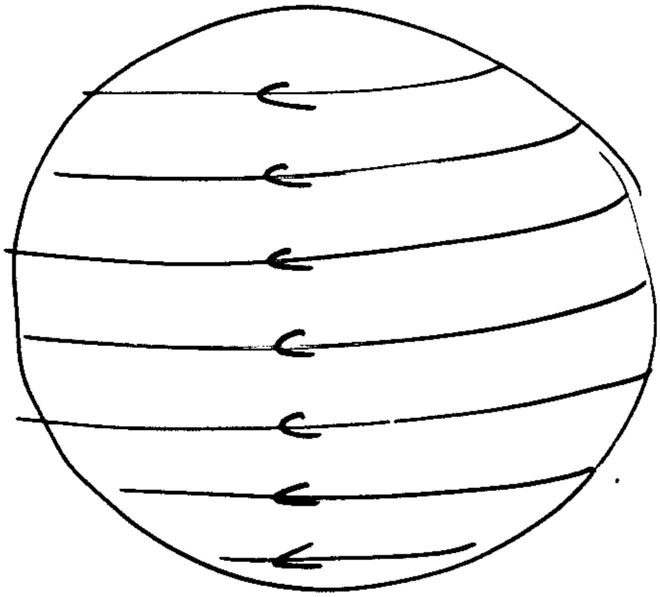
Unknown coefficients

$$\begin{vmatrix} u \end{vmatrix} = \begin{vmatrix} s_n^{mc,s} \\ t_n^{mc,s} \end{vmatrix}$$

$\vec{S}$: consoidal
(irrotationnel)



$\vec{T}$: toroidal
(solenoidal)



$$B_r\ (r=b) = \sum b_n^{mc,s}\ Y_n^{mc,s}$$

$$\dot{B}_r\ (r=b) = \sum \dot{b}_n^{mc,s}\ Y_n^{mc,s}$$

Equation (12) ($\vec{\nabla}_H \cdot \vec{u} \cos\theta = 0$) implies a relationship between t_n^m and s_n^m coefficients :

$$\left| u \right| = \left| \begin{matrix} s_n^m \\ t_n^m \end{matrix} \right| = Q \left| s_n^m \right| = Q \left| w \right|$$

The (linear) equation (3) can be written:

$$\left| \dot{b}_r \right| = M_b \left| u \right| = \underset{\overset{\uparrow}{\text{contains } |b_r|}}{M_b Q} \left| w \right|$$

Compute $|u|$ which minimizes

$$\left| \dot{b}_{r\,\text{obs}} - M_b Q\, w \right|^T C_e^{-1} \left| \dot{b}_r - M_b Q\, w \right|$$

and which is as regular - smooth, large wavelength - as possible, i.e which has as norm

$$\left|u^{T}\right| C_u^{-1} \left|u\right|$$

as small as possible.

Practically we will minimize

$$\left| \underset{obsr}{\dot{b}_r} - M_b Q w \right|^{T} C_e^{-1} \left| \dot{b}_r - M_b Q w \right| + \lambda \left|u\right|^{T} C_u^{-1} \left|u\right|$$

C_e^{-1}
C_u^{-1} Weighting matrices.

Standard inverse problem $\longrightarrow \left|u\right| \equiv \vec{u}$

$\downarrow$

Except for truncation difficulties.

$b_n^m \quad n = 1, 2 \dots \infty$

$\dot{b}_n^m \quad n = 1, 2 \dots \infty$

$s_n^m \quad n = 1, 2 \dots \infty$

Hulot, Le Mouël, Wahr, 1991

$$N_u \leqslant 4, 5$$

We can compute elementary tangentially geostrophic vectors:

$$\vec{T}_n^{\,o}$$

$$(14)\quad \begin{aligned} W_n^{mc} &= \vec{S}_n^{mc} - a_n^m \vec{T}_{n-1}^{ms} - b_n^m \vec{T}_{n+1}^{ms} \\ W_n^{ms} &= \vec{S}_n^{ms} + a_n^m \vec{T}_{n-1}^{mc} + b_n^m \vec{T}_{n+1}^{ms} \end{aligned}$$

Any tgt geostrophic flow can be written in the form:

$$(15)\quad \vec{u} = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \gamma_n^{mc} \vec{W}_n^{mc} + \gamma_n^{ms} \vec{W}_n^{ms} \right) + t_n^o \vec{T}_n^o$$

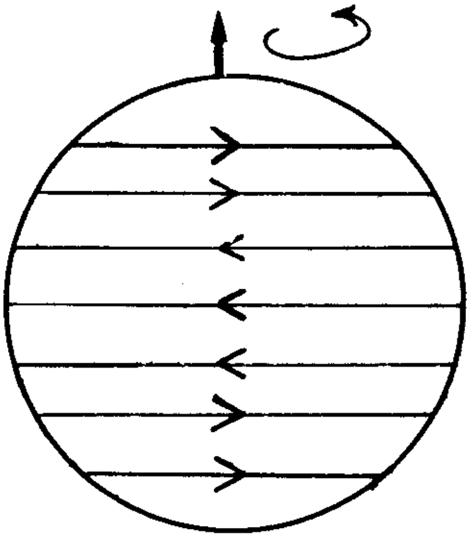
(14) and (15) show that the toroidal component of a tgt geostrophic flow is known when its poloidal part is ⟶

$$u = Q\,w$$

Examples of results.

