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New derivations of Darwin's theorem

and

Similarity of steady stratified flows

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New derivations of Darwin's theorem

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Two new derivations of Darwin's theorem on the equality of the added mass for translation of a body moving in an ideal fluid of infinite extent and the drift mass are given. The first is based on the idea of time lag, used by Rayleigh (1876), Ursell (1953), and Longuet-Higgins (1953) to study fluid drift. The second is truly elementary, relying only on the concept of continuity and Newton's second law of motion. A geometrical interpretation of the result in the first derivation is given, and a few examples are provided.

1. Introduction

Darwin's theorem (Darwin 1953) shows the equality of the added mass of a body in translation in an ideal fluid and the mass of the drift volume of the fluid at a section, as the body moves with constant velocity from the far right of the section to its far left. It is a beautiful theorem, for what it revealed was thitherto entirely unexpected and even today whoever encounters it for the first time still experiences the surprise and delight it affords.

In this paper two new derivations of Darwin's theorem are given. The first is based on the idea of time lag in steady irrotational flows, which allows Darwin's theorem to be obtained with simplicity and directness. At first I thought this idea was new, but it was pointed out to me that the idea originated with Lord Rayleigh (1876), who used it to study fluid drift in waves in a geometric way, but whose arguments (where he assumed two parallel streamlines near the bottom) are valid only for deep-water waves, as pointed out by Ursell (1953). It was Ursell (1953, p. 147) who first put Rayleigh's idea in analytical terms. Indeed (12) and (13) in this paper are quite reminiscent of Ursell's work. The idea of time lag was also used by Longuet-Higgins (1953) to steady fluid drift in space-periodic and solitary waves. However, neither Ursell nor Longuet-Higgins was concerned with Darwin's theorem, whereas this paper is.

The idea of time lag has also been quite explicitly used by Lighthill (1956). See, for instance, equation (46) on p. 42 of his article, which treated weak shear flows.

We shall derive Darwin's theorem for two-dimensional flows first. Then a geometrical interpretation will be given to the result obtained and a few examples provided. For the sake of completeness as well as to illustrate the usefulness of general stream functions, we shall derive Darwin's theorem for three-dimensional flows. Finally, an elementary proof of Darwin's theorem based on the concept of continuity and on Newton's second law will be given, without the explicit use of integral calculus, as well as an alternative form of Taylor's theorem (1928).

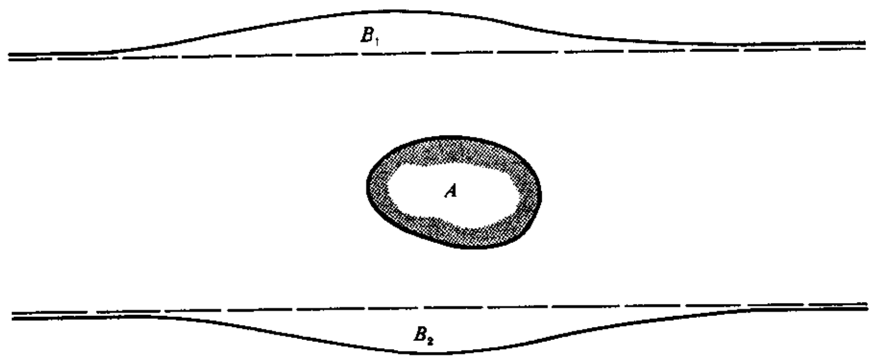

FIGURE 1. Sketch for the areas B_1, B_2, and A.

2. The two-dimensional case

As usual, the velocity potential ϕ' and the stream function ψ' of the irrotational flow caused by a body moving in an ideal fluid otherwise at rest are expressed in coordinates of a frame moving with the body. The velocity components in the directions of increasing x and y are, respectively,

$$u' = \phi'_x = \psi'_y, \quad v' = \phi'_y = -\psi'_x, \tag{1}$$

where subscripts indicate partial differentiation. The speed q' is defined by

$$q'^2 = u'^2 + v'^2. \tag{2}$$

Let the body move to the left (in the direction of decreasing x) with constant speed 1. Then the flow is steady with respect to the moving frame, and the velocity potential ϕ and the stream function ψ are given by

