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FERMI LIQUIDS AND SUPERFLUID FERMI SYSTEMS

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I. INTRODUCTION

The following lectures on Fermi liquid theory, superfluids and superconductivity cover the basic elements of Landau's theory of Fermi liquids and the BCS theory of superfluidity in Fermi systems. They are thought to provide a reference for what may nowadays be called the "standard" behavior of Fermi systems at low temperatures. Here the term standard is understood to characterize a situation where the low-lying fermionic excitations can be directly related to the basic fermions, in contrast to the scenarios of spin-charge separation, or gauge-field models, or anyon models, to name a few recent conjectures.

In view of the recent interest in "unconventional" anisotropic superconducting states in the context of high-T_c superconductivity (HTSC), the presentation of BCS theory given here will be sufficiently general to allow for a discussion of anisotropic states. In particular we will consider superfluid ^3He, superconductivity in heavy fermion compounds, and the d-wave state of HTSC.

There are a number of textbooks available, covering the first and larger part of these lectures. On phenomenological Fermi liquid theory, the texts by Baym and Pethick [1] and by Pines and Nozieres [2] will be useful. The microscopic underpinning of Fermi liquid theory is discussed in the classic book by Nozieres [3]. The theory of superfluid ^3He can be found in Refs. 4,5,6. Further references will be given at appropriate places in the text. Unconvential superconductivity in heavy fermion compounds is considered in the review by Sigrist and Ueda [7].

II. FERMI LIQUID THEORY

Systems of interacting fermions at low temperature have been of interest early on in the development of condensed matter theory. The most important example is the system of conduction electrons in metals. According to Sommerfeld's theory of metals [8], the conduction electrons behave like a gas of noninteracting fermions in spite of their mutual

Coulomb interaction. Thirty years later Landau put forth a phenomenological theory of interacting Fermi systems, the Fermi liquid theory or Landau theory, which was based on the new concept of quasiparticles [9]. It attempted to map the properties of Fermisystems at low temperature on to a dilute gas of strongly interacting thermal excitations. To some extent a microscopic justification of this picture was given by Landau and others, (see ref. 3), although a rigorous general mathematical proof is not available. After the discovery of high temperature superconductivity it has been suggested by Anderson [10] that Fermi liquid theory may not be applicable in this case. Following this conjecture a large number of studies of different kinds have been performed in order to prove or disprove it, so far without conclusive result.

In this first lecture, the main content of the phenomenological Fermi liquid theory will be presented.

A. The quasiparticle concept

Let us start by considering the noninteracting system first. Its energy eigenstates are completely described by the set of occupation numbers $N_{\vec{k}\sigma}$ of the single particle states $| \vec{k}\sigma >$. $N_{\vec{k}\sigma}$ can take the two values 0 or 1 only. Here we assume free fermions in eigenstates with momentum $\vec{k}$ and spin projection $\sigma(= \pm 1)$. It is convenient to define a smoothed distribution function $n_{\vec{k}\sigma}$ by averaging $N_{\vec{k}\sigma}$ over a group of neighboring states.

In the ground state all single particle states with momentum less than the Fermi momentum k_F are occupied and all other states are empty:

$$n_{\vec{k}\sigma}^{T=0} = \theta(k_f - k) = \begin{cases} 1 & k \leq k_F \\ 0 & k > k_F \end{cases} \tag{1}$$

The Fermi momentum is determined by the total number of particles in the given volume, i.e. the density n by

$$n = \sum_{\vec{k}\sigma} n_{\vec{k}\sigma}^{T=0} = \frac{k_F^3}{3\pi^2} \tag{2}$$

Let us now imagine that the interaction between the particles is turned on adiabatically. If the single-particle energy spectrum of the interacting system is in one-to-one correspondence with the Fermi-gas spectrum and if the ground state retains the full symmetry of the Hamiltonian the system is termed "normal" or more explicitly, a "normal Fermi liquid". In any ordered state such as a superconducting state or a magnetically ordered state, for example, a macroscopic fraction of the single particle degrees of freedom will be condensed into a macroscopic quantum state and the one-to-one correspondence of single-particle states is lost. Even in a normal Fermi liquid, the interaction will lead to the appearance of socalled collective modes. However, these bosonic excitations occupy a negligible fraction of phase space in the limit of low temperatures and therefore do not spoil the principal one-to-one correspondence of single-particle states, as we will see.

Therefore, the state of the Fermi liquid may again be described by the distribution function $n_{\vec{k}\sigma}$ of single particle excitations. These single particle excitations are called "quasiparticles". In particular, the ground state of the system is characterized by the distribution function $n_{\vec{k}\sigma}$ defined in (1).

The energy of a quasiparticle, $\epsilon_{\vec{k}\sigma}$, is defined as the amount of energy by which the total energy E of the system increases, if a quasiparticle is added to the unoccupied state $\vec{k}\sigma$;

$$\delta E = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}\sigma} \delta n_{\vec{k}\sigma} \tag{3}$$

where $\delta n_{\vec{k}\sigma}$ is the corresponding change of the distribution function. As a consequence of the interaction, the single particle energies depend on the state of the system, $\epsilon_{\vec{k}\sigma} = \epsilon_{k\sigma}\{n_{\vec{k}'\sigma'}\}$. The energy of a single low energy quasiparticle added to the groundstate may be parametrized as

$$\epsilon_{\vec{k}\sigma}\{n_{\vec{k}'\sigma'}^{T=0}\} = \mu + v_F(k - k_F) \quad , \quad | \epsilon_{k\sigma} - \mu | << \mu \tag{4}$$

where μ is the chemical potential,

$$v_F = \frac{k_F}{m^*} \tag{5}$$

is the Fermi velocity and m^* is the effective mass. Here the isotropy of the system has been used and the Fermi velocity is assumed to be finite. The effective mass m^* determines the density of states at the Fermi level

$$N_F = \frac{m^* k_F}{\pi^2} \quad , \text{ (for both spin projections)} \tag{6}$$

The effect of interactions with other excited quasiparticles on the energy of a specific quasiparticle may be expressed in terms of an effective two-particle interaction function or "Fermi-liquid" interaction $f_{\vec{k}\sigma\vec{k}'\sigma'}$

$$\delta\epsilon_{\vec{k}\sigma} = \sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'} \delta n_{\vec{k}'\sigma'} \tag{7}$$

where $\delta n_{k\sigma} = n_{k\sigma} - n^0_{k\sigma}$. At low temperatures, and for weakly excited systems we expect only a small number of quasiparticles on top of the ground st... It is then reasonable to approximate $f_{k\sigma\vec{k}'\sigma'}$, which itself depends on the distribution $n_{\vec{k}\sigma}$, by $f_{\vec{k}\sigma\vec{k}'\sigma'}\{n^{T=0}_{\vec{k}_1\sigma_1}\}$. To simplify the notation, we have assumed here that the spin quantization axes of all quasiparticle states is the z-axis. This precludes the discussion of transverse spin excitations. Note that $f_{\vec{k}\sigma\vec{k}'\sigma'} = f_{\vec{k}'\sigma'\vec{k}\sigma}$, since from (3) and (7) it follows that f is a second derivative of E w.r.t. $n_{\vec{k}\sigma}$.

