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SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

QUANTUM PHASE TRANSITIONS IN BOSONIC SYSTEMS

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These are preliminary lecture notes, intended only for distribution to participants.

QUANTUM PHASE TRANSITIONS IN BOSONIC SYSTEMS

EFFECT OF FINITE TEMPERATURE

I INTRODUCTION

II GENERAL SCALING IDEAS
THE "QUANTUM - CRITICAL" REGION.

III THE DILUTE BOSE GAS
[HALDANE GAP ANTIFERRO-MAGNETS IN A FIELD]

IV QUANTUM O(N) SIGMA MODEL
[TWO DIMENSIONAL ANTIFERROMAGNETS]

V CONSERVED QUANTITIES
[UNIFORM SUSCEPTIBILITY]

VI EFFECTS OF RANDOMNESS

QUANTUM SPIN GLASSES.

QUANTUM ROTORS

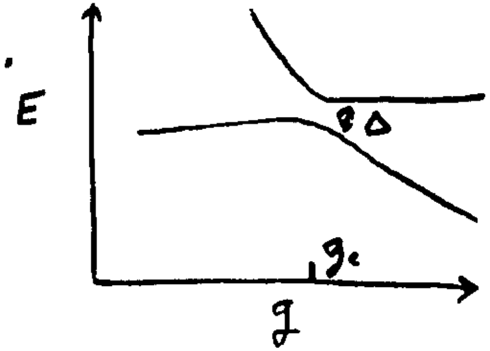
HEISENBERG SPIN GLASS

I INTRODUCTION

QUANTUM TRANSITION:

NON-ANALYTICITY IN GROUND STATE PROPERTIES AS A FUNCTION OF SOME COUPLING CONSTANT (g) AT $g = g_c$.

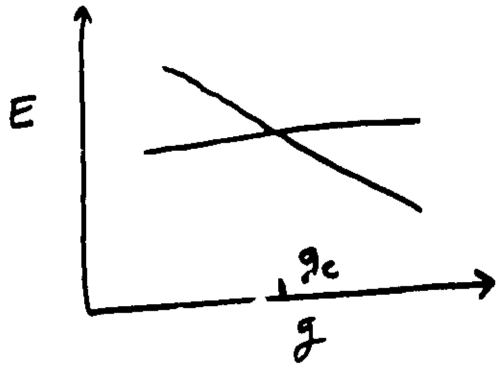
TYPE I (GENERIC)



AVOIDED LEVEL CROSSING

$\Delta \to 0$ AS SYSTEM SIZE $\to \infty$.

TYPE II



TRUE LEVEL CROSSING

$$\mathcal{H} = \mathcal{H}_0 + g Q$$

$$[\mathcal{H}_0, Q] = 0.$$

$(z\nu = 1)$

SECOND-ORDER QUANTUM TRANSITION

"DOMINANT" EXCITATIONS OCCUR ON A LENGTH SCALE ξ AND HAVE AN ENERGY $\sim \hbar / \xi_\tau$

$$\xi, \xi_\tau \to \infty \quad \text{AS} \quad g \to g_c$$

SIMPLE SCALING

$$\xi \sim |g - g_c|^{-\nu}$$

$$\xi_\tau \sim |g - g_c|^{-z\nu}$$

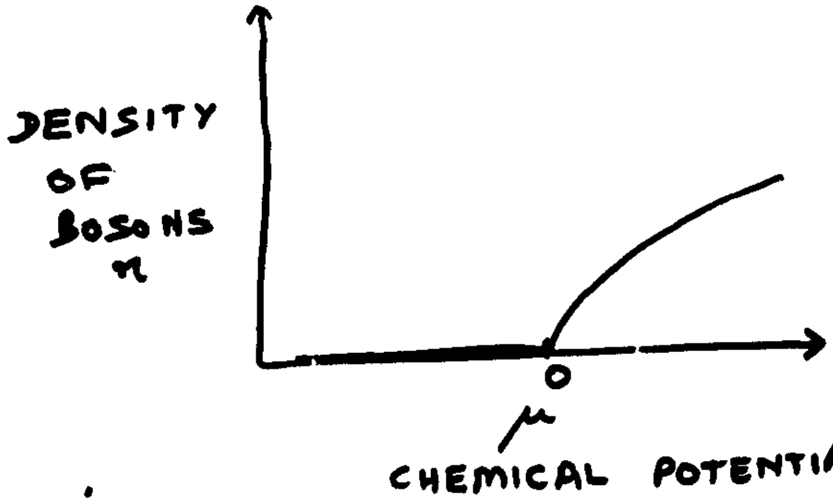
$$\xi_\tau \sim \xi^z$$

$\nu \to$ CORRELATION LENGTH EXPONENT

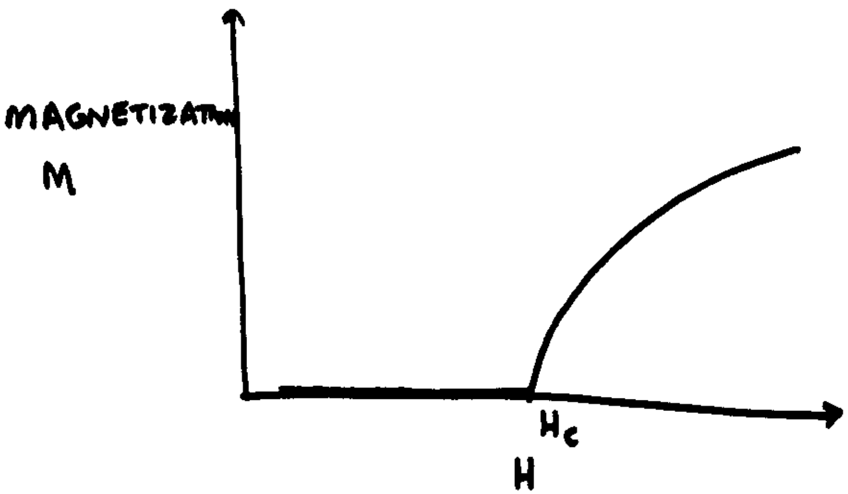
$z \to$ DYNAMIC CRITICAL EXPONENT.