$$\phi = x + \phi', \quad \psi = y + \psi'. \tag{3}$$

The velocity components are

$$u = \phi_x = \psi_y = 1 + u', \quad v = \phi_y = -\psi_x = v' \tag{4}$$

and the speed q is given by

$$q^2 = u^2 + v^2. \tag{5}$$

As is well known, the added mass of the body is given by

$$m_a = \rho \iint q'^2 \, dx \, dy, \tag{6}$$

where ρ is the density of the fluid and the integral is over the (infinite) area outside the body. Now consider the integral

$$I = \rho \iint_D [(u-1)^2 + v^2] \, dx \, dy, \tag{7}$$

where D is a domain (figure 1) bounded by two streamlines $\psi = \psi_B$ and $\psi = \psi_{-B}$, one above the body and the other below it. As these streamlines recede to infinity above and below, respectively, I approaches m_a.

Now, since

$$q^2 = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \tag{8}$$

we have

$$\frac{I}{\rho} = \iint (1-q^2)\,\mathrm{d}x\,\mathrm{d}y - 2\iint (u-q^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint \left(\frac{1}{q^2}-1\right)\mathrm{d}\phi\,\mathrm{d}\psi - 2\iint\left(\frac{u}{q^2}-1\right)\mathrm{d}\phi\,\mathrm{d}\psi = I_1 - 2I_2, \tag{9}$$

where all integrations are over D, and I_1 and I_2 are defined by the last equality sign. But

$$\frac{u}{q^2} = \frac{\partial x}{\partial \phi},$$

so that

$$I_2 = \iint \left(\frac{\partial x}{\partial \phi} - 1\right)\mathrm{d}\phi\,\mathrm{d}\psi = \int_{\psi_{-B}}^{\psi_B} (x-\phi)\,|_{-\infty}^{\infty}\,\mathrm{d}\psi = -\int_{\psi_{-B}}^{\psi_B} \phi'\,|_{-\infty}^{\infty}\,\mathrm{d}\psi = 0, \tag{10}$$

since $\phi' = 0$ at infinity for a body moving in infinite fluid. Therefore

$$I = \rho I_1. \tag{11}$$

In I_1,

$$\frac{\mathrm{d}\phi}{q} = \mathrm{d}s,$$

where $\mathrm{d}s$ is the distance along a streamline as ϕ changes by $\mathrm{d}\phi$. Thus

$$\frac{\mathrm{d}\phi}{q^2} = \mathrm{d}t, \tag{12}$$

where $\mathrm{d}t$ is the time required for a fluid particle to travel the distance $\mathrm{d}s$. The integral

$$I_3 = \int_{-\infty}^{\infty} \left(\frac{1}{q^2}-1\right)\mathrm{d}\phi \tag{13}$$

is then the difference between the time required by a fluid particle to go from $\phi = -\infty$ to $\phi = +\infty$ and the time required by a reference kinematic point moving with constant u ($=1$) to do the same. That is, it is the drift distance for a particle moving along any particular streamline in the steady flow (ϕ, ψ). (If the particle requires more time, and eventually its velocity is 1, the same as that of the reference point, it will never catch up, and will lag behind the reference point by the distance equal to 1 times I_3. This distance is the drift distance.) Hence I_1, being

$$\int_{\psi_{-B}}^{\psi_B} I_3\,\mathrm{d}\psi,$$

is the drift area (or drift volume per unit distance normal to the x, y plane). Then in the limit, as $\psi_B \to \infty$ and $\psi_{-B} \to -\infty$,

$$m_a = \lim \rho I_1. \tag{14}$$

That is, the drift mass is the added mass, per unit distance along the generatrix of the cylinder, which is the body under consideration. Thus Darwin's theorem is proved in a new, simple way.