It is now assumed that for isotropic systems with short range interaction the Fermi liquid interaction function is finite, and does only depend on the angle between the momenta $\vec{k}$ and $\vec{k}'$ and on the relative spin orientation of σ and σ', and hence may be parametrized as

$$f_{\vec{k}\sigma\vec{k}'\sigma'} = \frac{1}{N_F} \sum_{\ell=0}^{\infty} P_\ell(\hat{k}\cdot\hat{k}')[F^s_\ell + F^a_\ell \sigma\sigma'] \tag{8}$$

Here $\hat{k} = \vec{k}/\mid\vec{k}\mid$, $\sigma = \pm 1$, $P_\ell(x)$ are the Legendre polynomials, and F^s_ℓ and F^a_ℓ are the dimensionless spin-symmetric and spin-antisymmetric "Landau parameters", which characterize the effect of the interaction on the quasiparticle energy spectrum.

B. Thermodynamic properties

The equilibrium distribution function $n^0_{\vec{k}\sigma}$ at finite temperature T follows from the assumed one-to-one correspondence of single-particle states of the interacting and non-

interacting systems. The entropy density must have the same form as for the ideal gas

$$S = -\sum_{\vec{k}\sigma} \left[n_{\vec{k}\sigma} \ell n(n_{\vec{k}\sigma}) + (1 - n_{\vec{k}\sigma}) \ell n(1 - n_{\vec{k}\sigma}) \right] \tag{9}$$

The first law of thermodynamics must hold for any deviation $\delta n_{\vec{k}\sigma}$ from the equilibrium distribution $n^0_{k\sigma}$:

$$\delta E = T\delta S + \mu\delta n \tag{10}$$

Substituting δE from (3), $\delta S = \sum_{\vec{k}\sigma}(\delta S/\delta n_{\vec{k}\sigma})\delta n_{\vec{k}\sigma}$ from (9) and

$$\delta n = \sum_{\vec{k}\sigma} \delta n_{\vec{k}\sigma} \tag{11}$$

one finds $n^0_{\vec{k}\sigma}$ to be given b the Fermifunction

$$n^0_{\vec{k}\sigma} = \frac{1}{\exp(\epsilon_{\vec{k}\sigma} - \mu)/T + 1} \tag{12}$$

This is a complicated implicit equation for $n_{k\sigma}$ due to the dependence of $\epsilon_{k\sigma}$ on $\{n_{\vec{k}'\vec{\sigma}'}\}$.

We are now in a position to calculate the thermodynamic properties specific heat, compressibility and spin susceptibility. The derivative of the internal energy with respect to temperature yields the specific heat at constant volume (where $\frac{\partial}{\partial T}\sum_{\vec{k}\sigma} n_{\vec{k}\sigma} = 0$)

$$\begin{aligned} C_V = \frac{\partial E}{\partial T}\Big|_V &= \sum_{\vec{k}\sigma}(\epsilon_{\vec{k}\sigma} - \mu)\frac{\partial n_{\vec{k}\sigma}}{\partial T} \\ &= \sum_{\vec{k}\sigma}(\epsilon_{k\sigma} - \mu)\frac{\partial n_{\vec{k}\sigma}}{\partial \epsilon_{\vec{k}\sigma}}\Big[-\frac{\epsilon_{k\sigma} - \mu}{T} + \frac{\partial}{\partial T}(\epsilon_{k\sigma} - \mu)\Big] \end{aligned} \tag{13}$$

In the limit of low temperatures using the socalled Sommerfeld expansion we may replace $-(\partial n_{\vec{k}\sigma}/\partial\epsilon_{\vec{k}\sigma})$ by $\delta(\epsilon-\mu) - \frac{\pi^2}{6}(k_B T)^2 \frac{d^2}{d\epsilon^2}\delta(\epsilon-\mu)$. The term involving $\frac{\partial}{\partial T}(\epsilon_{k\sigma}-\mu)$ is contributing a correction to C_V of order T^3 and may be dropped. The leading term is linear in T, as for the free Fermi gas, and given by the (renormalized) density of states

$$C_V = \frac{\pi^2}{3} N_F T \tag{14}$$

The spin susceptibility χ follows from the spin polarization S_z in the presence of a magnetic field B

$$S_z = \sum_{\vec{k}\sigma} \sigma \delta n_{\vec{k}\sigma} = \chi B \qquad (15)$$

where $\delta n_{\vec{k}\sigma}$ is the linear change in the distribution function induced by the Zeeman energy shift $\Delta\epsilon_{\vec{k}\sigma} = -\mu_0 \sigma B$ in the bare quasiparticle energy. From

$$S_z = \sum_{\vec{k}\sigma} \sigma \frac{\partial n_{\vec{k}\sigma}}{\partial \epsilon_{\vec{k}\sigma}} \left[\delta\epsilon_{\vec{k}\sigma} - \mu_0 \sigma B\right] \qquad (16)$$

and $\delta\epsilon_{\vec{k}\sigma} = \sigma N_F^{-1} F_0^a S_z$ it follows that

$$\chi = \frac{\mu_0^2 N_F}{1 + F_0^a} \qquad (17)$$

χ is seen to be given by the Pauli susceptibility of a noninteracting Fermi gas of particles with mass m^*, "screened" by a polarization field proportional to the Landau parameter F_0^a.