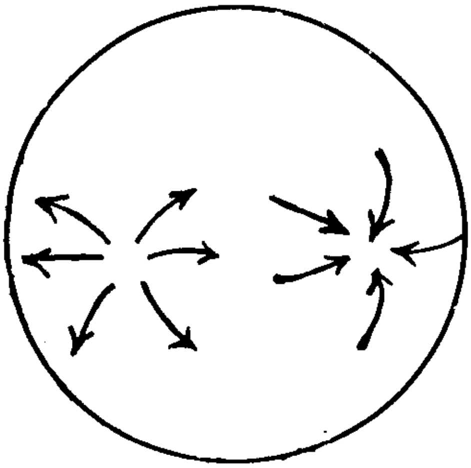
Le mouvement $\vec{u}$ est la somme de deux termes $\vec{u} = \vec{\theta} + \vec{v}$

$\vec{\theta}$ axisymétrique
toroidal



$\vec{v} = \vec{s} + \vec{t}$

n'a aucune
composante
axisymétrique



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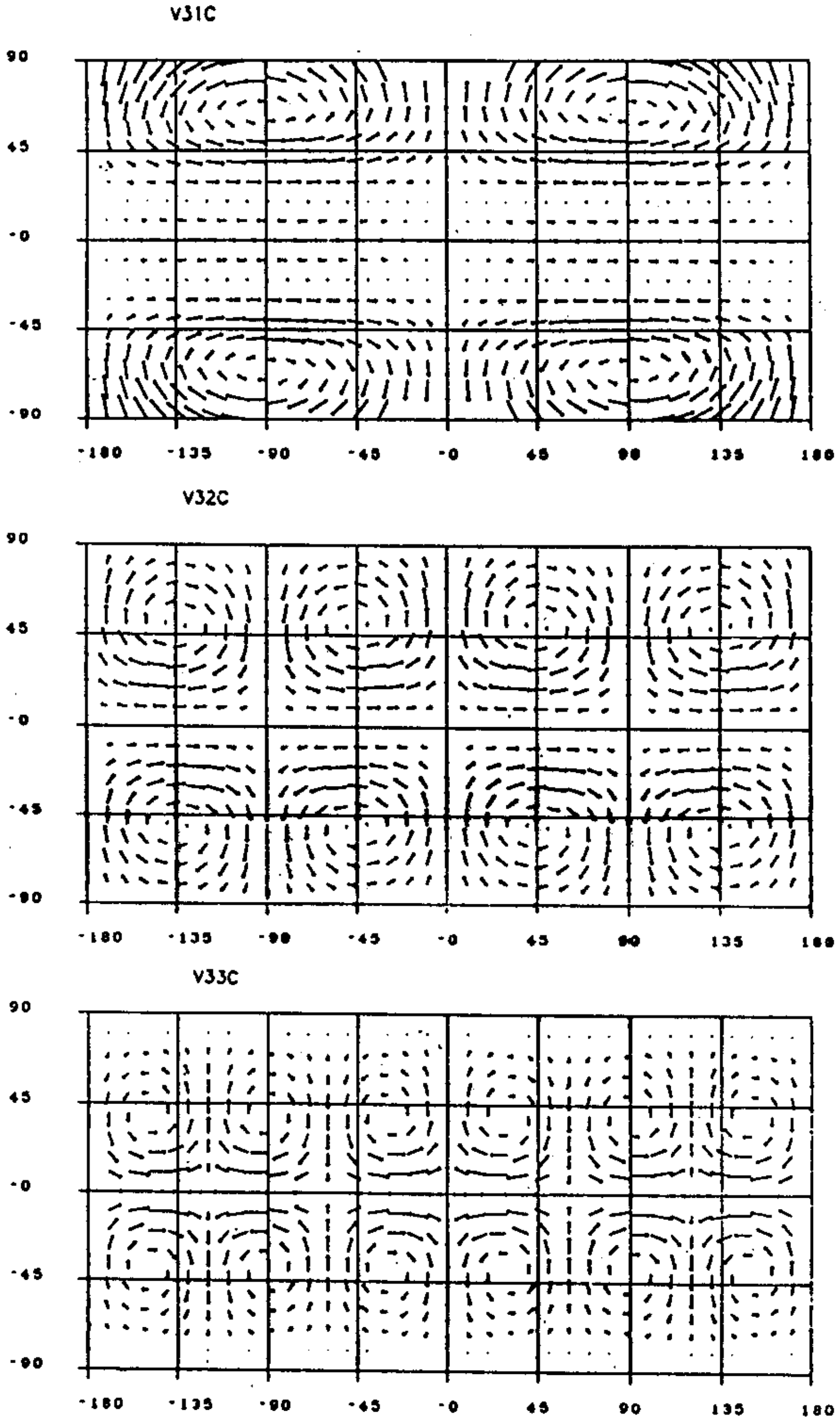
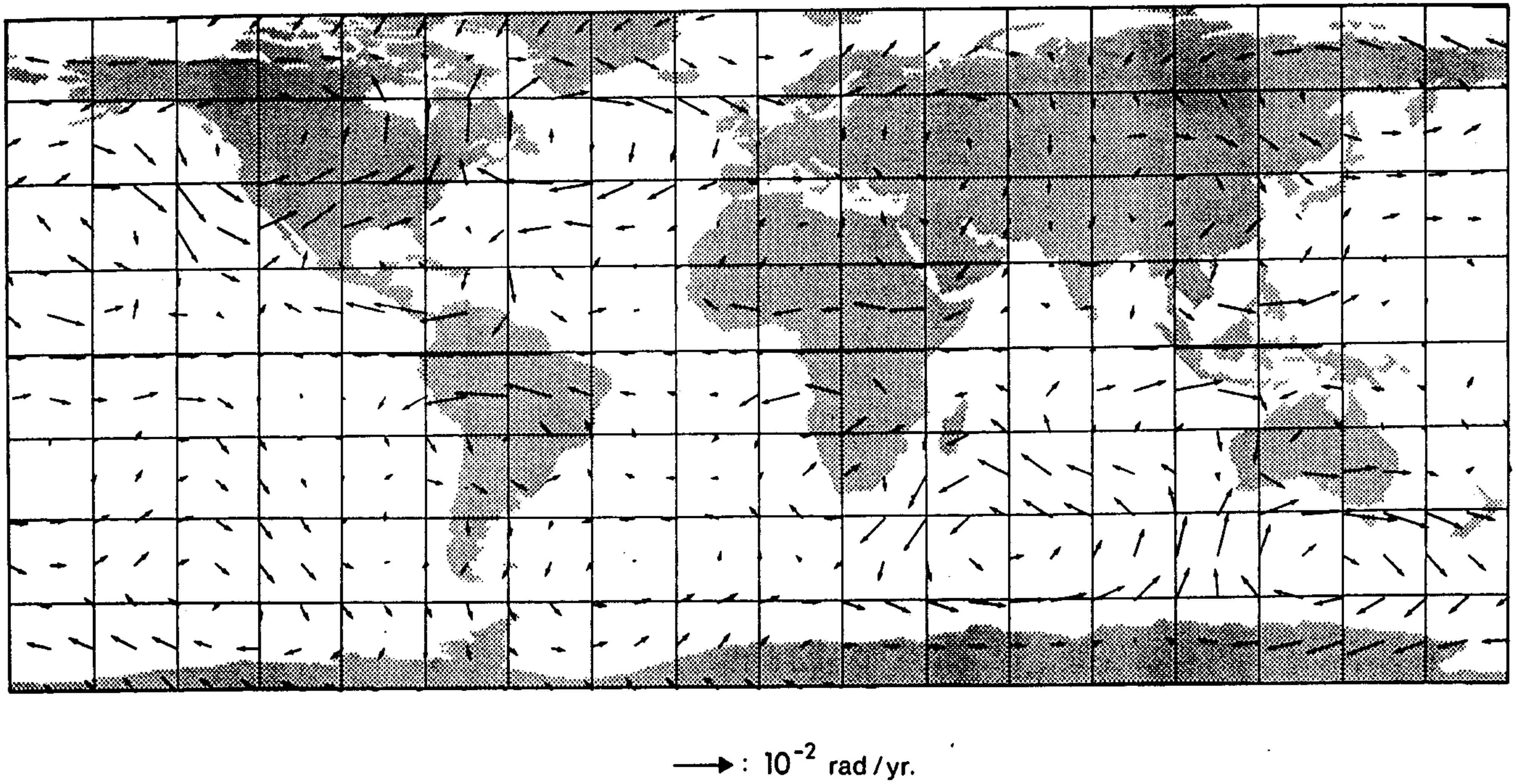
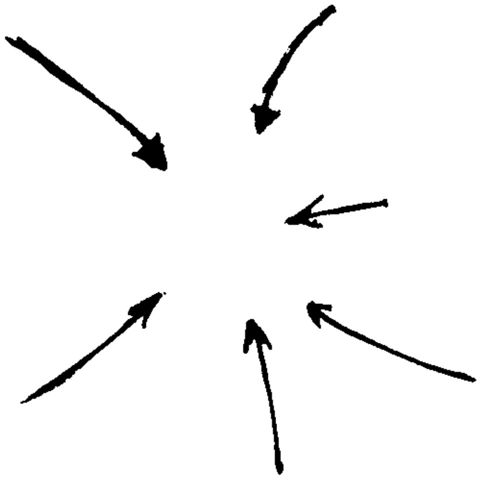


