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SPRING COLLEGE IN CONDENSED MATTER ON QUANTUM PHASES
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Background information and Notes for Lecture on

THE QUANTUM HALL EFFECT: THEORY- Part I

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These are preliminary lecture notes, intended only for distribution to participants.

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THE QUANTUM HALL EFFECTS

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1 Introduction

The quantum Hall effect [1] occurs in two-dimensional electronic systems (2 DES) [2] in strong perpendicular magnetic fields. It differs from many of the phenomena which are being discussed at this ASI in that it does not depend directly on quantum coherence for its occurrence. Quantum coherence plays an important supporting role rather that a starring role in *this* drama. At low temperatures and strong magnetic fields the magnetoresistance of two-dimensional (2 D) metals becomes extremely small and the Hall resistance nearly constant over certain ranges of carrier density and magnetic field strength. Although this effect seemed quite mysterious at the time of its discovery, the principle elements of the physics which is responsible are now understood. In these notes I hope to explain these elements and to point out some aspects of our current theoretical picture of the phenomenon which seem to be incomplete.

For orientational purposes it is useful to begin with a simple semi-classical description of transport in a 2 DES. In the presence of an electric field, an electron is accelerated for an average time τ_0, the relaxation time, before being scattered to a state which has, on average, zero velocity. In this picture the electrons have an average drift velocity

$$\vec{v}_D = -e\vec{E}\tau_0/m \tag{1}$$

where $\vec{E}$ is the electric field and m is the electron effective mass. The current density in an electric field is then given by

$$\mathbf{j} = -cn\vec{v}_D = \sigma_0\vec{E} \tag{2}$$

where

$$\sigma_0 = nc^2\tau_0/m \tag{3}$$

and n is the areal electron density. In the presence of a steady perpendicular magnetic field the Lorentz force must be added to the force from the electric field so that

$$\vec{v}_D = -c(\vec{E} + \vec{v}_D \times \hat{z}B/c)\tau_0/m \tag{4}$$

Quantum Coherence in Mesoscopic Systems
Edited by B. Kramer, Plenum Press, New York, 1991

and the current density is no longer parallel to the magnetic field. In eq. (4) $\hat{z}B$ is the magnetic field. Using eq. (4) for the drift velocity we find that the resistivity tensor ($\mathbf{j} \equiv \sigma\mathbf{E}, \mathbf{E} \equiv \rho\mathbf{j}, \rho = \sigma^{-1}$) is given by,

$$\rho_{xx} = \rho_{yy} = \sigma_0^{-1}, \tag{5}$$

$$\rho_{xy} = -\rho_{yx} = B/nec. \tag{6}$$

Note that the diagonal components of the resistivity tensor are unchanged by the magnetic field; i.e., there is no magnetoresistance. The effect of the magnetic field is to introduce an electric field, the Hall field, in the direction perpendicular to the current flow. The force on the electrons from the Hall field exactly cancels the Lorentz force which the electrons experience as they drift.

This simple theory does a good job of describing the effect of a weak magnetic field on transport. At strong fields, however, there are important corrections which are due to the magnetic quantization of the kinetic energy of 2D electrons. To go any further toward understanding magnetotransport in strong magnetic fields we must first consider the quantum treatment of an electron in a magnetic field. The Hamiltonian for a free electron in a magnetic field is

$$t = \pi^2/2m, \tag{7}$$

where

$$\boldsymbol{\pi} = -i\hbar\nabla + e\mathbf{A}/c \tag{8}$$

and $\mathbf{A}$ is the vector potential, related to the magnetic field by

$$\nabla \times \mathbf{A} = \mathbf{B}. \tag{9}$$

It follows from eq. (9) that

$$[\pi_x, \pi_y] = -i\hbar^2\ell^{-2} \tag{10}$$

and hence that

$$t = \hbar\omega_c(a^\dagger a + 1/2), \tag{11}$$

where $\omega_c = eB/mc$,

$$a^\dagger = \frac{\ell}{\sqrt{2}\hbar}(\pi_x + i\pi_y), \tag{12}$$

and $[a^\dagger, a] = 1$. Here $\ell = (\hbar c/eB)^{1/2}$ is the classical cyclotron orbit radius of an orbit with kinetic energy $\hbar\omega_c/2$.

The ladder operator commutation relations imply that the allowed kinetic energies are restricted to the discrete set of values $\hbar\omega_c(n + 1/2)$ where n is an integer. Classically, however, it is clear that a cyclotron orbit is not specified completely by its energy. The solution to the equation of motion for a classical electron in 2D moving in a magnetic field is

$$z(t) = C - iv(t)/\omega_c, \tag{13}$$

where $z(t) = x + iy$ is the electron position, $v(t) = v_x(t) + iv_y(t)$ is the electron velocity and $C = C_x + iC_y$ is the cyclotron orbit center. For a given orbit velocity, and hence a given orbit energy, the center is arbitrary. Quantum mechanically we might also expect that there be many possible states for one of the allowed kinetic energies, corresponding to different orbit centers. Expressing the orbit center as a quantum mechanical operator it is easy to show that

$$[C, C^\dagger] = 2\ell^2, \tag{14}$$

where we have substituted $\mathbf{v} = \pi/m$ in eq. (13). For the quantum case the orbit center cannot be precisely determined. From eq. (14) it is clear that we can define a set of ladder operators based on the orbit center operator just as we defined a set of ladder operator based on the orbit velocities. Let

$$b = C/\sqrt{2}\ell = \frac{2}{\sqrt{2}}\left((z + iA/B)/\ell + 2\ell\frac{\partial}{\partial z^*}\right) \tag{15}$$

so that $[b, b^\dagger] = 1$. (Here $A = A_x + iA_y$). Then $[a, b] = [a, b^\dagger] = 0$ and a complete orthonormal set of eigenfunctions of t is given by

$$\phi_{n,m} = \frac{(a^\dagger)^n(b^\dagger)^m\phi_{0,0}}{\sqrt{n!m!}} \tag{16}$$

where $a\phi_{0,0} = b\phi_{0,0} = 0$. The set of single-particle states with a given value for the kinetic energy is called a Landau level. The ladder operators $a, a^\dagger$ change the kinetic energy and are therefore inter-Landau level ladder operators, while the ladder operators $b, b^\dagger$ are closed in a subspace of any given kinetic energy and are called intra-Landau level ladder operators. The ability to separate operators into a factor associated with the cyclotron orbit velocity and a factor associated with the cyclotron orbit center will be very valuable when it comes to discussing the fractional quantum Hall effect.

Everything we have discussed to this point is independent of the choice of gauge. In the remaining portion of this section we carry the discussion a bit further for the gauge choice which has been most valuable for fractional quantum Hall effect studies, the symmetric gauge,

$$\mathbf{A} = (-By/2, Bx/2, 0). \tag{17}$$

For this choice

$$b^\dagger = \left(z^*/2\ell - 2\ell\frac{\partial}{\partial z}\right)/\sqrt{2} \tag{18}$$

and

$$a^\dagger = i\left(z/2\ell - 2\ell\frac{\partial}{\partial z^*}\right)/\sqrt{2}. \tag{19}$$

Note that $(\partial/\partial z)^\dagger = -(\partial/\partial z^*)$. Using the algebra of the ladder operators it is easy to show that

$$\phi_{n,m} = \frac{(-i)^n G^{m,n}(iz^*/\ell)\exp(-\frac{1}{4}|z|^2/4\ell^2)}{\sqrt{2\pi\ell}}, \tag{20}$$

where

$$G^{m,n}(\alpha) = \sqrt{n!/m!}\left(-i\alpha/\sqrt{2}\right)^{m-n} L_n^{m-n}\left(|\alpha|^2/2\right) \tag{21}$$

and $L_n^k(x)$is a generalized Laguerre polynominal. The areal density which can be accommodated by each Landau level is

$$n_1 = \sum_m |\phi_{n,m}|^2 = (2\pi\ell^2)^{-1} = B/\Phi_0, \tag{22}$$

where $\Phi_0 = hc/e$ is the electron's magnetic flux quantum. Equation (22) follows from eqs. (20), (21) and the frequently useful identity,

$$\sum_k G^{m,k}(\alpha_1)G^{k,m}(\alpha_2) = \exp(-\alpha_1^*\alpha_2/2)G^{n,m}(\alpha_1+\alpha_2). \tag{23}$$

The density measured in units of n_1 is referred to as the Landau level filling factor, $\nu \equiv 2\pi\ell^2 n$. This quantity plays an essential role in both the integer and fractional quantum Hall effects.

2 Incompressibility Implies Quantization

In Fig. 1 we show the dependence of the chemical potential of a 2 D gas of non-interacting spinless electrons on density at fixed magnetic field and zero temperature. The chemical potential is defined by

$$\mu \equiv \frac{\partial F}{\partial N} \tag{24}$$

where F is the free energy and N is the number of electrons. At zero temperature μ is the change in the ground state energy when one electron is added to the system. For free-electrons it follows that μ jumps by $\hbar\omega_c$ whenever a Landau level is filled, i.e. whenever ν is an integer. The densities at which the chemical potential jumps occur are proportional to magnetic field since n_1 is proportional to magnetic field. We show below that the quantum Hall effect will occur whenever the chemical potential jumps at a magnetic-field-dependent density. At densities where the chemical potential jumps the electron system is incompressible, since the compressibility is inversely proportional to

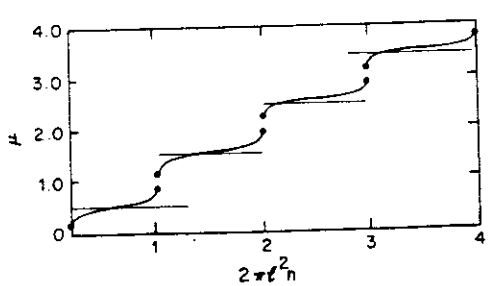

Figure 1. Density dependence of the chemical potential of a non-interacting 2 DES with and without disorder broadening of the Landau levels.

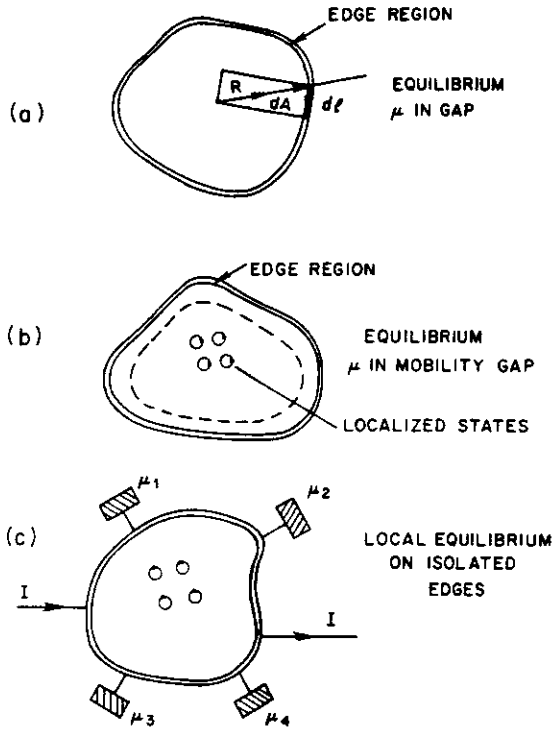

Figure 2. Derivation of the Streda formula. When the chemical potential has a gap in the thermodynamic limit the local properties of the system change with chemical potential only at the edge. The rate of change of edge current with chemical potential is related to the rate of change of electron density with magnetic field because the current operator is related to the derivative of the Hamiltonian with respect to magnetic field.

the density dependence of μ. It is useful to separate an explanation of the quantum Hall effects into: (1) an explanation of the fact that *incompressibility implies quantization* and (2) an explanation of the physical origin of the incompressibilities. For the integer ($\nu = i$) quantum Hall effect the chemical potential jumps are clearly due to the quantization of the kinetic energy of 2 D electrons by a magnetic field as discussed above. For interacting electrons and for electrons in an external periodic potential the electronic system can become incompressible at non-integral values of ν. In this section we will discuss the transport consequences of chemical jumps without reference to their origin. In the following sections we will turn our attention to aspects of the physics which are particular to the various possible origins of incompressibilities.

Consider a finite 2 D electron system with a confining potential of arbitrary shape as illustrated in Fig. 2. Assume that the chemical potential has a value which would be in a gap for a bulk system in the thermodynamic limit. We want to evaluate the

change in the magnetic moment of the system when the chemical potential of the system changes at constant magnetic field. The operator for the magnetic moment in the $\hat{z}$ direction is given by

$$\hat{M} = -\frac{\partial H}{\partial B} \tag{25}$$

where H is the Hamiltonian for electrons in an arbitrary external potential and with arbitrary electron-electron interactions.

$$H = \sum_i \frac{\pi_i^2}{2m} + V_{ext}(\mathbf{x}_i) + \sum_{i<j} V_{ee}(\mathbf{x}_i - \mathbf{x}_j) \tag{26}$$

Only the kinetic term in H depends on the magnetic field. Using eq. (8) in eqs. (26) and (25) we find that [3]

$$\hat{M} = \frac{1}{2c}\int d^2\mathbf{x}\, \mathbf{x} \times \hat{\mathbf{j}}(\mathbf{x}), \tag{27}$$

where $\hat{\mathbf{j}}(\mathbf{x})$ is the current density operator,

$$\hat{\mathbf{j}}(\mathbf{x}) = \frac{-e}{2m}\sum_i \left(\delta(\mathbf{x}-\mathbf{x}_i)\boldsymbol{\pi}_i + \boldsymbol{\pi}_i\delta(\mathbf{x}-\mathbf{x}_i)\right). \tag{28}$$

Because the chemical potential has been chosen so that it would lie in the gap for an infinite bulk system, when it changes the expectation value of the current density operator can change only at the system's edge. The change in the expectation value of the magnetic moment, δM, with a small change in chemical potential is

$$\delta\langle\hat{M}\rangle = \frac{1}{2c}\int d^2\mathbf{x}\, \mathbf{x} \times \delta\langle\hat{\mathbf{j}}(\mathbf{x})\rangle. \tag{29}$$

The integrand in eq. (29) is non-zero only in the edge region. Taking the position, $\mathbf{x}$, outside the integral when integrating across the edge (see Fig. 2), and using current conservation when integrating around the edge, eq. (29) simplifies to

$$\delta\langle\hat{M}\rangle = \frac{\delta I}{2c}\oint \mathbf{R} \times d\mathbf{l} = A\delta I_e/c, \tag{30}$$

where A is the area of the finite 2 D system and δI_e is the change in the current circulating around the edge. In the intermediate form of eq. (30) $\mathbf{R}$ is a position on the edge and $d\mathbf{l}$ points in the local direction of the change in edge current. Equation (30) provides us with a very general expression for the ratio of the change in the edge current to the change in the chemical potential.

$$\frac{\delta I_e}{\delta\mu} = \frac{c}{A}\left.\frac{\partial M}{\partial\mu}\right|_B = \frac{c}{A}\left.\frac{\partial N}{\partial B}\right|_\mu. \tag{31}$$

The second form for the right hand side of eq. (31) follows from the thermodynamic identity.