The density response to a change in chemical potential is obtained in complete analogy as

$$\frac{dn}{d\mu} = \frac{N_F}{1 + F_0^s} \qquad (18)$$

Thermodynamic stability requires that the susceptibilities χ and $\frac{dn}{d\mu}$ be positive, which leads to the requirements $F_0^{a,s} > -1$. A general analysis of the stability of the system with respect to any variation of $n_{\vec{k}\sigma}$ results in the stability conditions

$$F_\ell^{a,s} > -(2\ell + 1) \ , \quad \ell = 0, 1, .. \qquad (19)$$

For Galilei-invariant systems there exists a relation between the Landau parameter F_1^s and the effective mass ratio m^*/m. Consider a Galilei transformation to a system K' moving with velocity $-\vec{u}$. In K' the momenta and energies of quasiparticles are given by $\vec{k} + m\vec{u}$ and $\epsilon_{\vec{k}+m\vec{u}}$ and the distribution function is centered at $\vec{k} = m\vec{u}$, i.e. $n'_{\vec{k}\sigma} = n_{\vec{k}-m\vec{u}\sigma}$. We must have in linear oder in $\vec{u}$

$$\begin{aligned} \delta\epsilon_{\vec{k}\sigma} = \vec{k}\cdot\vec{u} &= \epsilon^0_{\vec{k}+m\vec{u}} - \epsilon^0_{\vec{k}} + \textstyle\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'} \left(n^0_{\vec{k}'-m\vec{u}} - n^0_{\vec{k}'}\right) \\ &= \tfrac{m}{m^*}\vec{k}\cdot\vec{u} + \textstyle\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'} \left(-\frac{\partial n^0}{\partial\epsilon_{\vec{k}'}}\right) m\vec{k}'\cdot\vec{u} \\ &= \tfrac{m}{m^*}\left[1 + \tfrac{F_1^s}{3}\right]\vec{k}\cdot\vec{u} \end{aligned} \qquad (20)$$

from which we get

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3} \qquad (21)$$

C. Transport properties.

1. Quasiparticle relaxation rate

So far we have been assuming that the quasiparticles are absolutely stable excitations. This is not the case, as may be suscepted from the fact that they interact strongly. The simplifying feature is that at low temperature $T << T_F$ there is only a small number of thermally excited quasiparticles around, which may act as interaction partners of a given quasiparticle.

The decay rate $\frac{1}{\tau}$ of a quasiparticle on top of the filled Fermi sea may be easily estimated in the following way. At low T, $\frac{1}{\tau}$ is dominated by binary collision processes, in which the considered quasiparticle in state $| 1 > = | \vec{k}_1\sigma_1 >$ scatters off a partner in state $| 2 >$, the two particles leaving in final states $| 3 >$ and $| 4 >$. The decay rate is given by the golden rule expression

$$\frac{1}{\tau_{\vec{k}_1\sigma_1}} = 2\pi \sum{}'_{234} | a(1,2;3,4) |^2 n_2^0 (1 - n_3^0)(1 - n_4^0) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \qquad (22)$$

where $a(1,2;3,4)$ is the transition amplitude, or scattering amplitude, and $n_2, 1-n_3, 1-n_4$ are distribution function factors describing the probability for state $| 2 >$ to be occupied and the final states to be empty. The summation over momenta and spins is restricted by momentum and spin conservation, requiring $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$ and $\sigma_1 + \sigma_2 = \sigma_3 + \sigma_4$. For a quick estimate at $T = 0$ we approximate $a(1,2,3,4)$ by a constant and the momentum sums by energy integrals ($\epsilon_i = \epsilon_{\vec{k}_i\sigma_i} - \mu$) and factors of density of states:

$$\begin{aligned} \frac{1}{\tau} &\sim | aN_F |^2 \tfrac{1}{\epsilon_F} \textstyle\int_{-\infty}^0 d\epsilon_2 \int_0^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \\ &= | aN_F |^2 \tfrac{1}{\epsilon_F} \textstyle\int_{-\epsilon_1}^0 d\epsilon_2 \int_0^{\epsilon_1+\epsilon_2} d\epsilon_3 = \tfrac{1}{2}(aN_F)^2 \tfrac{1}{\epsilon_F}\epsilon_1^2 \end{aligned} \qquad (23)$$

The quasiparticle decay rate is seen to be finite, i.e. the quasiparticle states are not energy eigenstates, but they are approaching true eigenstates in the limit $\epsilon_1 \to 0$, i.e. on the Fermi surface. Hence the quasiparticles are well-defined objects for not too large excitation energy, $\mid \epsilon_k - \mu \mid << \mu$.

A full evaluation of $\frac{1}{\tau}$ yields [1]

$$\frac{1}{\tau_k} = \left[1 + \left(\frac{\epsilon_k - \mu}{\pi T}\right)^2\right] \frac{1}{\tau_0} \tag{24}$$

where

$$\frac{1}{\tau_0} = \frac{\pi^3}{64} \frac{T^2}{\epsilon_F} < W > \tag{25}$$

and

$$\begin{aligned} < W > \quad &= \int_0^1 d\cos\frac{\theta}{2} \int_0^{2\pi} \frac{d\phi}{2\pi} W(\theta, \phi) \\ W(\theta,\phi) &= \mid A_0(\theta,\phi) \mid^2 + 3 \mid A_1(\theta,\phi) \mid^2 \end{aligned} \tag{26}$$

The quantities A_0 and A_1 are the dimensionless scattering amplitudes in the spin singlet and triplet channel, respectively, defined by

$$\begin{aligned} a(1,2;3,4) \quad &= \frac{1}{N_F}\left[A^s \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + A^a \vec{\tau}_{\sigma_1\sigma_3} \cdot \vec{\tau}_{\sigma_2\sigma_4}\right] \\ &= \frac{1}{2N_F}\left[- A_0 \tau^y_{\sigma_1\sigma_2}\tau^y_{\sigma_3\sigma_4} + A_1 (\tau^y \vec{\tau})_{\sigma_1\sigma_2} \cdot (\tau^y\vec{\tau})_{\sigma_3\sigma_4}\right] \end{aligned} \tag{27}$$

where

$$\begin{aligned} A^s &= \tfrac{1}{4}(3A_1 + A_0) \\ A^a &= \tfrac{1}{4}(A_1 - A_0) \end{aligned} \tag{28}$$

and $\vec{\tau}$ is the vector of Pauli matrices. In doing the momentum integrals we have used the fact that all the momenta $\vec{k}_1, \ldots \vec{k}_4$ are close the Fermi momentum and conserve momentum and $A_{0,1}$ may be taken to depend only on the orientation of the $\vec{k}_i$, leaving only two angular variables.