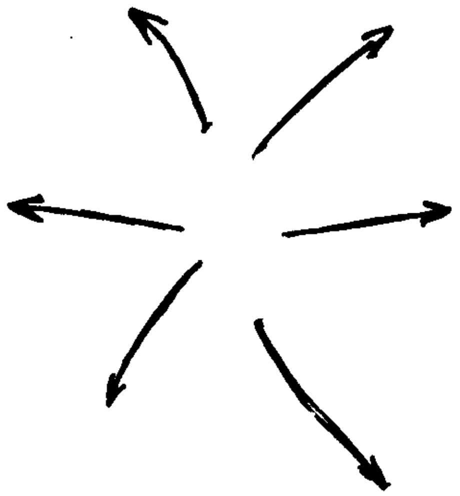
Fig. 1 (continued).

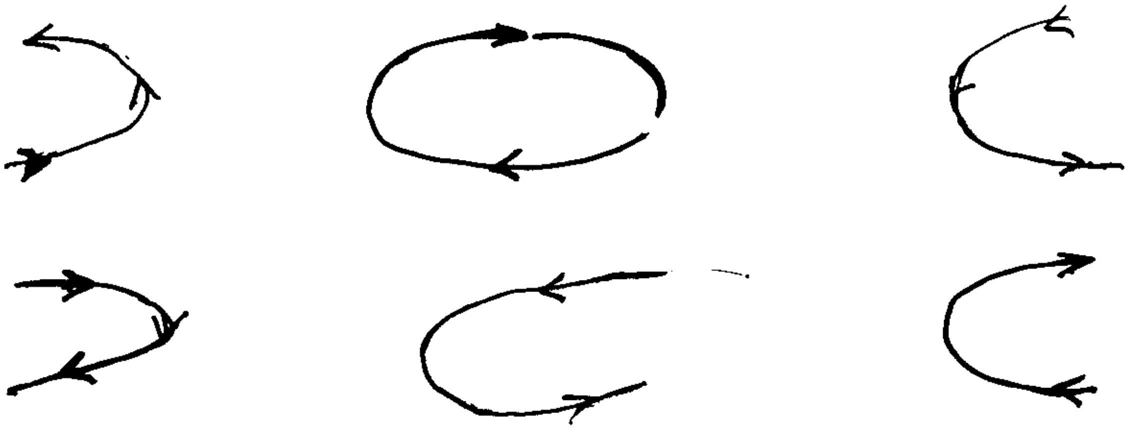
MOUVEMENT DU FLUIDE À LA SURFACE DU NOYAU.

MODÈLE POUR 1980









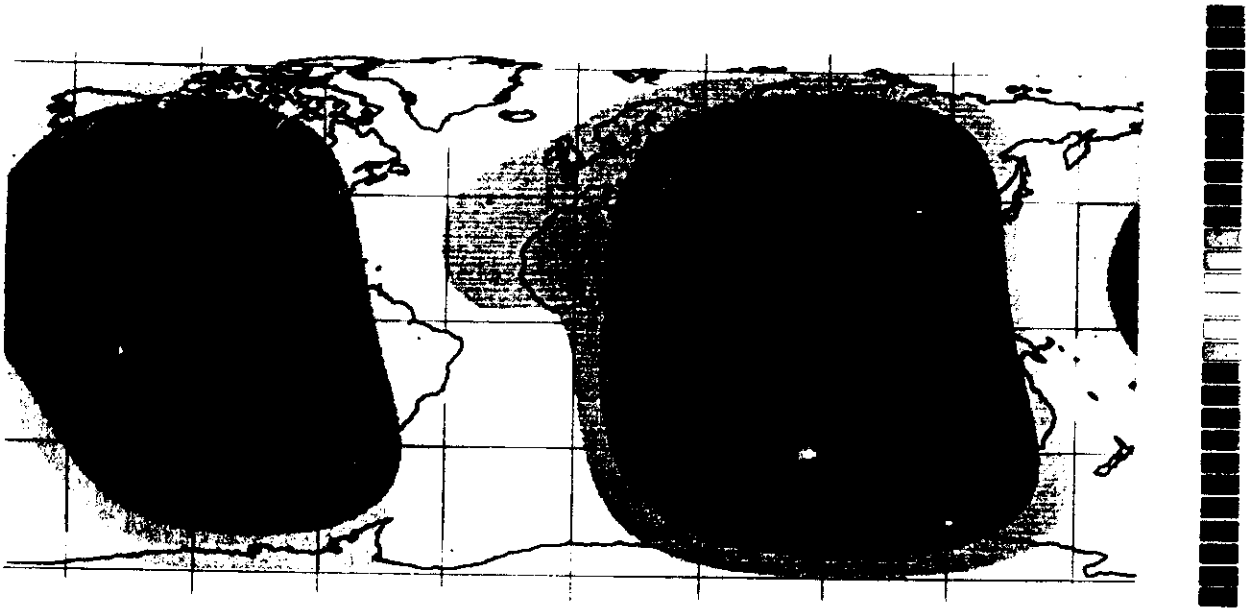
km/an.

DU NOYAU TERRESTRE

$\vec{u}$

Hulot, Le Mouël and Wahr, G.J.I (1992).

1980, UN MOUVEMENT POSSIBLE (PARTIE POLOIDALE)



1980, UN MOUVEMENT POSSIBLE (PARTIE TOROIDALE NON ZONALE)

following this criterion. It clearly comes out that the degrees of the flow larger than 8 are not at all constrained by the present SV data.

In order to evaluate more precisely the accuracy of the components of the flow with degree 1 to 8, we will derive a first, explicit model of the flow. Because the 1980 Main Field is very well defined (thanks to MAGSAT data) we decided to make our computation for the year 1980. We assume the flow is geostrophic (we use the tangentially geostrophic basis). Although, as already mentioned, this is not a key point (see the remark at the end of the section), the following results will therefore preferentially apply to geostrophic motions. The calculation is very similar to those done by Gire & Le Mouël (1990) and