3. Geometrical significance of the integral I_1

Considering I_1 again, we see that

$$I_1 = D - S, \tag{15}$$

where

$$D = \text{area of domain } D = \iint \frac{1}{q^2}\,\mathrm{d}\phi\,\mathrm{d}\psi = \iint \mathrm{d}x\,\mathrm{d}y,$$

$$S = \iint \mathrm{d}\phi\,\mathrm{d}\psi.$$

Obviously S is the area of the infinite strip of width $\psi_B - \psi_{-B}$, including the cross-sectional area A of the body. Thus

$$D - S = B_1 + B_2 - A, \tag{16}$$

where B_1 is the area bounded by the streamline $\psi = \psi_B$ above and $y = \psi_B$ below, and B_2 is the area bounded by $\psi = \psi_{-B}$ below and $y = \psi_{-B}$ above. The bounding lines do not cross if ψ_B and $-\psi_{-B}$ are sufficiently large. Let

$$B = B_1 + B_2.$$

Then in the limit, as $\psi_B \to \infty$ and $\psi_{-B} \to -\infty$,

$$\rho B = m_a + m, \tag{17}$$

where m is the mass of the fluid displaced by the body.

We do not have to go to the limit, however. From (16) we obtain that

$$\rho B = m_d + m, \tag{18}$$

where m_d is the drift mass between the streamlines $\psi = \psi_B$ and $\psi = \psi_{-B}$. It is this generalization and the geometric relation (16) that lead us to the results presented in the section below.

4. Examples of fluid drift

Consider the classical solitary wave, the solution for which was first given by Rayleigh (1914) and refined by subsequent authors. No exact solution exists. But the result given below is exact, not depending on the particulars of the solution.

Take the steady-flow solution (ϕ, ψ) for the solitary wave, and take

$$\psi_{-B} = 0$$

and ψ_B to be the ψ on the free-streamline. The velocity scale is the speed c of the solitary wave. Upon use of this scale, everything developed in §§2 and 3 stands. In this case $A = 0$ exactly, because there is no solid body in the fluid, and $B_2 = 0$, because $\psi = 0$ and $y = 0$ coincide. Hence upon dividing (18) by ρ, we have

$$B = B_1 = \frac{m_d}{\rho},$$

or the drift area is exactly equal to the area between the free surface and its horizontal asymptote, which is Ursell's result (1953), obtainable also by the consideration of continuity.

Internal solitary waves in two superposed fluid layers have been studied by Keulegan (1953), Long (1956), and Benjamin (1966). The wave may be one of elevation of the lower fluid (Case A), or one of depression of the lower fluid (Case B).

Upon application of (18), with $m = 0$, to the lower fluid in Case A, we see that again the drift area for the lower fluid is exactly equal to the area underneath the interface and above its horizontal asymptote. For the upper fluid the drift area is of exactly the same magnitude, but in the opposite direction (opposite to the direction of propagation of the solitary wave). The total drift area is then exactly zero. For Case B, the opposite is true. That is, the upper fluid drifts with the wave, and the lower fluid drifts in the opposite direction, the drift area for each layer being exactly equal to the area between the interface and its horizontal asymptote.

Obviously these results can be generalized to apply to solitary waves in a fluid system of many layers. But I shall refrain from doing so. Instead, I shall give some other examples.

Consider a circular cylinder of radius a moving with unit velocity to the left along the x-axis. As is well known, the stream function is given by

$$\psi = y - \frac{a^2 y}{x^2+y^2}, \tag{19}$$

the origin of Cartesian coordinates x and y being at the centre of the cylinder. Area B_1 is given by

$$B_1 = \int_{-\infty}^{\infty} (y-\psi_B)\,\mathrm{d}x = a^2 \int_{-\infty}^{\infty} \frac{y}{x^2+y^2}\,\mathrm{d}x, \tag{20}$$

in which y is a function of ψ_B and x, obtained by letting the ψ in (19) be ψ_B. As ψ_B increases indefinitely, we can replace the y in the second integral of (20) by ψ_B, committing thereby less and less error as ψ_B increases, and ultimately no error at all. Doing so, we obtain from (20) that

$$B_1 = \pi a^2.$$

Similarly $B_2 = \pi a^2$, so that

$$\rho B = \beta(B_1+B_2) = 2\rho\pi a^2.$$

Since m, the mass of the fluid displaced by the body, is $\rho\pi a^2$, it follows from (17) that the added mass is

$$m_a = \rho\pi a^2.$$

By letting $\psi_B \to \infty$ and $\psi_{-B} \to -\infty$, we obtain from (18) the same value for the total drift mass m_d, as expected.