The range of filling factors for which only edge states are at the chemical potential vanishes in the thermodynamic limit since the number of edge states scale with the system perimeter while the Landau level degeneracy scales with the system area. An essential part of von Klitzing's discovery was that the Hall conductance is precisely

constant and the magnetoresistance extremely small over a finite range of filling factors. To explain the occurrence of these plateaus, which allow the quantum Hall effect to be visible in large samples, we must invoke the phenomenon of localization. The states in the tails of the disorder broadened Landau levels, will be localized near minima or maxima in the random potential. For our finite system we will say that the chemical potential lies in a mobility gap whenever the bulk states are sufficiently localized to allow an unambiguous separation between bulk and edge contributions to the change in the local current density in eq. (29) (see Fig. 2) . In this case the bulk contribution to the magnetization, M_B satisfies

$$\left.\frac{\partial M_B}{\partial\mu}\right|_B = \left.\frac{\partial N_B}{\partial B}\right|_\mu, \tag{32}$$

where N_B is the contribution to the density from the bulk region. (In Fig. 2 the dashed line encloses the bulk region.) It follows that in the mobility gap

$$\partial^2 I_e\, \partial\mu^2 = \frac{c}{A}\frac{\partial}{\partial B}\left(\left.\frac{\partial N}{\partial\mu}\right|_B - \left.\frac{\partial N_B}{\partial\mu}\right|_B\right) = 0. \tag{33}$$

The last equality in eq. (33) is true in the thermodynamic limit since the contribution to the rate of change in particle number with chemical potential from the edge region will become negligible. Equation (33) says that the rate of change of edge current with chemical potential is constant within a mobility gap.

The above discussion was for a finite 2D system in equilibrium. When current leads are attached to the system, the edge is divided into two branches (see Fig. 2). Electrons injected into the system from the leads cannot communicate with the localized states in the bulk. The net current carried through the system, I, can be accommodated by establishing separate local equilibria on the two edges. The magnetoresistance, determined by the voltage difference between two points along the same edge, vanishes. The Hall conductance, G_H, is the ratio of the net current (i.e. the difference in edge currents) to the electrochemical potential difference between the two edges and it can be evaluated using eq. (31). (Since we incorporate any electric fields which may be present in the Hamiltonian the electrochemical potential difference is just e times the chemical potential difference.) If the chemical potential lies within a mobility gap associated with an incompressibility pinned to filling factor ν_0, the derivative appearing in eq. (31) is known and the Hall conductance is given by

$$G_H = \frac{\delta I_e}{\delta\mu} = e\nu/h. \tag{34}$$

From the discussion above we see that whenever the Fermi level lies in a gap the Hall conductance will be given by

$$G_H = ec\left.\frac{\partial n(eF)}{\partial B}\right|_\mu. \tag{35}$$

As far as we are aware, this formula was first derived for non-interacting electrons, by Streda [4] using the Kubo formula. The argument given above which leads to this formula has a more general validity and explains its origin in a more physical way. It is closely related to other [5,6] simple arguments for the quantization of the Hall conductance. As was emphasized early in the development of the quantum Hall effect

theory by Prange [7] the mysterious aspect of the integer quantum Hall effect is the fact that the free gas formula for G_H is exactly obeyed between Landau levels, despite the fact that localization reduces the number of current carrying states in a Landau level. The Streda formula shows that the Hall conductance in a mobility gap depends only on the magnetic field dependence of the carrier density when the Fermi level lies within the corresponding gap. In the case of the integer quantum Hall effect, for example, the Hall conductance depends only on the density which can be accommodated by each Landau level, something which is clearly preserved as long as the disorder is not so strong that different disorder broadened Landau levels overlap. (In fact it is believed that the integer quantum Hall effect can occur even when Landau levels overlap.)

3 The Integer Quantum Hall Effect

The integer quantum Hall effect is the simplest of the three quantum Hall effects which we will discuss. In this case it is possible to provide a concrete example which is useful in understanding the general discussion of the preceding section. The most elementary model which exhibits all the essential features of the quantum Hall effect is that of a strip of non-interacting electrons in an external potential which depends only on the x coordinate. The single-particle Hamiltonian for this system is given by

$$h = t + V(x), \tag{36}$$

where t is the kinetic energy operator discussed in section 1 and $V(x)$ is the external potential. We imagine $V(x)$ goes to infinity in some unspecified way at large and small x so that the 2 DES is confined to a strip of finite width, but is otherwise arbitrary. In discussing this problem it is convenient to choose a gauge, the Landau gauge, in which the vector potential is independent of the y coordinate,

$$\mathbf{A} = (0, Bx, o). \tag{37}$$

This allows us to choose eigenstates of h which have a plane-wave dependence on the y coordinate,

$$\psi(x,y) = \phi/x)\exp(ik_y y)/\sqrt{(L_y)}. \tag{38}$$

Here L_y is the length of the system in the $\hat{y}$ direction and, adopting periodic boundary conditions in this direction, the allowed values of k_y are separated by $2\pi/L_y$. For reasons which will become apparent we choose to label states by $X \equiv -\ell^2 k_y$ rather than by k_y. Substituting eq. (38) into eq. (36), we see that

$$h_x(X)\phi_{n,X}(x) = \varepsilon_{n,X}\phi_{n,X}(x), \tag{39}$$

where

$$h_x(X) = p_x^2/2m + \frac{1}{2}m\omega_c^2(x - X)^2 + V(x). \tag{40}$$

$\phi_{n,X}(x)$ provides the quantum description of a cyclotron orbit which is centered near $x = X$ and is free to drift along the $\hat{y}$ direction in response to the external potential. To calculate the Hall conductance in this circumstance we need only note that the total current flow is given by

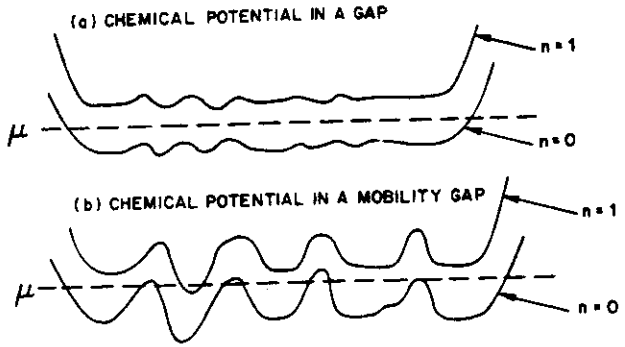

Figure 3. Derivation of the Streda formula for the simple case of non-interacting electrons in a disorder potential which depends on only one coordinate.

$$I_y = -e\sum_{n,X}\langle n, X|\pi_y|n, X\rangle/(mL_y) = -e\sum_{n,X}\langle n, X\left|\frac{\partial h_x}{\partial k_y}\right|n, X\rangle/(\hbar L_y). \tag{41}$$

The second form of eq. (41) is obtained from the first by comparing with eq. (40). Since adjacent values of X allowed by periodic boundary conditions are separated by $2\pi\ell^2/L_y$. Equation (41) leads to

$$I_y = \frac{e}{h}\sum_n \int dX \frac{\partial \varepsilon_n(X)}{\partial X}, \tag{42}$$

where we have invoked the Hellman-Feynman theorem to evaluate the expectation value. In eqs. (41) and (42) the sums and integrals are over occupied states. When the chemical potential lies in a gap between Landau levels the lower and upper limits of the integration over X in eq. (42) are at the left and right edge of the sample and in evaluating this integral we are led to introduce the edges explicitly, although we shall see that nothing depends on the specific model used to describe the edges (see Fig. 3). When an integer number, i, of Landau levels are full we may assume that, except at the edges, all states associated with that Landau level index are occupied. At either edge, we can assume that there is a well defined local chemical potential (equal to the energy of the highest occupied level at that edge). This chemical potential is what is probed when a voltage measurement is performed on that edge. Thus eq. (42) implies that

$$I_y = eN(\mu_R - \mu_L)/h = -ie^2V/h, \tag{43}$$

where V is the voltage difference from the left hand side to the right hand side of the sample. When the chemical potential lies in a mobility gap (see Fig. 3), it is clear from eq. (42) that the rate of change of I_e with changes in the local chemical potential at the edge is e/h for each Landau level whose edge states cross the Fermi level, even if the Landau level is not completely full or completely empty in the bulk. The quantized Hall current is carried by establishing local equilibria on the edges as discussed in the preceding section.

As implicitly assumed in the above discussion, the confining potential which defines the edges of the system causes $\varepsilon_n(X)$ to increase near the edges of the sample. The way in which this occurs depends on the confinement potential: two simple models for which

exact results can be obtained are the hard wall [8] case ($V(x) = \infty$ outside the sample) and the case of a parabolic confinement potential. From eq. (42) we see that this leads to large current densities flowing around the edge of a 2 D system. For a system with edges the increase in Landau level degeneracy with magnetic field is accomplished by having edge states adiabatically move toward the bulk with increasing magnetic field. The edge states energies decrease rapidly with magnetic field giving a paramagnetic contribution to the orbital magnetic moment [9,10] of the system which is comparable in size to the diamagnetic contribution from the bulk region even in the thermodynamic limit. The rate of change of the number of states in a Landau level with magnetic field is thus determined by the rate at which edge state energies change with magnetic field and, by eq. (41), to the current carried by the edge states within a given range of energies. This is a particular example of the more general argument given in the preceding section. The change in edge current for a given change in the local chemical potential is fixed by the rate of change of the number of occupied states at fixed chemical potential because the current operator is proportional to the derivative of the Hamiltonian with respect to the vector potential, and hence to the derivative of the Hamiltonian with respect to the magnetic field. The number of states below a gap cannot be altered by weak disorder.

For the model discussed above where the disorder potential is independent of the y coordinate, k_y is a good quantum number. There is a close analogy between the passage from eq. (41) to eq. (42) and the arguments used to justify Landauer type conductance formulas [11,12]. Indeed, Tthe Büttiker [12] formula for multi-probe conductance measurements is readily applied to the strong magnetic field regime [12,14]. This description of strong field transport allows the integer quantum Hall effect to be seen as a particularly simple case of the tremendous variety of quantum transport phenomena which can be understood in terms of the Büttiker formula [12,13]. The passage from weak fields to strong fields is a continuous one; the channel number evolves continuously from being the number of occupied transverse momenta to being the number of occupied Landau levels as magnetic depopulation of subbands occurs in a wire of finite width. From this point of view the unique aspect of the strong field regime is the spatial separation of left-going and right-going states and it is this which leads to the remarkable accurate quantization of transport coefficients. The Büttiker [12] conductance formula can also be used to explain experiments in which non-equilibrium distributions are created at the edges [15]. The understanding of these experiments depends on having a microscopic description of the electronic structure at the edge of the system. For the quantum Hall effect, on the other hand, a local equilibrium always exists at the edge and as we have emphasized the quantization follows solely from the magnetic-field dependence of the density at which an incompressibility occurs.

From our discussion of the quantum Hall effect we see that the Hall conductance should be independent of the Landau level filling factor as long as the chemical potential lies in a region where the localization length, ξ, is small compared with the system size. Since localization is dependent on the maintenance of phase-coherence this condition should be replaced at finite temperatures by the requirement that the Hall conductance will be quantized when ξ is smaller than the phase coherence length. The fact that there must be some extended states within each Landau level is required by the fact that the Landau level degeneracy depends on magnetic field.) The phase coherence length is the distance an electron diffuses between inelastic scattering events. In the strong magnetic field limit we are interested in the diffusive motion of the cyclotron orbit center since the kinetic energy of each cyclotron orbit is quantized. The diffusion

constant may be obtained from the Einstein relationship using the fact [16] that, in the absence of localization effects, the conductivity is $\sim e^2/h$. The result is given below and may be interpreted as follows. Classically a cyclotron-orbit center moves ballistically along equipotential contours with velocity

$$v = cE_{\rm loc}/B \tag{44}$$

in a direction perpendicular to the local electric field, $E_{\rm loc}$. The typical time between scattering events, τ, can be related to the Landau level width Γ by

$$\tau \sim \hbar/\Gamma. \tag{45}$$

Thus the typical distance moved between scattering events is

$$\sim cE_{\rm loc}\hbar/(B\Gamma) \sim \ell, \tag{46}$$

where we have noted that typical values of $E_{\rm loc}$ are about $\Gamma/(e\ell)$. Assuming a random distribution of local electric field directions the cyclotron orbit center moves diffusively with a diffusion constant given by

$$D \sim \ell^2/\tau \sim \Gamma\ell^2/\hbar, \tag{47}$$

and the phase coherence length is

$$L_\phi \sim \ell(\tau_{\rm inel}/\tau)^{1/2}. \tag{48}$$

(Note th . in contrast to the zero field case, L_ϕ decreases with τ.) As the temperature goes to zero, the inelastic scatterin rate vanishes as T^p and L_ϕ diverges as $T^{-p/2}$. Early experiments [17] on the integer quantum Hall effect showed that the width of the Hall plateaus in disorder systems approached a filling factor change of one at low temperatures, suggesting [18,19] that extended states exist at only one energy, E_c, within each disorder broadened Landau level. Recent numerical calculations [20] find that for E near E_c

$$\xi \sim (E - E_c)^{-\nu}, \tag{49}$$

where the critical exponent $\nu = 2.34 \pm 0.04$. This result is in agreement with the analytic result obtained by Mil'nikov and Sokolov [21], who included the effects of quantum mechanical tunneling between orbits along percolating equipotentials [22], and also in agreement with experiment [23,24] if the inelastic scattering rate is assumed to vanish as $T^2 (p = 2)$ at low temperatures.