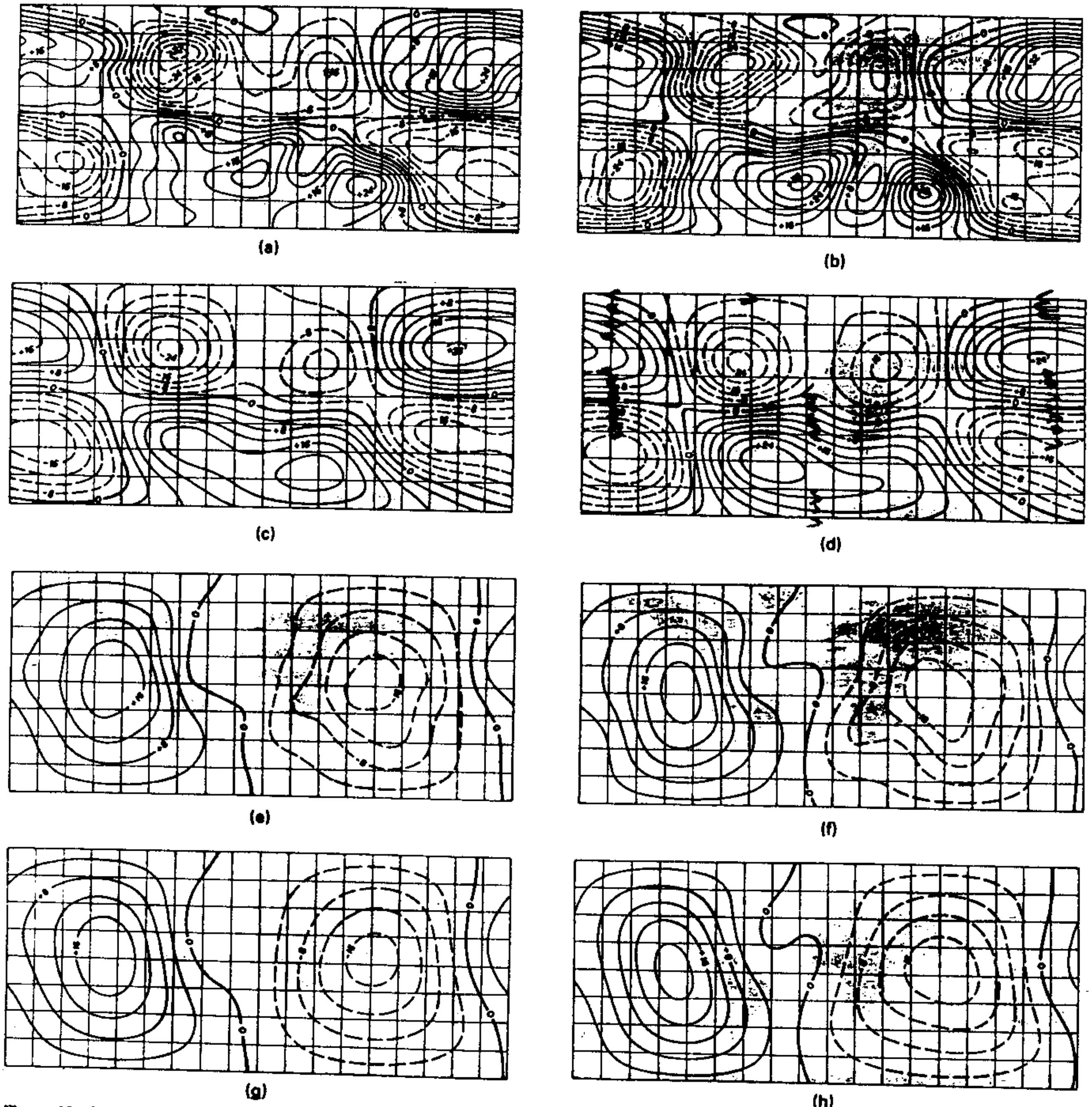
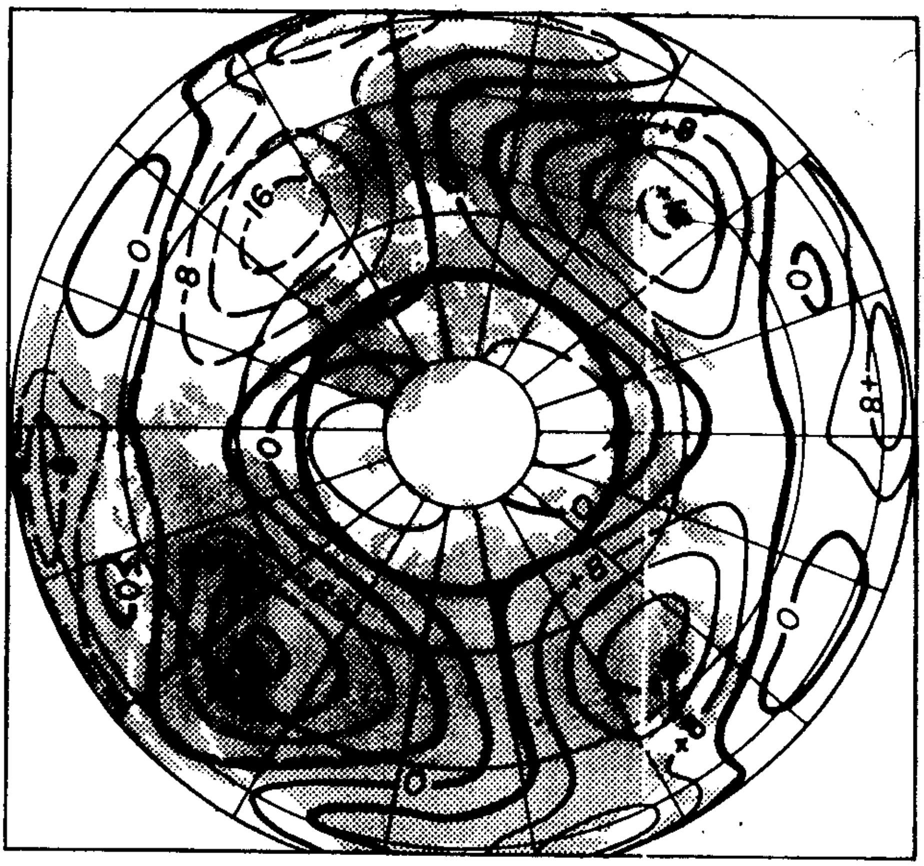


