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Patterns of ship waves II. Gravity-capillary waves

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PATTERNS OF SHIP WAVES*

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Abstract. Patterns of water waves created by a moving disturbance representing a moving body, floating or submerged, can be found by applying (1) the principle of stationary phase, (2) the principle that the phase lines are normal to the wave-number vector, and (3) the perception that the local phase velocity of the waves must be equal to the component of the velocity of the disturbance normal to the phase line. The three equations thus obtained are solved, and formulas for the phase lines are derived, which depend explicitly on the dispersion equation, and on that equation only. These formulas are applied to deep-water surface waves, surface waves in water of finite depth, internal waves, and capillary waves in thin sheets to obtain the wave patterns sufficiently far from the moving disturbance.

Finally, the patterns of the surface waves in deep water created by a moving body are determined, with the nonuniformity of the mean velocity of the fluid in the wake taken into account. The vorticity in the direction along the phase lines is shown to be small, so that the wave motion can still be assumed irrotational in a first approximation. The wave patterns differ from the Kelvin-wave pattern, as a result of the nonuniformity of fluid velocity in the wake.

1. Introduction. The pattern of gravity waves created by a moving disturbance in deep water was determined by Lord Kelvin (Sir W. Thomson, 1887) fully a century ago, by applying his principle of stationary phase to the well-known Cauchy–Poisson solution (see Lamb, 1945, pp. 429–434) for an instantaneous concentrated force (a concentrated impulse). But Kelvin's application of his own principle of stationary phase did not result in explicit formulas giving the wave pattern created by a moving disturbance, once the dispersion equation expressing the wave velocity in terms of the wave number is known, whatever the kind of wave—gravity wave in deep water or water of finite waves, internal waves, or capillary-gravity waves. These explicit formulas were given by Yih (1985). In this paper Yih's formulas will be presented and applied to gravity waves created by a moving disturbance, which can be regarded as representing a ship, floating or submerged. Results for gravity waves in deep water or water of finite depth and for internal waves will be presented graphically. Finally, the effects of nonuniformity of fluid velocity in the wake of the ship will be

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considered, and gravity-wave patterns will be shown, with the principal effect of this nonuniformity taken into account.

2. Formulas for phase lines in a ship-wave pattern. For simplicity, we can regard the moving disturbance to be a moving pressure distribution on the free surface, although this particularity does not affect the establishment of the formulas for determining the phase lines. This fact indicates that whatever the details of the disturbance may be, the wave pattern obtained will be the same, if the region under consideration is sufficiently far from the disturbance.

Let U be the speed of the disturbance, moving horizontally to the left. The direction opposite to the velocity of the disturbance is taken to be the direction of increasing x. The y-axis is normal to this direction but is also horizontal. The z-axis is directed verticallv upward. The wave-number components in the directions of increasing x and y will be denoted by ξ and η respectively, and

$$k^2 = \xi^2 + \eta^2. \tag{1}$$

For our purpose it is not necessary to give the formulas for the velocity distribution in the wave motion (or the velocity potential when it exists), or for the displacement of the free surface or interface. Such formulas are given for Kelvin's waves in Whitham (1972, p. 448, Eq. 13.56), for surface waves in water of finite depth and for gravity-capillary waves in Havelock (1908), and for internal waves in two superposed fluid layers by Hudimac (1961), Carrier and Baski (1963), and Yih (1985, Eqs. 30 and 31), among others. What is important for our purpose is that in these formulas there is always the exponential factor

$$\exp i(\xi x + \eta y) \tag{2}$$

in a double integral with respect to ξ and η.

The application of Kelvin's principle of stationary phase at any point (x, y) requires,[1] because of the factor (2),

$$\frac{y}{x} = -\frac{d\xi}{d\eta}. \tag{3}$$

The normal to any curve of constant phase has the slope η/ξ. The slope of the tangent to any curve of constant phase must then have the value $-\xi/\eta$, so that for that curve

$$\frac{dy}{dx} = -\frac{\xi}{\eta}. \tag{4}$$

The requirement that the wave velocity at any point must be equal to the component of the velocity of the disturbance normal to the wave front (or the phase line) is expressed by

$$\frac{U\xi}{k} = c(k), \tag{5}$$

[1]The Fourier integration is in the (ξ, η)-plane. The main contribution to the integral comes from the neighborhood of a ξ-η curve where a factor in the denominator vanishes. This vanishing is represented by (5). The integration in this neighborhood involves essentially an integration across this curve (giving what amounts to residues) and an integration along that curve subsequently. It is to this latter integration that the method of stationary phase applies. After this application ξ, η, and k are considered (slowly varying) functions of x and y. This understanding should be kept in mind from (3) onward.

where $c(k)$ denotes the wave velocity, dependent on k, the geometry (for instance, depth of water), and the physical parameters relevant to the problem, such as the gravitational acceleration g and the surface tension T. We are concerned in this paper with gravity waves mainly. We shall use U as the velocity scale, and a finite depth d as the length scale. If such a depth is not available we shall use U^2/g as the length scale. Then ξ, η, k, x, and y are dimensionless. The dimensionless form of (5) is now

$$\xi = F(k), \tag{6}$$

where $F(k) = kc(k)$, and the function $F(k)$ may depend on other parameters, such as the Froude number, as well as k, because $c(k)$ does.

Since (6) is an equation between ξ and k, it is convenient to write (3) and (4) in terms of ξ and k explicitly. A brief calculation gives, upon use of (1),

$$\frac{y}{x} = -\frac{1}{d\eta/d\xi} = \frac{(k^2-\xi^2)^{1/2}}{k(dk/d\xi)-\xi}, \tag{7}$$

$$\frac{dy}{dx} = -\frac{\xi}{(k^2-\xi^2)^{1/2}}. \tag{8}$$

Finally, upon using (6), we have

$$\frac{y}{x} = -\frac{F'(k^2-F^2)^{1/2}}{k-FF'}, \tag{9}$$

$$\frac{dy}{dx} = -\frac{F}{(k^2-F^2)^{1/2}}. \tag{10}$$

We can write

$$y = -f(k)F'(k^2-F^2)^{1/2}, \tag{11}$$

$$x = f(k)(k-FF'), \tag{12}$$

which satisfy (9), and endeavor to determine $f(k)$. From (11) and (12) we obtain

$$dy = -\frac{1}{(k^2-F^2)^{1/2}}[fF'(k-FF') + (fF')'(k^2-F^2)], \tag{13}$$

$$dx = f(k-FF')' + f'(k-FF'). \tag{14}$$

These, together with (10), give

$$fF(k-FF')' + f'F(k-FF') = fF'(k-FF') + (fF')'(k^2-F^2). \tag{15}$$

It is satisfying that this equation can always be explicitly integrated, for it can be rewritten as

$$fF(k-FF')' + (fF)'(k-FF') = 2fF'(k-FF') + (fF')'(k^2-F^2), \tag{16}$$

and this can be immediately integrated to (a = constant of integration)

$$fF(k-FF') = fF'(k^2-F^2) + a, \tag{17}$$

or

$$f = \frac{a}{k(F-kF')}. \tag{18}$$

Thus,

$$y = -\frac{aF'}{k(F - kF')}(k^2 - F^2)^{1/2}, \tag{19}$$

$$x = \frac{a(k - FF')}{k(F - kF')}. \tag{20}$$

These are the parametric equations for the curves of constant phase, when the eikonal equation is in the form of (6), *and it is remarkable that they can be explicitly given in terms of* $F(k)$. The function $F(k)$ can easily be obtained for any kind of waves and any geometry in a two-dimensional wave motion (or one-dimensional propagation). Once it is given, (19) and (20) can be applied to find the wave pattern directly. Thus they are very useful and convenient.

2.1. *Determination of the critical angle* ϕ_c. The critical angle ϕ_c is determined by

$$\frac{d^2\eta}{d\xi^2} = 0, \quad \text{or} \quad \frac{d}{d\xi}\frac{k\,dk/d\xi - \xi}{(k^2 - \xi^2)^{1/2}} = 0. \tag{21}$$

When (6) is the eikonal equation, this becomes

$$\frac{d}{dk}\left[\frac{k - FF'}{F'(k^2 - F^2)^{1/2}}\right] = 0. \tag{22}$$

Given F, this can be solved for k. With k known and equal to k_c (say), ξ is known from (6), η is known, and x and y are known, so that

$$\phi_c = \tan^{-1}\left.\frac{y}{x}\right|_{k_c}. \tag{23}$$

If there is more than one root of (22), take the greatest of the values of ϕ corresponding to these roots to be ϕ_c. This determines the vertex angle $(2\phi_c)$ of the wedge in which waves can be found. Equation (22), too, is an important result. One expects to find waves only in some wedge where

$$-\phi_c < \phi < \phi_c.$$

3. Applications.

3.1. *Gravity waves in deep water, or Kelvin waves.* For Kelvin waves the dimensionless eikonal equation is (the length scale for this case is U^2/g)

$$\xi^2 = k. \tag{24}$$

Then

$$F(k) = k^{1/2} \quad \text{and} \quad F'(k) = \frac{1}{2}k^{-1/2}, \tag{25}$$

$$y = -\frac{a}{k^2}(k^2 - k)^{1/2}, \tag{26}$$

$$x = \frac{a(2k - 1)}{k^{3/2}}. \tag{27}$$

Following Lamb (1945, p. 433), we define θ as the angle of inclination of the normal to any curve of constant phase, so that, along any curve of constant phase

