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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.755/5

Workshop on Fluid Mechanics

(7 - 25 March 1994)

Shear Dispersion

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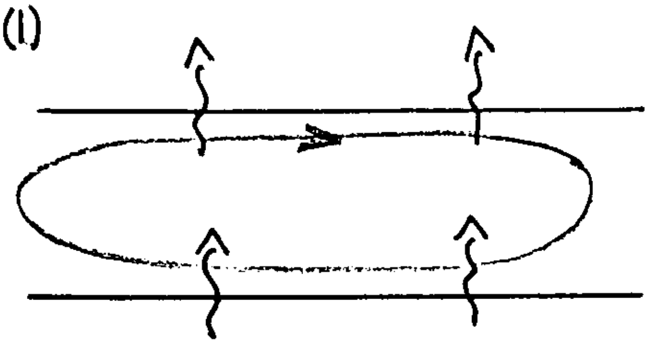
These are preliminary lecture notes, intended only for distribution to participants

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449

Other Examples

(i)



Fixed flux convection

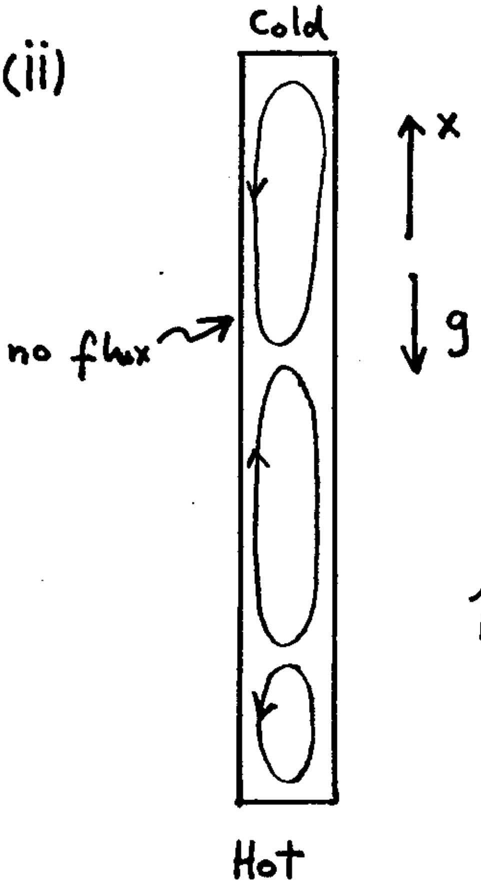
CHAPMAN + PROCTOR, JFM 101

$$f_t = -f_{xx} - \mu^2 f_{xxxx} + (f_x^3)_x$$

↑ Shear Dispersion

The most unstable wavenumber is $k = 0$.

(ii)