Figure 10. See Section 5.2. Scale: 10^{-4} rad^2 yr^{-1}. Parallels are shown every 20° between −80° and 80° and meridians are shown every 20°, Greenwich meridian being at the centre of the picture. Dashed lines for negative values, full lines for positive values. (a) Toroidal scalar of the 'typical' flow (see text). (b) Toroidal scalar of the second flow (see text). (c) Toroidal scalar of the 'typical' flow truncated at degree 4. (d) Toroidal scalar of the second flow truncated at degree 4. (e) Poloidal scalar of the 'typical' flow. (f) Poloidal scalar of the second flow. (g) Poloidal scalar of the 'typical' flow truncated at degree 4. (h) Poloidal scalar of the second flow truncated at degree 4.

LE MOUVEMENT SYMETRIQUE EVITE LA GRAINE

Hulot, le Mouël and Jault, J.G.G. (1990)



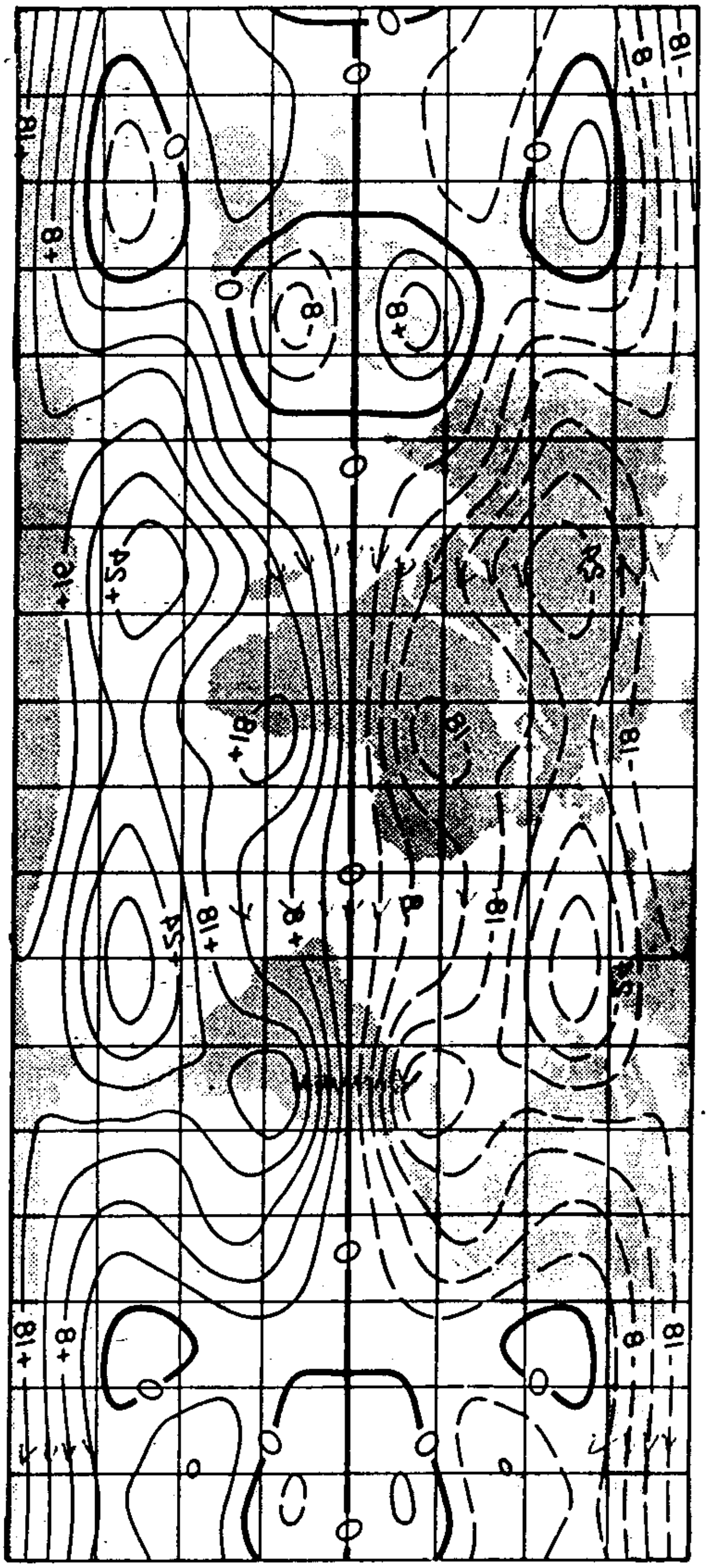
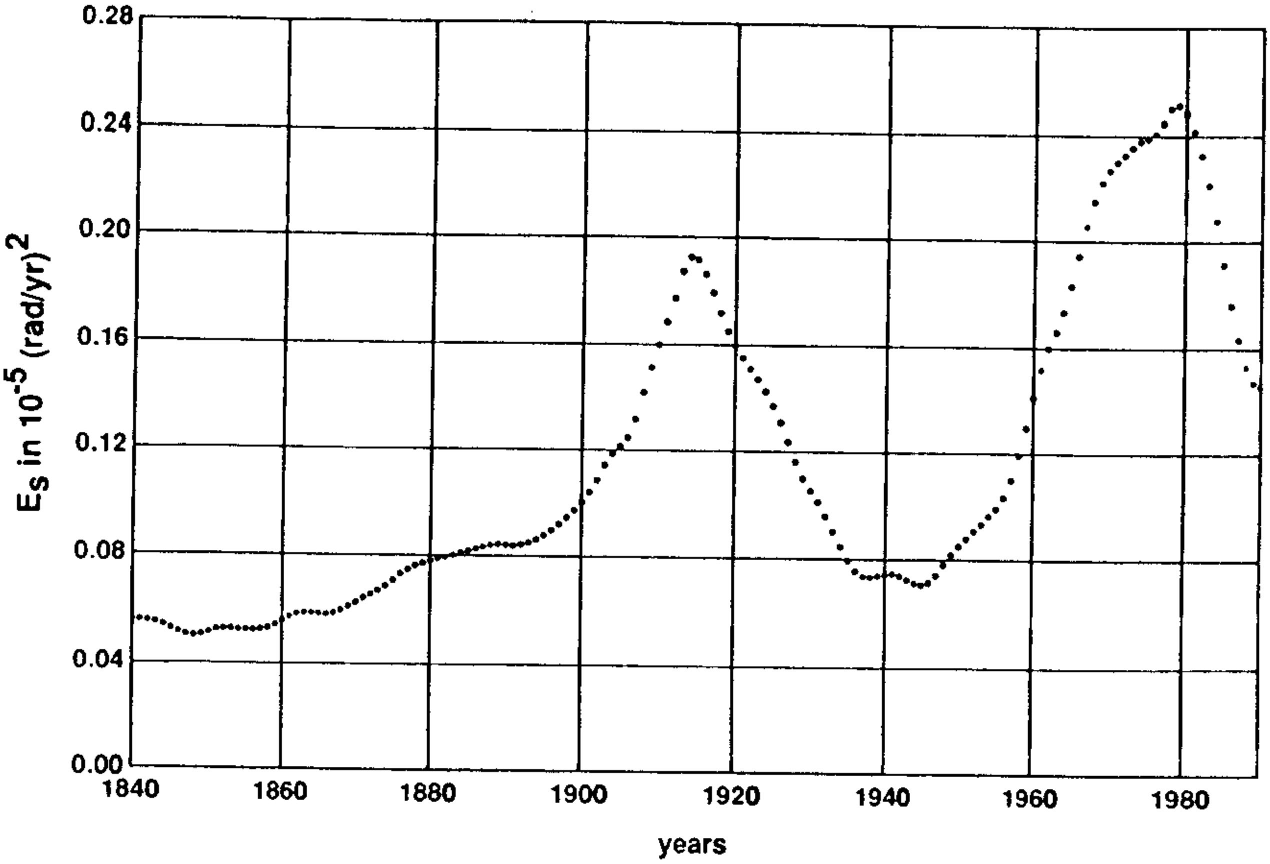
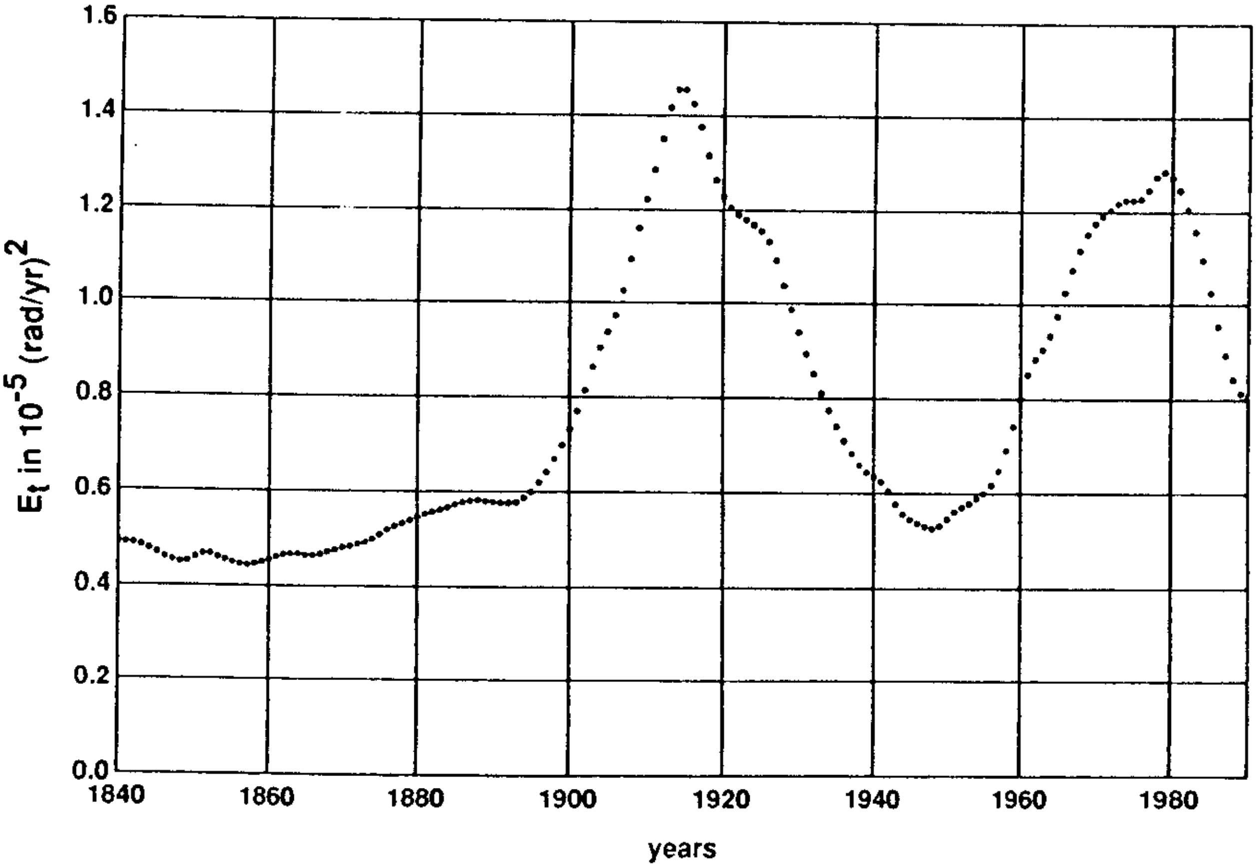