Another example is provided by the stream function

$$\psi = y' - \frac{a^2 y'}{x'^2+y'^2},$$

where the flow in the x', y' plane is the flow past a circular cylinder. If the coordinates (x, y) and (x', y') are related by

$$y = y' - \frac{b^2 y'}{x'^2+y'^2}, \quad x = x' + \frac{b^2 x'}{x'^2+y'^2},$$

we have the well-known result that the flow in the x, y plane is that past an elliptic cylinder with semi-major axis $a + b^2/a$ and semi-minor axis $a - b^2/a$. We have then

$$y - \psi = \frac{(a^2-b^2)\,y'}{x'^2+y'^2}.$$

If we replace y' by ψ_B and x' by x, and integrate with respect to x, in the limit, as ψ_B tends to ∞, we obtain without error

$$B_1 = \int_{-\infty}^{\infty} (y-\psi)\,\mathrm{d}x = (a^2-b^2)\,\pi.$$

Similarly B_2 has the same value, and

$$\rho B = \rho(B_1+B_2) = 2\rho\pi(a^2-b^2) = m_a+m = m_d+m.$$

But

$$m = \rho\pi\left(a+\frac{b^2}{a}\right)\left(a-\frac{b^2}{a}\right) = \rho\pi\left(a^2-\frac{b^4}{a^2}\right).$$

Hence

$$m_a = m_d = \rho\pi\left(a-\frac{b^2}{a}\right)^2,$$

as is well known.

Note that Darwin proved this theorem (as I do here also) only for a fluid of infinite extent. It can be generalized to apply to a semi-infinite fluid bounded by a single plate to which the velocity of the immersed body is parallel. But it is not true for a restricted fluid, such as the fluid between two parallel plates, in which a body moves. In such cases the I_2 in (10) is not zero, and therefore Darwin's theorem does not hold, because ϕ' is not zero at infinity, as Ursell (1953) pointed out in the case of the solitary wave, and as can be shown easily in the case of the fluid bounded by two parallel plates. For such a case (9) needs to be carefully re-examined, for there is a subtle point involving the interpretation of the integral S in (15), which no longer represents the area of the infinite strip including the area A, as stated after (15), but contains an additional part that can be easily shown to be $-I_2$. Then, since B vanishes in this case, (9) becomes, upon use of (15),

$$I = -\rho A-\rho I_2, \quad \text{or} \quad -\rho I_2 = m_a+\rho A,$$

as can be shown independently by using a control volume and applying Newton's second law and Bernoulli's theorem (for unsteady flows with the body moving and the fluid at rest at infinity).

5. The three-dimensional case

We shall now return to a new proof of Darwin's theorem for three-dimensional flows, based on the idea of time lag. This proof has some incidental merit in demonstrating the usefulness of stream functions for three-dimensional flows (Yih 1957, 1979). Let these be denoted by ψ and χ. Then the velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \operatorname{grad}\psi \times \operatorname{grad}\chi. \tag{21}$$

Thus

$$q^2 = u^2+v^2+w^2 = \frac{\partial(\phi,\psi,\chi)}{\partial(x,y,z)} = J, \tag{22}$$

$$\frac{u}{q^2} = \frac{1}{J}\frac{\partial(\psi,\chi)}{\partial(y,z)} = \frac{\partial x}{\partial\phi}, \tag{23}$$

in which u, v, and w are the velocity components in the directions of increasing x, y, and z, respectively, with the velocity *at infinity* being

$$u = 1, \quad v = 0, \quad w = 0.$$

We shall not assume any axial symmetry. But it is still useful to define

$$r = (y^2+z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\frac{y}{z}, \tag{24}$$

because, at $x = \pm\infty$,

$$\psi = \tfrac{1}{2}r^2, \quad \chi = \theta. \tag{25}$$

Since the body is assumed to move with speed 1 (in the direction of decreasing x), the added mass is, as is well known,

$$m_a = \rho \iiint q'^2 \,dx\,dy\,dz, \tag{26}$$

q' being the speed of the fluid for the (unsteady) flow caused by the body in the fluid otherwise at rest. The integral in (26) is carried over the entire space occupied by the fluid. We now take D to be the domain between $\psi = 0$ and $\psi = \psi_0 = \frac{1}{2}r_0^2$, from $x = -\infty$ to $x = +\infty$, and consider the integral