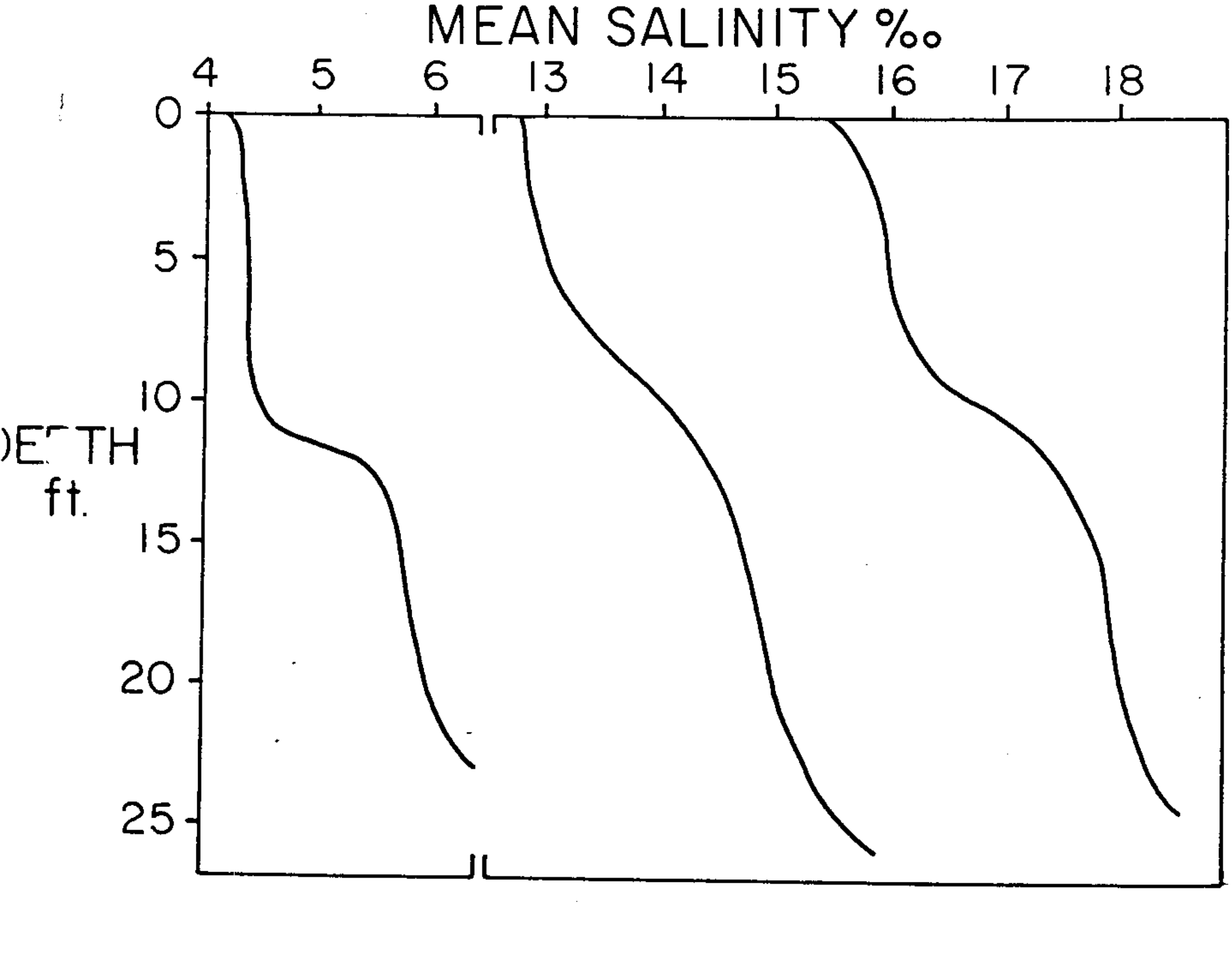


Again the most unstable wavenumber is $k = 0$.