EXAMPLES

1. DILUTE BOSE GAS



2. GAPPED ANTIFERROMAGNET IN A FIELD



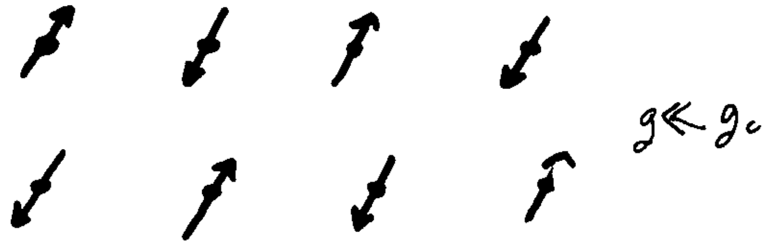
3. ANTIFERROMAGNETS.

~~Quantum Phase Transitions~~

Let us consider the possible ground states of a clean two-dimensional quantum Heisenberg antiferromagnet

$$H = \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

- Magnetic long-range-order
  Each spin has a definite orientation in spin space.



- Spin-fluid
  Quantum fluctuations destroy correlations between spins. Ground state can be visualized as a sea of singlet bonds.

Let this transition occur as a function of an all-purpose coupling constant g, at the critical value $g = g_c$. The constant g is a function of ratios of the J_{ij}

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4. FERMI LIQUID

TO

SPIN-DENSITY WAVE,

FERROMAGNET

SUPERCONDUCTOR

⋮

8

Quantum Criticality

Consider the $T = 0$ system at $g = g_c$

CONTINUUM SCALE-INVARIANT FIELD THEORY IN
d SPACE AND 1 TIME DIMENSION

$$\exp\left(-\frac{i}{\hbar}\int dt H\right)$$

analytic continuation to imaginary time

$it = \tau$

STATISTICAL MECHANICS IN d+1 SPACETIME DIMENSIONS

$$\exp\left(-\frac{1}{\hbar}\int d\tau H\right)$$

possibly with anisotropic scaling and complex weights. Typical time scales are a power z of spatial scales.

- What happens at a finite T?

d+1 dimensional quantum field theory at
finite temperature
$\mathrm{Tr}\exp\left(-H/k_BT\right)$

analytic continuation to imaginary time

$$\mathrm{Tr}\exp\left(-\frac{1}{\hbar}\int_0^{L_\tau} d\tau H\right)$$

Statistical mechanics in a box of size
$\infty^d \times L_\tau$ $(L_\tau \equiv \hbar/(k_BT))$
with periodic boundary conditions along the time
direction.
Properties specified by the theory of FINITE-SIZE
SCALING

EFFECT OF RENORMALIZATION GROUP TRANSFORMATION

1) Integrate out short-distance degrees of freedom

2) Rescale distances and times

$$x' = x/s$$
$$\tau' = \tau/s^2$$
$$T' = T s^2$$

3) Define new coupling constant

$$g' = R[g, u]$$
$$u' = R[g, u].$$

omogeneity relationship for free energy

$$\mathcal{F} = -\frac{k_B T}{V} \ln Z$$

$$\mathcal{F} = s^{-d-3}\Big[\mathcal{F}'(g', u') + \underbrace{\mathcal{G}(g', u')}_{\text{free energy of modes integrated out.}}\Big]$$

Iterate process $[s = e^{l}]$

$$\mathcal{F} = e^{-(d+3)l}\,\mathcal{F}(g(l), u(l), T(l)) + \int_0^{l} e^{-(d+3)l'}\,\mathcal{G}(g(l'), u(l'), T(l'))\,dl'$$

In the limit $l \to \infty$, first term is not important

Near critical point

$$g(l) = (g - g_c)\,e^{l/\nu}$$

$$T(l) = e^{3l}\,T$$

$$u(l) = e^{-\phi l}\,(u - u^*)$$

$$\mathcal{F} = \int_0^{\infty} dl\, e^{-(d+3)l}\,\mathcal{G}\left(e^{3l}T,\ e^{l/\nu}\delta g,\ e^{-\phi l}\delta u\right)$$

define $x = e^{3l}T$

$$= \frac{T^{1+d/3}}{3}\int_T^{\infty}\frac{dx}{x}\left(\frac{1}{x}\right)^{1+d/3}\mathcal{G}\left(x,\ \delta g\left(\frac{x}{T}\right)^{1/3\nu},\ \delta u\left(\frac{x}{T}\right)^{-\phi/3}\right)$$

As $T \to 0$ lower limit can be set to 0. (Infrared divergent terms lead only to regular terms in T).

Finally (13)

$$\mathcal{F} = T^{1+d/z}\, \Phi\left(\frac{|\delta g|^{z\nu}}{T}\right)$$

UNLESS:-

g has a singular dependence on δu

THEN

if $g \sim \frac{1}{\delta u}$

and $\delta u(\ell) = e^{-\theta \ell}\, \delta u$

we get

$$\mathcal{F} = T^{1+\frac{d-\theta}{z}}\, \Phi\left(\frac{|\delta g|^{z\nu}}{T}\right)$$

$\theta \rightarrow$ VIOLATION OF HYPERSCALING EXPONENT.

IN SUMMARY (14)

$$\mathcal{F} = T^{1+(d-\theta)/z}\, \Phi(T\xi_\tau)$$

SIMILARLY FOR OTHER OBSERVABLES

$$\chi(k,\omega) = T^{-\frac{(2-\eta)}{z}}\, \tilde{\Phi}\left(T\xi_\tau, \frac{k}{T^{1/z}}, \frac{\omega}{T}\right).$$