4 The Quantum Hall Effect in a Periodic Potential

The integer quantum Hall effect occurs because of gaps which occur at integer Landau level filling factors, rather than at constant density. (The reader should realize that there is something unusual about a gap which occurs at a magnetic-field dependent density.) In this section we discuss a qualitatively different example of the quantum Hall effect in which the gap occurs neither at constant density nor at constant Landau level filling factor. The gaps in this case are those which occur when the 2 DEG is placed in both a constant magnetic field and a periodic potential. The effect of the periodic potential is

both to broaden a Landau level and to open up an int icate array of gaps within a each Landau level.

The main features of the quantum Hall effect in a periodic potential can be understood on the basis of qualitative arguments. For a 2D system in a strong magnetic field and a periodic potential, there are two length scales. One is the magnetic length, ℓ, introduced earlier which was important in the integer quantum Hall effect. The second length scale is the period of the external potential. As we learned above, we can understand the quantum Hall effect if we understand the magnetic-field dependence of the densities at which gaps occur in the spectrum. In the case of a free gas, gaps occur at integer filling factors. The effect of the periodic potential is to allow other gaps to occur.

It is clear from a simple scaling argument that the filling factor at which a gap occurs can depend only on the ratio of the two length scales in the problem. This ratio is conveniently parameterized in terms of the quantity

$$\alpha = 2\pi\ell^2/A_0, \tag{50}$$

where A_0 is the unit cell area of the periodic potential. To determine the spectrum of the system we use translational periodicity, applying periodic boundary conditions to a finite system whose volume can ultimately be taken to infinity. It will not be necessary for our purposes to describe in detail how this is done in the presence of a magnetic field. We merely note that it is possible to apply periodic boundary conditions to the system, only if the number of units cells of the periodic potential in the system is an integer and if $A/(2\pi\ell^2) \equiv N_L$, is an integer. (Here A is the area of the system.) The first condition is obvious and is required at zero magnetic field as well. The second condition is due to the magnetic field , and is required even in the absence of a periodic potential; in that case it requires N_L, the number of states per Landau level, to be an integer in a finite system. The main features of the intricate gap structure can be understand from these observation and the scaling argument.

It follows from the above that in the thermodynamic limit,

$$\sigma \equiv \frac{\partial n(\varepsilon_F)}{\partial B}/(2\pi\ell^2/B) \tag{51}$$

and

$$s \equiv -A_0^2\frac{\partial n(\varepsilon_F)}{\partial A_0} \tag{52}$$

are integers when ε_F lies in a gap. (ε_F is the Fermi energy which is the zero temperature limit of the chemical potential.) σ is the change in the number of states below the gap when the magnetic field changes by an amount sufficient to change N_L by one, while s is the change in the number of states below the gap when A_0 is changed by an amount sufficient to increase the number of unit cells in the system by one. Note that the Streda formula implies that

$$G_H = (e^2/h)\sigma. \tag{53}$$

The change in the filling factor at which a gap occurs when A_0 and the magnetic field change c an be expressed in terms of σ and s. Using eqs. (50) and (51), in $\nu = 2\pi\ell^2 n$ gives,

$$\delta\nu = [\sigma - \nu]\frac{\delta B}{B} - \alpha s\frac{\delta A_0}{A_0}. \tag{54}$$

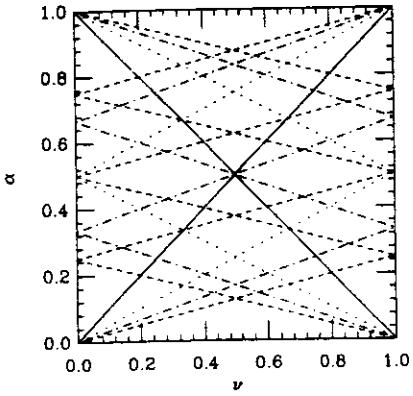

Figure 4. Dependence of the filling factor at which a gap occurs on the number of magnetic flux quanta per unit cell of an external periodic potential. All gaps but one will go to zero at the points of intersection. The Hall conductivity when the chemical potential lies in a gap is e^2/h times the value of ν at which the line intersects $\alpha = 0$.

Similarly the change in α for a change in B and A_0 is

$$\delta\alpha = -\alpha(\delta B/B + \delta A_0/A_0). \tag{55}$$

It follows that the change in the filling factor at which a gap occurs at constant α is given by

$$\delta\nu = [\sigma - \nu + \alpha s]\delta B/B. \tag{56}$$

However, it follows from the scaling argument that the filling factor at which a gap occurs cannot change if α does not change and hence that the filling factor at which a gap occurs is related to σ and s by

$$\nu = \sigma + \alpha s. \tag{57}$$

Equation (57) is a deceptively simple equation from which many conclusions can be drawn. Consider the situation when α has an arbitrary rational value, $\alpha = q/p$. Then any ν at which a gap occurs, using the fact that σ and s in eq. (57) are integers, must also occur at a rational filling factor with the same denominator, $\nu = t/p$. Thus each Landau level is split into p subbands when $\alpha = q/p$. Each gap in the spectrum characterized by two integer quantum numbers σ and s and follows the straight line in ν, α space dictated by eq. (57). A countably infinite number of such lines intersect at any rational value of ν and α, characterized by the integers t, q and p, and their values of σ and s are given by

$$(s, \sigma) = (s_0 + kp, \sigma_0 - kq), \tag{58}$$

where k is an integer (see Fig. 4). The allowed values of s are separated by p and the allowed values of σ are separated by$q$.

All the above conclusions can be drawn without specifying the periodic potential, which determines which of the possible values of σ is selected at a given value of α and ν. The case of a weak periodic potential of the form

$$V(x) = V_0(\cos(2\pi \quad /a) + \cos(2\pi y/a)) \tag{59}$$

is equivalent to the model studied numerically ba Hofstader [25] and the complicated recursive structure he discovered in the spectrum, *The Hofstader Butterfly*, is equivalent to the statement [26,27,28] that the correct value of s at any gap for this model is the smallest possible one [29], i.e. the one obeying $|s| \leq p$. This is not a general result ; in particular the gap structure is quite different in the case of a hexagonal external potential [30,31]. From our discussion it is clear that the quantum Hall effect should provide a sensitive probe of the exotic gap structure in a periodic potential and a magnetic field. The prospects of observing exotic transport effects in artificially structured [32,33] 2 D system in the future are promising.

It is easy to verify that in the case of an external periodic potential the values of the integral quantum number related to the Hall conductance, σ are very different from ν and, in particular can have large and even negative values within any Landau level. Thus when a periodic potential is present the quantum Hall effect can occur at fractional filling factors but the filling factor dependence of the gap is always such that the Hall conductance is an integral multiple of e^2/h. In the next section we discuss gaps which originate from electron-electron interactions and occur at fixed fractional Landau level fillings at any field, and hence yield a fractional Hall conductance.

We have followed Streda [34] in discussing the quantum Hall effect periodic potential in terms of the Streda formula. Thouless et al. [26] obtained equivalent results starting directly from the Kubo formula. These authors found that the Hall conductivity is related to the change of the phase of the Bloch wavefunction on moving around the perimeter of the Brillouin zone, which must be an integer (σ) times 2π. The Hall conductance quantum number is thus related to a topological invariant of a mapping of the Brillouin zone by the Bloch wavefunction. It has subsequently been realized that the Hall conductance may be regarded as a topological invariant in more general circumstances [35]. The fact that the Hall conductance in a periodic potential turns out to always be an integral multiple of e^2/h despite gaps at fractional filling is then seen to be a very general feature of the quantum Hall effect. Fractional values of quantized Hall conductance can only be produced by electron-electron interactions and, as we will see, they are accompanied by a fractionalization of the electron charge.

5 The Fractional Quantum Hall Effect

We have learned that the quantum Hall effect will occur for a 2 D system in a magnetic field whenever there is a gap, i.e. whenever the chemical potential has a discontinuity as a function of density at fixed magnetic field. The integer quantum Hall effect is caused by the quantization of the allowed kinetic energy values for an electron moving in a plane perpendicular to a steady magnetic field. In the case of an electron system moving in a periodic potential we learned in the last section that gaps open up within Landau levels. Even in this case, however the Hall conductance is an integer multiple of e^2/h. The fractional quantum Hall effect [28,36,37,38] occurs near a certain set of fractional values of ν and has plateaus in the Hall conductance equal to $\nu e^2/h$. We saw in the last section that a fractionally quantized Hall conductance is not possible for

non-interacting electrons, even in an external potential which produces gaps within each Landau level. Our object here will be to explain how electron-electron interactions can give rise to chemical potential discontinuities which are pinned to *fractional* Landau level filling factors.

We will restrict our discussion to the extreme quantum limit in which the Landau level degeneracy is large enough that all electrons can be accommodated within the lowest Landau level and the Landau level separation is large enough that quantum mechanical Landau level mixing by disorder or by interactions can be ignored. Since each electron present has the same kinetic energy, only electron-electron interactions can produce the gap which we require to explain the quantum Hall effect. We will see that the kinetic energy quantization indirectly can produce gaps at fractional filling factors, because the restriction to a single Landau level puts powerful filling-factor-dependent restrictions on the properties of the many-electron wavefunctions. According to eq. (20), the single-particle wavefunctions in the lowest Landau level in the symmetric gauge are

$$\phi_m(\mathbf{x}) = \frac{z^{*m}\exp(-|z|^2/4)}{\sqrt{2^{m+1}m!\pi}} \tag{60}$$

where $z = x + iy$, we have dropped the Landau level index on the wavefunction and we have adopted the magnetic length, ℓ, as the unit of length for the discussion of the fractional quantum Hall effect. (In what follows we will redefine $z \equiv x - iy$.) Note that these wavefunctions describe electrons located within one magnetic length of a circle centered on the origin and enclosing an area

$$\pi\langle m|, \ |^2|m\rangle = 2\pi\ell^2(m+1)\,. \tag{61}$$

Any many-electron wavefunction formed entirely within the lowest Landau level must be a sum of products of one-electron orbitals for each coordinate which are of the form given by eq. (61). It follows that the many electron wavefunction must take the form

$$\Psi[z] = \left(\prod_{k=1}^{N}\exp\left(-|z_k|^2/4\right)\right)P[z] \tag{62}$$

where $P[z]$ is a polynomial, and in the $N \to \infty$ limit an analytic function, in each of the z_k's [39]. N is the number of electrons. At zero temperature the chemical potential equals the change in the ground state energy when one electron is added to the system. Since any state of the form of eq. (62) has the same kinetic energy, $T = N\hbar\omega_c/2$, the discontinuity in the chemical potential responsible for the fractional quantum Hall effect must come from electron-electron interactions. We know that the electron-electron interaction is increasingly repulsive at short distances, so the ground state will be determined, qualitatively, by minimizing the probability of electrons being close together. For an isotropic system of identical particles the pair correlation function, which measures the probability of two electrons being at a certain separation compared to the same probability in an uncorrelated system of the same density, is given by

$$g(|\mathbf{x}_1 - \mathbf{x}_2|) = n^{-2}N(N-1)\prod_{k=3}^{N}\int d^2z_k|\Psi[z]|^2\,. \tag{63}$$

Substituting eq. (62) into eq. (63) gives for the case of interest,

$$g(|z_1 - z_2|) \sim \exp\left(-(|z_1|^2 + |z_2|^2)/2\right) \prod_{k=3}^{N} \int d^2 z_k \exp\left(-|z_k|^2\right) P^*[z] P[z] \,. \tag{64}$$

It will prove convenient to replace the coordinates z_1 and z_2 by a relative coordinate, $\zeta \equiv z_2 - z_1$, and a mean coordinate, $\bar{z} \equiv (z_2 + z_1)/2$. As noted in eqs. (63) and (64) the dependence of g on the mean coordinate is expected to be removed by the integrations over $d^2 z_k$ for an isotropic fluid state, such as is generally expected in an electron gas [40]. Since $P[z]$ is an analytic function of z_1 and z_2, it will be an analytic function of ζ, and we can expand

$$P[z] = \sum_p \zeta^{2p+1} F_p(\bar{z}, z_3, \ldots, z_N) \,. \tag{65}$$

Note that because of the anti-symmetry requirement on the many-electron wavefunction for electrons only odd powers of ζ appear in eq. (65). Substituting eq. (65) into eq. (64) gives

$$g(|\zeta|) = \exp\left(-|\zeta|^2/4\right) \sum_{p,p'} |\zeta|^{*(2p'+1)} \zeta^{2p+1} f_{p',p} \tag{66}$$

where $f_{p',p}$ is given by an integral over the other coordinates. For an isotropic system g can depend only on the magnitude and not on the orientation of the separation ζ. It follows that $f_{p',p}$ must be zero when p is not equal to p' and hence that

$$g(|\zeta|) = \exp(-|\zeta|^2/4) \sum_p |\zeta|^{2(2p+1)} f_{p,p} \,. \tag{67}$$

Equation (67) tells us that for any isotropic state formed in the lowest Landau level of a 2 DEG, the pair correlation function must vanish as an odd power of $|\zeta|^2$ at small $|\zeta|$. The fractional quantum Hall effect is due to a connection between the small-separation behavior of g and the Landau level filling factor, which we now establish. Assume that $g(|\zeta|)$ varies as $|\zeta|^{2m}$ at small $|\zeta|$. It follows from the argument leading to eq. (67) that $P[z]$ has $(z_1 - z_2)^m$ as a factor and hence, since all particles are identical, that $P[z]$ has

$$P_m[z] \equiv \prod_{i<j} (z_i - z_j)^m \tag{68}$$

as a factor. For a wavefunction representing a large but finite number of electrons, N, the maximum power to which z_1 (or any other coordinate) appears in $P[z]$ is therefore

$$M_1 \geq m(N-1) \tag{69}$$

and hence the area occupied by the wavefunction, according to eq. (61), is

$$A \geq 2\pi\ell^2 m(N-1). \tag{70}$$

It follows from eq. (70) that in the thermodynamic limit $A/2\pi\ell^2 N = \nu^{-1} \geq m$. Thus as the electron density is increased at constant magnetic field so that the filling factor crosses $\nu = 1/m$ we go from a regime where it is possible to form states with $g(|\zeta|) \sim |\zeta|^{2m}$ to a regime where $g(|\zeta|)$ vanishes only a $|\zeta|^{2(m-1)}$. This qualitative change in the ability of electrons to avoid each other causes a jump in the chemical potential when the filling factor crosses $1/m$ and, invoking the Streda formula, also causes the Hall conductance to be quantized at $e^2/(mh)$ at filling factor $1/m$.