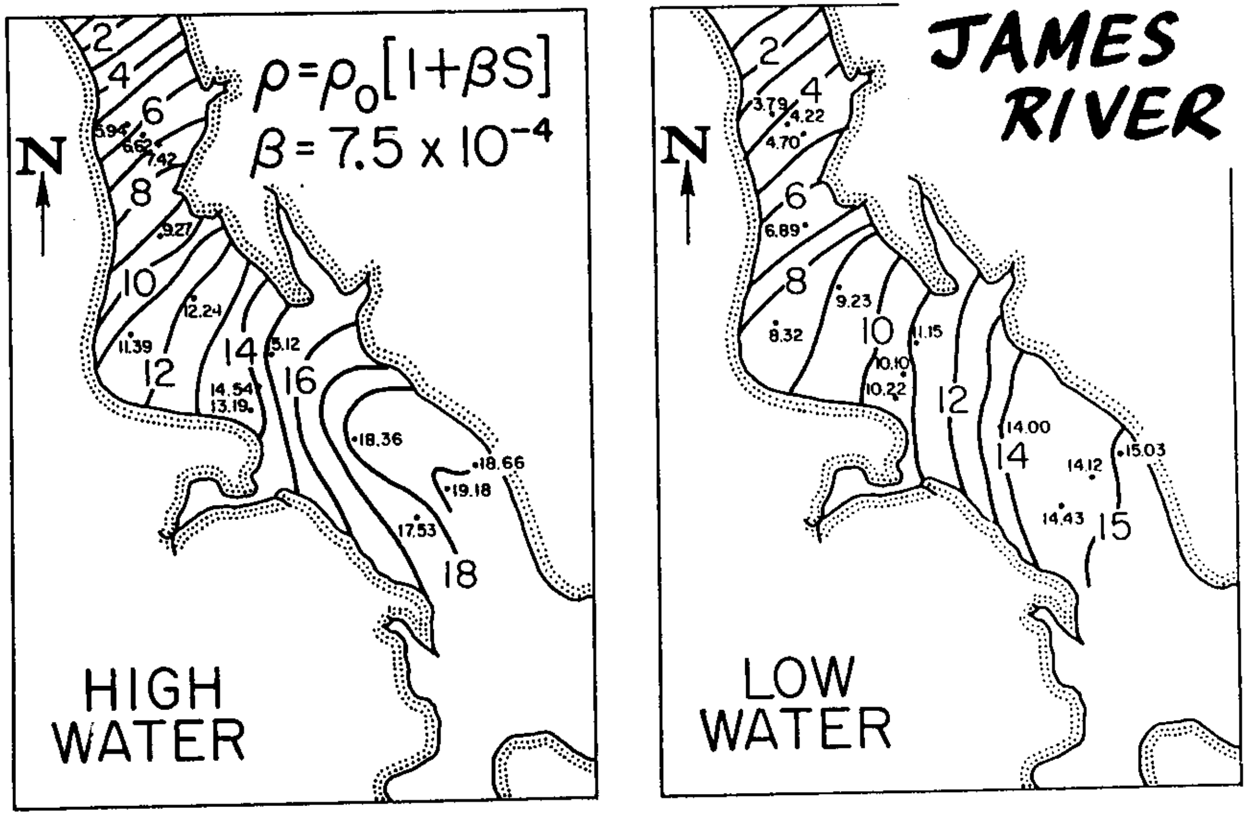
$$F_t = F_{xx} + (A^2)_x$$

$$p A_t = A_{xx} + (r + F_x)A$$

Cessi + Young JFM 237

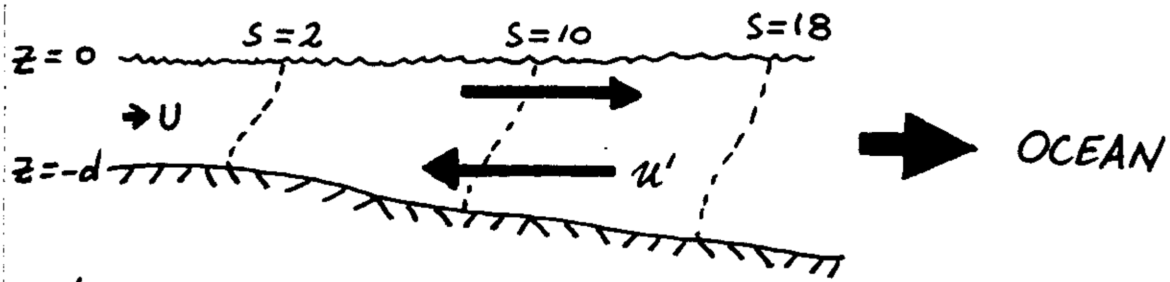


MEAN SALINITY ‰
4
5
6
13
14
15
16
17
18
0
5
10
15
20
25
DEPTH
ft.