$$\frac{dy}{dx} = -\frac{\xi}{\eta} = -\cot\theta. \tag{28}$$

Because of (24),

$$k^2 - k = k^2 - \xi^2 = \eta^2,$$

so that

$$y = -\frac{a\eta}{k^2} = -\frac{a}{\xi^3}\frac{\eta}{\xi}. \tag{29}$$

Now (28) can be written as

$$\pm\cot\theta = \frac{\xi}{(k^2 - \xi^2)^{1/2}} = \frac{\xi}{(\xi^4 - \xi^2)^{1/2}} = \frac{1}{(\xi^2 - 1)^{1/2}} \tag{30}$$

so that

$$\xi = \sec\theta, \tag{31}$$

and (29) becomes

$$y = -a\cos^3\theta\tan\theta = -a\sin\theta\cos^2\theta. \tag{32}$$

Similarly, (27) becomes

$$x = \frac{a(2k-1)}{k^{3/2}} = \frac{a(2\xi^2 - 1)}{\xi^3} = a\cos\theta(2 - \cos^2\theta),$$

or

$$x = a\cos\theta(1 + \sin^2\theta). \tag{33}$$

Equations (32) and (33) are exactly the equations (Lamb, 1945, p. 434) for the Kelvin curves of constant phase for surface waves. The wave pattern is shown in Fig. 1, upon taking a equal to 1, 2, 3, etc. Since on the centerline $k = 1$, because the length scale is U^2/g and on the centerline $U = c$, the increment in a, indicating a wavelength on the centerline, corresponds to $2\pi U^2/g$ when converted to dimensional length. (Recall wave number $= 2\pi$/wavelength.) The cusps occur at $\xi = (3/2)^{1/2}$, giving a ϕ_c of 19°28′. The flow near the cusps requires better resolution. This was provided by Ursell (1960).

3.2. *Gravity waves in water of finite depth.* Let the depth of the water be d. Then, as is well known, in *dimensional* terms the phase velocity $c(k)$ is given by

$$c^2 = \frac{g}{k}\tanh kd.$$

In dimensionless terms, with d as the length scale and U the velocity scale, this becomes

$$c^2 = N_F^{-2}k^{-1}\tanh k, \tag{34}$$

where N_F is the Froude number defined by

$$N_F^2 = \frac{U^2}{gd}. \tag{35}$$

Then (6) takes the form

$$\xi = N_F^{-1}(k\tanh k)^{1/2}, \tag{36}$$

the right-hand side of which is $F(k)$. Using this $F(k)$ in (19) and (20), we obtain the wave patterns shown in Figs. 2 for

$$N_F = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.5,\ 2.0,\ \text{and } 5.0,$$

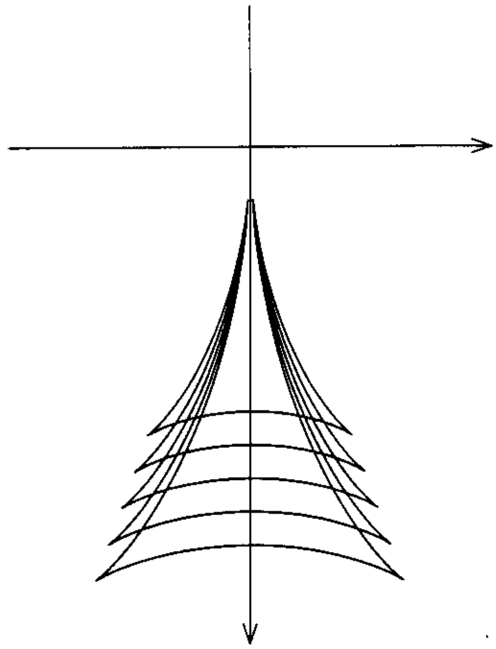

FIG. 1. Kelvin-wave pattern.

respectively. On the centerline, $\xi = k = k_0$, and (36) gives

$$N_F^2 = \frac{1}{k_0} \tanh k_0. \tag{37}$$

The wavelength of transverse waves is $2\pi d/k_0$ on the centerline. Given the Froude number N_F and the depth d, k_0 is determined from (37) and the wavelength on the centerline is known. The patterns in the figures should be read with this in mind. The fact that the patterns have been obtained by assigning integral values to a in (19) and (20) is of little importance since the scale used in plotting the figures is arbitrary.

The most important feature of Fig. 2 is that only when the Froude number N_F is less than 1 are there transverse waves. This is because the wave velocity c is bounded by $(gd)^{1/2}$, so that when N_F exceeds 1 no waves, however long, can be stationary at the centerline, so that no transverse waves can exist. When N_F exceeds 1, the phase lines at large distances from the disturbance appear to approach asymptotes.

The k_c (k at the cusps), defined by (23) when $N_F < 1$, and the ϕ_c giving the half angle of the wedge in which waves exist, depend on the value of N_F. We note that when transverse waves exist, k increases from k_0 on the centerline to k_c at the cusp, and then increases monotonically along the phase line of a divergent wave. When transverse waves do not exist, k increases along such a line as $|y|$ decreases; i.e., as the centerline is approached. The same is true for internal waves.

3.3. *Internal waves.* Let us consider gravity waves in two superposed fluids. The upper fluid has density ρ and depth h, and the lower fluid has density $\rho'(> \rho)$ and infinite depth. Let the velocity potential perturbations (from the mean flow of

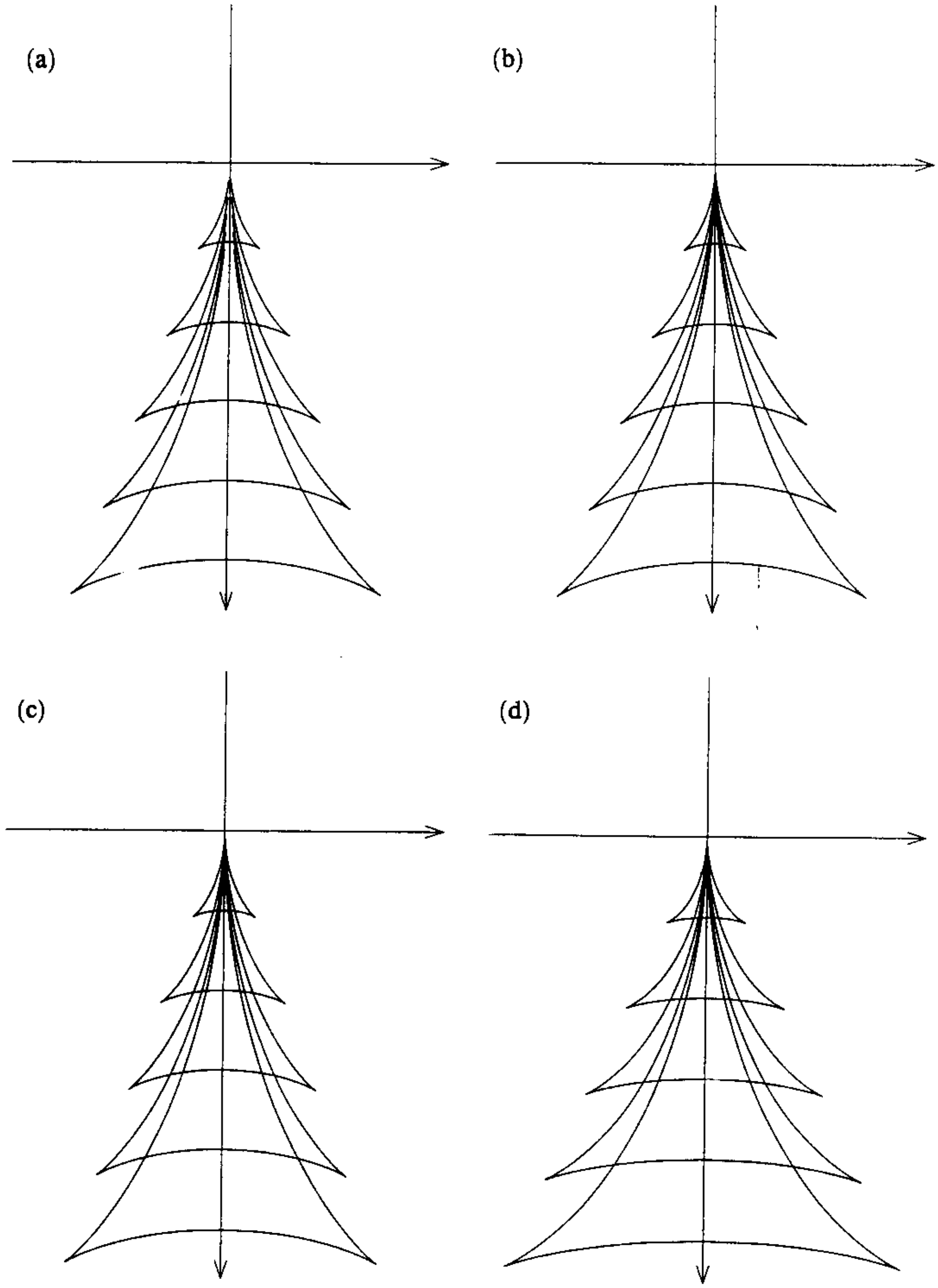

FIG. 2. (a)–(d). Pattern of ship waves in water of finite depth. (a) $N_F = 0.2$, (b) $N_F = 0.4$, (c) $N_F = 0.6$, (d) $N_F = 0.8$.