The dimensionless quantity

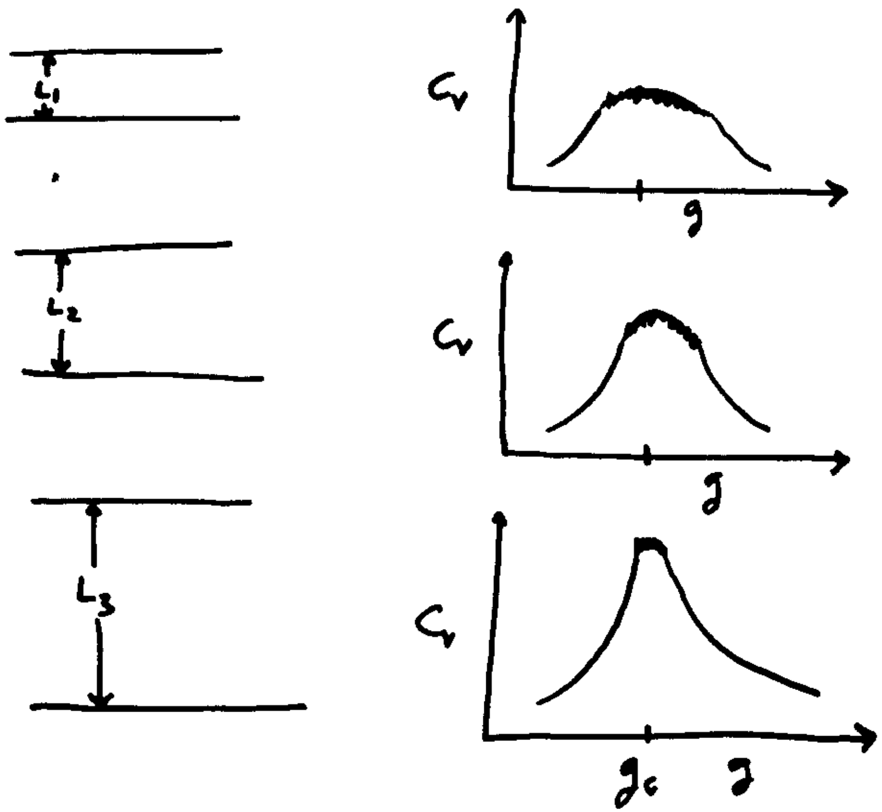
$$T\xi_\tau = \xi_\tau / L_\tau$$

is crucial.

FINITE SIZE SCALING IN STATISTICAL MECHANICS

2D ISING MODEL ON A $(L \times \infty)$ LATTICE

$$Z = \sum_{\sigma_i} \exp\left(+ \frac{1}{g} \sum_{\langle ij \rangle} \sigma_i \sigma_j \right)$$



Let $\xi_\infty \sim |g - g_c|^{-\nu}$ be the correlation length in $\infty \times \infty$ sample

$L < \xi_\infty$

PROPERTIES ARE SENSITIVE TO L BUT INSENSITIVE TO g

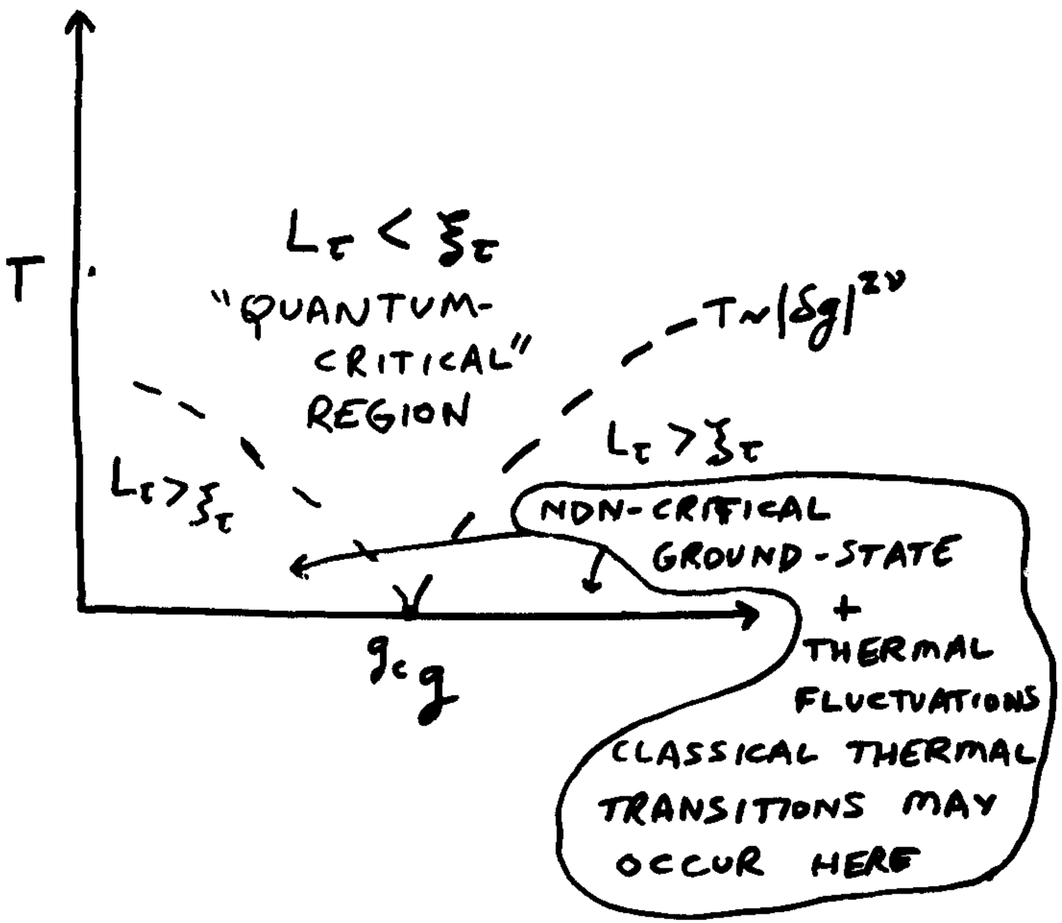
$L > \xi_\infty$

PROPERTIES DEPEND UPON g BUT ARE INSENSITIVE TO L

RETURN TO QUANTUM PROBLEM

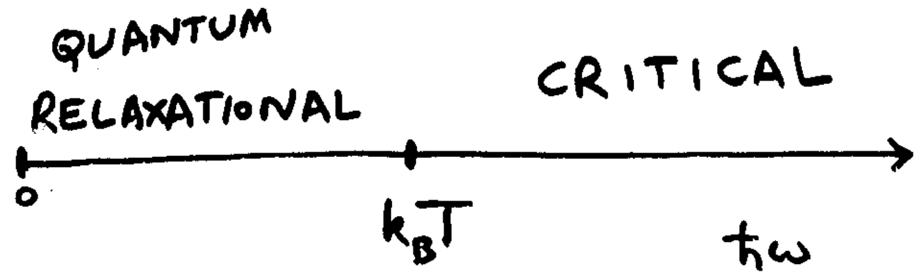
$$L \rightarrow L_\tau = \hbar / k_B T$$

