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Workshop on Fluid Mechanics

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Derivation of Darwin's theorem in its weak form

and

Selective withdrawal

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Derivation of Darwin's theorem in its weak form:

We have

$$I = I_1 - 2 I_2 ,$$

where

$$I = \iint \rho [(u-1)^2 + v^2] dx dy ,$$

$$I_1 = \iint \rho (1 - q^2) dx dy ,$$

$$I_2 = \iint \rho (u - q^2) dx dy ,$$

$$q^2 = u^2 + v^2 .$$

We can write

$$I_1 = \rho \iint \left(\frac{1}{q^2} - 1 \right) d\phi \, d\psi$$

$$I_2 = \rho \iint \left(\frac{u}{q^2} - 1 \right) d\phi \, d\psi = \rho \iint \left(\frac{\partial \chi}{\partial \phi} - 1 \right) d\phi \, d\psi$$

$$= -\rho \iint \frac{\partial \phi'}{\partial \phi} d\phi \, d\psi , \quad \phi' = \phi - \chi .$$

For large distances from the body

$$\phi = x + \phi' = x + \frac{a^2 (lx + my)}{r^2} , \quad r^2 = x^2 + y^2 .$$

$$\psi = y - \frac{a^2 (ly - mx)}{r^2} .$$

Domain of integration:

D_A: $\quad x = -x_0, \quad x = x_0, \quad \psi = -\psi_A, \quad \psi = \psi_A$

$$m_a = m_d + 8\rho a^2 \arctan \frac{\psi_A}{x_0}$$

Let

$$\psi_A = Na \quad (N >> 1),$$
$$x_0 = N^3 a.$$

Then

$$\underset{\substack{\uparrow \\ \text{for entire} \\ \text{fluid}}}{m_a} = \underset{\substack{\uparrow \\ \text{for domain} \\ D_A}}{m_d} + O\left(\frac{1}{N^2}\right) \longleftarrow \text{Darwin theorem in its Weak Form}$$

For 3-dim. flows, let ψ be the one (of the two stream functions ψ and χ) that corresponds to the Stokes stream function. (Axisymmetry not assumed!) Use a domain D_A defined as above, and let $\psi(x_0, r_0) = \psi_A$. Then, if

$$r_0 = Na \quad (N >> 1),$$
$$x_0 = N^{2.5} a,$$

we have, similarly,

$$\underset{\substack{\uparrow \\ \text{for entire} \\ \text{fluid}}}{m_a} = \underset{\substack{\uparrow \\ \text{for } D_A}}{m_d} + O\left(\frac{1}{N^3}\right). \longleftarrow \text{Darwin theorem in its Weak Form}$$

Selective Withdrawal

Craya's pioneering work (1949)

Richardson's formula (Phil. Mag. 1920)

$$\frac{dz}{dw} = [3g' G(w)]^{-1/3} \left\{ [1 - G'^2(w)]^{1/2} + i\, G'(w) \right\},$$

$G'(w)$ & $[1 - G'^2(w)]^{1/2}$ real on free surface

$$g' = \frac{\Delta\rho}{\rho} g$$

Check:

$$y = (3g')^{-1/3} \int \frac{G'}{G^{1/3}}\, dw = \frac{1}{2} \left(\frac{9G^2}{g'} \right)^{1/3} \quad (+\text{ const.})$$

$$\left| \frac{dw}{dz} \right|^2 = (3g' G)^{2/3}$$

$-2g'y + \left| \frac{dw}{dz} \right|^2 =$ const., as required for the case of flowing fluid of density ρ over stagnant fluid of density $\rho + \Delta\rho$.

Craya took

$$G = \frac{2q'}{\pi} \exp\left(\frac{\pi}{2q'} w \right).$$

Then

$$CB = l = \frac{1}{(3g')^{1/3}} \, \frac{3}{2}\left[G(w)\right]^{2/3}\Big|_{-\infty}^{0} = \left(\frac{9q'^2}{2\pi^2 g'}\right)^{1/3}$$

$$= \left(\frac{9}{2\pi^2}F'^2\right)^{1/3} h . \qquad \left(F'^2 = \frac{q'^2}{g' h^3}\right)$$

h = distance of sink above free surface at ∞.

$$i\,\frac{BP}{l} = \int_1^{\infty}\left[\left(\frac{1}{t^3}-1\right)+i\right]dt$$