JAMES RIVER
$\rho = \rho_0[1+\beta S]$
$\beta = 7.5 \times 10^{-4}$
N
2
4
6
8
10
12
14
16
18
5.94
6.62
7.42
9.27
12.24
11.39
5.12
14.54
13.19
18.36
18.66
19.18
17.53
HIGH WATER
N
2
4
6
8
10
12
14
15
3.79
4.22
4.70
6.89
9.23
8.32
11.15
10.10
10.22
14.00
14.12
15.03
14.43
LOW WATER

ESTUARINE DYNAMICS

— SHEAR DISPERSION OF DENSITY



$u' \sim 5 \text{ cm s}^{-1}$

$U \sim 1 \text{ mm s}^{-1}$

$$\rho \approx \rho_0 (1 + \beta \bar{S}) \Rightarrow p_x \sim g \rho_0 \beta \bar{S}_x z$$

$$0 \approx - p_x + \mu u'_{zz} \Rightarrow u' \sim g \beta \bar{S}_x d^3 / \nu$$

$$u' \bar{S}_x = \kappa S'_{zz} \Rightarrow S' \sim (g \beta \bar{S}_x)^3 d^5 / \nu \kappa$$

$$\left.\begin{array}{l}\text{Salt Flux due} \\ \text{to buoyancy driven} \\ \text{flow}\end{array}\right\} = \int_{-d}^{0} u' S' dz \sim \frac{(g\beta \bar{S}_x)^3 d^9}{\nu^2 \kappa}$$

$$= D_{eff}(\bar{S}_x)\, \bar{S}_x$$

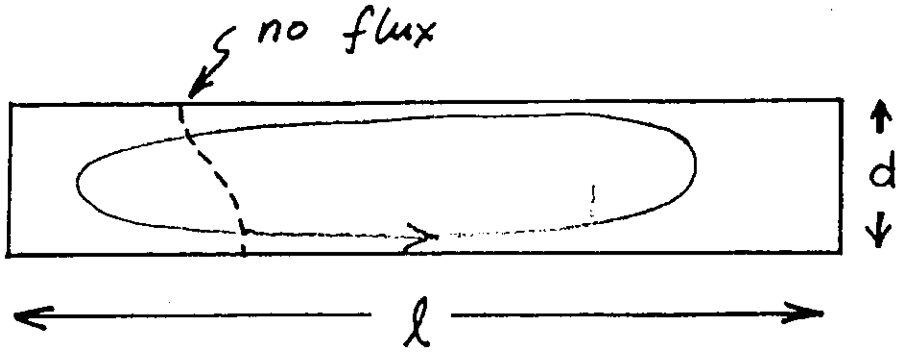
Steady State Balance

$$U \bar{S} - D_{eff} \bar{S}_x = 0$$

GODFREY, Estuarine + Coastal Marine Sci. 11 (1980)

Shear Dispersion of Density

ERDOGAN & CHATWIN, JFM 29



$\frac{d}{\ell} \ll 1$

Mixing of density in a shallow cavity takes place on two time scales

- rapid vertical mixing ($\frac{d^2}{\kappa}$):

  ➡ $\rho(x,z,t) \approx \bar{\rho}(x,t)$

- slow horizontal mixing produced by shear dispersion:

$$\bar{\rho}_t = \kappa \bar{\rho}_{xx} + \alpha (\bar{\rho}_x^3)_x$$

$$D_{eff} = \frac{u^2 d^2}{D_{mol}} \Rightarrow \alpha \sim \frac{d^8}{\nu^2 \kappa}$$

$$u \sim \frac{g \bar{\rho}_x}{\rho_0} \frac{d^3}{\nu}$$

Systematic Improvement of Taylor's Result.