velocity c) for the two layers be ϕ and ϕ', respectively. The displacement of the free surface is denoted by ζ and that of the interface by ζ'. All quantities will be dimensional until otherwise stated. The phase velocity c will be determined from a calculation for stationary waves, that is, for a coordinate system moving with the waves to the left. Then the kinematic condition at the free surface is (the origin of z being at the interface)

$$c\zeta_x = \phi_z \quad \text{at } z = h, \tag{38}$$

and the dynamic condition there is the Bernoulli equation

$$g\zeta + c\phi_x = 0 \quad \text{at } z = h. \tag{39}$$

Combination of (38) and (39) gives

$$s\phi_{xx} = -\phi_z \quad \text{at } z = h \tag{40}$$

for the free surface, where

$$s = \frac{c^2}{g}. \tag{41}$$

At the interface, the kinematic conditions are

$$c\zeta'_x = \phi_z \quad \text{and} \quad c\zeta'_x = \phi'_z \quad \text{at } z = 0; \tag{42}$$

and the dynamic condition is

$$\rho g\zeta' + \rho c\phi_x = \rho' g\zeta' + \rho' c\phi'_x \quad \text{at } z = 0. \tag{43}$$

Defining

$$\beta = \frac{\rho' - \rho}{\rho} \tag{44}$$

and substituting (42) into (43), we obtain

$$\frac{\beta}{s}\phi_z = -(\beta + \mathrm{i})\phi'_{xx} + \phi_{xx} \quad \text{at } z = 0. \tag{45}$$

Equations (42) can be combined into

$$\phi_z = \phi'_z \quad \text{at } z = 0. \tag{46}$$

One last condition is

$$\phi' = 0 \quad \text{at } z = -\infty. \tag{47}$$

Taking

$$\phi = (Ae^{kz} + Be^{-kz})\cos kx, \qquad \phi' = Ce^{kz}\cos kx, \tag{48}$$

which satisfies (47), since k is assumed positive, we use (40), (45), and (46) to eliminate the constants A, B and C, and obtain from a straightforward calculation

$$(ks - 1)[ks(2 + \beta + \beta e^{-2kh}) - \beta(1 - e^{-2kh})] = 0. \tag{49}$$

The root

$$s = \frac{c^2}{g} = \frac{1}{k} \tag{50}$$

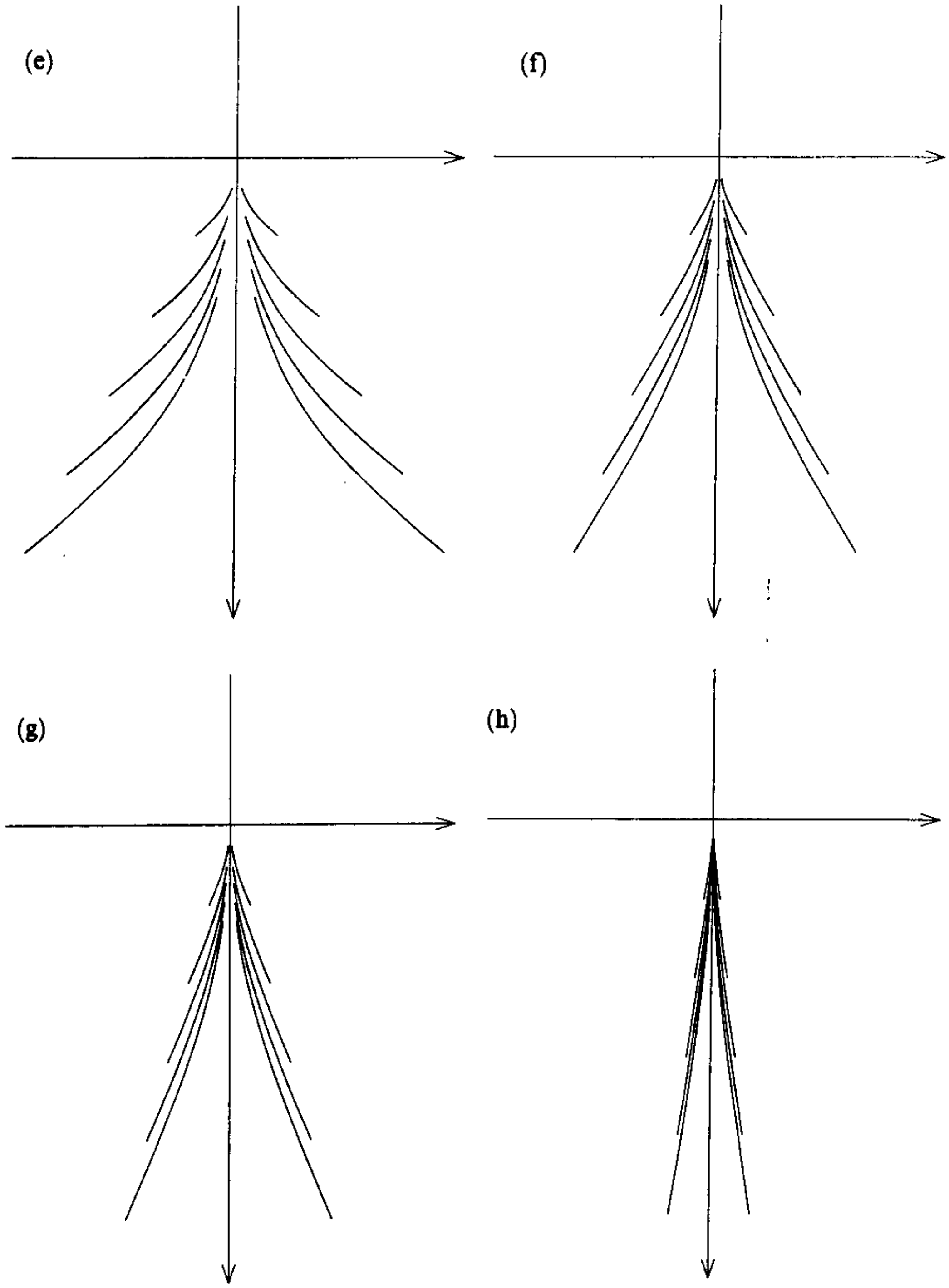

FIG. 2. (e)–(h). Pattern of ship waves in water of finite depth. (e) $N_F = 1$, (f) $N_F = 1.5$, (g) $N_F = 2$, (h) $N_F = 5$.

corresponds to irrotational wave motion of the entire fluid, with no vortex sheet even at the interface, as is well known (Yih 1960, and 1980, pp. 60–62). This wave motion has the Kelvin-wave pattern presented in subsection 3.2. The other root of (49) is

$$c^2 = \frac{\beta g}{k} \frac{e^{2kh} - 1}{(2+\beta)e^{2kh} + \beta}, \tag{51}$$

and corresponds to predominantly internal waves. Substituting (51) into (5), using h as the length scale, we obtain the dimensionless equation

$$\xi = N_F^{-1} \left(\frac{\beta \gamma k}{\alpha} \right)^{1/2}, \tag{52}$$

$$\alpha(k) = (2+\beta)e^{2k} + \beta, \tag{53}$$

$$\gamma(k) = e^{2k} - 1, \tag{54}$$

and N_F is the Froude number now defined by

$$N_F^2 = \frac{U^2}{gh}.$$

We have done the calculations for the internal-wave patterns for $\beta = 0.04$. The patterns are shown in Fig. 3 for

$$N_F = 0.01,\ 0.05,\ 0.1,\ 0.15,\ 0.2,\ 0.5,\ 1,\ \text{and } 2,$$

respectively. Again transverse waves exist only if N_F is sufficiently small. The critical N_F can be calculated from (52) upon putting ξ equal to k, letting k approach zero (to get the longest wavelength possible), and taking the limit. Doing so, we obtain the critical Froude number

$$(N_F)_c = \frac{\beta}{1+\beta}. \tag{55}$$

When $N_F > (N_F)_c$, no transverse waves are possible. For $\beta = 0.04$, $(N_F)_c$ is approximately 0.2. That is why in Fig. 3(e) no transverse waves appear.

As for surface waves in water of finite depth, the angle ϕ_c which the line of cusps makes with the centerline, defined by (23), depends on N_F. This dependence is evident upon examination of Figs. 3(c) and 3(d). From Fig. 3(e) onward to Fig. 3(h), the wave region narrows as the phase lines, which appear to approach asymptotes, make smaller and smaller angles with the centerline.

3.4. *Capillary waves on a thin sheet.* For a fluid sheet of thickness $2h$, there are two modes of capillary waves. For the one mode, the sheet deforms as a whole antisymmetrically, with hardly any change in thickness. In this mode the waves are nondispersive. For the other mode, the sheet deforms symmetrically, and the dimensional dispersive equation is, upon neglect of gravity effects and on the assumption that the wavelength anywhere is much greater than h,

$$c^2 = \frac{Th}{\rho} k^2,$$

where T is the surface tension, ρ is density of the fluid, and c and k are the dimensional phase velocity and wave number, respectively. (See Whitham, 1972, p.