$t = \exp\,(\pi w/3q')$.

$$BP = \int_1^{\infty}\left(1-\sqrt{1-\frac{1}{t^3}}\right)dt = 0.293$$

$$l + BP = h = 1.293\,l$$

$$\left(\frac{9F'^2}{2\pi^2}\right)^{1/3} = \frac{1}{1.293} . \qquad F' = \left(\frac{1}{1.293}\right)^{3/2}\frac{\sqrt{2}\,\pi}{3} = 1.01 .$$

Craya's estimates:

1. No boundary (or vertical boundary):

   $F' \cong \frac{3}{2}(1.01) = 1.52.$

   Underestimate O.K.

   $F'^2 \sim 2.31 \times 4 = 9.24$ Compare with the accurate value of 12.622 obtained by Tuck & v. Broeck (1984)

2. Horizontal Boundary

   $F' = \frac{3}{4}(1.01) = 0.75$ compare with v. Broeck & Keller's accurate (?) value of 1.

Underestimate strange.

2a

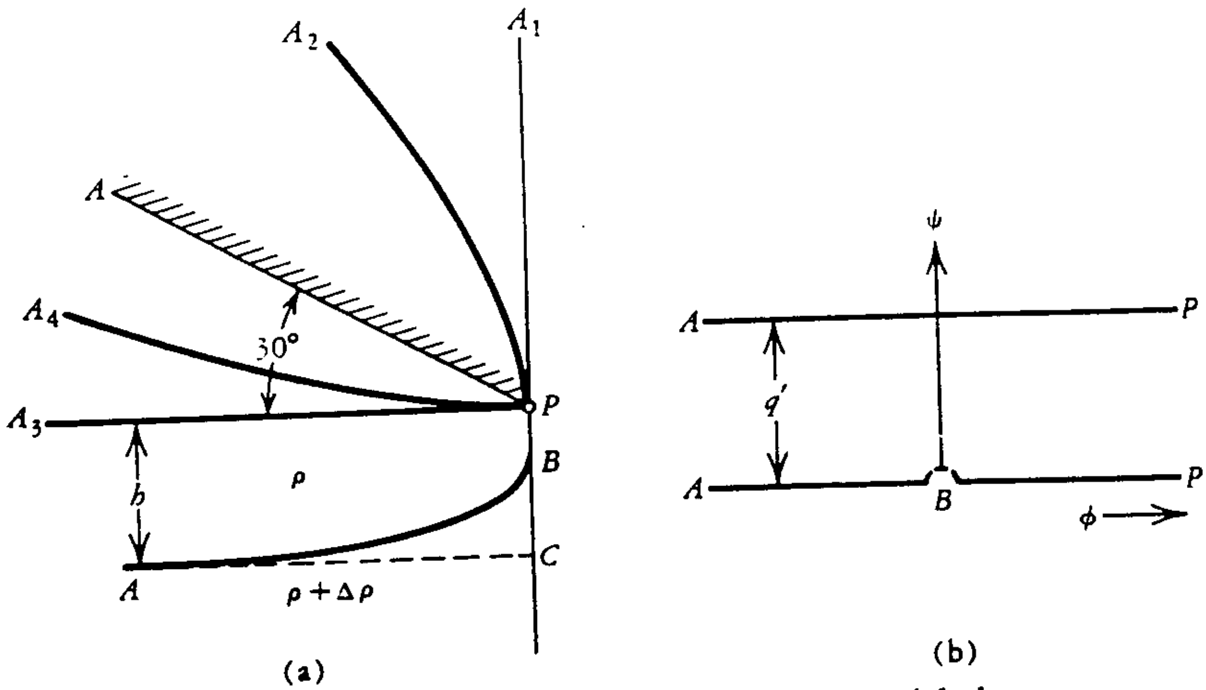

FIGURE 30. (a) Physical plane. (b) Complex-potential plane.