$$\xi_\infty \longrightarrow \xi_\tau \sim |g - g_c|^{-z\nu}$$



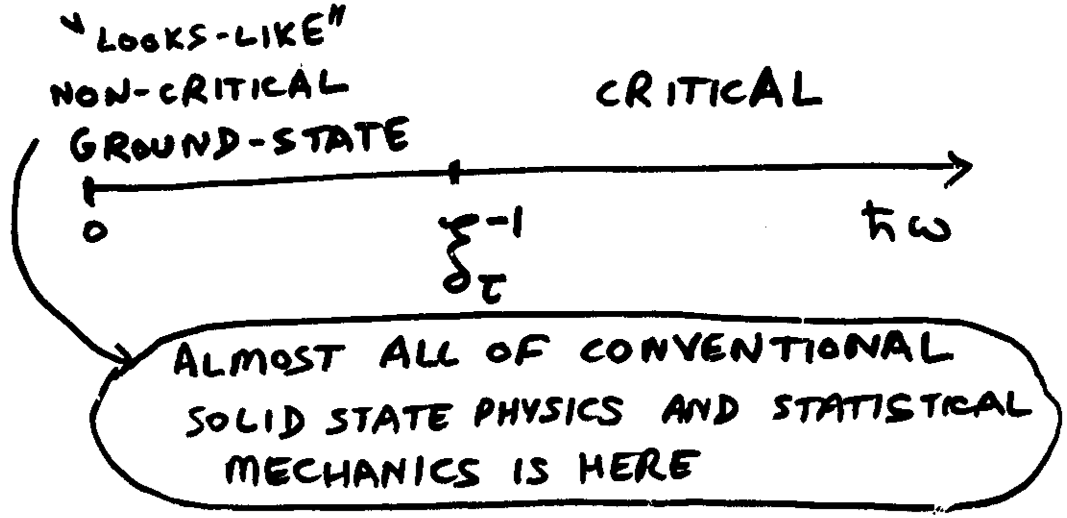
QUANTUM-CRITICAL REGION

$$L_\tau < \xi_\tau$$



OTHER REGIONS

$$L_\tau > \xi_\tau$$



First example of a quantum phase transition

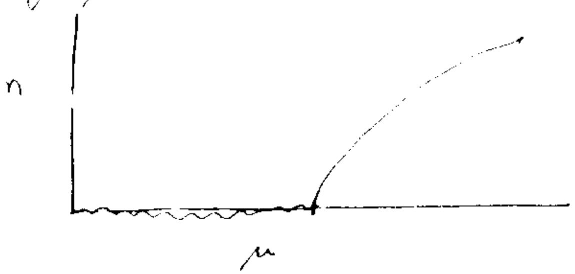
DILUTE BOSEGAS WITH REPULSIVE INTERACTIONS

$$H = \int\left[-\frac{\hbar^2\, \psi^\dagger_{(r)}\nabla^2\psi(r)}{2m} - \mu\, \psi^\dagger_{(r)}\psi(r)\right] d^d r + \frac{1}{2}\int d^d r\, d^d r'\; \psi^\dagger(r)\,\psi^\dagger(r')\, v(r-r')\; \psi(r')\,\psi(r)$$

$\psi \rightarrow$ Boson annihilation operator

$v > 0 \longrightarrow$ repulsive interactions.

Follow density $n = \langle|\psi|^2\rangle$ as a function of μ at $T=0$



For $\mu<0$, $n=0$ clearly

$\mu>0$ $\quad n\neq 0$

Non-analyticity at $\mu=0$

$\Rightarrow$ Quantum phase transition at $\mu=0$ $T=0$

Renormalization group analysis

Co-herent state path integral

$$Z = \int \mathcal{D}\psi\, \exp\left(\frac{S}{\hbar}\right)$$

$$S = \int_0^{\hbar/k_BT} d\tau \int d^d x \left[\hbar\, \psi^\dagger\frac{\partial\psi}{\partial\tau} + \frac{\hbar^2}{2m}\psi^\dagger\nabla^2\psi + \mu\,\psi^\dagger\psi - u\,|\psi^\dagger\psi|^2\right]$$

Rescaling transformation

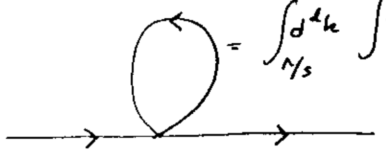
$x' = x/s \qquad \tau' = \tau/s^z$

$\psi' = \psi\, s^{(d+z-2+\eta)/2}$

$\mu' = \mu\, s^2$

$u' = u\, s^{4-2\eta-d-z}$

Integrate out high momentum modes



$$= \int_{\Lambda/s}^{\Lambda} d^d k \int d\omega \, \frac{1}{-i\omega + \Lambda^2} = 0 .$$

Graph vanishes at $T = 0$.

In fact, all corrections to Green's function vanish and self energy $\Sigma = 0$ exactly.

$\Rightarrow$ No renormalizations to $\psi^* \frac{\partial \psi}{\partial \tau}$, $\psi^+ \nabla^2 \psi$, $\psi^+ \psi$ terms

$\Rightarrow$ RG equations $(s = e^{\ell})$

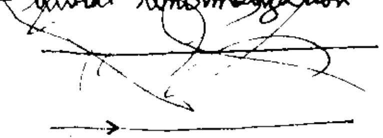
$$\boxed{z = 2 \qquad \eta = 0 \qquad \text{exact}}$$

$$\boxed{\frac{d\mu}{d\ell} = 2\mu \qquad \text{exact}}$$

$\hookrightarrow$ Implies $\nu = \frac{1}{2}$

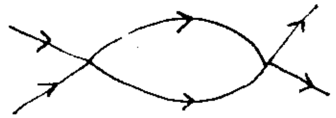
~~Eqn for u,~~

~~Non-trivial renormalization~~



Eqn for u

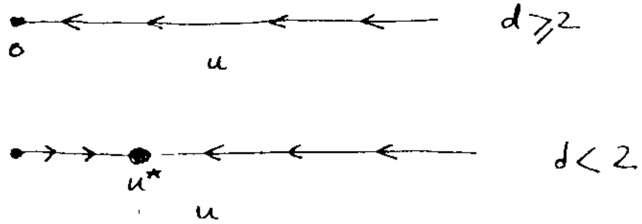
Non-trivial renormalization



$$\int_{\Lambda/s}^{\Lambda} d^d k \int d\omega \, \frac{1}{-i\omega + \Lambda^2} \, \frac{1}{i\omega + \Lambda^2}$$

$$\frac{du}{d\ell} = (2-d) u - \frac{m K_d \Lambda^{d-2}}{\hbar^2} u^2$$

$$\left[K_d = \frac{S_d}{(2\pi)^d} \right].$$



For $d \geq 2$ interactions are "irrelevant"

$d < 2$ " " "relevant"

$d = 2$ is the upper critical dimension.