The parametrization used here is the one introduced by Abrikosov and Khalatnikov [see Ref. 1]: θ is the angle between $\vec{k}_1$ and $\vec{k}_2$, and ϕ is the angle enclosed by the planes $(\vec{k}_1, \vec{k}_2)$ and $(\vec{k}_3, \vec{k}_4)$.

2. Kinetic equation

In the presence of slowly varying disturbances in space and time, the system may be described by a quasiclassical distribution function $n_{\vec{k}\sigma}(\vec{r}, t)$. This is possible as long as the energy and momentum of the quanta of the external field ω, q are much less than the typical energy and momentum of the quasiparticles, i.e. $\omega << T$, $q << \frac{T}{v_F}$. The distribution function satisfies the kinetic equation

$$\partial_t n_{\vec{k}\sigma} + \vec{\nabla}_k \epsilon_{\vec{k}\sigma} \cdot \vec{\nabla}_r n_{\vec{k}\sigma} - \vec{\nabla}_r \epsilon_{\vec{k}\sigma} \cdot \vec{\nabla}_k n_{\vec{k}\sigma} = I\{n_{\vec{k}\sigma}\} \tag{29}$$

The left-hand side describes the dissipationless flow of quasiparticles in phase space. It goes beyond the Boltzmann equation for a dilute classical gas in that the quasiparticle energy $\epsilon_{\vec{k}}(\vec{r}, t)$ is itself a quantity dependent on position and time, due to its dependence on the distribution function as given by (7). As we will see, this gives rise to the appearance of collective modes, as well as interesting nonlinear effects (which we will not discuss).

On the r.h.s. of (29) we have the socalled collision integral I, which describes the abrupt change of momentum and spin of quasiparticles in a collision process. It is given by $I_{k\sigma} = -n_{\vec{k}\sigma}/\tau^{noneq}_{\vec{k}\sigma}$, which is the number of quasiparticles in state $\mid \vec{k}\sigma >$ decaying per unit time. The nonequilibrium relaxation rate $1/\tau^{noneq}_{\vec{k}\sigma}$ is obtained from (22) by replacing $n_{\vec{k}_i\sigma_i}$ and $\epsilon_{\vec{k}_i\sigma_i}$ by their nonequilibrium counterparts.

The collision integral is zero, when multiplied by $1, \vec{k}, \sigma, \epsilon_{\vec{k}\sigma}$ and summed over $\vec{k}, \sigma$, expressing the conservation of particle number, momentum, spin, energy, in the collision process. Conversely, the collision integral vanishes if $n_{\vec{k}_i}$ has the form of a socalled local equilibrium distribution function

$$n^{\ell e}_{\vec{k}\sigma} = \left[\exp \frac{\epsilon^{\ell e}_{\vec{k}\sigma}(\vec{r}, t) - \mu_\sigma(\vec{r}, t) - \vec{k} \cdot \vec{u}(\vec{r}, t)}{T(\vec{r}, t)} + 1\right]^{-1} \tag{30}$$

where $\mu_\sigma(\vec{r}, t)$ is a local spin dependent chemical potential, $T(\vec{r}, t)$ is the local temperature , $\vec{u}(\vec{r}, t)$ is a local fluid velocity and $\epsilon^{\ell e}_{\vec{k}\sigma}(\vec{r}, t)$ is the corresponding local quasiparticle energy. The values of these potentials are determined by the local densities. The effect of collisions is to drive $n_{\vec{k}\sigma}$ towards a local equilibrium form.

Ofen the applied external field causing a particular transport phenomenon is weak and one is allowed to linearize in the deviation of the distribution function from its equilibrium value or else a particular local equilibrium value. The resulting linearized and Fourier transformed kinetic equation is given by

$$(\omega - \vec{v}_k \cdot \vec{q})\delta n_{\vec{k}}(q,\omega) + \vec{v}_k \cdot \vec{q}\frac{\partial n_k^0}{\partial \epsilon_k}\delta\epsilon_{\vec{k}} = i\delta I \tag{31}$$

where $\delta\epsilon_k$ is defined as

$$\delta\epsilon_{k\sigma}(q,\omega) = \sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'}\delta n_{\vec{k}'\sigma'}(q,\omega) \tag{32}$$

In principle, $f_{\vec{k}\sigma\vec{k}'\sigma'}$ is a function of q and ω, and the limit $q,\omega \to 0$ is understood here. (We will come to the subleties of this limit later.) For a charged system with long range Coulomb interaction, a classical or Hartree-type interaction part must be separated out, such that

$$f_{\vec{k}\sigma\vec{k}'\sigma'}(q) = \frac{4\pi e^2}{q^2} + \tilde{f}_{\vec{k}\sigma\vec{k}'\sigma'} \tag{33}$$

which affects only the $\ell = 0$, s channel.

3. Zero sound

As we have seen, a quasiparticle is embedded in an effective medium formed by the other quasiparticles. It may be expected that this medium has a certain stiffness or elasticity, and will provide a restoring force for spatially and temporally varying components of the distribution function. Of course, the components with large Landau parameters will be dominant. Let us assume that the Landau parameter F_0^s is large and positive, as is actually the case in liquid ^{3}He, and let us drop all other F_ℓ's. We will also drop collision effects for the moment. The kinetic equation takes the form

$$(\omega - \vec{v}_k \cdot \vec{q})\delta n_{\vec{k}}(q,\omega) + \vec{v}_k \cdot \vec{q}\frac{\partial n_k^0}{\partial \epsilon_k}\Big[N_F^{-1}F_0^s \sum_{\vec{k}'\sigma'} \delta n_{\vec{k}'\sigma'} - \delta\mu^{ext}\Big] = 0 \tag{34}$$

where an external potential $-\delta\mu^{ext}$ has been assumed. We may solve this equation to get the density response function

$$\frac{\delta n}{\delta\mu^{ext}} = \frac{\chi_0(q,\omega)}{1 + F_0^s\chi_0(q,\omega)} \tag{35}$$

where

$$\chi_0(q,\omega) = \sum_{\vec{k}\sigma} \frac{\vec{v}_k \cdot \vec{q}}{\omega - \vec{v}_k \cdot \vec{q} + i0}\frac{\partial n_k^0}{\partial \epsilon_k} \tag{36}$$

This is the well-known RPA form of the density response function in the limit $q << k_F$. In order to show that the denominator of (36) vanishes for given q at a certain ω, we expand χ_0 for large ω:

$$\chi_0(q,\omega) \simeq \sum_{\vec{k}\sigma} \Big(\frac{\vec{v}_k \cdot \vec{q}}{\omega}\Big)^2 \frac{\partial n_k^0}{\partial \epsilon_k} \simeq -\frac{1}{3}\Big(\frac{v_F q}{\omega}\Big)^2 \tag{37}$$

Hence $\frac{\delta n}{\delta\mu^{ext}}$ has a pole at

$$\omega = \sqrt{F_0^s/3}\; v_F q \tag{38}$$

corresponding to a collective mode of the system with the characteristics of a density wave, called "zero sound".