$$I = \rho \iiint_D [(u-1)^2 + v^2 + w^2]\,dx\,dy\,dz.$$

Then the development is exactly as that following (7), and we obtain

$$\begin{aligned}\frac{I}{\rho} &= \iiint (q^2 - 2u + 1)\,dx\,dy\,dz \\ &= \iiint \left(\frac{1}{q^2} - 1\right) d\phi\,d\psi\,d\chi - 2\iiint \left(\frac{u}{q^2} - 1\right) d\phi\,d\psi\,d\chi \\ &= I_1 - 2I_2,\end{aligned} \tag{27}$$

where I_1 and I_2 are defined by the last equality sign, and all integrations are over the domain D. Again, for an unrestricted fluid,

$$I_2 = \iiint \left(\frac{\partial x}{\partial \phi} - 1\right) d\phi\,d\psi\,d\chi = \iint_{(\psi,\chi)} (x-\phi)_{-\infty}^{\infty}\,d\psi\,d\chi = 0,$$

and we have

$$\frac{I}{\rho} = I_1.$$

In the limit,

$$\rho I_1 = m_a.$$

But, as before, I_1 is the drift volume. Hence we have given a new proof of Darwin's theorem for three-dimensional flows using the idea of time lag.

6. A proof of Darwin's theorem without calculation

Darwin showed in his paper (1953) that the drift volume V_D for three-dimensional flows (the formula for two-dimensional flows then follows directly) is given by

$$V_D = -\iiint \phi'_x \,dx\,dy\,dz, \tag{28}$$

where the integration is carried over the entire fluid domain, and where the sign convention of (1) regarding ϕ has been adopted. I have used a minus sign in (28) to make V_D positive, since in this paper the body is assumed to move toward the left (i.e. in the direction of decreasing x). It is evident that the integral is the total momentum of the fluid divided by ρ, and, if the body is moving with unit velocity to the left, the right-hand side of (28), with the minus sign included, is obviously the added mass divided by ρ, upon consideration of acceleration of the body. So the drift mass is the added mass, and this fact is the substance of Darwin's theorem.

Darwin's theorem is not only true; it is beautiful as well. Part of the reason for the delight it gives is its unexpectedness. And yet one could wonder whether it is a fortuitous truth stating a fortuitous equality, or whether the equality it states is dictated by kinematics and dynamics in so simple and direct a way that it is obvious. Evidently if the latter is true it has hitherto not been recognized, for Darwin's theorem is widely regarded as difficult to grasp, and its unexpectedness (for this writer at least) seems to indicate that the equality of drift mass and added mass is fortuitous. But then how could such a general equality, regardless of shape of the body, be only fortuitously true? One could argue, of course, that any truth demonstrated mathematically is not fortuitous, that it has mathematical necessity. Yet mathematical necessity is not mechanical necessity, and, upon learning of Darwin's theorem, one is always left wondering why it must be true mechanically.

I shall now show, without the explicit use of integral calculus, that Darwin's theorem is to be expected on the basis of continuity and Newton's second law.

Consider the domain D shown in figure 1, bounded by two streamlines (or a stream surface if the flow is three-dimensional) and the body. For convenience, and convenience only, I shall treat the flow as two-dimensional. But every statement that follows can be made applicable to three-dimensional flows by the change of a word here and there (e.g. the word 'area' to 'volume'). The geometry of D will be called the 'pattern'. The pattern moves with the body, though the fluid at infinity is at rest.

Now, at $x = 0$ and $t = 0$, let the intersection of D with the y-axis (or the y, z plane in three-dimensional flows) be dyed blue, and let the streamlines shown in figure 1 be dyed red at $t = 0$. Furthermore, let the body move left from $x = \infty$. After the body has moved to the far left, the blue line will have drifted left, and the drift area is the area swept by the blue line from its initial position to its final position. The red lines (but not the particles on them) move with the body and are made of the same fluid particles.

Consider the fluid mass to the left of the blue line and bounded by the red lines and the body. There is no flow across either the blue lines or the red lines since they are material lines, and there is no flow at $x = -\infty$. The fluid area just described must then be constant and equal to the area initially to the left of the blue line (and bounded by the red lines), when the body was at $x = \infty$. Thus†

$$B - A = C, \tag{29}$$

where $B \equiv B_1 + B_2$, and C is the drift area between the red lines. (Recall the definitions of B_1, B_2, and A.)