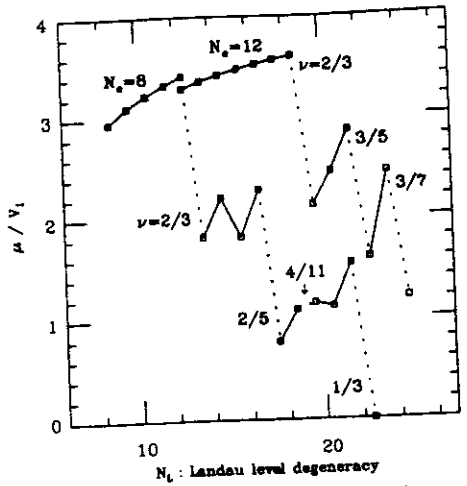

Figure 5. Finite-size exact-diagonalization estimate of the chemical potential's dependence on the number of states in a Landau level for eight and twelve electrons on the surface of a sphere.

At $\nu = 1/m$ the ground state wavefunction corresponds to the analytic function $P_m[z]$ (eq. (68)). It was first introduced by Laughlin [41]. When the electrons are in this state we see by inspection that any pair of electrons are always in a state of relative angular momentum at least m. As first emphasized by Haldane [42], the electron-electron interaction within the lowest Landau level is completely specified by a set of pseudopotential parameters, $\{V_m\}$, which give the interaction strength in each relative angular momentum channel. (Only odd values of m are relevant if the electrons are completely spin polarized.) An extremely attractive picture of the fractional quantum Hall effect can be given in terms of these pseudopotential parameters. For filling factors less than $1/m$, where we have shown that it is possible to form states in which no pair of electrons has a relative angular momentum less than m, there are zero-energy eigenstates in models which have repulsion only in channels of relative angular momentum less than m. At filling factor $1/m$ our discussion shows that there is only one state, the Laughlin state, which has zero energy for such a model. Thus associated with the chemical potential jump which occurs as the filling factor crosses $1/m$, there is an excitation gap which occurs when the filling factor equals $1/m$. In Fig. 5 we show the chemical potential as a function of Landau level filling factor for a system in which electrons repel each other only when they have relative angular momentum one as estimated from finite-system exact diagonalization studies [43].

The pseudopotential parameters for realistic electron-electron interaction decrease monotonically with m. Exact diagonalization studies for small systems, pioneered by Yoshioka et al. [44], have provided convincing evidence that the realistic interactions are

sufficiently close to the simple pseudopotential models lescribed above, that the ground states maintain a nearly quantitative similarity.

To explain the plateaus which occur in connection with the fractional quantum Hall effect we must also consider states which occur at filling factors close to $1/m$. The expectation is that some type of localization behavior must occur which is analogous to the localization of solutions to the one-body Hamiltonian responsible for the plateaus in the integer quantum Hall effect. For example, consider the case in which the area of the system is larger than that at filling factor $1/m$ by $2\pi\ell^2$ and the disorder potential has a repulsive peak at some point, which we take to be the origin. Laughlin [41] suggested that the ground state in this circumstance could be obtained from the ground state at filling factor $1/m$ by increasing the single-particle angular momentum labels of all the occupied states by one. This would give a state in which the $m = 0$ single-particle state is never occupied. It is easy to show that away from the origin this state has the same correlations as the Laughlin fluid state, while near the origin there is a decrease in the charge density with a total deficiency of charge equal to e/m. One can form similar states in which there is an excess of charge equal to e/m at some point in space. The picture of the situation on the plateaus near $\nu = 1/m$, is therefore as follows. The ground state may be thought of as consisting of a Laughlin fluid, plus some number of quasi-holes or quasi-particles localized [45] near maxima or minima of the disorder potential. The novel feature in the fractional case is that these quasi-particles have *fractional* charge. The fractional charge also leads to the possibility that the quasi particles may be most conveniently described as having neither Fermi or Bose statistics, but rather an intermediate *fractional* statistics [46].

To avoid giving the reader the incorrect impression that everything is now explained we must emphasize that the fractional quantum Hall effect also occurs at filling factors not equal to $1/m$ (see Fig. 5). Some of these fractions can be explained by invoking the particle-hole symmetry which exists in the strong magnetic field limit. For example, a fractional quantum Hall effect occurs at filling factor 2/3, due to the formation of the 1/3 Laughlin state in the holes of a full Landau level. This notion can be generalized so that, for example the 2/5 fractional effect can be explained as being due to the formation of a Laughlin fluid in the *fractionally charged* quasi-particles of the $\nu = 1/3$ Laughlin fluid. It should be recognized, however, that the understanding of these so-called hierarchy states is not yet nearly as complete as our understanding of the Laughlin states.

For a non-interacting metal in the absence of a magnetic field, the excited states which are coupled to the ground state by the one-particle operators relevant to most experiments are particle-hole excitations in which an electron in one of the single particle states inside the Fermi surface is promoted to a single-particle state outside the Fermi surface. For an interacting metal the particle-hole excitations still exist and their excitation energies are, for the most part, only slightly changed. However the excitation energy of one particular linear combination of particle-hole excitations is changed qualitatively, especially at long wavelengths. This linear combination is formed by operating on the ground state with the density operator,

$$\hat{\rho}(\mathbf{k}) = \sum_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i) \,. \tag{71}$$

This state is the plasmon state in which the electrons are collectively oscillating with wavevector k. For a 2 D electron system in a magnetic field,

$$\hat{\rho}(\mathbf{k}) = \sum_i A_i(k)B_i(k) \tag{72}$$

where $k = k_x + ik_y$,

$$A_i(k) = \exp\left(-k^* a_i^\dagger/\sqrt{2}\right)\exp\left(ka_i/\sqrt{2}\right) \,, \tag{73}$$

and

$$B_i(k) = \exp\left(-ik^* b_i/\sqrt{2}\right)\exp\left(-ikb_i^\dagger/\sqrt{2}\right) \,. \tag{74}$$

In eq. (72) we have separated the density operator into a factor involving inter-Landau-level transitions ($A_i(k)$) and a part involving intra-Landau-level transitions ($B_i(k)$). Equation (72) is obtained from eq. (71) by using eqs. (18), (19) and the fact that the inter-Landau-level and intra-Landau-level ladder operators commute. We see that in a magnetic field, the density operators creates particle-hole excitations which involve a change of Landau level index as well as particle-hole excitations within a Landau level. In a strong magnetic field the Landau level separation, $\hbar\omega_c$, is a large energy and the electron-electron interaction will be important in coupling only particle-hole excitations with the same change of Landau level index. Of particular importance to the fractional quantum Hall effect is the coupling of intra-Landau-level excitations since they all have zero excitation energy, in the absence of interactions. We may expect that a collective mode is formed, at least at long wave lengths, from the intra-Landau level particle-hole excitations. The excitation energy for collective modes may be estimated by a generalization of the theory used by Feynman [47] to estimate the collective phonon-roton excitation spectrum in superfluid ^{4}He. We assume that the collective mode state is given approximately by the projection of $\hat{\rho}(k)|\Psi_0\rangle$ onto the lowest Landau level, i.e. by the part of the density wave excitation which does not produce an increase in kinetic energy;

$$|\Psi(k)\rangle = \frac{\bar{\rho}(k)|\Psi_0\rangle}{\sqrt{\langle\Psi_0|\,|\Psi_0\rangle}} \tag{75}$$

where the projected density wave operator $\bar{\rho}(k)$ is given by

$$\bar{\rho}(k) = \sum_i B_i(k) \,. \tag{76}$$

and $|\Psi_0\rangle$ is the ground state wavefunction. The excitation energies of these magnetoroton [48,49] states are given by

$$E_{MR}(k) = \langle\Psi(k)|\bar{V}|\Psi(k)\rangle - E_0 \tag{77}$$

where E_0 is the ground state energy and $\bar{V}$ is the projection of the electron-electron interaction onto the lowest Landau level,

$$\bar{V} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} V(q)\bar{\rho}(-q)\bar{\rho}(q) \,. \tag{78}$$

In eq. (78), $V(q)$ is the Fourier transform of the electron-electron interaction. $E_{MR}(k)$ is readily evaluated by expressing the energy difference in eq. (77) in terms of commutators, expressing $\bar{V}$ in terms of projected density operators using eq. (78), and using the following expression for the commutator of projected density operators:

$$[\bar{\rho}(k_1), \bar{\rho}(k_2)] = (\exp(k_1^* k_2/2) - \exp(k_1 k_2^*/2))\, \bar{\rho}(k_1 + k_2) \tag{79}$$

Equation (79) follows from the commutation relations of the intra-Landau-level ladder operators. The result for the excitation energies is

$$E_{MR}(k) = \int \frac{d^2q}{(2\pi)^2} V(q) \exp(-|q|^2/2) \left(\exp((q^* k - k^* q)/2) - 1\right) \left(\bar{s}(q) - \bar{s}(k+q)\right) / \bar{s}(k) \tag{80}$$

where $\bar{s}(k) = \exp(|k|^2/2) \bar{s}(k)$ and

$$\bar{s}(k) = s(k) - \left(1 - \exp(|k|^2/2)\right) , \tag{81}$$

the projected static structure factor, is the difference between the static structure factor of the partly filled state and the structure factor when the Landau level is filled. Small system calculations have verified that collective density modes do indeed exist within a partly filled Landau level *when the ground state is incompressible* and that their dispersion is given extremely accurately by eq. (80). As expected $\lim_{k\to 0} E_{MR}(k)$ is finite and $E_{MR}(k)$ has a minimum where $\bar{s}(k)$ has a maximum, analogous to the roton minimum in Helium.

In two dimensions it is possible for particles to have statistics intermediate between those of Fermi and Bose particles [50]. In fact the Fermi electrons in a two dimensional electron gas can be *equivalently* considered to be bosons provided that a contribution is added to the Hamiltonian which corresponds to an odd number of magnetic flux quanta piercing the system at the position of each particle. Girvin and MacDonald [51] showed that when the particles at $\nu = 1/m$ are considered to be bosons they show quasi-long range order at zero temperature if and only if the ground state is incompressible. This property has been verified by numerical calculations [52] and has been exploited to develop Landau-Ginzburg theories [53] for the fractional quantum Hal. states. The incompressible states of the fractional quantum Hall effect thus exhibit a kind of Bose condensation in which the objects condensing consist of an electron and its associated flux quanta. Very recently there has been a great deal of interest in the possibility that the ground state of high-temperature superconductors may be related to fractional Hall incompressible states [54]. The superconducting property of these states would then emerge as a consequence of the long-range order discussed above. It may turn out that it is this aspect of the surprising incompressible states which are responsible for the fractional Hall effect which has the greatest impact on other sub-fields of physics.

6 Conclusions

In these notes we have discussed only the simplest version of the fractional quantum Hall effect where all the electrons are in the lowest orbital Landau level of the conduction band of a semiconductor. The theory has been generalized to higher orbital Landau levels [55] and to electrons in the valence band [56,57]. The most intricate and interesting

complications, however, come when more than one Landau level has its energy close enough to the chemical potential to be relevant. This happens, for example, when the Zeeman energy is small enough that the spin degree of freedom becomes important. Such a *two component* system can also be created by fabricating a system in which two 2 D electron layers are in close proximity. The richness of the fractional quantum Hall effect in multi-component systems has been anticipated theoretically [58,59]. Continuing technical advances in material fabrication techniques have recently made it possible to study such systems experimentally [60,61]. This area of study is likely to bring new surprises in the future.