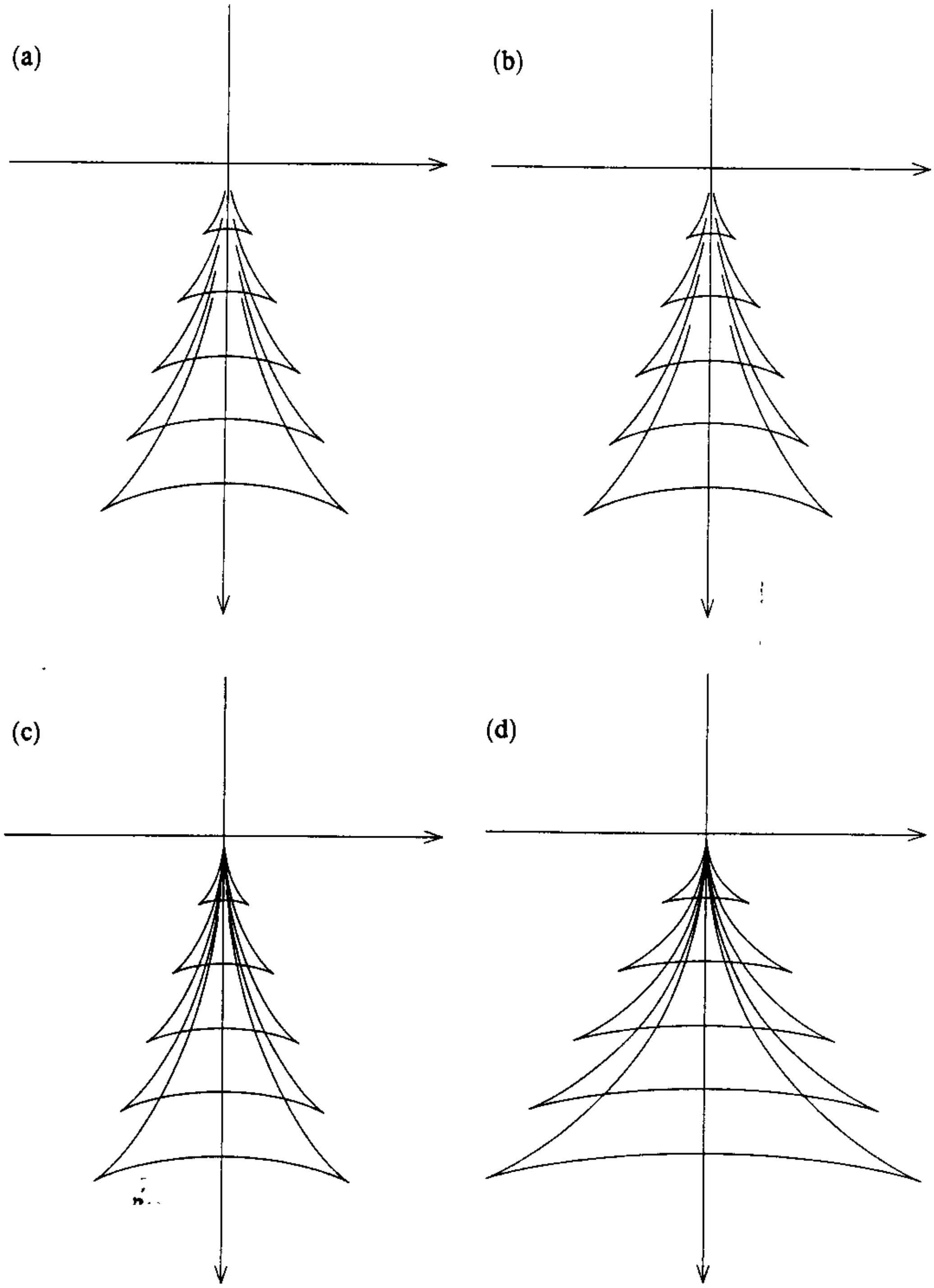

FIG. 3. (a)–(d). Pattern of internal waves created by a moving body.
(a) $N_F = 0.01$, (b) $N_F = 0.05$, (c) $N_F = 0.1$, (d) $N_F = 0.15$.

405.) We shall use h as the length scale and $(\rho h^3/T)^{1/2}$ as the time scale. Then the dimensionless dispersion equation becomes

$$c = k, \quad \text{or} \quad F(k) = ck = k^2.$$

Putting this into (11) and (12), we have

$$y = -\frac{2a}{k}(1-k^2)^{1/2},$$
$$x = -\frac{a(1-2k^2)}{k^2},$$

in which the square root may be positive or negative, while k is restricted to values less than 1.

It is a very simple matter to show that the x and y given above satisfy a parabolic relation

$$\frac{x}{a} = 1 - \left(\frac{y}{a}\right)^2. \tag{56}$$

The source of the disturbance is at $x = 0 = y$, at which a must be taken to be zero. Along the centerline ($y = 0$) *in front* of the disturbance, the dimensionless k is 1. That is to say, the dimensionless wavelength λ (in units of h) is 2π. Hence to show the wave pattern one must take successively

$$a = 2\pi, \ 4\pi, \ 6\pi, \text{ etc.}$$

Whitham (1972, pp. 415–416) gave a more indirect derivation of the pattern of capillary waves in thin sheets, and obtained "roughly parabolic crests" in his Fig. 12.7. From our derivation here it is clear that the crests are not merely roughly parabolic, but exactly parabolic.

We note that Whitham's Fig. 12.7 (Whitham 1972, p. 416) is for the disturbance moving to the left. Our formula (56) is for the disturbance moving to the right. Otherwise our pattern for capillary waves is the same as his.

4. Gravity waves in the wake of a moving body. The wake behind a moving body is an important effect of viscosity, for it is ultimately related to the boundary layer on the body. Again let U^2/g be the length scale and use x and r as dimensionless coordinates, r being the radial distance from the x-axis. The velocity distribution in the wake can take a variety of forms, depending on the shape of the body and the Reynolds number, as is well known. Using U again as the velocity scale, the dimensionless velocity can be represented, without serious error, by a class of profiles as follows (α and β are not the same as in subsection 3.3):

$$u = 1 - \alpha x^{-2/3}\exp(-\beta r^2 x^{-2/3}). \tag{57}$$

The momentum flux across any section of constant x below the free surface is, for the mean flow given by (57),

$$\frac{\pi\alpha}{2\beta}\rho U^6 g^{-2}, \tag{58}$$

and this is equal to the viscous drag force $C_D A \rho U^2$, if A is a cross section of the body and C_D the coefficient of drag. This equality is based on the neglect of the

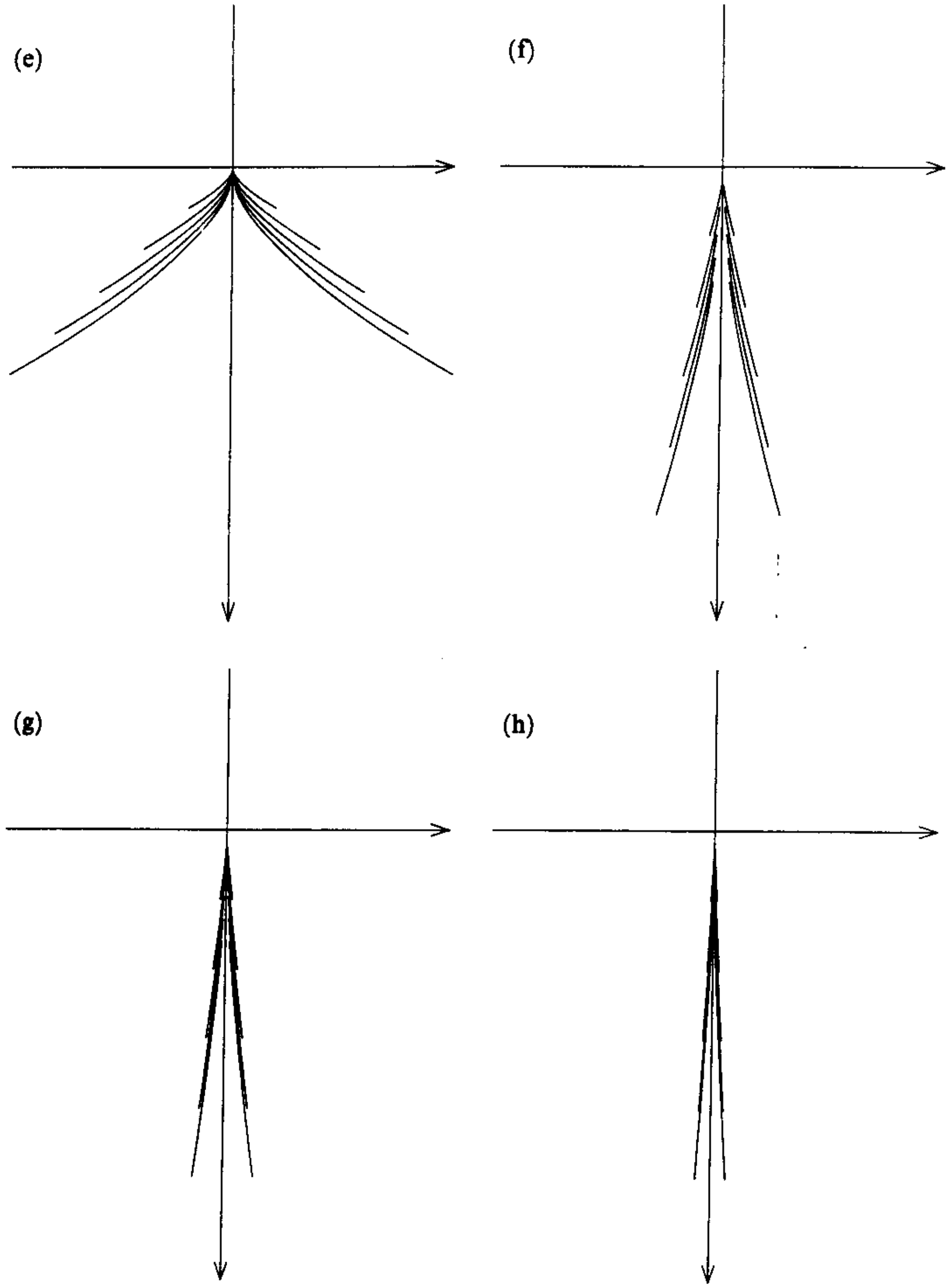

FIG. 3. (e)–(h). Pattern of internal waves created by a moving body.
(e) $N_F = 0.2$, (f) $N_F = 0.5$, (g) $N_F = 1$, (h) $N_F = 2$.

longitudinal stress across any section of constant x, and gives

$$\frac{\alpha}{\beta} = \frac{2}{\pi} C_D F_A^{-4}, \tag{59}$$

where F_A is a Froude number based on a linear dimension $A^{1/2}$, defined by

$$F_A^4 = \frac{U^4}{g^2 A}. \tag{60}$$

In our calculation we have taken

$$C_D = 0.05, \tag{61}$$

$$\frac{\alpha}{\beta} = \frac{1}{20}. \tag{62}$$

This corresponds to

$$F_A = \left(\frac{2}{\pi}\right)^{1/4} = 0.89, \tag{63}$$

not an unreasonable number. Other values of C_D and F_A can be taken, resulting in other values of α/β.