Now consider the domain D at any time. As the body moves left, the centre of gravity of D moves also. There is no flow at infinity and the 'pattern' moves with the body, so that the area B and the area A (occupied by the body, which is a fluid hole) move with the body. Hence the amount of fluid moving with the same mean x-velocity as the body is $\rho(B - A) = \rho C$, which then must be the added mass, as the time-rate of change of the momentum of the fluid in D is equal to the force imparted it by the body when it accelerates, if the x-component of the force from integrating the pressure on the red lines is ignored‡ as it can be ignored when the red lines recede to $y = \pm\infty$. Thus the drift mass must be equal to the added mass upon consideration of continuity and Newton's second law, and Darwin's theorem can be expected on these principles.

It is regrettable that we can no longer ask Sir Charles what inspiration led him to his discovery. I think it was unlikely that the considerations I have just presented

† The same arguments can be applied to viscous fluids to obtain the same result, which can also be otherwise established.

‡ The force arising from pressure at the ends of D (where $x = \pm\infty$) is zero.

went through his mind. These considerations are mere hindsights, and his theorem stands as an interesting example of the essential inexplicability and intractability of inspiration of the human mind. However, my attempts here perhaps serve to make his theorem more graspable and therefore more satisfying to his readers.

7. Connection between Darwin's theorem and Taylor's theorem

Equation (17) is in effect Taylor's theorem (1928). To save space, I shall consider only two-dimensional flows here. The three-dimensional counterpart of the development can be established without difficulty.

It is clear that

$$B = B_1 + B_2 = \int_{-\infty}^{\infty} (y_1 - \psi_B)\,\mathrm{d}x + \int_{-\infty}^{\infty} (\psi_{-B} - y_2)\,\mathrm{d}x, \tag{30}$$

where y_1 is y on the streamline $\psi = \psi_B$ and y_2 is y on $\psi = \psi_{-B}$, the B's being defined in §3. Equation (30) can be written as

$$B = -\int_{-\infty}^{\infty} \psi'_B\,\mathrm{d}x + \int_{-\infty}^{\infty} \psi'_{-B}\,\mathrm{d}x. \tag{31}$$

As $\psi_B \to \infty$ and $\psi_{-B} \to -\infty$, we obtain from (17) and (31)

$$\rho\left[-\lim_{y\to\infty} \int_{-\infty}^{\infty} \psi'\,\mathrm{d}x + \lim_{y\to-\infty} \int_{-\infty}^{\infty} \psi'\,\mathrm{d}x\right] = m + m_a. \tag{32}$$

By taking an infinite strip bounded externally by

$$\phi = \phi_B \quad \text{and} \quad \phi = \phi_{-B} \tag{33}$$

and internally by the surface of the body, we can obtain (see Appendix)

$$\rho\left[\lim_{x\to\infty} \int_{-\infty}^{\infty} \phi'\,\mathrm{d}y - \lim_{x\to-\infty} \int_{-\infty}^{\infty} \phi'\,\mathrm{d}y\right] = m + m_a. \tag{34}$$

The left-hand side of (32) and (34) are an alternative expression of Taylor's expression obtained from the singularities inside the body. On the other hand (32) is (17), (34) is an alternative form of (17), and (17) is closely related to the proof of Darwin's theorem. Thus, although Darwin was thinking of drift mass and Taylor was not, the grounds they traversed, a quarter of a century apart, were not far from each other.

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Appendix

To show (34), consider the domain D' bounded by the curves (33) externally and by the body internally. Then

$$\begin{aligned}\frac{I}{\rho} &= \iint_{D'} \left(\frac{1}{q^2} - \frac{u}{q^2}\right) \mathrm{d}\phi\,\mathrm{d}\psi - \iint_{D'} \left(\frac{u}{q^2} - 1\right) \mathrm{d}\phi\,\mathrm{d}\psi \\ &= \iint_{D'} \mathrm{d}x\,\mathrm{d}y - \iint_{D'} \mathrm{d}x\,\mathrm{d}\psi - \int_{-\infty}^{\infty} [(x_B - \phi_B) - (x_{-B} - \phi_{-B})]\,\mathrm{d}\psi. \end{aligned} \tag{A 1}$$

The first integral on the right-hand side is the (infinite) area of D', the second integral is the area of the infinite strip bounded by (33), including the body of Area A. Hence their difference is $-A$, and upon multiplication by ρ, and recalling that $m = \rho A$ and $I \rightarrow m_a$ as $\phi_B \rightarrow \infty$ and $\phi_{-B} \rightarrow -\infty$, we have, in the limit, (34), since $\phi' = \phi - x$.