Mercer & Roberts (SIAM J. Appl. Math. '90)

Exact Dynamics:

$$c_t + u\, c_x = D \nabla^2 c + S$$

↑ source e.g. $S = c_0(\underset{\sim}{x})\,\delta(t)$

$$(U, C) \equiv \frac{1}{A} \int (u, c)\, dA$$

Use center manifold theorem:

$$C_t + U C_x - D_{eff} C_{xx} - D_3 C_{xxx} \cdots = S + E_1 S_x + \cdots$$

↑ Source

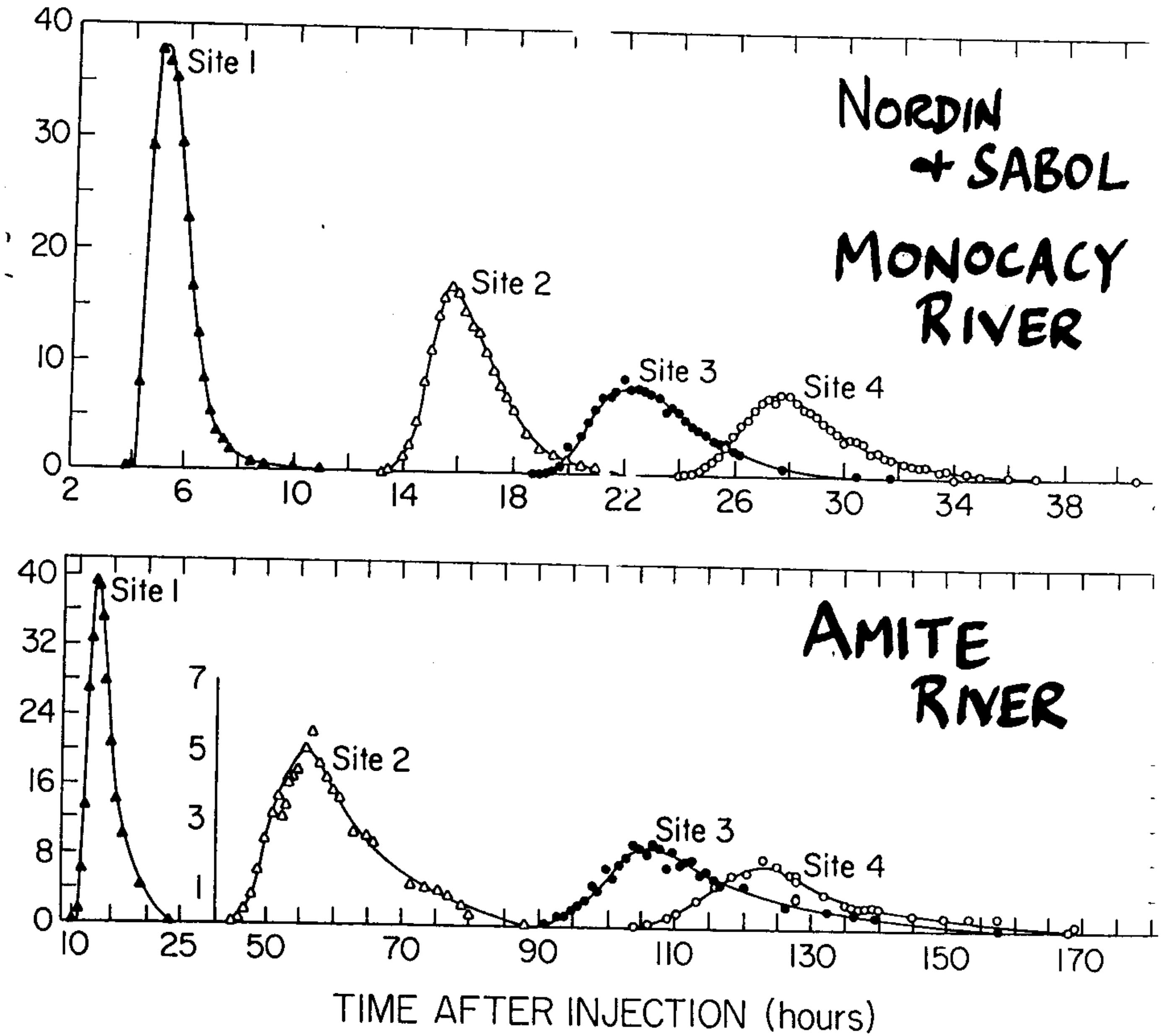
This is an expansion about $k = 0$ i.e. long horizontal length scales.