On the free surface (57) becomes

$$u = 1 - \alpha x^{-2/3} \exp(-\beta y^2 x^{-2/3}). \tag{64}$$

We shall assume this velocity to be prevailing in the fluid, and furthermore shall assume the wave motion to be irrotational. This requires some justification and explanation. Assuming (64) throughout the fluid amounts to assuming that the wave motion does not penetrate the fluid significantly to distances below the free surface where u is very different from that given by (64). This is true only if x is large. As to the irrotationality of the wave motion, one justifies it on two grounds.

(i) The length scale of the variation of u for large values of x is large, and therefore much greater than the wavelengths in the wave region.

(ii) All the vorticity lines for the flow (57) are circles in planes normal to the x-axis. Near the free surface they are nearly vertical, whereas the local wave motion, assumed irrotational in the plane containing the wave-number vector and a vertical line, can become rotational only if the local wave motion bends the vertical vorticity lines of the mean flow in the direction of the phase lines. But any bending of the vorticity lines by the local wave motion would be in the direction normal to the phase lines. Thus the irrotationality of the local wave motion is hardly affected.

We can use equations (3) and (4) again, but must modify (6) to

$$u\xi = F(k). \tag{65}$$

The calculation is one of step-by-step computation, as k is increased from k_0 (now not constant) on the centerline toward k_c (at the cusp) and beyond. The k_0 for the transverse waves on the centerline is determined from (64).

We have performed the calculations for

$$\alpha = 1/8, 1/4, 1/2, 3/4,$$
$$\beta = 2.5, 5, 10, 15.$$

The results are shown in Fig. 4, in which (a) is for $(\alpha, \beta) = (1/8, 2.5)$, etc. In starting the lines of the same phase, we start with $x = 8$ on the centerline, calculate the wavelength of the transverse wave there, and mark off the next point on the centerline of the same phase, and use that point to start the calculation for the next line of the same phase, and so on. From Fig. 4 it can be seen that near the centerline the transverse waves bent back toward positive x more pronouncedly than in the Kelvin-wave pattern, but become straight sooner. The curvature of the transverse waves at the centerline is larger when there is a wake.

For the Kelvin-wave pattern,

$$\xi = \xi_c = (3/2)^{1/2}$$

at the cusp, and $\phi_c = 19°28'$. In our case, since u in (65) is variable, it is not obvious that ϕ_c has the same value. However, our calculations seem to give the same value. An explanation of this on analytical grounds goes as follows. It can easily be verified that for *any fixed* value of u the maximum value of $d\eta/d\xi$ occurs at $\xi = (3/2)^{1/2}$, and gives $\phi_c = 19°28'$. Now fix the x and y in u, and perform the calculation for the phase lines as if u were constant. A cusp is encountered at some point (x_c, y_c). If these are not the same as the fixed x and y, the cusp has no real significance. But if they happen to be the same, then, since the calculation in a neighborhood of that point is now valid, it must be a cusp in the wave pattern. This can be seen in the following way. Start from any cusp point (x_c, y_c), calculate $u(x_c, y_c)$, fix u at this value, then use the equations of Section 2 to trace out the two branches of the phase line originating from the cusp. Let us call this line A. Then starting from the same cusp, use the actual u in the step-by-step numerical computation in this section to trace out the two branches of the phase line originating from the cusp. Let us call this line B. Lines A and B are not the same, of course, but they have the same cusp and are nearly coincident near the cusp.

5. Range of k values for the figures. For the benefit of those who wish to reproduce the wave patterns given in the figures of this paper, we supply the ranges of the values of the dimensionless k for each of the figures, except Figures 4(a)–4(d), in Table 1.

TABLE 1

Values for $k_{\min}$ and k at cusp for the figures.

Figure	1	2a	2b	2c	2d	2e-2h
$k_{\min}$	1	25	6.24995	2.7554	1.37458	1.5
k at cusp	$(3/2)^{1/2}$	37.432	9.39995	4.0574	1.74508	

Figure	3a	3b	3c	3d	3e-3h
$k_{\min}$	196.0784	7.84313	1.91763	0.6113	0.3
k at cusp	294.3604	11.74913	2.72963	0.7933	

For Figures 4, the minimum k for each of the phase lines is for the transverse wave at the centerline, and this minimum k is determined by the equality of the local wave velocity of the transverse waves at the centerline and the local speed of the fluid in the wake. Because of the nonhomogeneity of fluid velocity in the wake (even apart from

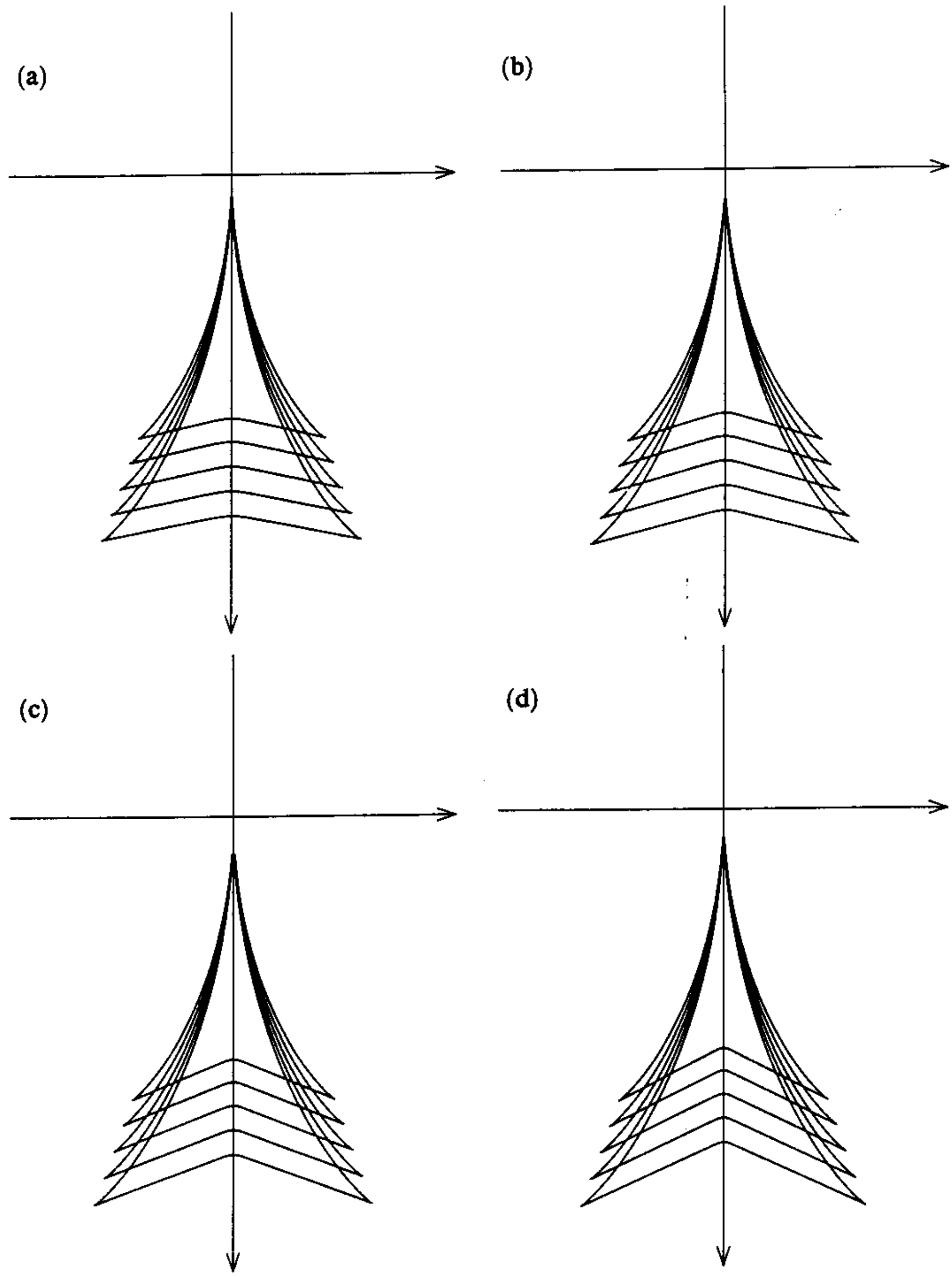

FIG. 4. (a)–(d). Pattern of gravity waves in the wake of a ship. (a) $\alpha = 1/8, \beta = 2.5$, (b) $\alpha = 1/4, \beta = 5$, (c) $\alpha = 1/2, \beta = 10$, (d) $\alpha = 3/4, \beta = 15$. The phase difference between consecutive phase lines is approximately 1 radian.

wave effects), the spacing of crests at the centerline cannot be determined simply, and can only be approximated by computing the locations of many intermediate phase lines. For this reason only phase lines, which are not necessarily crests, are shown in Figure 4, to give a general idea of the wave pattern. The maximum k for all the figures, including Figure 4, is infinity. We have simply stopped at a large enough k when the position of the disturbance has become too near to draw phase lines distinctly. We believe this description will be sufficient to guide anyone wishing to reproduce our figures.