Free energy

(I) $d > 2$

u is "dangerously-irrelevant"

Free energy $\sim 1/u$

$\Rightarrow \theta = d - 2.$

Using $z = 2$, $\nu = 1/2$ we then have
$z\nu = 1$

$$\mathcal{F} = T^2 \, \Phi\left(\frac{\mu}{T}\right)$$

or at $T = 0$

$$\mathcal{F} \sim \mu^2$$

$$\Rightarrow \quad n = \frac{\partial \mathcal{F}}{\partial \mu} \sim \mu$$

cf. result of Lee and Yang for $d = 3$ hard sphere Bose gas

$$n = \frac{2m\mu}{\hbar^2} \frac{1}{8\pi a} + \ldots \mathcal{O}(\mu^2)$$

> Note – Co-eff $\frac{n}{\mu}$ depends on details of microscopic interactions

(II) $d < 2$

$u \to u^*$ at fixed point

$u - u^*$ is not dangerously irrelevant as $\mathcal{F} \sim \frac{1}{u^*}$

So $\theta = 0$

and

$$\mathcal{F} = k_B T \left(\frac{2mk_BT}{\hbar^2}\right)^{d/2} \Phi\left(\frac{\mu}{k_BT}\right)$$

Φ is a non-trivial scaling function which depends on u^*

$\Rightarrow$ Φ is <u>completely universal</u>, with no arbitrary scale-factors, and independent of <u>microscopic interactions</u>.

At $T = 0$

$$n = \frac{\partial \mathcal{F}}{\partial \mu} = \left(\frac{2m\mu}{\hbar^2}\right)^{d/2} \mathcal{C}$$

$\mathcal{C} \to$ universal number

Test of scaling results in $d=1$.

It can be shown that in $d=1$, $\boxed{u^* = \infty}$ and no other terms become relevant.

Critical theory → IMPENETRABLE BOSE GAS

↓ equivalent to free fermions.

$$k_F^2 = \frac{2m\mu}{\hbar^2}$$

• At $T=0$

$$\mathcal{F} = \int_{-k_F}^{k_F} \frac{dk}{2\pi}\,\frac{\hbar^2 k^2}{2m} - \mu \int_{-k_F}^{k_F} \frac{dk}{2\pi}$$

$$= -\left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{2\mu^{3/2}}{3\pi} \qquad \left[\sim \mu^{1+d/z} \quad \checkmark\right]$$

$$n = -\frac{\partial \mathcal{F}}{\partial \mu} = \left(\frac{2m\mu}{\hbar^2}\right)^{1/2} \frac{1}{\pi} \Rightarrow \boxed{C = \frac{1}{\pi}}$$

Scaling function

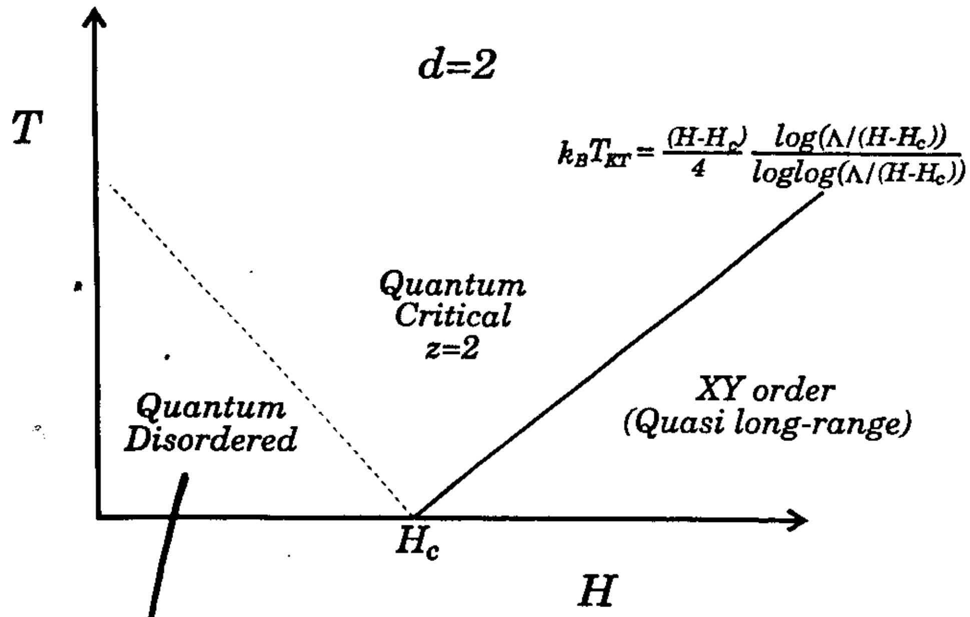
$$\mathcal{F} = k_B T \left(\frac{2 m k_B T}{\hbar^2}\right)^{1/2} \Phi\left(\frac{\mu}{k_B T}\right)$$

$$\Phi(t) = -\frac{1}{\pi} \int_0^\infty \ln\left(1 + e^{t - x^2}\right) dx$$

FINITE TEMPERATURE PHASE DIAGRAM

$z\nu = 1.$

↳ CROSSOVERS ARE STRAIGHT LINES.