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Introduction To The (Fractional) Quantum Hall Effect

Allan MacDonald
Indiana University

OUTLINE

Lecture 1 - General Introduction

2DEG - Incompressibility ⇒ Quantization - 1-body Electron in Magnetic Field - Localization in a Strong Magnetic Field - Periodic Potentials

Lecture 2 - Basic Physics of Fractional Hall Effect

2-body problem - Haldane pseudopotentials - Laughlin wavefunctions - Fractional charges - - Plasma analogy - Anyons & Boson ODLRO & Frac. Stat. - Quasiparticles

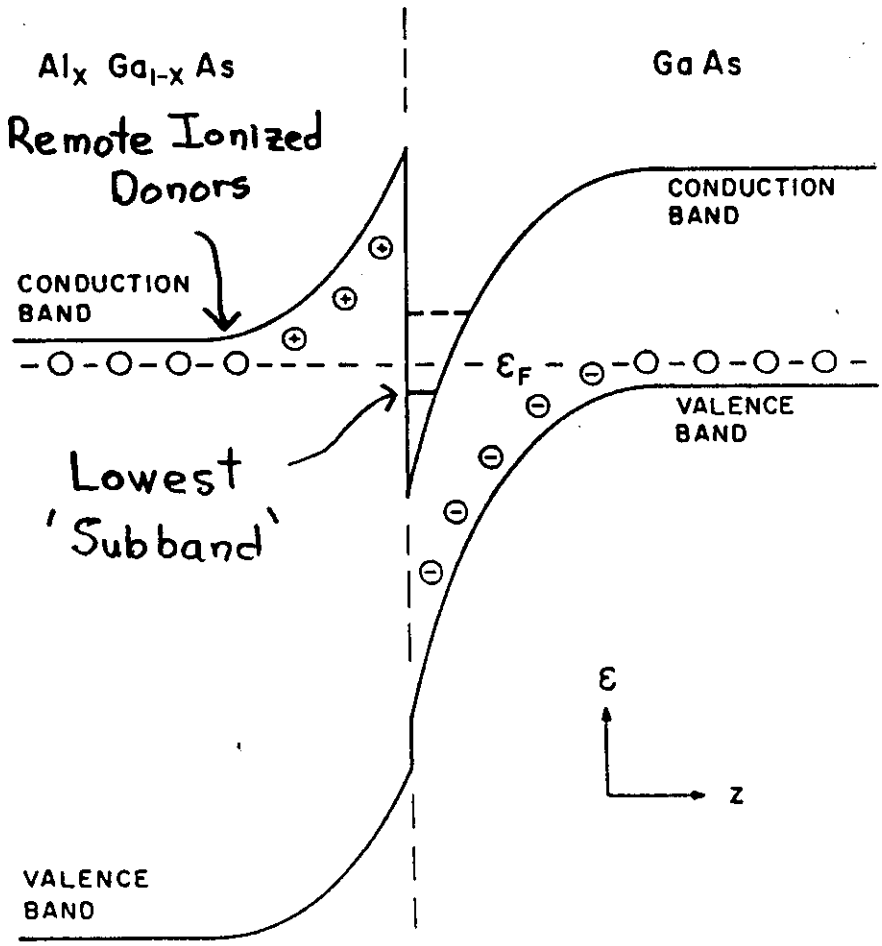
Lecture 3 - Collective Modes & Hierarchy States

projected structure factors & moment sum rules - SMA & magnetorotons - magnetoplasmons - - 'conventional' hierarchy - CS hierarchy - 'composite-fermion' hierarchy - compressible states

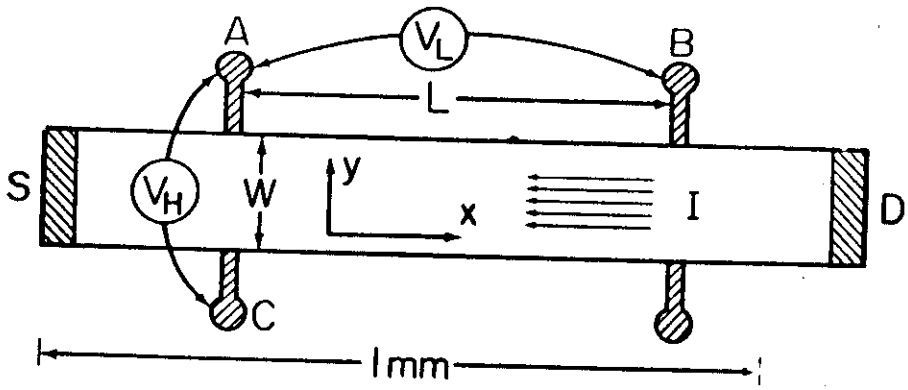
Lecture 4 - Wigner Crystal State & - 2, 3 or Many Component Systems

WC ground state - reentrant crystallization - elementary excitations - spin degree of freedom - layer degree of freedom - spontaneous interlayer coherence - commensurate-incommensurate transition - hidden symmetries in electron-hole systems - excitonic superfluidity

Two-Dimensional Electron Gas



Hall Bar



$$R = V_L / I$$

$$R_H = V_H / I$$

NAIVE (DRUDE) THEORY OF MAGNETOTRANSPORT

- $\vec{v}_D = \dfrac{\vec{F}\tau}{m^*}$

$$= -\frac{e\tau}{m^*}\left[\vec{E} + \frac{\vec{v}_D}{c} B \times \hat{z}\right]$$

- $\vec{J} = -ne\vec{v}_D$

$$\Rightarrow$$

$$\vec{E} = \underbrace{\frac{m^*}{ne^2\tau}}\vec{J} + \underbrace{\frac{B}{nec}}\hat{z}\times\vec{J}$$

$R \sim \rho_{xx} = \rho_0$ $\quad$ $R_H \sim \rho_{xy} \propto B$

Discovery of Quantum Hall Effect

VOLUME 45, NUMBER 6 PHYSICAL REVIEW LETTERS 11 AUGUST 1980

New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance

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(Received 30 May 1980)

Measurements of the Hall voltage of a two-dimensional electron gas, realized with a silicon metal-oxide-semiconductor field-effect transistor, show that the Hall resistance at particular, experimentally well-defined surface carrier concentrations has fixed values which depend only on the fine-structure constant and speed of light, and is insensitive to the geometry of the device. Preliminary data are reported.

PACS numbers: 73.25.+i, 06.20.Jr, 72.20.My, 73.40.Qv

In this paper we report a new, potentially high-accuracy method for determining the fine-structure constant, α. The new approach is based on the fact that the degenerate electron gas in the inversion layer of a MOSFET (metal-oxide-semiconductor field-effect transistor) is fully quantized when the transistor is operated at helium temperatures and in a strong magnetic field of order 15 T.[1] The inset in Fig. 1 shows a schematic diagram of a typical MOSFET device used in this work. The electric field perpendicular to the surface (gate field) produces subbands for the motion normal to the semiconductor-oxide interface, and the magnetic field produces Landau quantization of motion parallel to the interface. The density of states $D(E)$ consists of broadened δ functions[2]; minimal overlap is achieved if the magnetic field is sufficiently high. The number of states, N_L, within each Landau level is given by

$$N_L = eB/h, \tag{1}$$

where we exclude the spin and valley degeneracies. If the density of states at the Fermi energy, $N(E_F)$, is zero, an inversion layer carrier cannot be scattered, and the center of the cyclotron orbit drifts in the direction perpendicular to the electric and magnetic field. If $N(E_F)$ is finite but small, an arbitrarily small rate of scattering cannot occur and localization produced by the long lifetime is the same as a zero scattering rate, i.e., the same absence of current-carrying states occurs.[3] Thus, when the Fermi level is between

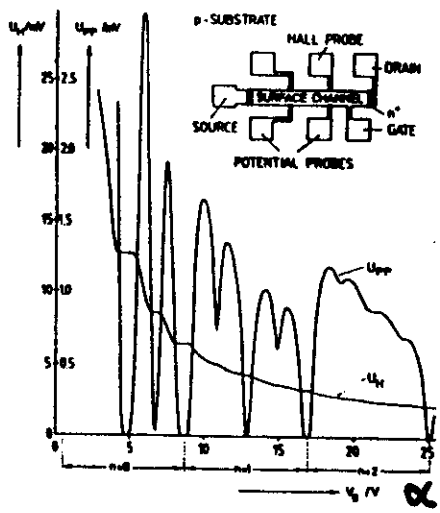
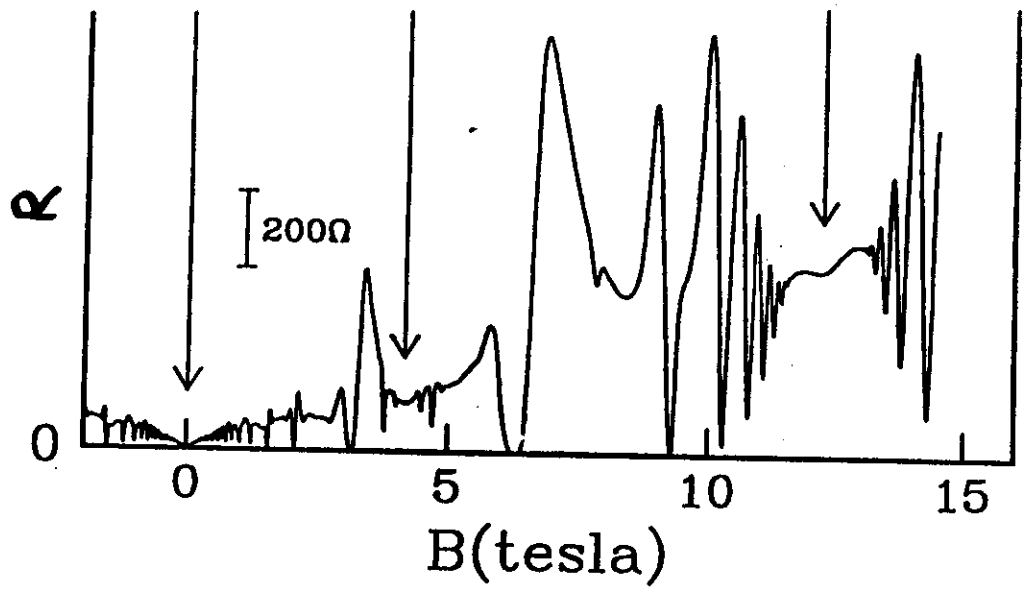


FIG. 1. Recordings of the Hall voltage U_H, and the voltage drop between the potential probes, U_{pp}, as a function of the gate voltage V_g, at $T = 1.5$ K. The constant magnetic field (B) is 18 T and the source drain current, I, is 1 μA. The inset shows a top view of the device with a length of $L = 400$ μm, a width of $W = 50$ μm, and a distance between the potential probes of $L_{pp} = 130$ μm.

†1985 Nobel Prize

Quantum Hall Effect - 1994



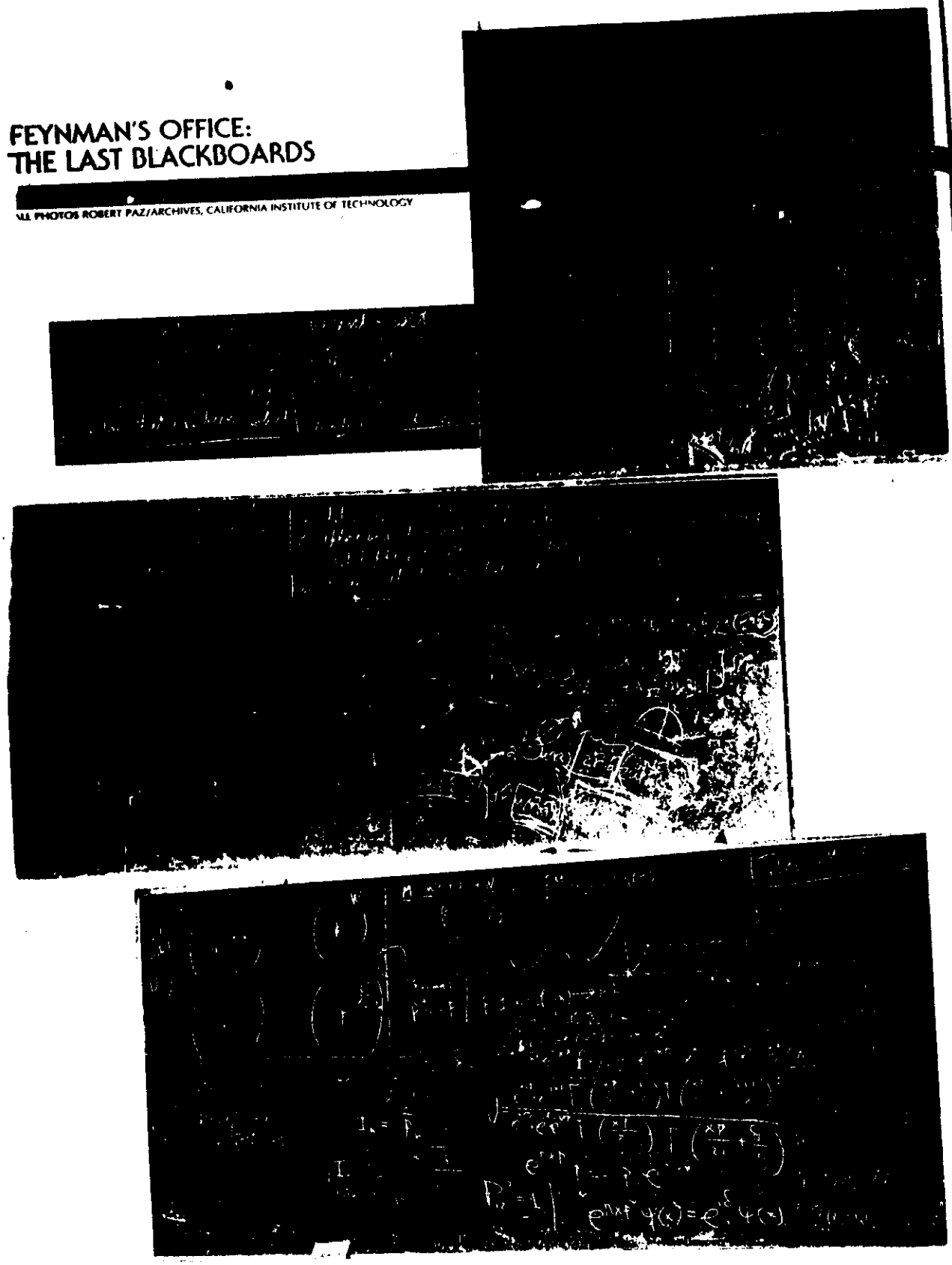
After Kang et al. PRL 71, 3850 (1993)

⟶

- Higher Mobility Samples
- Lower Temperatures

FEYNMAN'S OFFICE: THE LAST BLACKBOARDS

ALL PHOTOS ROBERT PAZ/ARCHIVES, CALIFORNIA INSTITUTE OF TECHNOLOGY



量子ホール効果

吉岡 大二郎　Daijiro Yoshioka

85年度のノーベル物理学賞は量子ホール効果の発見に対して，クリッツィンク(Klaus von Klitzing)に与えられた。この量子ホール効果とは，半導体の界面に実現された2次元電子系が，極低温，強磁場の下で示す奇妙なふるまいである。この現象は，これを利用すると微細構造定数が精度よく求まること，基礎的な標準抵抗として利用できること，場合によっては特別な電子状態の存在が可能となることなどから，物性物理のみならず，素粒子物理の分野からも注目を集めている。