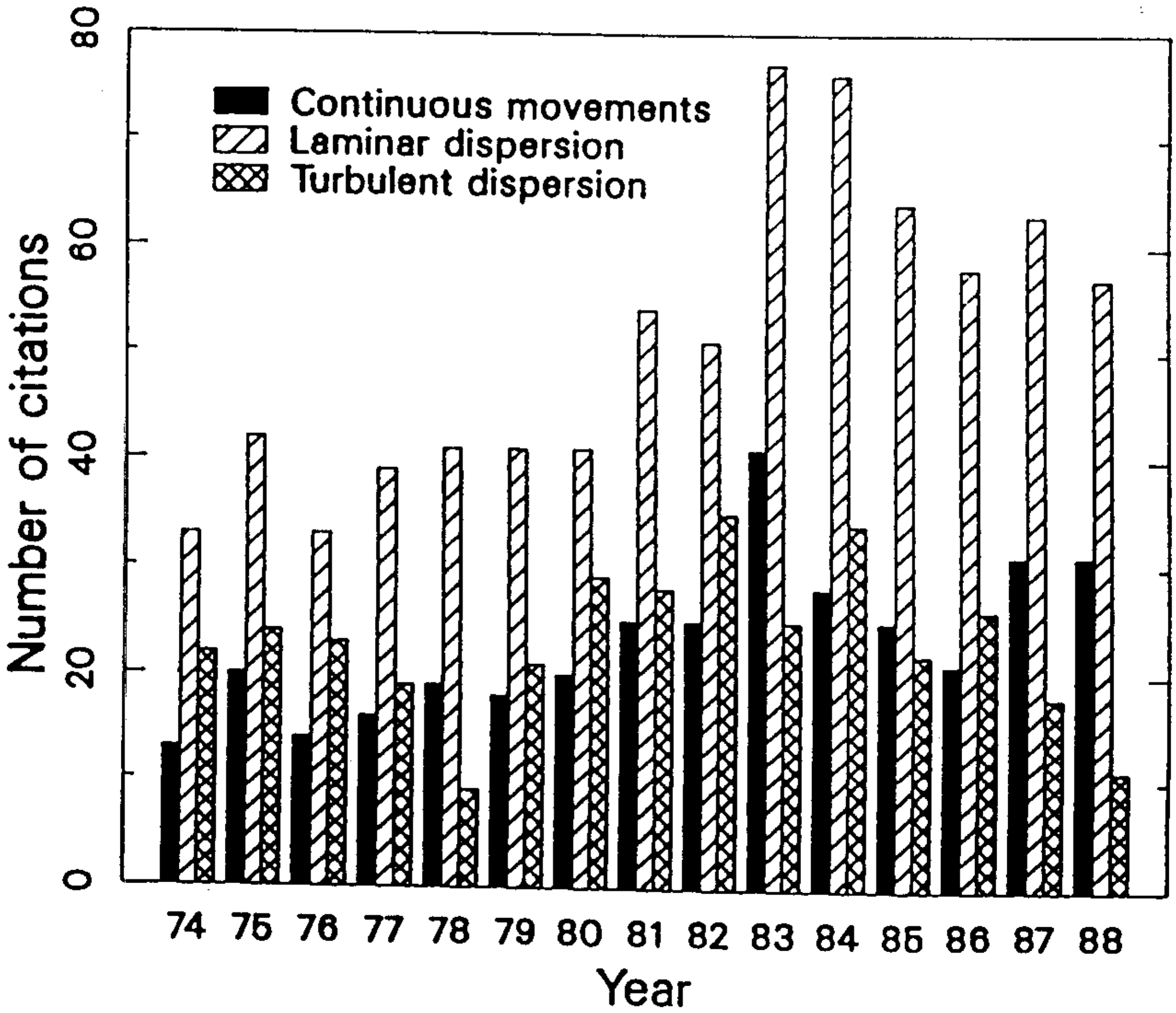
The moments (and cumulants)