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PATTERNS OF SHIP WAVES II. GRAVITY-CAPILLARY WAVES*

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In a previous paper [4], Yih's formulas [3] were used to obtain patterns of gravity waves, or of capillary waves in a thin fluid sheet, created by a moving disturbance. In this paper the effects of surface tension are taken into account in finding the patterns of capillary-gravity waves in deep water with a free surface created by a moving disturbance, and much more extensive results than those of Rayleigh [2] have been obtained. The most important feature of the waves is that there are capillary waves *behind* the disturbance, which have very short wavelengths at high values of the speed U of the disturbance and which are confined to a wedge of an angle that decreases as U increases. Of interest too is the existence of two cusps in the phase lines on either side of the centerline at high values of U (relative to a minimum wave velocity defined in the paper) for those waves which are entirely behind the disturbance.

1. Introduction. Explicit formulas for phase lines of any kind of dispersive waves created by a point disturbance moving in a fluid with a free surface were given by Yih [3]. These formulas are in terms of the parameter k, which is the local wavenumber. The point disturbance is an idealized representation of a ship, for instance, so that the formulas are useful for determining the pattern of waves far enough away from the ship. Yih's formulas were used by Yih and Zhu [4] to obtain patterns of ship waves in deep water (Kelvin waves), in water of finite depth, in a stratified ocean, and in the wake of a ship, as well as patterns of waves in a thin sheet caused by a moving point disturbance. But capillary-gravity waves were not treated in [4]. It will be treated in this paper, and many patterns of capillary-gravity waves caused by a moving disturbance will be presented.

The main reason for giving capillary-gravity waves a closer examination is that the treatment by Rayleigh [2], as quoted in Lamb ([1], pp. 469–471), is very sketchy and calls for a new calculation after more than a century, especially in view of the relevance of the problem to remote sensing. As will be seen, one important feature of capillary-gravity waves caused by a moving disturbance is that there are (predominantly) capillary waves *behind* the disturbance. This point has not been stressed in

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Lamb's book, but explains the presence of short waves within a narrow wedge which are often found in photographs obtained by remote sensing in the wake of a ship.

2. Analysis. Let the point disturbance move with speed U in the horizontal direction of decreasing x. The y-axis is also horizontal, and is normal to the x-axis. As in [4], ρ denotes the density of the fluid, g denotes the gravitational acceleration, T denotes surface tension, and we shall continue to use U^2/g as the length scale, so that the local wavenumber k will continue to be measured in units of g/U^2. The x and y components of the wavenumber vector k will again be denoted by ξ and η, so that

$$\xi^2 + \eta^2 = k^2. \tag{1}$$

With this in mind, the requirement that the local wave velocity must be equal to the component of the velocity of the disturbance normal to the wave front (or the phase line) is

$$\xi = F(k) = k(1/k + \sigma k)^{1/2}, \tag{2}$$

which is Eq. (6) in [4] for the present problem. The σ in (2) is

$$\sigma = \frac{Tg}{\rho U^4} = \frac{1}{4}\left(\frac{c_{\min}}{U}\right)^4, \tag{3}$$

where $c_{\min} = 2(Tg/\rho)^{1/2}$ is the minimum value of the wave velocity c calculated from the dimensional dispersion equation, given by

$$c^2 = \frac{g}{\hat{k}} + \frac{T}{\rho}\hat{k}, \tag{4}$$

in which $\hat{k}$ denotes the dimensional local wavenumber. The $\hat{k}$ for $c_{\min}$ is

$$\hat{k}_{cr} = \{\rho g/T\}^{1/2}. \tag{5}$$

If $\hat{k} > \hat{k}_{cr}$, or

$$k > (1/\sigma)^{1/2},$$

the waves are predominantly capillary waves. If

$$k < (1/\sigma)^{1/2},$$

the waves are predominantly gravity waves. The word "predominantly" may from time to time be omitted in the rest of this paper for brevity.

Yih's formulas for phase lines are (see Eqs. (19) and (20) in [1])

$$y = -\frac{aF'}{k(F - kF')}(k^2 - F^2)^{1/2}, \tag{6}$$

$$x = \frac{a(k - FF')}{k(F - kF')}, \tag{7}$$

where a is a constant of integration. The k in (1) will be treated as positive. Since y is real, (6) demands that

$$k^2 - F^2 \geq 0, \tag{8}$$

or

$$\sigma k^2 - k + 1 \leq 0. \tag{9}$$

That means

$$k_{\min} \le k \le k_{\max}, \tag{10}$$

where

$$k_{\min} = \frac{1-(1-4\sigma)^{1/2}}{2\sigma}, \tag{11}$$

$$k_{\max} = \frac{1+(1-4\sigma)^{1/2}}{2\sigma}. \tag{12}$$

Equations (11) and (12) show that there are no waves if $\sigma \ge \frac{1}{4}$. This condition can be written as $U/c_{\min} \le 1$, by virtue of (3).

Equation (7) contains the factor

$$h(k) = k - FF' = -\tfrac{3}{2}\sigma k^2 - k - \tfrac{1}{2}. \tag{13}$$

This can be written as

$$h(k) = -\tfrac{3}{2}\sigma(k-k_1)(k-k_2), \tag{14}$$

where

$$k_1 = \frac{1}{3\sigma}[1-(1-3\sigma)^{1/2}], \tag{15}$$

$$k_2 = \frac{1}{3\sigma}[1+(1-3\sigma)^{1/2}]. \tag{16}$$

Thus

$$h(k) > 0 \quad \text{if} \quad k_1 < k < k_2,$$

$$h(k) < 0 \quad \text{if} \quad k \le k_1 \text{ or } k \ge k_2.$$

The sign of $h(k)$ affects the sign of x through (7).

Another quantity of which the sign is important is

$$q(k) = F - kF' = \frac{1}{2(k+\sigma k^3)^{1/2}}(1-\sigma k^2). \tag{17}$$

It is evident that

$$q(k) \ge 0 \quad \text{if} \quad k < k_3 = \sigma^{-1/2},$$

$$q(k) < 0 \quad \text{if} \quad k > k_3 = \sigma^{-1/2}.$$

The sign of $q(k)$ affects the sign of y through (6), and the sign of x through (7).

It can be shown (the demonstration is omitted here), that

$$k_1 \le k_{\min} \le k_3 \le k_2 \le k_{\max}. \tag{18}$$

Hence k_1 has no significance, since a k less than $k_{\min}$ results in no waves.

Finally, the k-values for the cusps, denoted by k_{c1} and k_{c2}, are determined from

$$\frac{d^2\eta}{d\xi^2} = 0, \tag{19}$$

or

$$\frac{d}{d\xi}\frac{k\,dk/d\xi - \xi}{(k^2-\xi^2)^{1/2}} = 0.$$

With (1), this becomes, after a brief calculation,

$$\frac{d}{d\xi}\frac{2h(k)\xi}{(1+3\sigma k^2)(k^2-\xi^2)^{1/2}} = 0. \tag{20}$$

This can only be solved numerically once σ is given, and we shall list the values of k_{c1} and k_{c2} in Table 1 for various values of σ.

TABLE 1. Important values of k and values of γ and ϕ.

$\frac{U}{c_{\min}}$	k	Waves of Sets 2 and 3				
		$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$
10	k_{st}			1.000025		
	k_{c1}			1.500253		
	k_{c2}			78.41496		
	k_{en}	196.3524	192.7049	189.6653	186.0177	182.3702
6	k_{st}			1.000193		
	k_{c1}			1.501962		
	k_{c2}			28.06051		
	k_{en}	70.02272	67.82574	65.84847	63.65150	61.23482
4	k_{st}			1.000978		
	k_{c1}			1.510122		
	k_{c2}			12.30653		
	k_{en}	30.71992	29.34138	27.96284	26.38736	24.71341
2	k_{st}			1.016133		
	k_{c1}			1.808496		
	k_{c2}			2.535961		
	k_{en}	7.590197	7.180394	6.715951	6.224187	5.623143
1.8	k_{st}			1.025022		
	k_{en}	6.152701	5.798128	5.416279	4.979881	4.461658
1.5	k_{st}			1.054960		
	k_{en}	4.276073	4.304920	3.759316	3.466488	3.087534
1.1	k_{st}			1.279578		
	k_{en}	2.248937	2.089277	1.941023	1.809875	1.695823

$\frac{U}{c_{min}}$	k	Waves of Set 1					γ	ϕ
		$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$		
10	k_{st}			39999			6000	5.71°
	k_{c1}							
	k_{c2}							
	k_{en}	4816.684	7045.430	8716.984	10070.15	11104.93		
6	k_{st}			5183			750	9.59°
	k_{c1}							
	k_{c2}							
	k_{en}	613.7659	899.9819	1114.644	1278.196	1421.304		
4	k_{st}			1023			150	14.47°
	k_{c1}							
	k_{c2}							
	k_{en}	127.1359	182.6319	224.2538	255.9658	283.7139		
2	k_{st}			62.98387			10	29.98°
	k_{c1}							
	k_{c2}							
	k_{en}	11.07910	13.27845	15.37935	16.57748	17.89709		
1.8	k_{st}			40.96538			10	
	k_{en}	8.962947	10.68722	11.99766	13.10193	14.06678		
1.5	k_{st}			19.19504			10	
	k_{en}	6.057674	7.027547	7.732909	8.291320	8.732171		
1.1	k_{st}			4.576822			1	
	k_{en}	2.588232	2.752151	2.907442	3.054106	3.187829		

3. Procedure of Computation. There are three sets of waves created by the moving disturbance:

1. Set 1. Capillary waves *ahead* of the disturbance.
2. Set 2. Largely gravity waves behind the disturbance.
3. Set 3. Capillary waves *behind* (!) the disturbance.