$\mu = H - H_c$

FIG 2

DILUTE THERMALLY EXCITED BOSONS

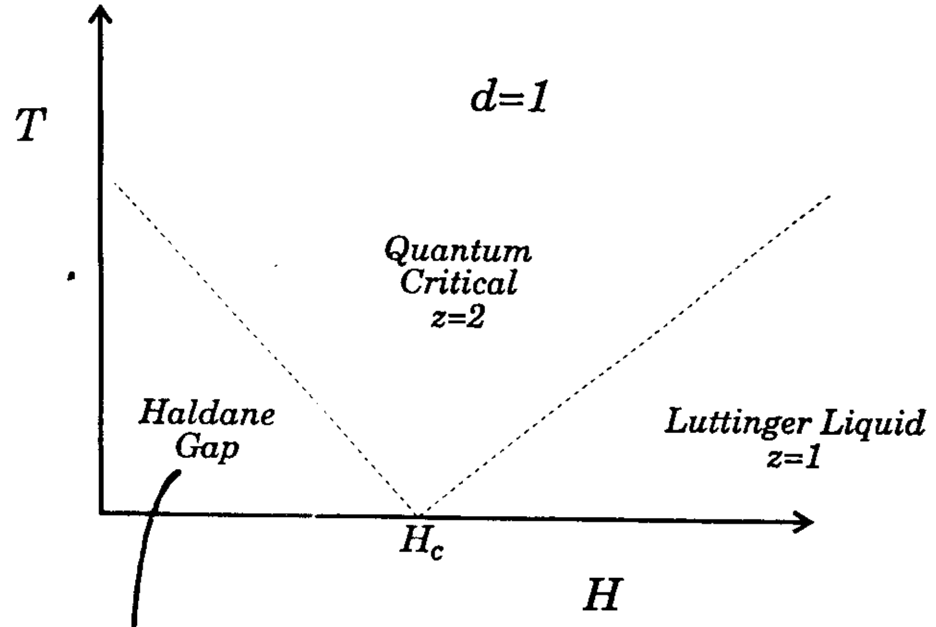
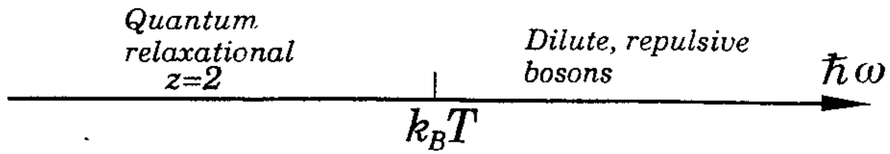


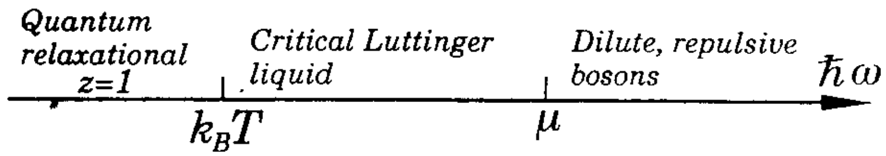
FIG 1

DILUTE THERMALLY EXCITED BOSONS

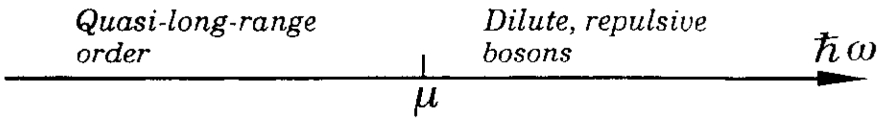
$$\boxed{|\mu| \ll k_B T\ ;\ d=1,2}$$



$$\boxed{\mu \gg k_B T\ ;\ d=1}$$



$$\boxed{\mu \gg k_B T\ ;\ d=2}$$



ALL CROSSOVERS FIG 3 DESCRIBED BY

$$G_R(k,\omega) = \frac{1}{k_B T}\,\tilde{\Phi}\left(\frac{\hbar\omega}{k_B T}, \frac{\hbar k}{\sqrt{2 m k_B T}}, \frac{\mu}{k_B T}\right)$$

$\tilde{\Phi} \rightarrow$ FULLY UNIVERSAL FUNCTION.

Computation of $\tilde{\Phi}$ in $d=1$.

Can be reduced to (numerical) solution of complicated integro-differential equations (Korepin - Slavnov).

Illustrate two special cases

(1) Equal time correlation function

Structure factor

$$S(k) = \int dx\, G(x)\, e^{ikx}$$

$$G(x) = \langle \psi(0)\, \psi^{+}(x) \rangle$$

$$= \left\langle \psi_F(0) \exp\left(i\pi \int_0^x dx' \psi_F^{+}(x') \psi_F(x') \right) \psi_F^{+}(x) \right\rangle$$

Jordan-Wigner phase factor

fermion annihilation operator

$$= \langle 0 | \hat{G}_F (1 - 2\hat{G}_F)^{-1} | x \rangle \det(1 - 2\hat{G}_F)$$

where $\langle x | \hat{G}_F | x' \rangle = \displaystyle\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{e^{(\hbar^2k^2/2m - \mu)/k_BT} + 1}$

Expression can be obtained by expanding exponent, using Wick's theorem for free fermions, and then resumming the series.

Result

$$S(k) = \mathcal{B}\left(\frac{\hbar k}{\sqrt{2mk_BT}}, \frac{\mu}{T} \right).$$