In addition to this pole contribution the absorptive part of the density response, $\text{Im}\{\frac{\delta n}{\delta\mu^{ext}}\}$, is characterized by a continuum of particle-hole excitations. Their contribution follows from

$$\text{Im}\{\chi_0(q,\omega)\} = \frac{\pi}{2}N_F\Big(\frac{\omega}{v_F q}\Big)\theta(v_F q - |\omega|) \tag{39}$$

One finds that the collective mode branch is outside the p-h continuum for $F_0^s > 0$. For $-1 < F_0^s < 0$ the collective mode is overdamped: the collective excitation may decay in p-h pairs.

Zero sound modes may exist for any component ℓ, s or a, of $\delta n_{\vec{k}\sigma}$, provided the corresponding Landau parameter is sufficiently large. For a charged system, the zero sound mode develops a gap and is identical to the plasma mode as is seen by substituting (33) into (35). Of course, the plasma frequency $\omega_{pl} = (4\pi e^2 n/m)^{1/2}$ is usually beyond the regime of validity of Fermi liquid theory.

Let us now have a brief look at the effect of collisions. The characteristic frequency $1/\tau$ separates the collision dominated hydrodynamic regime, where $\omega << 1/\tau$, and the "collisionless" mean-field regime, where $\omega >> 1/\tau$. A simple phenomenological description captures the main trends (in the case $F_0^s >> 1$). We begin with the conservation laws for particle number

$$\omega \delta n = \vec{q} \cdot \vec{j} \tag{40}$$

and momentum

$$\omega m \vec{j} = \vec{q} \delta P - \vec{q} \cdot \overleftrightarrow{\Pi} \tag{41}$$

where $\vec{j} = \sum_{\vec{k}\sigma} \vec{v}_k [\delta n_{\vec{k}\sigma} - (\partial n_k^0 / \partial \epsilon_k) \delta \epsilon_{\vec{k}\sigma}]$, is the number current density, which follows from the kinetic equation (31). The pressure change δP may be expressed in terms of the density change by $\delta P = \frac{\partial P}{\partial n} \delta n$, where the thermodynamic derivative $(\partial P/\partial n) = m c_1^2$, is connected with c_1, the usual hydrodynamic (or "first") sound velocity. In the hydrodynamic limit, the longitudinal part of the stress tensor is given in terms of the shear viscosity η as

$$\hat{q} \cdot \overleftrightarrow{\Pi}^D \cdot \hat{q} = -i \frac{4}{3} \frac{\eta}{n} \vec{q} \cdot \vec{j} \tag{42}$$

since the bulk viscosity is negligible in a Fermi liquid. At higher frequencies, $\overleftrightarrow{\Pi}$ deviates from $\overleftrightarrow{\Pi}^D$, but relaxes towards $\overleftrightarrow{\Pi}^D$:

$$\omega \overleftrightarrow{\Pi} = -\frac{i}{\tau_\eta} (\overleftrightarrow{\Pi} - \overleftrightarrow{\Pi}^D) \tag{43}$$

The relaxation time is taken to be the one defined by the kinetic expression for the shear viscosity $\eta = \frac{1}{5} m n v_F^2 \tau_\eta$. Substituting (42) into (43) and (43) into (41), and finally (41) into (40), one obtains a dispersion relation for density waves

$$\omega^2 = c_1^2 q^2 \Big(1 - \frac{4}{15} \frac{v_F^2}{c_1^2} \frac{i\omega\tau_\eta}{1 - i\omega\tau_\eta}\Big) \tag{44}$$

The sound absorption $\alpha = -\mathrm{Im}\{q\}$ follows as

$$\alpha(\omega) = \frac{2}{15} \frac{\omega}{c_1} \frac{v_f^2}{c_1^2} \begin{cases} \omega\tau_\eta \ , & \omega\tau_\eta << 1 \\ \frac{1}{\omega\tau_\eta} \ , & \omega\tau_\eta >> 1 \end{cases} \tag{45}$$

In the hydrodynamic regime, $\alpha \sim \omega^2/T^2$, whereas in the zero sound regime, $\alpha \sim T^2$. One can show that in the extreme low temperature quantum regime, $\omega > T$, $\alpha \sim \omega^2$.

D. Quasiparticle scattering amplitude and effective pair interaction

The quasiparticle scattering amplitude $a(1,2;3,4)$ may be related to the Landau parameters in the limit of forward scattering. In order to demonstrate this it is useful to parametrize a in a different form. Instead of the scattering of particles $1,2$ into $3,4$ we may as well consider the scattering of particle 1 and hole 3 into particle 4 and hole 2, or else, taking into account momentum conservation, the scattering of particle-hole pair $(\vec{k} + \vec{q}/2 \ , \ \vec{k} - \vec{q}/2)$ into p-h pair $(\vec{k}\,' + \vec{q}/2 \ , \ \vec{k}\,' - \vec{q}/2)$. Taking all momenta on the Fermi surface, the relevant two variables are then (i) the momentum transfer q (ii) the "Landau" angle, θ_L, enclosed by $\vec{k}$ and $\vec{k}\,'$. The scattering amplitude $a_{\vec{k}\vec{k}'}(q)$ may be shown to be equal to the change of q-th Fourier component of the quasiparticle energy $\delta\epsilon_{\vec{k}}(\vec{q})$, induced by the addition of a particle-hole pair with momenta $(\vec{k}\,' + \frac{\vec{q}}{2} \ , \ \vec{k}\,' - \frac{\vec{q}}{2})$.