The first set is the well-known fish-line waves. The starting k-value for this set, denoted by k_{st} in Table 1, is just k_{max}. There are no cusps in the phase lines for this set, and for each phase line the k-values decrease from k_{max}, at which $y = 0$ and x is negative, to k_2, at which $x = 0$, and then to k_3 at which x is positive and both x and y are infinite. In plotting the phase lines, one cannot reach k_3, of course, and we have stopped at an ending k, denoted by k_{en}, and given numerically for all values of c_{min}/U and for Sets 1 and 3.

Set 2 corresponds to Kelvin waves, except the effect of surface tension has been taken into account. For this set one starts with a k_{st} equal to k_{min}, at the centerline $y = 0$ and a value of x given by (7), once a is given. One then proceeds along a transverse phase line to the first cusp on either side of the centerline, where k is k_{c1}. Then one increases k toward k_{c2}, in the process tracing out the phase line corresponding to the divergent part of the Kelvin waves, except that one does not reach the origin (where the disturbance is) but reaches the second cusp instead, at which k is k_{c2}.

Then one traces another divergent wave, of Set 3, as one increases k from k_{c2} to k_3, in the process tracing out the phase line that is almost straight, and diverges from the second cusp toward infinity (with positive x), asymptotically making an angle ϕ with the centerline. The angle ϕ is recorded in Table 1 for all cases, and is, incidentally, the same for Set 1 and Set 3. That is, the asymptotes of phase lines for Set 1 at infinity also make the same angle ϕ with the centerline. The angle ϕ is determined analytically by putting $k = k_3 = \sigma^{-1/2}$ in

$$\frac{dy}{dx} = \tan\phi = -\frac{F}{(k^2 - F^2)^{1/2}}, \tag{21}$$

with F defined by (2). Equation (21) is equation (6) of [4]. It is immediately clear that, as σ decreases to zero, k_3 approaches infinity and ϕ approaches zero—a result of great significance to remote sensing.

It is clear from the foregoing description that

$$k_3 < k \le k_{\max}$$

for Set 1,

$$k_{\min} \le k \le k_{c2}$$

for Set 2, and

$$k_{c2} \le k < k_3$$

for Set 3. For Set 2 (Kelvin waves with surface tension taken into account)

$$k_{\min} \le k \le k_{c1}$$

for transverse waves, and

$$k_{c1} \le k \le k_{c2}$$

for divergent waves.

4. Results. The results for various values of σ, represented by $U/c_{\min}$, are given in Table 1 and the figures. In Table 1, the results for Sets 2 and 3 are presented together, and those for Set 1 are presented separately. For $U/c_{\min}$ equal to or less than 1, there are no waves, as mentioned in Sec. 2.

The various k-values given and the angle ϕ in Table 1 have been explained in Sec. 3. Since the wavelength for Set 2 at the centerline is enormously greater than the wavelength for Set 1 at the centerline, it is inconvenient to present the entire wave pattern in a single figure with the same length scale. For this reason, for all cases except $U/c_{\min} = 1.1$ (see Figure 1), the phase lines for Set 1 are presented separately from those for Sets 2 and 3. The ratio of the length scale for Sets 2 and 3 to that for Set 1 is denoted by γ, and given in Table 1. In reading the figures, then, one must, for all cases except $U/c_{\min} = 1.1$, imagine the parts b to be magnified γ times in one's mind. It is also helpful to keep in mind that along the centerline ($y = 0$) consecutive phase lines for Set 1 and Set 2 are spaced at one wavelength apart, and

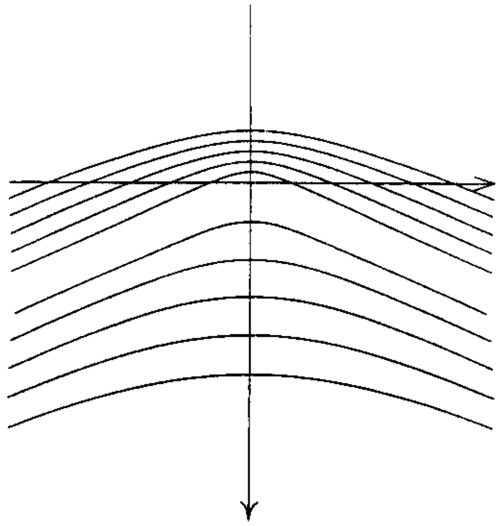

FIG. 1. Pattern of gravity-capillary waves at $U/c_{\min} = 1.1$.

that the wavelength for Set 1 (predominantly capillary waves) is very much smaller than that for Set 2 (predominantly gravity waves).

From Table 1, one sees that for transverse waves of Set 2 the k-values are of order 1. These are predominantly gravity waves. For divergent waves of Set 2, k increases from k_{c1} to k_{c2}, and k_{c2} may be considerably greater than k_{c1} if $U/c_{\min}$ is greater than 4. Hence the divergent waves of Set 2, which have their counterpart in Kelvin waves (for which only gravity is taken into account), become more and more capillary waves as k_{c2} is approached for $U/c_{\min}$ greater than 4 (which is not a sharp boundary, and is cited here only because it is one value of $U/c_{\min}$ chosen in Table 1 which seems to divide large values of k_{c2}/k_{c1} from modest ones of order 1).

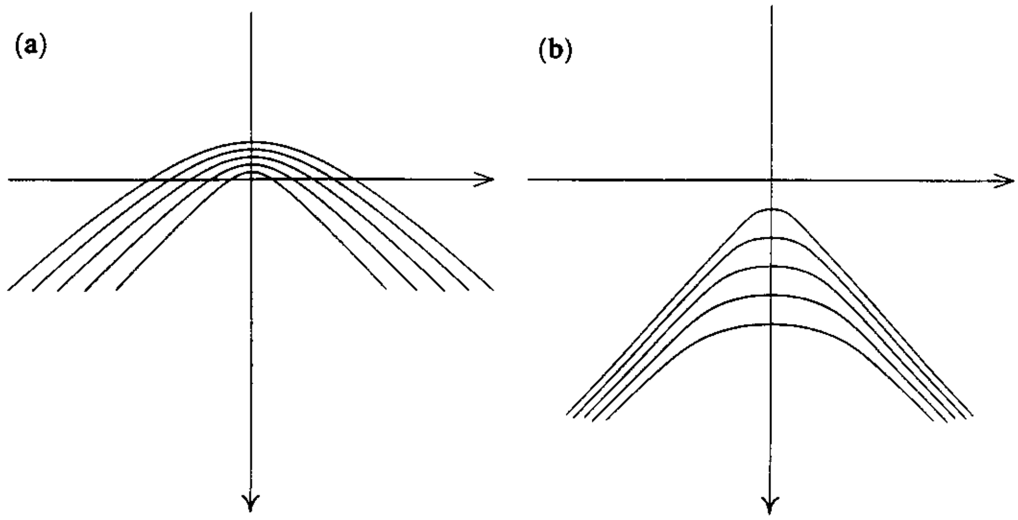

FIG. 2. Pattern of gravity-capillary waves at $U/c_{\min} = 1.5$.
(a) Set 1, (b) Sets 2 and 3 merged.

For $U/c_{\min} \geq 4$, the waves of Set 3 are predominantly capillary waves. For smaller values of $U/c_{\min}$ (see, e.g., Figure 2), one can only say that as the phase lines depart more and more from the center (where $y = 0$), the waves they represent become more and more capillary waves.

As noted before, the waves of Set 1 are predominantly capillary waves, and it is emphasized again that as the phase lines approach $y = \pm\infty$, they become increasingly straight lines that make the same angle ϕ with the centerline as the asymptotes of the phase lines of Set 3, as k_3 is approached (or as y approaches $\pm\infty$).

The most important part of the results is that there are capillary waves with wave fronts making a smaller and smaller angle ϕ with the centerline as $U/c_{\min}$ is increased. This has been observed in photographs obtained by remote sensing, and has been a point of keen interest in naval circles.

From Table 1, one sees for the case $U/c_{\min} = 2$ that k_{c2} is not equal to k_{c1}. Therefore the phase lines for Sets 2 and 3 in Fig. 4b should show a loop as in Fig. 5b. The loop is too small to be seen, and Fig. 4b instead shows a discontinuity in slope at a point near where the loop should be. This loop disappears when $U/c_{\min} = 1.8$ (which may be taken as the limiting value of $U/c_{\min}$ below which there is no loop), which may be compared with the tentative value 2 of Lamb ([1], p. 471, footnote 1). When the loop disappears, the slope at the juncture of the transverse waves of Set 2 and the divergent waves of Set 3 should be continuous. Figure 3b (for $U/c_{\min} = 1.8$) shows a slight but detectable discontinuity at that juncture. That discontinuity should not be there, and is a consequence of the finite-difference calculation when Δk, the increment of k, is not small enough for the neighborhood of the juncture.