Numerical results for $\mathcal{B}$ $\longrightarrow$

$S(k)$ in $d=1$

$S(k) = \mathcal{B}\left(\frac{\hbar k}{\sqrt{2mk_BT}}, \frac{\mu}{k_BT}\right)$

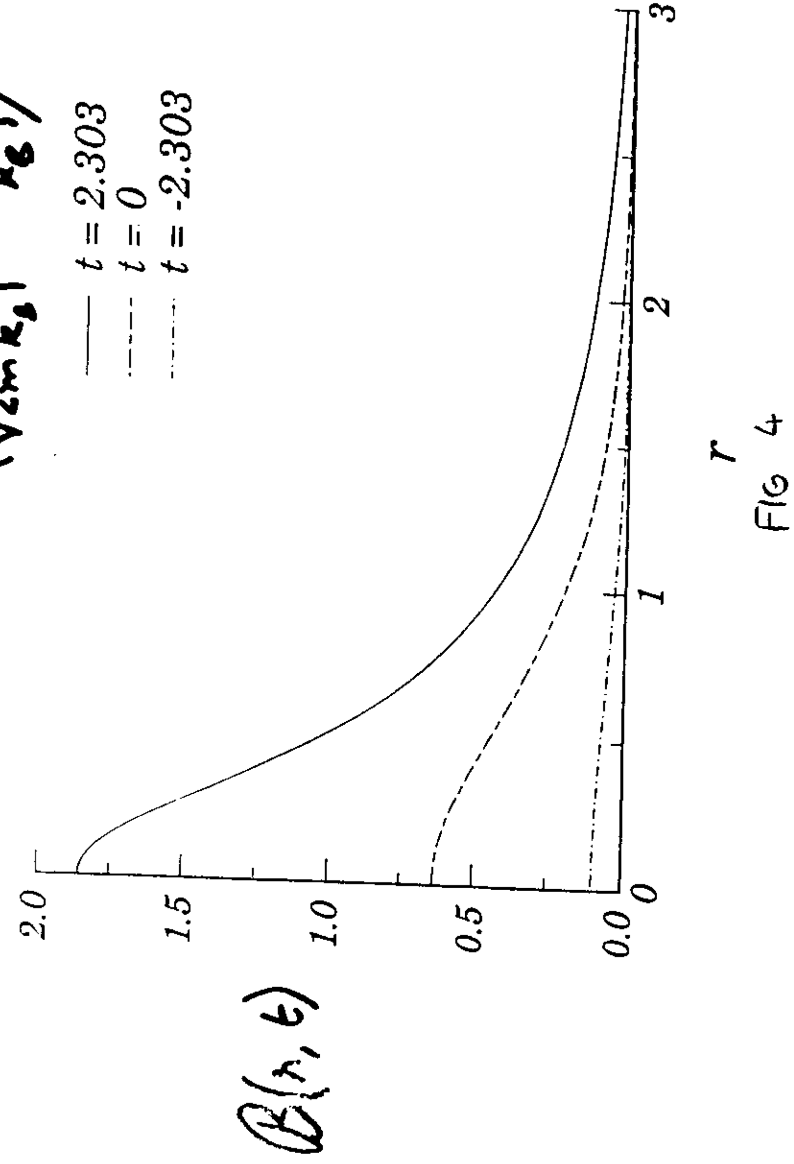


FIG 4

(II) ~~[illegible]~~ Dynamic correlation function in Luttinger liquid ~~[illegible]~~ region

$\mu \gg k_BT$, ~~[illegible]~~ scan $\hbar\omega$ from $\ll k_BT$ to $\gg k_BT$.

At $T=0$, bosonization technology can be ~~[illegible]~~ used to obtain

$$G(x\,\tau) = \langle \Psi(x,\tau)\Psi^{+}(0,0)\rangle$$

$$= \frac{D}{(x^2 + c^2\tau^2)^{1/4}}$$

[$c = v_F$ Fermi velocity of fermions]

Finite T: $z=1$ critical theory is Lorentz and conformally invariant.

Conformal invariance maps $T=0$ problem onto finite T problem

Infinite $z = x + i\tau$ plane $\Rightarrow w = \frac{1}{2\pi T}\ln z \Rightarrow$ $\hbar/k_BT$

Using conformal transformation

$$G(x\tau)\Big|_{\text{finite } T} = \frac{D}{2^{1/4}}\left(\frac{2\pi k_B T}{\hbar c}\right)^{1/2} \frac{1}{\left(\cosh\frac{2\pi k_B T x}{\hbar c} - \cos\frac{2\pi k_B T \tau}{\hbar}\right)^{1/4}}$$

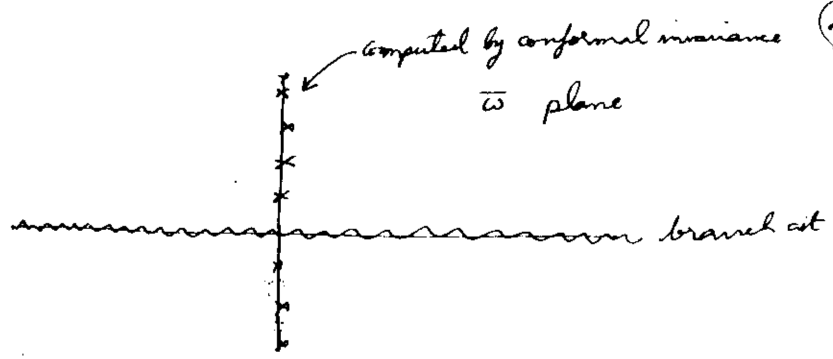
Take F.T.

$$\chi(q, i\omega_n) = \int dx\, e^{ikx} \int_0^{\hbar/k_B T} d\tau\, e^{-i\omega_n \tau}\, G(x,\tau)$$

$$= \frac{D}{c}\left(\frac{\hbar c}{k_B T}\right)^{3/2} A_L\left(\frac{i\hbar\omega_n}{k_B T}, \frac{\hbar c k}{k_B T}\right)$$

$$A_L(i\bar\omega, \bar k) = \frac{1}{2\sqrt{2\pi}}\frac{\Gamma(3/4)}{\Gamma(1/4)} \frac{\Gamma\left(\frac{1}{8} + \frac{|\bar\omega| + i\bar k}{4\pi}\right)\Gamma\left(\frac{1}{8} + \frac{|\bar\omega| - i\bar k}{4\pi}\right)}{\Gamma\left(\frac{7}{8} + \frac{|\bar\omega| + i\bar k}{4\pi}\right)\Gamma\left(\frac{7}{8} + \frac{|\bar\omega| - i\bar k}{4\pi}\right)}$$

To get dynamic correlation functions we must analytically continue to physical frequencies. Note: poles of Γ fns are in unphysical Riemann sheet → the only singularity of $A_L(\bar\omega, \bar k)$ in complex $\bar\omega$ plane is the branch cut on real $\bar\omega$ axis



Poles are in second sheet.

Excitation spectrum

$\hbar\omega,\ \hbar ck \gg k_B T$

propagating "phonons" with lifetime $\sim k_B T$

$\hbar\omega,\ \hbar ck \ll k_B T$

overdamped phonons.