$$a_{\vec{k}\vec{k}'}(q) = \Big(\frac{\delta\epsilon_{\vec{k}}(\vec{q})}{\delta n_{\vec{k}'}(\vec{q})}\Big)_{\text{total}} \tag{46}$$

including all rearrangements in states $\vec{k}\,'' \neq \vec{k}, \vec{k}\,'$. By contrast, the Fermi liquid interaction is defined by

$$f_{\vec{k}\vec{k}'} = \Big(\frac{\delta\epsilon_{\vec{k}}}{\delta n_{k'}}\Big)_{q=0, n_{\vec{k}''}} \tag{47}$$

which is the energy change $\delta\epsilon_k$ induced by adding a quasiparticle in state $| \vec{k}\,' >$, keeping the occupation of all other states fixed. Using the kinetic equation, we can write

$$\Big(\frac{\delta\epsilon_{\vec{k}}}{\delta n_{\vec{k}'}}\Big)_{\text{total}} = \Big(\frac{\delta\epsilon_{k}}{\delta n_{\vec{k}'}}\Big)_{n_{k''}} - \sum_{\vec{k}'' \neq \vec{k}'} \Big(\frac{\delta\epsilon_{\vec{k}}}{\delta n_{\vec{k}''}}\Big)_{n_{k'''}} \frac{(\vec{v}_{\vec{k}''} \cdot \vec{q})(\frac{\partial n_{\vec{k}''}}{\partial \epsilon_{\vec{k}''}})}{\omega - \vec{v}_{\vec{k}''}\vec{q}} \Big(\frac{\delta\epsilon_{\vec{k}''}}{\delta n_{\vec{k}'}}\Big)_{\text{total}} \tag{48}$$

which yields in the limit $\omega = 0$, $q \to 0$

$$a_{\vec{k}\vec{k}'}(0) = f_{\vec{k}\vec{k}'} + \sum_{\vec{k}''} f_{\vec{k}\vec{k}''} \Big(\frac{\partial n_{\vec{k}''}}{\partial \epsilon_{k''}}\Big) a_{\vec{k}''\vec{k}'}(0) \tag{49}$$

This integral equation may be solved by decomposition into a Legendre series:

$$a_{\vec{k}\vec{k}'}^{s,a}(0) = \frac{1}{N_F} \sum_{\ell=0}^{\infty} A_\ell^{s,a}(0) P_\ell(\hat{k} \cdot \hat{k}\,') \tag{50}$$

with the result

$$A_\ell^{s,a}(0) = \frac{F_\ell^{s,a}}{1 + F_\ell^{s,a}/(2\ell+1)} \tag{51}$$

Thus we have found that the forward scattering amplituide (limit $q \to 0$) is completely determined by the Landau parameters. Since we expect the F_ℓ's to vanish rapidly with increasing ℓ, a model form which keeps only the first few parameters F_0^s, F_1^s, F_0^a, etc., will give a reasonable approximation for $A(q, \cos\theta_L)$ at $q = 0$.

Had we taken the opposite limit, $q = 0$, $\omega \to 0$ in (48), the result would have been very different: $\lim_{\omega\to 0} \lim_{q\to 0}(\delta\epsilon_{\vec{k}}/\delta n_{\vec{k}'})_{\text{total}} = f_{\vec{k}\vec{k}'}$.

The form of $A(q, \cos\theta_L)$ for large momentum transfer q is to some extent determined by exchange symmetry, which requires the singlet (triplet) scattering amplitude to be even (odd) under exchange of the momenta of the particles in the initial state,

$$A_J(\theta, \phi) = (-1)^J \; A_J(\theta, \phi + \pi), \quad J = 0, 1 \tag{52}$$

or in terms of the q, θ_L parametrization

$$A_J(q, \theta_L) = (-1)^J A_J(\tilde{q}, \tilde{\theta}_L) \;, \quad J = 0, 1 \tag{53}$$

where in terms of variables $z = \frac{1}{2}(\frac{q}{k_f})^2 - 1$ and $y = \cos\theta_l\,(1 - z)$ the exchange transformation takes the (linear) form

$$\begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{54}$$

The latter transformation has a line of fixed points

$$3z^* + y^* = -1 \tag{55}$$

i.e. $q = \tilde{q}$ and $\theta_L = \tilde{\theta}_l$ along this line in the $(q, \cos\theta_l)$ plane. It follows that the triplet amplitude A_1 is zero along this line. At the point $q = 0$, $\theta_l = 0$ the condition $A_1(q = 0, \theta_L = 0) = 0$ yields a sum rule for the $A_\ell^{s,a}(0)$:

$$\sum_{\ell=0}^{\infty}(A_\ell^s(0) + A_\ell^a(0)) = 0 \tag{56}$$

The scattering amplitude may be expanded in terms of a complete set of orthonormal functions with definite exchange symmetry [4,11]:

$$X_{\ell k}(\theta, \phi) = \left[(k+1)(2\ell+1)\right]^{1/2}(-)^\ell P_\ell(\cos\phi)\left[\sin\frac{\theta}{2}\right]^{2\ell} P_{k-\ell}^{(2\ell+1,0)}(\cos\theta) \tag{57}$$

where $P_\ell(x)$ and $P_{k-\ell}^{(2\ell+1,0)}(x)$ are the Legendre polynomials and the Jacobi polynomials of index $(2\ell+1, 0)$ and degree $k - \ell$. These functions obey the orthogonality relations

$$\frac{1}{2}\int_{-1}^{1} d\cos\theta \int_{-1}^{1} d\cos\phi \;\sin^2\frac{\theta}{2}\; X_{\ell k}(\theta,\phi)X_{\ell' k'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{kk'} \tag{58}$$

The parity of $X_{\ell k}$ under the exchange operation is obviously $(-1)^\ell$. Therefore, the singlet and triplet amplitudes may be expanded in terms of the even ℓ and odd ℓ functions, respectively:

$$\begin{aligned} A_0(\theta, \phi) &= \textstyle\sum_{k=0}^{\infty} \sum_{\ell\le k}^{\text{even}} a_{\ell k} X_{\ell k}(\theta, \phi) \\ A_1(\theta, \phi) &= \textstyle\sum_{k=1}^{\infty} \sum_{\ell\le k}^{\text{odd}} a_{\ell k} X_{\ell k}(\theta, \phi) \end{aligned} \tag{59}$$

Using the available experimental information on the $F_\ell^{s,a}$ and additional information on angular averages of the type $< A_1 A_2 >$ from various transport measurements (see the expression for $1/\tau_k$, (25)) the first few coefficients $a_{\ell k}$ may be determined, assuming the higher coefficients to be zero.