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Similarity of steady stratified flows

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Steady flows of an incompressible, inviscid, and non-diffusive fluid of variable density in a gravitational field are first considered. By a transformation it is shown conclusively that there are infinitely many flows with the same flow pattern, provided the density gradients of these flows at any section (e.g. far upstream) differ only by a multiplicative constant. These flows have identical local internal Froude numbers at all corresponding points of the flows and, hence, identical local Richardson numbers. They are therefore dynamically similar. Every time a solution for one stratification is obtained, one has in fact obtained the solutions for infinitely many stratifications.

The creation of vorticity in steady stratified flows is then examined, and it is shown that this creation can be divided into two parts, one part being entirely due to the inertial effect and the other originating from the gravity effect of density variation. Finally, compressibility is considered and the results on similarity of stratified flows and on vorticity and circulation are extended to apply to steady flows of gases stratified in entropy.

1. Similarity of steady stratified flows of an incompressible fluid

For an incompressible and non-diffusive fluid stratified in density, the equation of incompressibility is

$$D\rho/Dt = 0, \tag{1}$$

where, since only steady flows are considered,

$$D/Dt = u_\alpha \partial/\partial x_\alpha. \tag{2}$$

In (1) and (2), ρ is the density, x_1, x_2, and x_3 are Cartesian co-ordinates, and u_1, u_2, and u_3 are the corresponding velocity components. The summation convention is used in (2). The equation of continuity is, by virtue of (1),

$$\partial u_i/\partial x_i = 0. \tag{3}$$

If the fluid is also assumed inviscid, the equations of motion are

$$\rho u_\alpha \,\partial u_i/\partial x_\alpha = -\partial p/\partial x_i - g\rho\delta_{i3} \quad (i = 1, 2, 3), \tag{4}$$

where p is the pressure, g is the gravitational acceleration acting in the direction of decreasing x_3, and δ_{i3} is the Kronecker delta.

Let the density be put in the form

$$\rho = \rho_0 + \rho_1(x_1, x_2, x_3), \tag{5}$$

where ρ_0 is a constant, and consider another stratified fluid with the density distribution

$$\hat{\rho} = \hat{\rho}_0 + \hat{\rho}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3), \tag{6}$$

where $\hat{\rho}_0$ is another constant and the circumflexes on the x's denote the co-ordinates for the flow with density $\hat{\rho}$. Since two flows can be similar only if the geometry of the boundaries are similar, we denote the length scales of the two flows by L and $\hat{L}$, and write

$$m = \hat{L}/L. \tag{7}$$

A point in the flow with length scale L is said to be corresponding to a point in the flow with length scale $\hat{L}$ if the dimensionless co-ordinates (measured in units of L or $\hat{L}$) of the two points are identical. For dynamical similarity to exist, we must have

$$\hat{\rho}_1/\rho_1 = r \tag{8}$$

at corresponding points of the two flows, r being a positive constant.

The question is then posed: Can a flow have the density distribution ρ and be similar to a given flow with the density distribution $\hat{\rho}$? We shall show that the answer is in the affirmative.

If the solution with distribution $\hat{\rho}$ has velocity components $\hat{u}_i$, we have

$$\hat{\rho}\hat{u}_\alpha \partial\hat{u}_i/\partial\hat{y}_\alpha = -\partial\hat{\pi}/\partial\hat{y}_i - \hat{L}g\hat{\rho}_1\delta_{i3}, \tag{9}$$

where

$$\hat{\pi} = \hat{p} + \hat{\rho}_0 g x_3, \quad \hat{y}_i = x_i/\hat{L}. \tag{10}$$