Strong interaction between thermally excited phonons.

$$\chi(k,\omega) = \frac{D}{c}\left(\frac{\hbar c}{k_B T}\right)^{3/2} \mathcal{A}\left(\frac{\hbar\omega}{k_B T}, \frac{\hbar c k}{k_B T}\right)$$

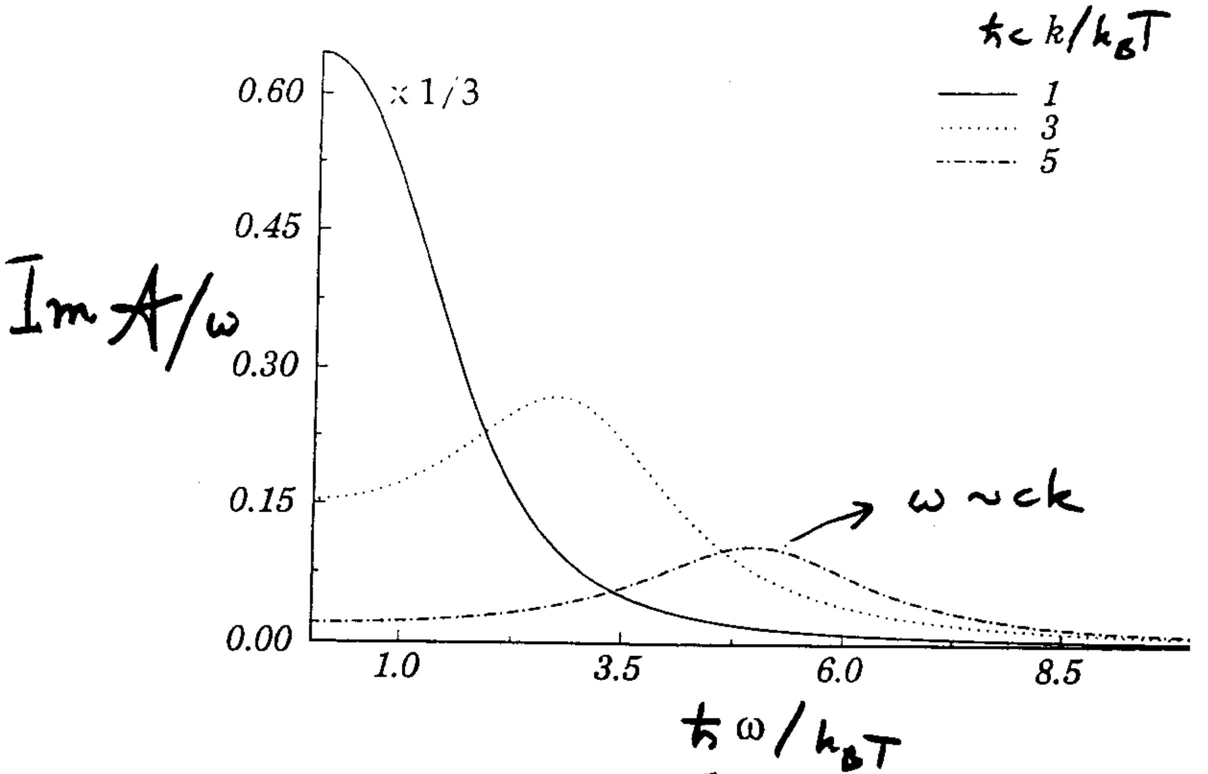


FIG 5

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$$\chi_L(\omega) = \int \frac{dk}{2\pi} \chi(k,\omega)$$

$$\mathrm{Im}\,\chi_L(\omega) = \frac{D}{\sqrt{c\omega}} F_L\left(\frac{\hbar\omega}{k_B T}\right)$$

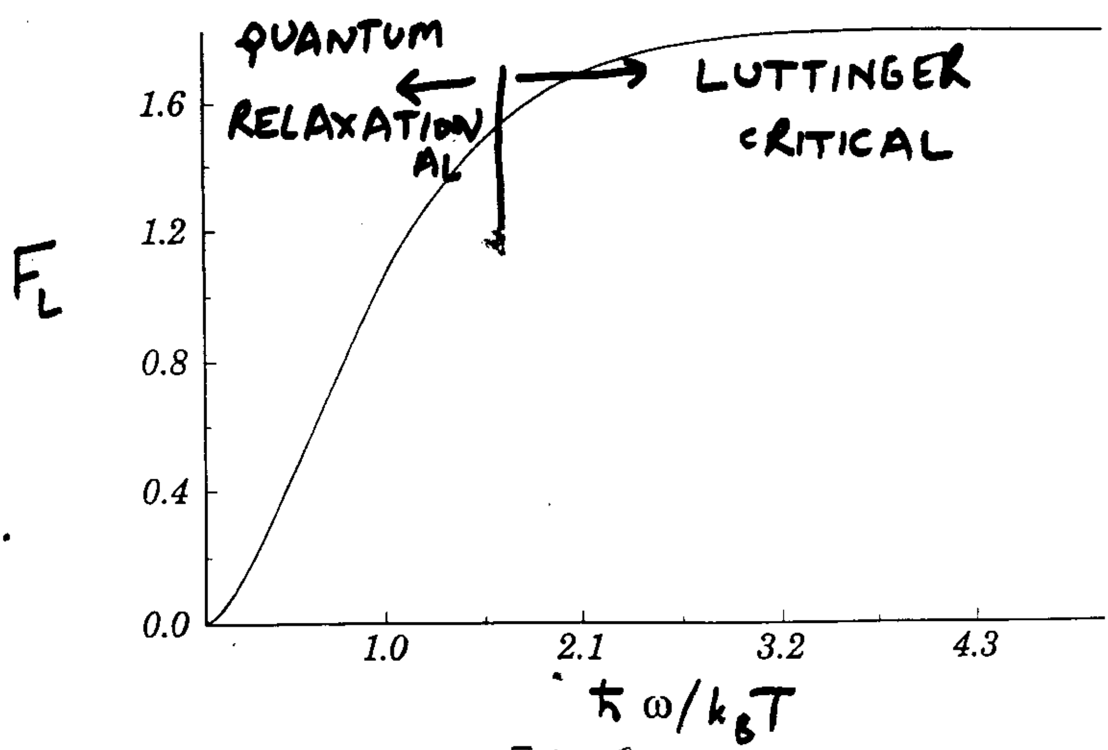

FIG 6

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