¿From the general form of the scattering amplitude, we may extract information on the interaction of pairs of particles with momenta $\vec{k}, -\vec{k}$ (corresponding to scattering angles $\theta = \pi$, ϕ arbitrary), relevant for superconductivity. As we will discuss in the next lecture, the scattering amplitude has a singularity if the instability towards a superconducting ground state occurs, which is the case if one or several angular momentum components V_ℓ of the pair interaction

$$V_{\vec{k},\vec{k}'} = \sum_{\ell=0}^{\infty}(2\ell+1)V_\ell P_\ell(\hat{k}\cdot\hat{k}') \tag{60}$$

are negative (attractive interaction). We note that $\vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$, and $\vec{k}' = \frac{1}{2}(\vec{k}_3 - \vec{k}_4)$, and therefore $\hat{k}\cdot\hat{k}' = \cos\phi$. For sufficiently small coupling constants, $\lambda_\ell = \frac{1}{2}N_F V_\ell < 1$, the components of the scattering amplitude are approximately equal to the pair interaction, except for a small region around the singularity. The pair coupling constants λ_ℓ may then be extracted from the scattering amplitude by

$$\lambda_\ell = \frac{1}{4}\int_{-1}^{1} d\cos\phi \;\; P_\ell(\cos\phi) \;\; A_J(\pi,\phi) \tag{61}$$

where $J = 0$ or 1 for ℓ even or odd. Employing the expansion (59) of A_J one finds

$$\lambda_\ell = \frac{1}{4}(2\ell+1)^{-1/2}\sum_{k=\ell}^{\infty}(k+1)^{1/2}(-1)^k a_{\ell k} \tag{62}$$

Let us use this general result to obtain a rough estimate of the pair interaction in liquid ^{3}He. The three Landau parameters F_0^s, F_0^a, F_1^s determine the weight factors $a_{k\ell}$ of the three lowest eigenfunctions: $a_{00} = -\frac{4}{3}A_1^s - 4A_0^a$, $a_{01} = \sqrt{2}(\frac{4}{3}A_1^s + A_0^s + A_0^a)$, $a_{11} = \sqrt{\frac{2}{3}}(-A_0^s - A_0^a)$. This yields the following explicit expressions for the pair coupling constants:

$$\begin{aligned}\lambda_0 &= -\frac{1}{2}(A_0^s + 3A_0^a) - A_1^s \\ \lambda_1 &= \frac{1}{6}(A_0^s + A_0^a)\end{aligned} \tag{63}$$

Substituting the values $F_0^s \cong 10$ to 100, $F_1^s \cong 6$ to 15 and $F_0^a \simeq -0.7$ for low to high pressures, respectively, one finds $\lambda_0 \simeq 3/2$ and $\lambda_1 \simeq -1/3$. Thus the simple estimate yields a clear indication for attractive pair interaction in the p-wave channel.

III. SUPERFLUID FERMI LIQUIDS: THE CASE OF ^{3}HE

Superfluidity is a property of a macroscopically occupied quantum state. In a Fermi system a macroscopic occupation of a quantum state can occur for compound objects with bosonic character, like pairs, quadruples, etc. One should expect the most likely compound to be formed for a one-component system to be a pair of particles. One can distinguish two limiting cases: (i) the case of preformed pairs of extension less or at most comparable to the interparticle distance (ii) the case of weakly bound pairs, of extension much larger than the interparticle distance. In case (i) the system behaves at energies less than the binding energy (which must be larger than the Fermi energy) as a Bose system, i.e. there will be a Bose-Einstein transition into a superfluid state. A good example is liquid ^{4}He. In case (ii) the bound state of energy much less than the Fermi energy will be broadened into a resonance and the system will behave like a normal Fermi liquid down to a temperature T_c below which a coherent "condensate" of pairs will form. This is the case of BCS theory [12], which we will consider now in the example of liquid ^{3}He.

A. Cooper instability

We will be interested in Fermi systems with largely repulsive interactions, such as the repulsive hard core interaction between two He atoms or the Coulomb repulsion between electrons. In this situation we can not expect an attractive interaction component to be very strong, such as to produce a bound state of two isolated particles in three dimensions (actually, the necessary condition, $|V_0| > \hbar^2/mr_0^2$, where V_0 and r_0 are strength and range of the attractive potential, would require $|V_0| \gtrsim \epsilon_F$, for a shortrange potential with $r_0 \lesssim k_F^{-1}$). However, as discovered by Cooper [13], the fact that in a degenerate Fermi system the occupied states in the Fermi sea are not available as final states in the scattering process changes the situation dramatically. Let us consider the two-particle Schrvdinger equation in momentum representation

$$\left(\xi_{\vec{P}/2+\vec{k}} + \xi_{\vec{P}/2-\vec{k}} - E\right)\psi_{\vec{k}} = -\sum{}'_{\vec{k}'} V_{\vec{k}-\vec{k}'}\psi_{\vec{k}'} \tag{64}$$

where $\xi_{\vec{k}} = \frac{k^2}{2m} - \mu$, $\vec{P}/2 \pm \vec{k}$ are the momenta of the two particles and $V_{\vec{k}-\vec{k}'}$ is the pair interaction. The summation over final states $\vec{P}/2 \pm \vec{k}\,'$ is restricted by the conditions $|\,\vec{P}/2 \pm \vec{k}\,'\,| > k_F$ due to the Pauli principle. It is clear that the lowest energy eigenvalue is obtained for zero center of mass momentum $\vec{P} = 0$.

In order to determine a possible bound state, it is convenient to expand $V_{\vec{k}-\vec{k}'}$ in terms of Legendre polynomials

$$V_{\vec{k}-\vec{k}'} = \sum_{\ell=0}^{\infty}(2\ell + 1)V_\ell(k, k\,')P_\ell(\hat{k}\cdot\hat{k}') \tag{65}$$

and to model the coefficients V_ℓ as

$$V_\ell(k, k\,') = \begin{cases} V_\ell & ; \quad |\,\xi_k\,|\,,\ |\,\xi_{k'}\,| << \epsilon_c << \epsilon_F \\ 0 & \text{else} \end{cases} \tag{66}$$

The separable integral equations for each ℓ

$$(2\xi_k - E)\psi_\ell(k) = -V_\ell N(0)\int_0^{\epsilon_c} \psi_\ell(k\,')d\xi_{k'} \tag{67}$$

may be easily solved. One observes that for each ℓ-channel with attractive interaction, $V_\ell < 0$, there exists a bound state with energy

$$E_\ell = -2\epsilon_c \exp\left(-\frac{2}{N(0)\,|\,V_\ell\,|}\right) \tag{68}$$

The reason is that due to the blocking of states in the Fermi sea the formation of the bound state involves particles in states near the Fermi energy, where the density of states is large and approximately independent of energy, whereas in the usual two-particle problem the relevant states are near zero energy, where the density of states vanishes as $\sqrt{E}$. The Cooper pair problem is thus similar to the usual two particle problem in two dimensions, where the density of states is finite for $E \to 0$ and exactly one bound state is known to exist for arbitrarily small attraction.