物性物理学は天然に存在する物質のさまざまな性質を調べ，それを理解し説明することから始まった。しかし最近の物性物理では，新しい物質を人工的につくり，その性質を調べたり，実現したい性質をもつように人工的な物質をつくることが試みられている。また，極端な環境のなかに物質を置いて，そこでの性質を調べることも行われている。このように物性物理の研究範囲が広がるにつれて，予想もつかなかったことが，次々と現れてきた。これから紹介する量子ホール効果は，人工的につくられた半導体界面の2次元電子系が，強磁場，超低温のもとで示す性質であり，まさに典型的な物性物理最前線の研究テーマである。またそこで見つかった現象がまったく予想外のものであったため，多くの研究者の注目を集めた。

量子ホール効果での予想外のことの1つは，ふつう物性の測定値は3桁の精度があれば十分であり，それ以上精度をあげてもあまり意味はないと思われているのだが，この場合には実に7桁の精度を問題にすることができるということである。しかも今の場合には，適当な大きさの半導体をもってきて，2桁程度の精度で磁場をかけ，適当に温度を下げて電流を流し，電流と電圧だけを7桁の精度で測定すれば，微細構造定数が7桁の精度で測れるのである。またこの現象を逆に使えば標準抵抗として使うこともできる。

もう1つの意外なことは，量子ホール効果のなかでもとくに“分数量子ホール効果”とよばれる現象の原因を調べるうちに，まったく新しいタイプの基底状態が発見されたことである。電子の集団は絶対零度では数種類の秩序正しい基底状態におちつく。強磁性状態とか，超伝導状態などはそのような秩序状態の例であって，これらの状態には，それを特徴づける秩序パラメーターというものが存在している。しかし今回発見された2次元電子系の基底状態にはそのような秩序パラメーターは存在しない。それにもかかわらずこの基底状態には奇妙な秩序があり，整然とした階層構造がみられている。

以下では，この変わった現象を説明するために，まず2次元電子をどのように実現するのか，量子ホール効果とはどのような現象なのか，それがどのように説明されるのか，をみていくことにしよう。

2次元電子系

電子は3次元空間に存在している。しかし3次元空間を自由に動ける電子は，金属中の伝導電子や，プラズマ中の電子などだけで，絶縁体中の電子や，気体分子中の電子は原子核のポテンシャルにつかまっていて，勝手には動けないから，これらの電子は0次元の電子と考えてもよい。同様に，ある平面上は自由に動けるが，面から離れる方向にはポテンシャルが高く逃げだせないようになっているときには2次元電子と考えられる。

このような2次元電子系が実現できるようになったのは，主に半導体技術の進歩によっている。よく知られ，実用にもなっている2次元電子系はSi-MOSとよばれる構造で実現され，これは電界効果トランジスター(FET)に使われているが，ここでは半導体であるシリコン(Si)と絶縁体である酸化シリコン(SiO_2)の境界面に電子をつかまえるしくみになっている。境界面

秩序パラメーター
強磁性体の自発磁化，超伝導体の超伝導成分などのように，秩序状態のみで有限の値をとり，秩序状態を特徴づける量。

Free 2D Electron in a Magnetic Field

- Classical -

- Complex Number Notation ($z = x + iy$

  $V = V_x + iV_y$ ...)

- Equations of Motion

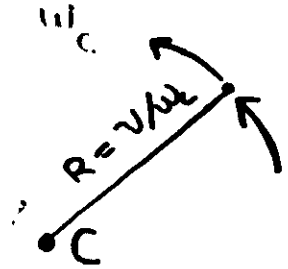
$$\left.\begin{aligned} m\ddot{x} &= -\frac{eB}{c}\dot{y} \\ m\ddot{y} &= \frac{eB}{c}\dot{x} \end{aligned}\right\} \quad \ddot{z} = i\omega_c \dot{z}$$

- Solution of EQM

$$\dot{z} = v_0 e^{i\omega_c t}$$

$$\downarrow$$

$$z = C - \frac{i v_0 e^{i\omega_c t}}{\omega_c}$$



Free 2D Electron in a Magnetic Field
- Quantum -

- Hamiltonian

$$\mathcal{H} = \frac{\vec{\Pi}^2}{2m} \qquad \vec{\Pi} = -i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}$$

$$[\Pi_x, \Pi_y] = -\frac{i\hbar e}{c}\hat{z}\cdot(\vec{\nabla}\times\vec{A}) = -i\frac{\hbar^2}{\ell^2} \quad \leftarrow \ell^2 = \frac{\hbar c}{eB}$$

$$2\pi\ell^2 B = \Phi_0$$

- Ladder Operators

$$a^\dagger \equiv \frac{\ell/\hbar}{\sqrt{2}}(\Pi_x + i\Pi_y) \qquad [a, a^\dagger] = 1$$

$$\mathcal{H} = \frac{\hbar\omega_c}{2}(aa^\dagger + a^\dagger a) \quad \leftarrow \text{Kinetic Energy Quantization}$$

($\omega_c \equiv eB/mc$)

- Orbit Center Degree of Freedom

$$C = x + \frac{i\Pi_y}{m\omega_c} \qquad [C_x, C_y] = i\ell^2$$

$$b \equiv \frac{1}{\ell}(C_x + iC_y) \qquad [b, b^\dagger] = 1$$

$$[a, b] = [a^\dagger, b] = [\mathcal{H}, b] = 0$$

Free 2D Electron in a Magnetic Field
- Quantum -

- Eigenstates

$$|n,m\rangle = \frac{(a^+)^n (b^+)^m}{\sqrt{n!\,m!}} |0,0\rangle$$

$$\varepsilon_n = \hbar\omega_c \left(n + \tfrac{1}{2}\right) \longleftarrow \text{Landau level degeneracy}$$

- Symmetric Gauge $\longleftarrow$ $\vec{A} = \frac{B}{2}(-y, x, 0)$

$$b = \frac{1}{\sqrt{2}}\left(\frac{z}{2\ell} + 2\ell\,\frac{\partial}{\partial \bar{z}}\right)$$

$$b^+ = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell} - 2\ell\,\frac{\partial}{\partial z}\right)$$

$$a^+ = \frac{i}{\sqrt{2}}\left(\frac{z}{2\ell} - 2\ell\,\frac{\partial}{\partial \bar{z}}\right)$$

$$a = \frac{-i}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell} + 2\ell\,\frac{\partial}{\partial z}\right)$$

$$\psi_{0,0} = \frac{1}{\sqrt{2\pi\ell^2}} e^{-z\bar{z}/4\ell^2}$$

$$\psi_{0,m} = \frac{1 \cdot \bar{z}^m e^{-z\bar{z}/4\ell^2}}{\sqrt{2\pi\ell^2 2^m m!}}$$

i — phase factor choice

Hence $z = \sqrt{2}\,\ell\,(b + a^+)$

$\bar{z} = \sqrt{2}\,\ell\,(b^+ + a)$

Useful Identities

- **Plane-Wave Matrix Elements**

$$\langle n', m' | e^{i\vec{k}\cdot\vec{r}} | n, m \rangle$$

$$= \langle n', m' | e^{-i\bar{k}(b+a^\dagger)/\sqrt{2}} e^{-ik(b^\dagger+a)/\sqrt{2}} | n, m \rangle$$

$$= e^{-|k|^2/2} \underbrace{\langle n' | e^{-i\bar{k}a^\dagger/\sqrt{2}} e^{-ika/\sqrt{2}} | n \rangle}_{= G_{n',n}(\bar{k})} \underbrace{\langle m' | e^{-ikb^\dagger/\sqrt{2}} e^{-i\bar{k}b/\sqrt{2}} | m \rangle}_{\equiv G_{m',m}(k) \equiv \langle m' | B(k) | m \rangle e^{+|k|^2/2}}$$

where

$$G_{m',m}(k) = \left(\frac{m!}{m'!}\right)^{1/2} \left(\frac{-ik}{\sqrt{2}}\right)^{m'-m} L_m^{m'-m}\left(\frac{k\bar{k}}{2}\right)$$

↑ generalized Laguerre polynomials

- **Inversion**

$$G_{m',m}(-k) = (-)^{m'-m} G_{m',m}(k)$$

- $\frac{\text{Area}}{(2\pi\ell^2)} \sim$ momentum

$$\pi \langle n, m | \bar{z} z | n, m \rangle = 2\pi\ell^2 (n+m+1)$$

$$\ll |z|^2 \cdot |z|^2 \gg \ll \langle |z|^2 \rangle^2$$

Useful Identities

- Wavefunctions

$$\langle \vec{r} | n,m \rangle \equiv \varphi_{n,m} = \frac{e^{-|z|^2/4}}{\sqrt{2\pi}} G_{m,n}(iz)$$

- Matrix Products

$$\sum_{\ell} G_{m',\ell}(k_1) G_{\ell,m}(k_2) = \sum_{\ell} \langle m' | e^{-ik_1 b^\dagger/\sqrt{2}} e^{-i\bar{k}_1 b/\sqrt{2}} | \ell \rangle$$

$$\times \langle \ell | e^{-ik_2 b^\dagger/\sqrt{2}} e^{-i\bar{k}_2 b/\sqrt{2}} | m \rangle$$

$$= e^{-\bar{k}_1 k_2/2} G_{m',m}(k_1 + k_2)$$

- Hermitian Conjugate

$$\overline{G}_{m',m}(k) = G_{m,m'}(-k)$$

- Landau Level Degeneracy

$$\sum_m |\varphi_{n,m}|^2 = \frac{e^{-|z|^2/2}}{2\pi} \sum_m G_{n,m}(-i\bar{z}) G_{m,n}(i\bar{z})$$

$$= \frac{1}{2\pi} G_{n,n}(0) = \frac{1}{2\pi\ell^2} \Rightarrow \frac{N}{A} = \frac{[redacted] B}{[redacted] \Phi_0}$$

$$\Rightarrow N = N_\phi$$

Useful Identities

- Full Landau Level Rule

$$\sum_{m=0}^{\infty} G_{m,m}(k) = \int d\vec{r}\; e^{-i\vec{k}\cdot\vec{r}}\, e^{|k|^2/2}\,(2\pi\ell^2)^{-1} = N_\phi\, \delta_{\vec{k},0}$$

- Orthogonality

$$\int d^2\vec{k}\; e^{-|k|^2/2}\; G_{m',m}(k)\, G_{n',n}(\bar{k})$$

$$= \overline{\varphi}_{n',m'}(0)\, \varphi_{n,m}(0) = \frac{\delta_{m',m'}\,\delta_{m',n}}{2\pi}$$

INCOMPRESSIBILITY

- $\kappa \equiv -\frac{1}{V}\frac{\partial V}{\partial P}$ ← relative volume change with pressure

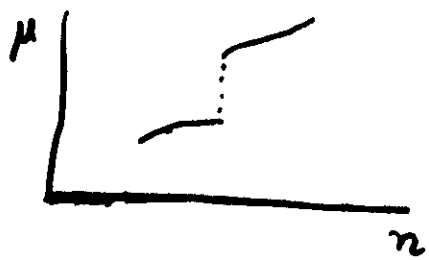
- $\kappa^{-1} \equiv -V\frac{\partial P}{\partial V} = V\frac{\partial^2 E}{\partial V^2}$

but $E = N\epsilon(N/V)$

$$\Rightarrow \kappa^{-1} = n^2 \frac{d^2}{dn^2}(n\epsilon(n)) \quad \leftarrow n = \frac{N}{V}$$

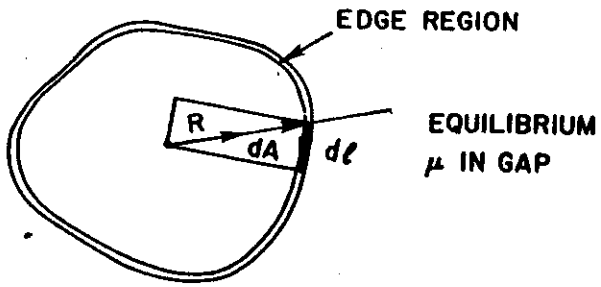
$$= n^2 \frac{d\mu}{dn}$$

$\Rightarrow$ INCOMPRESSIBILITY $\equiv$ 'CHARGE GAP'

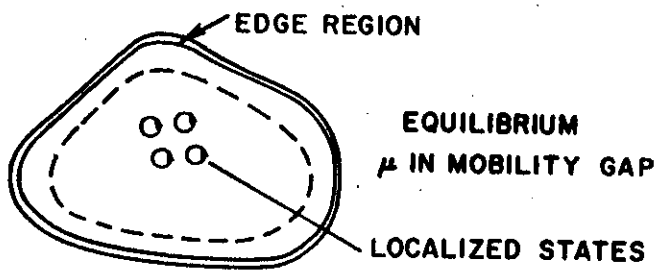


INCOMPRESSIBILITY ⇒ QUANTIZATION

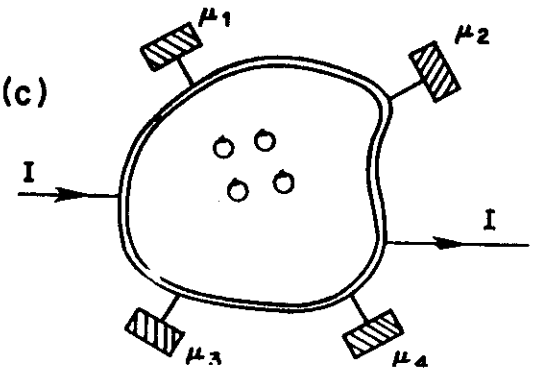
$$\delta M = \frac{A}{c}\,\delta I \qquad \text{(a)}$$



$$\frac{\delta I}{\delta \mu} = \frac{c}{A}\left.\frac{\partial M}{\partial \mu}\right|_B = \frac{c}{A}\left.\frac{\partial N}{\partial B}\right|_\mu \qquad \text{(b)}$$



(c)



LOCAL EQUILIBRIUM ON ISOLATED EDGES

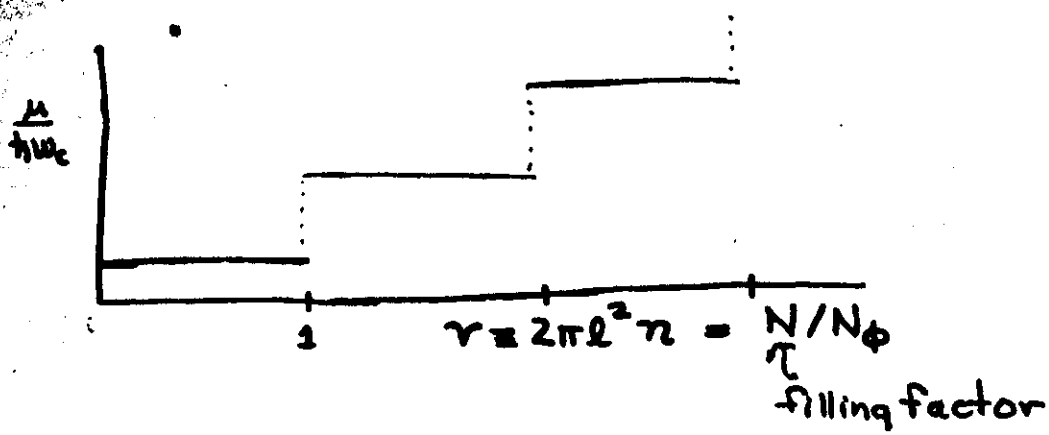
$$R = \frac{(\mu_2 - \mu_1)}{I} = 0$$

$$R_H = \frac{(\mu_2 - \mu_4)}{I} = \frac{A}{c\left.\frac{\partial N}{\partial B}\right|_\mu}$$ ← Streda formula

! Need Charge Gap at a density !