$$m_j(t) \equiv \int dV\, x^j\, c(x, y, z, t)$$

$$M_j(t) \equiv \int dx\, x^j\, C(x, t)$$

agree apart from exponentially decaying transients.


Nordin + Sabol
Monocacy River
Site 1
Site 2
Site 3
Site 4
40
30
20
10
0
2
6
10
14
18
22
26
30
34
38
Amite River
Site 1
Site 2
Site 3
Site 4
40
32
24
16
8
0
7
5
3
1
10
25
50
70
90
110
130
150
170
TIME AFTER INJECTION (hours)


Continuous movements
Laminar dispersion
Turbulent dispersion
Number of citations
0
20
40
60
80
74
75
76
77
78
79
80
81
82
83
84
85
86
87
88
Year

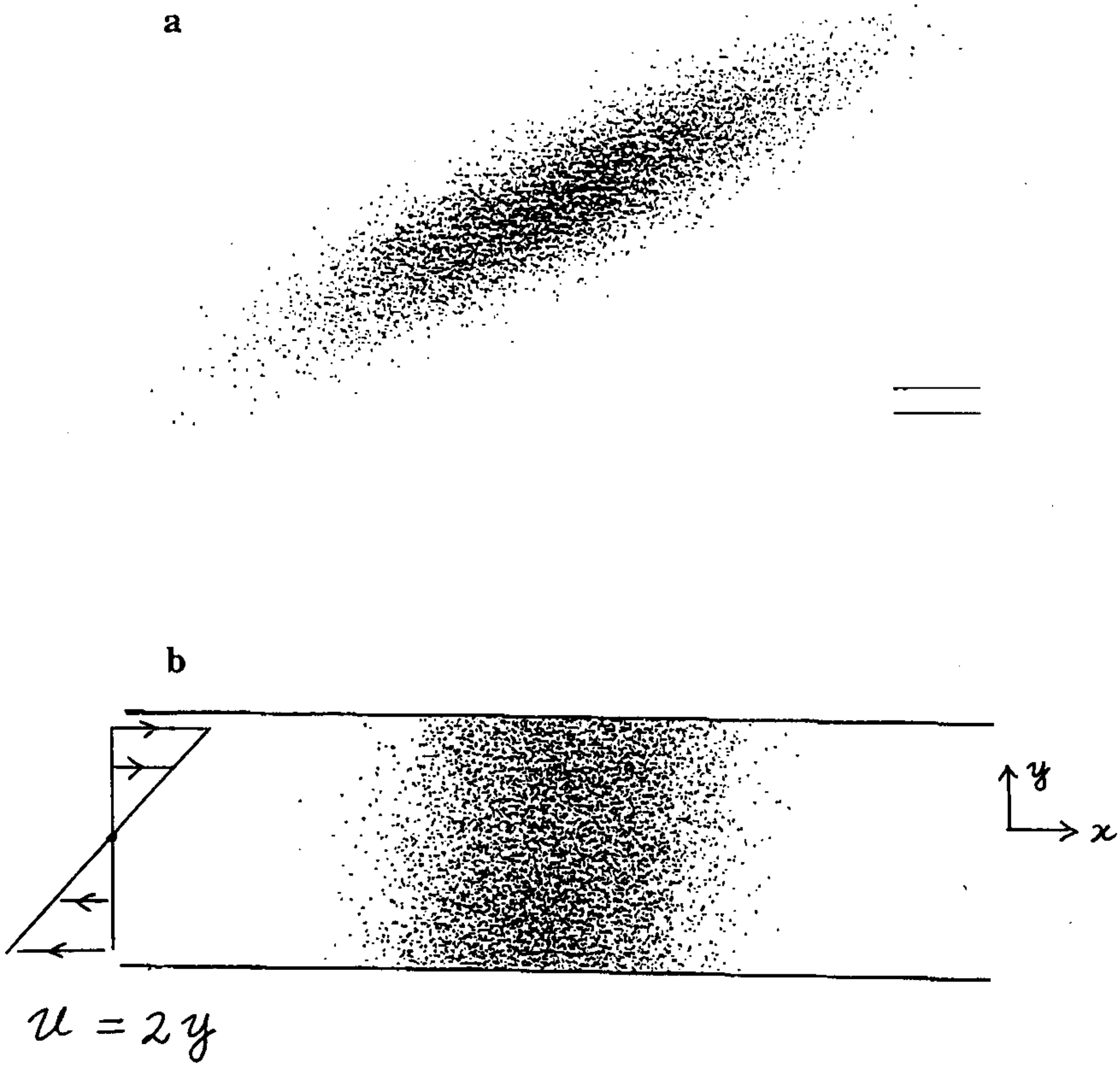


Fig. 1. Dispersion of an ensemble of diffusive particles by the flow $u = 2y$. In an (a) unbounded domain; (b) bounded domain. The channel shown in (b) is reproduced in the lower right-hand corner of (a)

FROM SCOTT JONES

An Informal Derivation

Exact tracer conservation

$$c_t + u\, c_x = D\, \nabla^2 c$$

where $u = 2U\left(1-\frac{r^2}{a^2}\right)$ and $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$

Sectional average