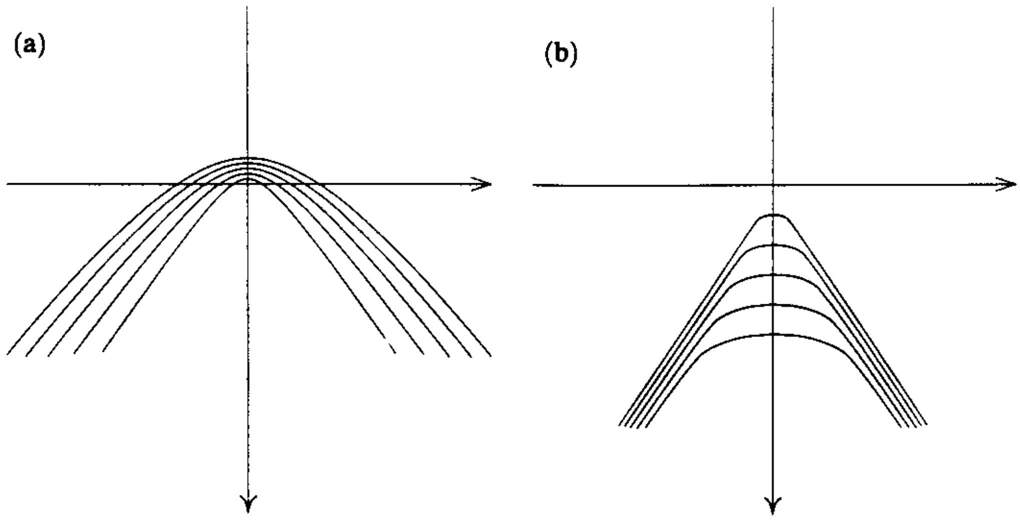

FIG. 3. Pattern of gravity-capillary waves at $U/c_{\min} = 1.8$. (a) Set 1, (b) Sets 2 and 3 merged.

Finally, we note that the square roots in (2) and (6) involve ambiguities in sign. The square root in (6) can be positive or negative, so that both positive and negative y-values are allowable. The sign of the square root in (2), which has a consequence on the sign of x given by (7), is chosen so that waves of Set 1 start in front of the disturbance before they wrap around it to positive values of x, and that waves of Sets 2 and 3 are *behind* the disturbance, so that x is always positive for waves of these sets.

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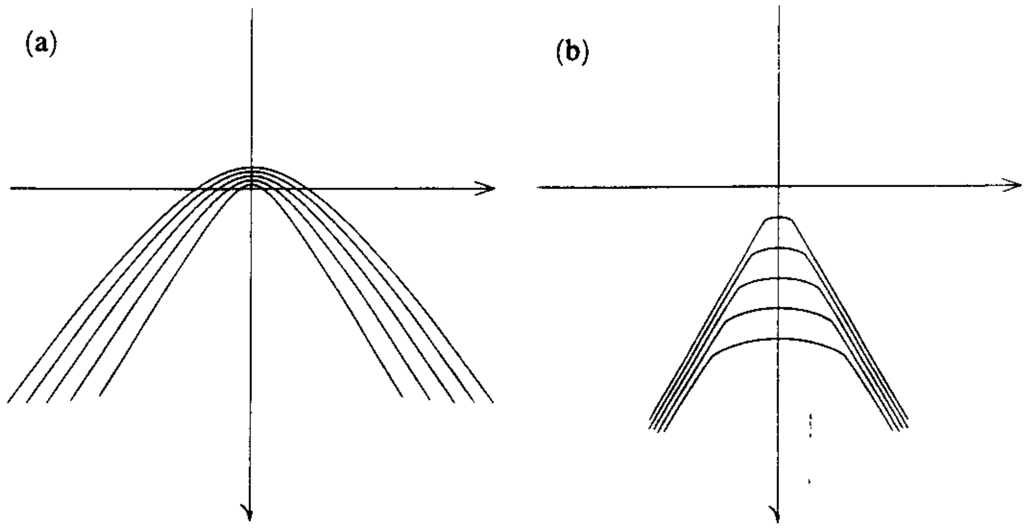

FIG. 4. Pattern of gravity-capillary waves at $U/c_{\min} = 2$.
(a) Set 1, (b) Sets 2 and 3 (loop with cusps invisible).

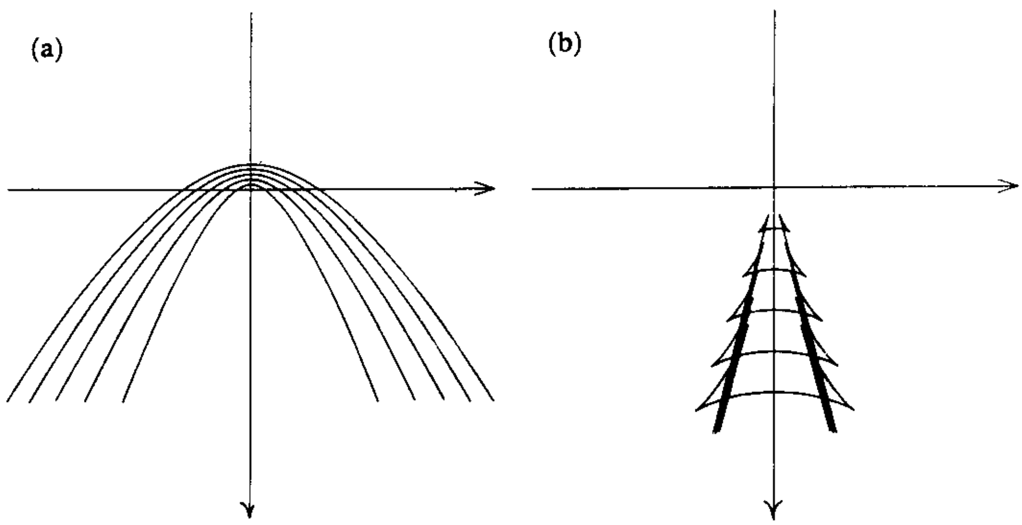

FIG. 5. Pattern of gravity-capillary waves at $U/c_{\min} = 4$.
(a) Set 1, (b) Sets 2 and 3.

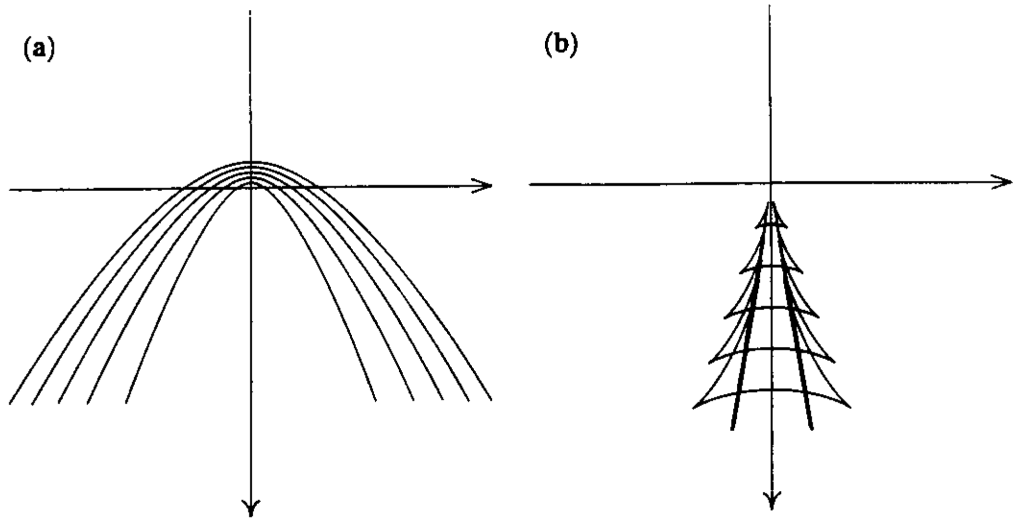

FIG. 6. Pattern of gravity-capillary waves at $U/c_{\min} = 6$.
(a) Set 1, (b) Sets 2 and 3.

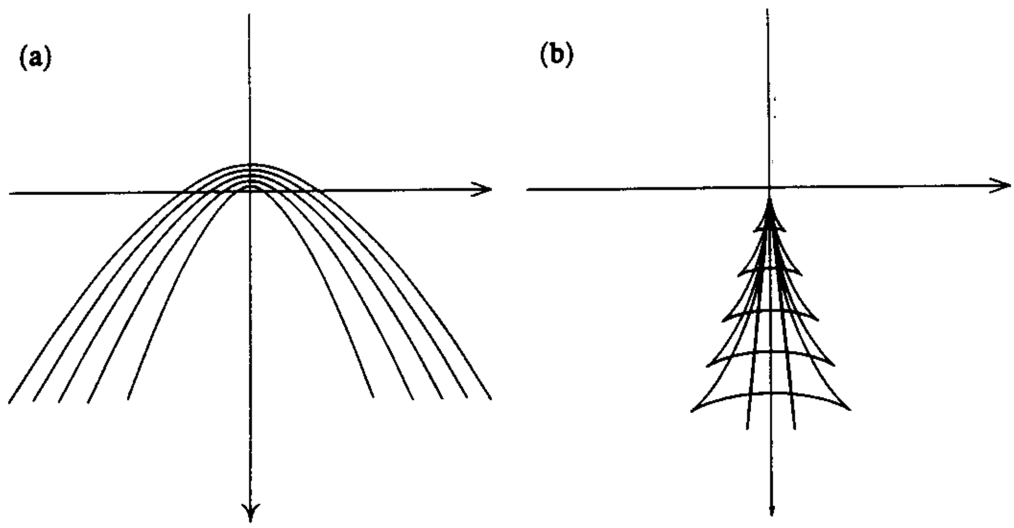

FIG. 7. Pattern of gravity-capillary waves at $U/c_{\min} = 10$.
(a) Set 1, (b) Sets 2 and 3.

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