We also have

$$\hat{u}_\alpha \partial\hat{\rho}/\partial\hat{y}_\alpha = 0 \tag{11}$$

and

$$\partial\hat{u}_i/\partial\hat{y}_i = 0. \tag{12}$$

Now let (actually an arbitrary constant can be added to π or $\hat{\pi}$)

$$u_i = (\hat{\rho}/rm\rho)^{\frac{1}{2}}\hat{u}_i, \quad \pi = \hat{\pi}/rm, \quad y_i = x_i/L, \tag{13}$$

where π is defined by

$$\pi = p + \rho_0 g x_3. \tag{14}$$

Then, the first equation in (13), and (5), (6), and (8) guarantee that

$$u\,\partial\rho/\partial y_\alpha = 0, \tag{15}$$

provided (11) is satisfied. Furthermore, obviously

$$u_\alpha \partial\hat{\rho}/\partial y_\alpha = 0. \tag{16}$$

Equations (7), (8) and (13) allow us to write†

$$\rho u_\alpha \partial u_i/\partial y_\alpha = (\hat{\rho}/rm)\hat{u}_\alpha \partial\hat{u}_i/\partial\hat{y}_\alpha. \tag{17}$$

Thus (9) can be written as

$$rm\rho u_\alpha \partial u_i/\partial y_\alpha = -rm\,\partial\pi/\partial y_i - r\hat{L}g\rho_1\delta_{i3}, \tag{18}$$

† Remember that $y_i = \hat{y}_i$ at corresponding points, so that $\partial/\partial y_\alpha = \partial/\partial\hat{y}_\alpha$. This would be even clearer if we had used $\hat{x}_i = mx_i$.

which, after division by rm, is

$$\rho u_\alpha \partial u_i/\partial y_\alpha = -\partial \pi/\partial y_i - Lg\rho_1 \delta_{i3}. \tag{19}$$

This is exactly (4) with the Cartesian co-ordinates in dimensionless form. Thus, if (9) is satisfied by $\hat{u}_i$ and $\hat{\pi}$, u_i and π given by (13) satisfy (19), or (4).

Also, because of (15) and (16), (3) is satisfied if (12) is. Hence we have proved what we set out to prove. The boundary conditions, if they are kinematical, are identical since the boundary geometries are identical, and, if they are dynamical (such as at density discontinuities), are also identical since dynamic boundary conditions are natural boundary conditions derivable from the differential equations. Hence the boundary conditions are satisfied by the flow (u_1, ρ, p) if they are satisfied by the flow $(\hat{u}_i, \hat{\rho}, \hat{p})$.

If we define the local internal Froude number F at any point of the flow by

$$F^2 = \rho u_\alpha u_\alpha / g|\nabla \rho| L^2 \tag{20}$$

(and similarly for the flow with density $\hat{\rho}$), and the local Richardson number Ri at any point of the flow by

$$Ri = g|\nabla \rho|/\rho|\nabla q|^2, \quad q^2 = u_\alpha u_\alpha, \tag{21}$$

(and similarly for the flow with density $\hat{\rho}$), then we can say that the two flows have identical local internal Froude numbers at corresponding points (i.e. for $y_i = \hat{y}_i$), and consequently the same local Richardson numbers at these points. The flow patterns are also similar, by virtue of the first equation in (13). The two flows are indeed similar geometrically, kinematically, and dynamically.

But since ρ_0 and r are arbitrary, we are not merely treating one flow similar to the flow $\hat{\rho}$; we are treating a doubly infinite family of flows, all of which are similar to the flow for $\hat{\rho}$, and hence to one another. This result is new, and I think it is very useful for laboratory simulation of natural phenomena. Note that density discontinuities are not ruled out. But, for similarity to exist between any two flows, they must occur at corresponding places, and, wherever they occur, their ratio must be the same constant (denoted by r in this paper). We note that when the density variation is small as compared with the mean density, the factor $(\hat{\rho}/\rho)^{\frac{1}{2}}$ in (13) can be replaced by $(\hat{\rho}_0/\rho_0)^{\frac{1}{2}}$ (and by 1 if water is used in the laboratory to model lakes or oceans), and the effect of density variation, important at low Froude numbers, is embodied in the factor r and is entirely associated with gravity.

In conclusion, to ensure similarity, the requirements expressed by (8) and the first equation in (13) must be satisfied at corresponding sections somewhere, say far upstream. Note that ρ does not have to be proportional to $\hat{\rho}$ at corresponding points.

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