Recent (1984) Work of Tuck & vanden Broeck

Key transformations:

(1) $\exp f = 4t(t+1)^{-2}$, $\quad f = \phi + i\psi$

(2) $z(t) = -\frac{i}{t+1}\sum_0^\infty b_j t^j$, $\quad z = x + iy$

On F.S.,
$$t = e^{-i\theta}, \quad 0 \le \theta \le \pi .$$

As $t \to 0$, $z \to -ib_0$, $|f| \to \infty$, with
$$f \to \log(z + ib_0)$$

Scales:

Length Scale $\lambda = \left(\frac{m^2}{8\pi^2 g}\right)^{1/3}$

Velocity Scale $(mg/\pi)^{1/3}$

Complex-Potential Scale: $m/2\pi$

Then
$$F^2 = \frac{m^2}{gh^3} = \frac{m^2}{g\left(\frac{h}{\lambda}\right)^3}\frac{1}{\lambda^3} = \frac{m^2}{g\lambda^3}\, b_0^{-3} = 8\pi^2 b_0^{-3}$$

(At sink: $z = -ib_0$ dim-less; $z = -ih$ dimensional)
$$\therefore \frac{h}{\lambda} = b_0 .$$

Free-Surface Condition.

$$y + |f'(z)|^2 = 0$$

or

(3) $$y + \left|\frac{t-1}{t+1}\right|^2 \left|\frac{dz}{dt}\right|^{-2} = 0, \quad t = e^{-i\theta}, \ |t| = 1.$$

Put (1) into (3). Obtain

$$P(\theta; b_j) = 0; \qquad 0 \le \theta \le \pi$$

(4) $$P(\theta; b_j) = Y(\theta) + 4\sin^2\theta\,[A^2(\theta) + B^2(\theta)]^{-1}$$

(5) $$Y(\theta) = \sum_{j=0}^{\infty} b_j \left[-\frac{1}{2}\cos j\theta - \frac{1}{2}\tan\frac{\theta}{2}\sin j\theta\right],$$

(6) $$A(\theta) = \sum_{j=0}^{\infty} b_j \left[(j-1)\cos j\theta + j\cos(j-1)\theta\right]$$

(7) $$B(\theta) = \sum_{j=0}^{\infty} b_j \left[(j-1)\sin j\theta + j\sin(j-1)\theta\right].$$

Certain constraints on the b's.

In order that $P = 0$ at $\theta = \pi$ $(|z| = \infty)$, (4) & (5) give

(8) $$\boxed{\sum_{j=0}^{\infty} (-1)^j (2j-1)\, b_j = 0.}$$

In order that $P=0$ at $\theta=0$,

Either

(9) $$\sum_{j=0}^{\infty} b_j = 0$$

or

(10) $$\boxed{\sum_{j=0}^{\infty} (2j-1)\, b_j = 0.}$$

As $\theta \to 0$ or $t \to 1$, (1) implies

$$f \to -\frac{1}{4}(t-1)^2 + \frac{1}{4}(t-1)^3 + O(t-1)^4,$$

while

$$z = z_1 + (t-1) z_1' + \frac{1}{2}(t-1)^2 z_1'' + \frac{1}{6}(t-1)^3 z_1''' + O(t-1)^4,$$

for some (imaginary) coefficients z_1', z_1'' etc.

Now (3) can be satisfied at $t=1$ only if either z_1 or z_1' is zero. In the former case (9) holds (see (2)), & , as $z \to 0$,

$$f \to -\frac{1}{4} z^2/(z_1')^2 + O(z^3)$$ Stag. Point.

In the latter case (10) holds, and

$$f \to \underbrace{-\frac{1}{2}(z-z_1)/z_1''}_{\text{local Unif. stream}} + \underbrace{K}_{\text{Real}}\,(z-z_1)^{3/2} + O(z-z_1)^2$$

$\therefore$ Bifurcating streamlines at $t=1$, or $z=z_1$.

Procedure:

There are $N+1$ unknowns

$$b_0,\ b_1,\ \cdots b_N.$$

Pick $N-1$ points at $\theta_1, \theta_2, \ldots, \theta_{N-1}$.
Then there are

$$N-1+\underset{\substack{\uparrow \\ (8)\ \&\ (10)}}{2} = N+1 \text{ equations}$$

Solve numerically.

Results:

$$b_0 = 1.84257$$

$$F^2 = 12.622$$

TABLE 1

N	b_0
5	1.86935
10	1.84223
15	1.84260
20	1.84256
25	1.84257

TABLE 2

j	b_j
0	1.84257
1	0.41325
2	0.55982
3	-0.06731
4	0.01766
5	-0.00613
6	0.00248
7	-0.00111
8	-0.00053
9	-0.00027
10	0.00014