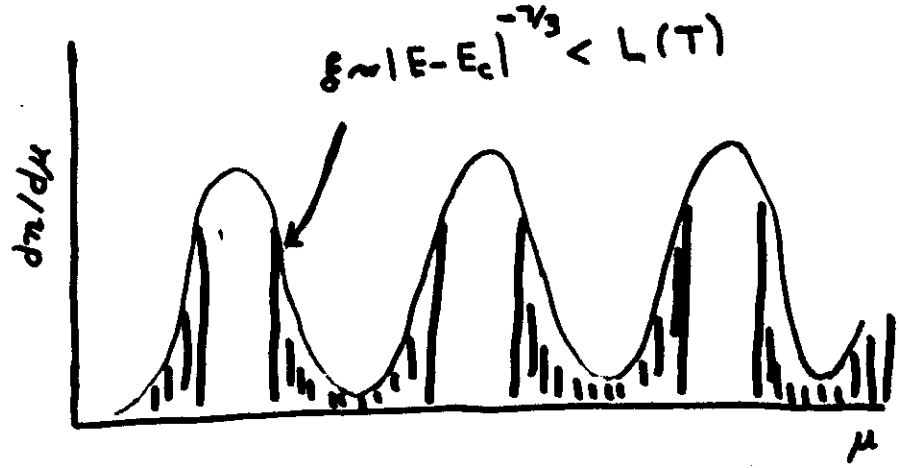
which depends on B

Integer Quantum Hall Effect



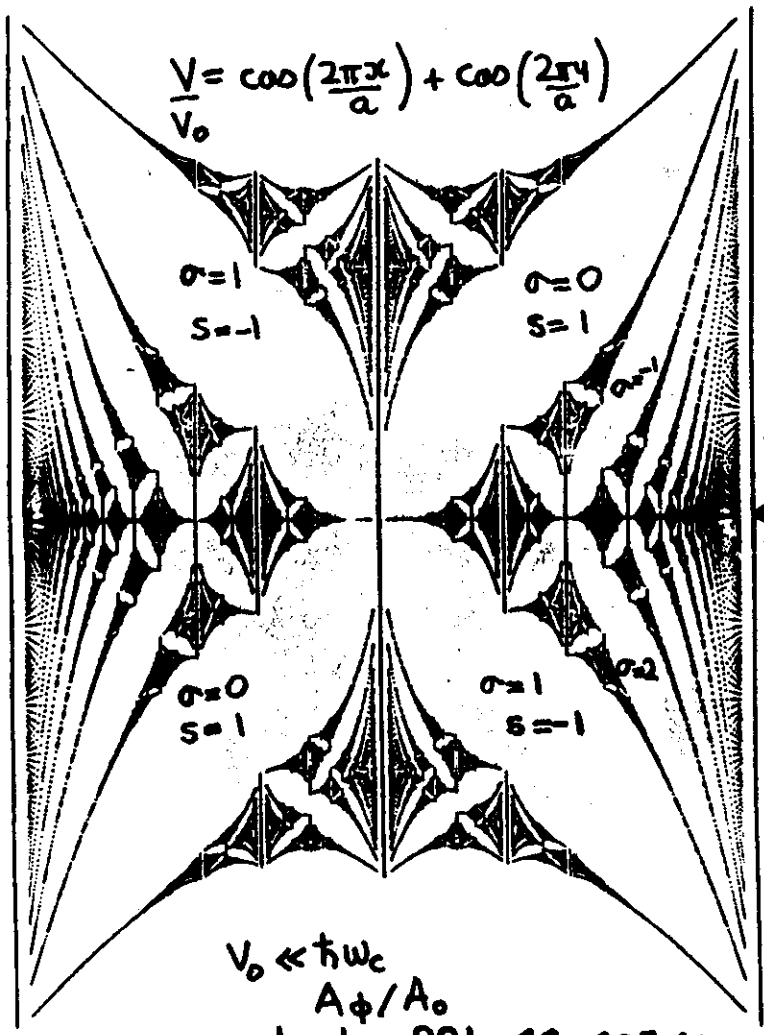
- $G_H = R_H^{-1} = \frac{e^2}{h} \cdot i \qquad i = 1, 2, 3, \ldots$

- Localization physics



! N.B. ← no perturbation theory for disorder

Periodic Potential



S as small as possible

Thouless et. al., PRL 49, 405 (1982)

Figure 4.1. Energy bands obtained by Hofstadter (1976) for a square lattice vs. q/p, the inverse of the flux per unit cell (running horizontally from 0 to 1). Energy bands are plotted vertically and shown for all values of p up to 50. In principle, there are p bands but for even p the two central bands touch one another.

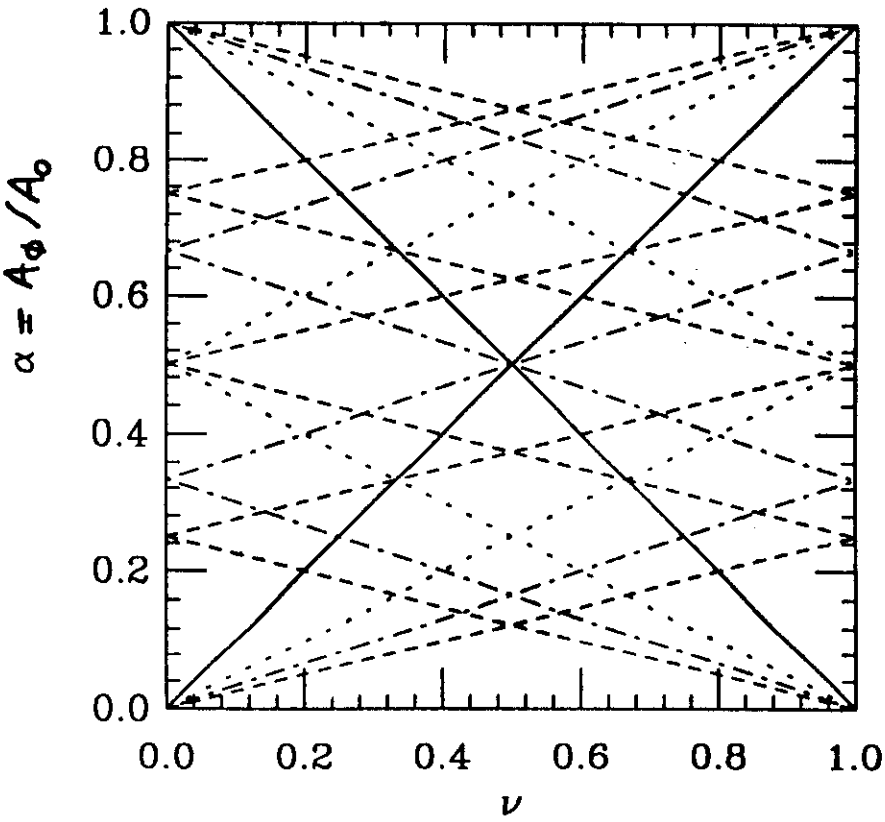
$$n = \frac{\sigma}{A_\phi} + \frac{s}{A_0} \qquad A_\phi = \frac{\Phi_0}{B} = \frac{q}{p} A_0$$

σ, s TKNN integers

$$\Rightarrow r = \sigma + s\frac{A_\phi}{A_0} \Rightarrow r = \frac{t}{p} \text{ where } t = \sigma p + s q$$

Subband Trajectories

17/4 e 45



$$(\sigma, s) = (\sigma_0, s_0) + (-k q_0, k p)$$

$$t = \sigma_0 p + s_0 q_0$$

$$\sigma_H = \frac{e^2}{h}\sigma_0 \leftarrow \text{Streda formula}$$

Haldane Pseudopotentials

.- two body problem -

Center of Mass & Relative Ladders

$$b_R^+ = \frac{b_1^+ + b_2^+}{\sqrt{2}} \longrightarrow \ell_R = \frac{\ell}{\sqrt{2}}$$ COM ladder operators

$$b_r^+ = \frac{b_1^+ - b_2^+}{\sqrt{2}} \longrightarrow \ell_r = \sqrt{2}\,\ell$$ relative ladder operators

Hamiltonian

$$H = \hbar\omega_c (a_1^+ a_1 + a_2^+ a_2) + V(\vec{r}_1 - \vec{r}_2)$$

$$\longrightarrow \hbar\omega_c (a_R^+ a_R + a_r^+ a_r) + \sum_m{}' |m\rangle_{rr}\overbrace{\langle m|V|m\rangle_r}^{V_m = \text{Haldane Pspot}}{}_r\langle m|$$

($\langle m|V|m\rangle_r$: LLL projection)

$$\longrightarrow \nrightarrow \text{odd}$$

$$\sum_m{}' V_m P_m^{1,2}$$ ← relative angular momentum projector

Energy Scale Cascade

$$V_1 \gg \underbrace{V_3 \gg V_5 \cdots}$$

$\longrightarrow 0$ in hard core model

Digression on Haldane Pseudopotentials

- Lowest Landau level case

$$V_m = {}_r\langle m|V|m\rangle_r = \int \frac{d^2\vec{q}}{(2\pi)^2} V(\vec{q})\, {}_r\langle m| e^{i\vec{q}\cdot\vec{r}} |m\rangle_r$$

$$= \int \frac{d^2\vec{q}}{(2\pi)^2} V(\vec{q})\, e^{-q^2} L_m(q^2)$$

$$\text{Coulomb } V(q) = \frac{2\pi e^2}{q} \Rightarrow V_m = \frac{e^2}{\ell}\frac{\sqrt{\pi}}{2}\left(\frac{(2m-1)!!}{2^m m!}\right)$$

Inverse

$$V(\vec{q}) = 4\pi \sum_m V_m L_m(q^2)$$

- Mapping from Higher Landau Levels

$$\langle n,m_1' : n,m_2' | e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)} | n,m_1 : n,m_2 \rangle$$

$$= [L_n(q^2/2)]^2 \langle m_1', m_2' | e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)} | m_1, m_2 \rangle$$

$\Rightarrow$ $V(\vec{q})$ projected on n-th LL

$\equiv [L_n(q^2/2)]^2$ projected on 0-th LL

Hence influence of

- ... tilted fields
- ... anisotropy
- ... valence band

Haldane Pseudopotentials for Coulomb Model

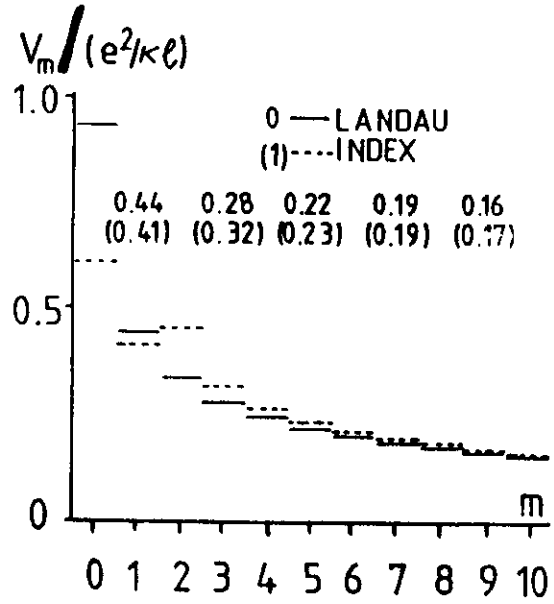


Figure 8.1 Pseudopotentials V_m (pair energies) for the pure Coulomb interaction and Landau indices $n=0$ and *1*.

$$V_m = \frac{e^2}{\kappa\ell}\frac{\sqrt{\pi}}{2}\cdot\left[\frac{(2m-1)!!}{2^m\, m!}\right]$$

Laughlin Wavefunction

(Maximum Density $E=0$ Eigenfunction of Hard-Core Model)

- Lowest Landau Level Condition

$$\Psi[z] = P(z_1, z_2, \cdots, z_N) \prod_k \exp(-|z_k|^2/4)$$

(P: Polynomial)

- Zero-Energy Condition

$$P[z] \propto (z_1 - z_2)^3 (z_1 - z_3)^3 \cdots (z_{N-1} - z_N)^3$$

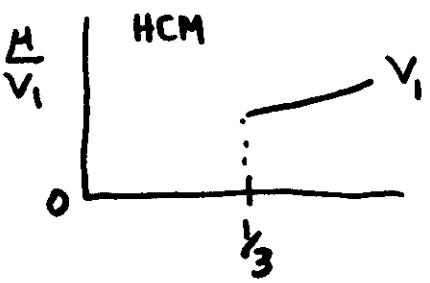
$$\Rightarrow \quad M_1 > 3(N-1)$$

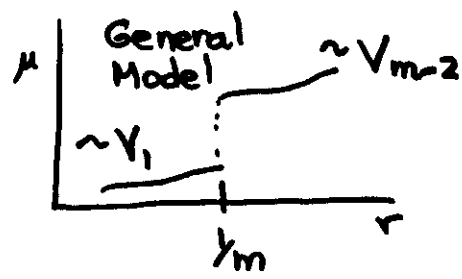
(M_1: degree of $p(z_1 : [z_2, \cdots, z_N])$)