$$\overline{\theta} \equiv \frac{1}{A}\int \theta\, dA \quad , \quad (U,C) = \frac{1}{A}\int (u,c)\, dA$$

where $dA = dy\,dz$

Averaged equation

$$C_t + U C_x + (\overline{u'c'})_x = D\, C_{xx}$$

"Fluctuation" equation

$$c'_t + U c'_x + (u'c' - \overline{u'c'})_x + D(c'_{xx} + c'_{yy} + c'_{zz}) = -u' C_x$$

exact solution : $c' = \mathcal{G} * [-u' C_x]$

$\Rightarrow$ nonlocal transport

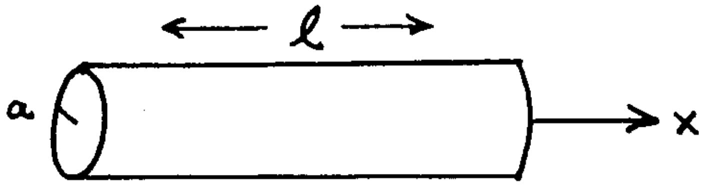
asymptotic solution : $D(c'_{yy} + c'_{zz}) = -u' C_x$

$\Rightarrow$ $c' \sim C_x$ $\rightarrow$ local transport

(N.B. $P_e \equiv \frac{aU}{D} \gg 1$) $\quad + \; c' \sim \frac{1}{D}$

SHEAR DISPERSION

G.I. TAYLOR, 1953



Poiseuille flow: $u = 2U\left(1 - \frac{y^2+z^2}{a^2}\right)$

Dispersion of tracer in a long pipe is a two stage process:

FAST: $T \sim a^2/D$

$$c(x,y,z,t) = C(x,t) + c'(x,y,z,t)$$
$$\approx C(x,t)$$
$$\equiv \frac{1}{A}\int c(x,y,z,t)\,dA$$

SLOW: $T \sim \ell^2/D_{eff}$

$$C_t + UC_x = D_{eff}\,C_{xx}$$

$$D_{eff} = D + \underbrace{\frac{U^2a^2}{48D} \gg D}_{\text{N.B.}}$$