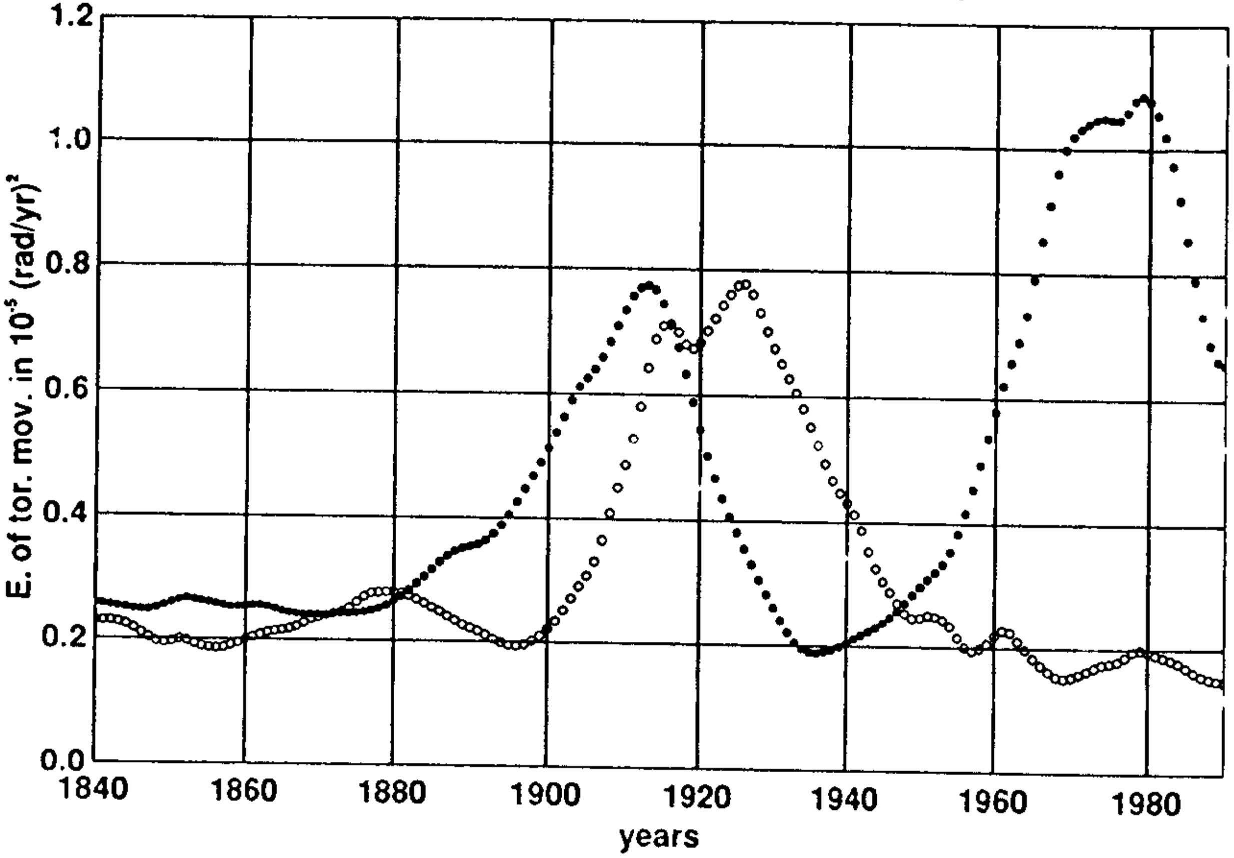
Fig. 2(a). Symmetrical part of the toroidal flow, the scalar of which is represented by Fig. 1(c). Scale: 10^{4} rad.yr^{-1}. Contour interval: $4 \cdot 10^{4}$ rad.yr^{-1}. Dashed lines for negative values, full lines for positive values.



Energy of poloidal movement



Energy of toroidal non-zonal movement



Symmetry proposed by Hulot et al. (1990)