- Area - Degree Connection

$$A \propto 2\pi \ell^2 M_1 \underset{\text{minimum area}}{\gtrsim} 3N A_\phi \Rightarrow \nu^{-1} = 3$$

$$\Rightarrow \quad \Psi_m[z] = \prod_{i<j} (z_i - z_j)^m \prod_k \exp(-|z_k|^2/4)$$





Fractional Charge

(Laughlin)

Azimuthal Field $\Rightarrow$ Radial Current

$$E_\phi = \frac{1}{c}\frac{d\phi}{dt}\frac{1}{2\pi R} \qquad j_R = \nu\frac{e^2}{h}\cdot\frac{1}{c}\frac{d\phi}{dt}\cdot\frac{1}{2\pi R}$$

$\Rightarrow$ Fractional Charge

$$Q = 2\pi R\int j_R\,dt = \nu\frac{e^2}{h}\cdot\frac{1}{c}\cdot\frac{hc}{e} = \nu e \qquad \left(\frac{hc}{e} = \Phi_0\right)$$

MacDonald & Girvin
PRB 34, 5639 (1986)

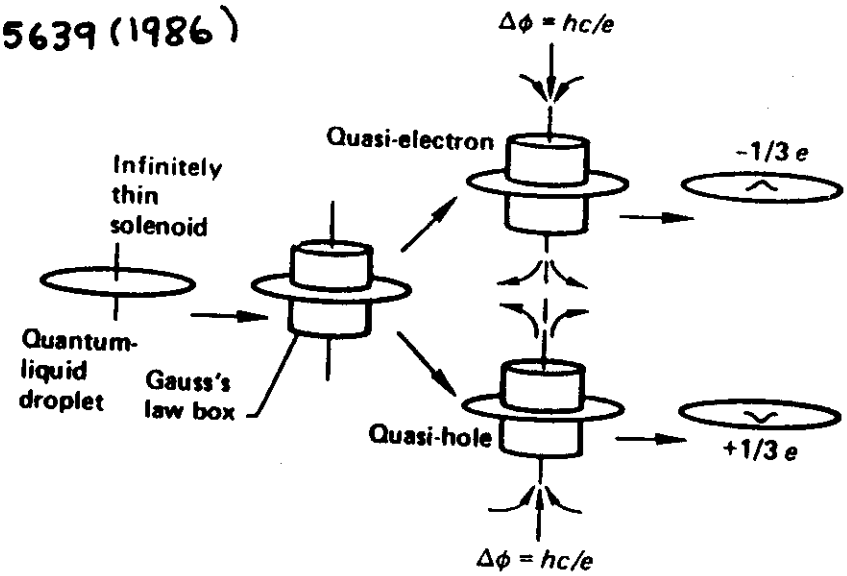


Figure 7.8 Illustration of thought experiment generating fractionally charged quasiparticles. This shows the charge to be $\pm 2\pi\ell^2$ times the ground-state charge density, regardless of the form of the inter-electronic repulsions, so long as the ground state is non-degenerate.

$$\Psi^L_{qh} = \prod_k z_k \prod_{i<j}(z_i - z_j)^m \prod_a \exp(-|z_a|^2/4)$$

Plasma Analogy

- Laughlin State

$$|\Psi_L|^2 = e^{-U_0}$$

$$\Rightarrow U_0 = +m\Big(\sum_{i<j} 2\ln|z_i - z_j|\Big) + \sum_k \frac{|z_k|^2}{2} \quad \Rightarrow \rho_B = \frac{1}{2\pi}$$

$$\Rightarrow 2\pi n = \nu = \frac{1}{m}$$

& uniform!!

- Laughlin quasihole state

$$|\Psi_{qh}^L|^2 = e^{-U}$$

$$\Rightarrow U = U_0 + \sum_k 2\ln|z_k|$$

$$\Rightarrow 1 + m\delta Q_e = 0$$

$$\Rightarrow \delta Q_e = -\frac{1}{m}$$

o.k. ✓

Digression: Anyons & Statistical Transmutation

- Anyons

$\theta = 2n\pi \Rightarrow$ boson

$\theta = (2n+1)\pi \Rightarrow$ fermion

$$\Psi(z_2, z_1, z_3, \cdots, z_N) = e^{\pm i\theta} \Psi(z_1, z_2, z_3, \cdots, z_N)$$

Makes sense in 2D

Canright, Girvin, Science 247, 1197 (1990)

- Statistical Transmutation

(Wilczek, PRL 49, 957 (1982))

$$\Psi' = \prod_{\substack{j \neq k \\ j > k}} e^{i\theta_{jk}\alpha} \Psi \quad \leftarrow \quad \theta' = \theta + \alpha$$

$$H' = \frac{1}{2m} \sum_j \left(\vec{p}_j + \frac{e}{c}\vec{A}_j + \frac{e}{a}\vec{a}_j \right)^2 + U$$

external mag. field

statistical mag. field

$$\vec{a}_j = -\frac{\hbar c}{e} \cdot \alpha \cdot \sum_{k \neq j} \vec{\nabla}_j \theta_{jk} \Rightarrow \oint_k \vec{a}_j \cdot d\vec{\ell} = -\Phi_0 \cdot \alpha$$

$$\vec{B}_j = \vec{\nabla} \times \vec{a}_j = -\frac{\Phi_0}{a} \cdot \alpha \cdot \sum_{k \neq j} \delta(\vec{r}_j - \vec{r}_k) \leftarrow \text{flux-tube attachment}$$

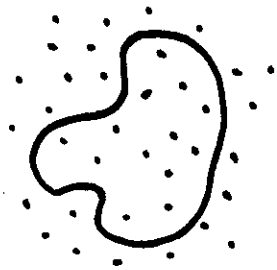
Quasiparticle Statistics

- Single Quasihole

$$\Psi_{z_0}[z] = \prod_i (z_i - z_0)\, \Psi_L[z] = |\Psi_{z_0}[z]|\, e^{i\varphi[z]}$$

$$\Rightarrow$$

$$\oint dz_0 \frac{d\varphi[z]}{dz_0} = -i \oint \frac{\Psi'_{z_0}[z]}{\Psi_{z_0}[z]}\, dz_0 = 2\pi N_0$$



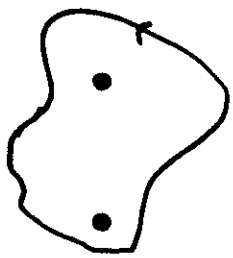
N_0 = # of particles inside loop

- Quasiparticle Statistics

Phase Change for quasiparticle enclosure

$$= 2\pi \cdot \nu$$

$$\Rightarrow \theta_{q,p} = \nu \pi$$



Arovas, Schrieffer, Wilczek, Zee
Nucl. Phys. 251, 117 (1985)

Digression: Lowest Landau Level Density Matrices

- Number Density

$$n(\vec{r}) = \sum_{m',m} \bar{\psi}_{m'}(z)\,\psi_m(z)\,\langle c^\dagger_{m'} c_m\rangle_0$$

$$\to \frac{1}{2\pi}\sum_{m',m} \frac{r^{m+m'} e^{i\theta(m-m')}}{\sqrt{2^{m+m'} m!\, m'!}}\,\langle c^\dagger_{m'} c_m\rangle_0\; e^{-r^2/2}$$

isotropic $\langle c^\dagger_{m'} c_m\rangle_0 \propto \delta_{m',m}$

$$\to \frac{1}{2\pi}\sum_m \frac{1}{m!}\left(\frac{r^2}{2}\right)^m e^{-r^2/2}\,\langle n_m\rangle_0 r = \frac{r}{2\pi}$$

- uniform $\langle n_m\rangle$ ← independent of m

- Density Matrix

$$\rho(z,\bar{z}') = \sum_{m',m} \bar{\psi}_{m'}(z')\,\psi_m(z)\,\langle c^\dagger_{m'} c_m\rangle_0$$

uniform system ↓

$$\frac{\nu}{2\pi}\sum_m \frac{1}{m!}\left(\frac{z\bar{z}'}{2}\right)^m e^{-|z|^2/4} e^{-|z'|^2/4}$$

phase factor ↙

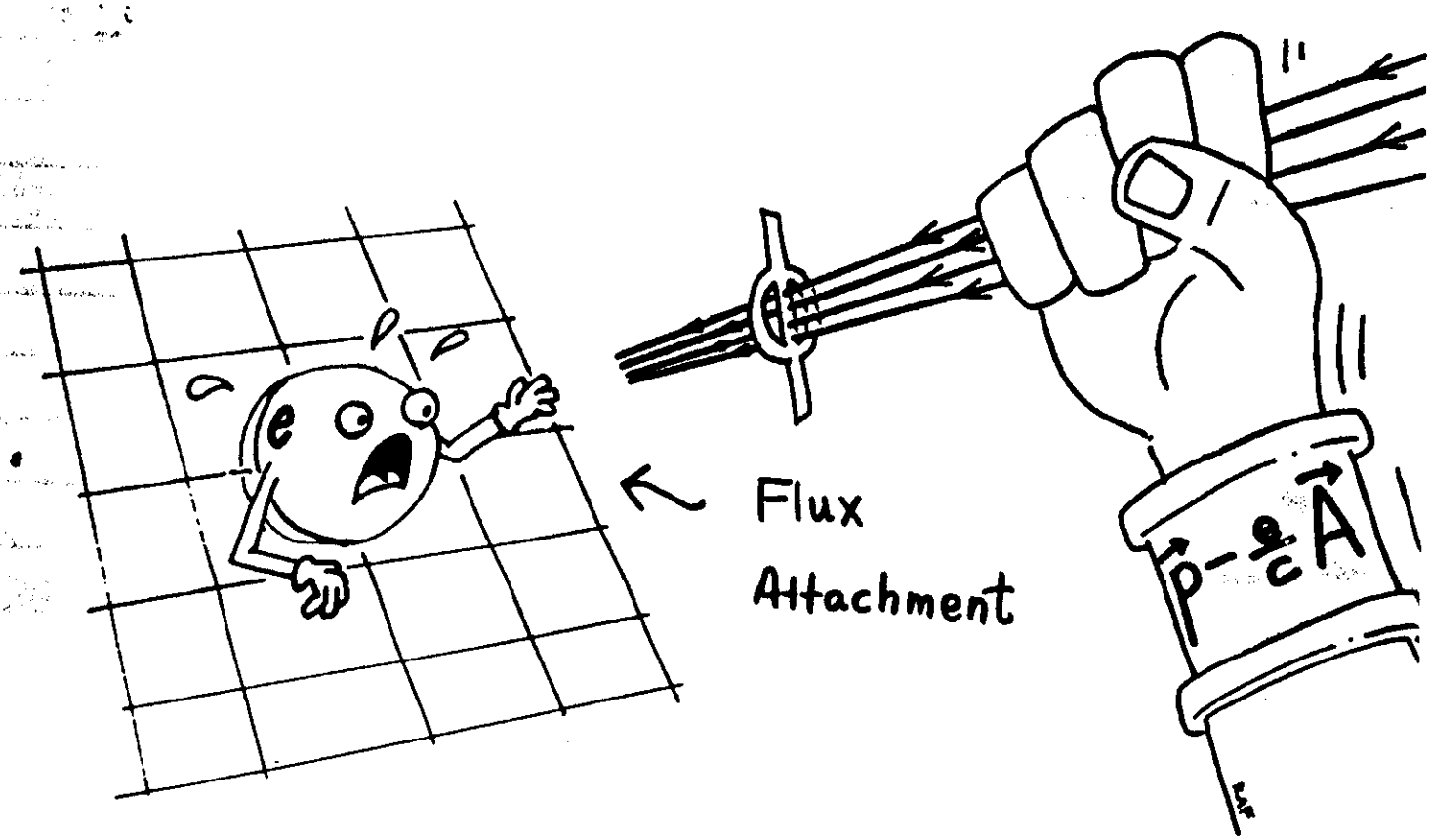
$$= \frac{\nu}{2\pi}\exp\left(-\frac{|z-z'|^2}{4}\right)\exp\left((z\bar{z}' - z'\bar{z})/4\right)$$

BOSONIZATION

Fig. 6

$$\Psi'_m = \prod_{j>k} e^{-im\theta_{jk}} \Psi_m$$

$m = 3$ for $\nu = 1/3$



Dan Arovas
Ph.D Thesis
UCSB

BOSON ODLRO

Density Matrix - first Quantization

$$\rho(z,z') = N\int d^2z_2 \cdots \int d^2z_N\ \psi(z,z_2,\cdots,z_N)\,\bar{\psi}(z',z_2,\cdots,z_N)$$

$$\sim e^{-|z-z'|^2/4}$$

Boson Density Matrix

$$o'(z,z') = N\int d^2z_2 \cdots \int d^2z_N\ \psi'(z;[z])\,\bar{\psi}'(z';[z])$$

$$= N\int d^2z_2 \cdots \int d^2z_N \prod_{1<k} |z-z_k|^m |z'-z_k|^m e^{-|z_0|^2/4} e^{-|z'|^2/4}$$

$$\prod_{\ell<m} |z_\ell - z_m|^{2m} \prod_{2<j} e^{-|z_j|^2/2}$$

$\sim$ N-1 plasma with impurities at z & z'

$$\sim |z-z'|^{-m/2}$$ Girvin, MacDonald, PRL 58, 1252 (1987)

$\Rightarrow$ quasi long-range order

Quantum Hall Effect $\simeq$ Boson Superfluidity

Chern-Simons-Landau-Ginzburg Theory

Read - PRL 62, 86 (1989)

Zhang, Hansson, Kivelson - PRL 62, 82 (1989)

Fisher, Zhang, Lee ...