We conclude that in the case of one or several attractive interaction components V_ℓ, the Fermi sea is unstable against formation of Cooper pairs. The pairs may be expected to form in the angular momentum channel L with the strongest attraction. In the case of ^{3}He this is the $L = 1$, or p-wave component.

B. Mean field theory

The Hamiltonian of the interacting Fermi system

$$H - \mu N = \sum_{\vec{k}\sigma}\xi_k a^+_{\vec{k}\sigma}a_{\vec{k}\sigma} + \frac{1}{2}\sum_{\substack{\vec{k},\vec{k}',\vec{q}\\ \sigma,\sigma'}} V_{\vec{k}-\vec{k}'}a^+_{\vec{k}'\sigma}a^+_{-\vec{k}'+\vec{q}\sigma'}a_{-\vec{k}\sigma'}a_{\vec{k}+\vec{q}\sigma} \tag{69}$$

may be approximated in terms of the mean fields

$$\begin{aligned} &< a^+_{k\sigma}a_{k'\sigma'} > = \delta_{kk'}\delta_{\sigma\sigma'}n_{k\sigma} \\ &< a_{-k\sigma'}a_{k'\sigma} > = \delta_{kk'}F_{k\sigma\sigma'} \end{aligned} \tag{70}$$

expressing the possibility of a "pair condensate", as

$$\begin{aligned} H_{MF} - \mu N = &\sum_{\vec{k}\sigma}\xi_k a^+_{k\sigma}a_{k\sigma} + \tfrac{1}{2}\sum_{\vec{k}\sigma\sigma'}\left(\Delta^*_{\vec{k}\sigma\sigma'}a_{-\vec{k}\sigma'}a_{\vec{k}\sigma} + a^+_{k\sigma}a^+_{-k\sigma'}\Delta_{k\sigma'\sigma}\right) \\ &- \tfrac{1}{2}\sum_{\vec{k}\sigma\sigma'}\Delta^*_{\vec{k}\sigma\sigma'}F_{\vec{k}\sigma\sigma'} \end{aligned} \tag{71}$$

where the "off-diagonal energy" or "gap function" has been introduced as

$$\Delta_{\vec{k}\sigma\sigma'} = \sum_{\vec{k}'} V_{\vec{k}-\vec{k}'}F_{\vec{k}'\sigma\sigma'} \tag{72}$$

Since H_{MF} is a bilinear form in the field operators $a_{k\sigma}$, $a^+_{k\sigma}$, it may be diagonalized by a linear transformation, mixing particles and holes in states $\vec{k}$ and $-\vec{k}$, the socalled Bogoliubov-Valatin transformation. The new fermion operators $b_{\vec{k}\sigma}$, $b^+_{\vec{k}\sigma}$, defined by

$$b_{\vec{k}\sigma} = \sum_{\sigma'}\left(u_{\vec{k}\sigma\sigma'}a_{\vec{k}\sigma'} - v_{\vec{k}\sigma\sigma'}a^+_{-\vec{k}\sigma'}\right) \tag{73}$$

are chosen such that terms $b_{\vec{k}}b_{-\vec{k}}$ vanish in the transformed Hamiltonian. One finds for the simplest cases of (i) unitary $v_{k\sigma\sigma'}$ and (ii) diagonal $v_{k\sigma\sigma'}$ and $u_{k\sigma\sigma'}$

$$\begin{aligned} u_{\vec{k}\sigma\sigma'} &= \delta_{\sigma\sigma'}\left[2(\xi_{\vec{k}\sigma} + E_{\vec{k}\sigma})/E_{\vec{k}\sigma}\right]^{1/2} \\ v_{\vec{k}\sigma\sigma'} &= -\Delta_{\vec{k}\sigma\sigma'}\left[2(\xi_{\vec{k}\sigma} + E_{\vec{k}\sigma})E_{\vec{k}\sigma}\right]^{-1/2} \end{aligned} \tag{74}$$

and

$$H_{MF} - \mu N = \sum_{\vec{k}\sigma} \left\{ E_{\vec{k}\sigma} b^+_{\vec{k}\sigma} b_{\vec{k}\sigma} - \frac{1}{2}\Big[\frac{(\Delta_k \Delta^+_k)_{\sigma\sigma}}{\xi_{k\sigma} + E_{k\sigma}} + (\Delta^+_k F_k)_{\sigma\sigma}\Big]\right\} \tag{75}$$

The energy eigenvalue $E_{\vec{k}\sigma}$ is found as

$$E_{\vec{k}\sigma} = \left[\xi^2_{\vec{k}\sigma} + (\Delta_{\vec{k}} \Delta^+_{\vec{k}})_{\sigma\sigma}\right]^{1/2}. \tag{76}$$

The mean field Hamiltonian (75) describes independent fermions, the socalled Bogoliubov quasiparticles (BQP). The number of BQP's is not conserved. The energy spectrum $E_{\vec{k}\sigma}$ has a gap at the Fermi surface, which is in general anisotropic.

The expectation values $n_{\vec{k}\sigma}$ and $F_{\vec{k}\sigma\sigma'}$ may be obtained from the inverse B-V-transformation, and using

$$\begin{aligned} &< b^+_{\vec{k}\sigma} b_{\vec{k}\sigma'} > = \delta_{\vec{k}\vec{k}'} \delta_{\sigma\sigma'} f_{\vec{k}\sigma} \\ &< b_{\vec{k}\sigma} b_{\vec{k}'\sigma'} > = 0 \end{aligned} \tag{77}$$

where

$$f_{\vec{k}\sigma} = f(E_{\vec{k}\sigma}) = \frac{1}{\exp(E_{\vec{k}\sigma}/T) + 1} \tag{78}$$

These relations follow from the fact that in the approximation (75) for the Hamiltonian the Bogoliubov quasiparticles form a noninteracting Fermi gas. One finds

$$\begin{aligned} n_{\vec{k}\sigma} &= \frac{1}{2}\Big(1 + \frac{\xi_{\vec{k}\sigma}}{E_{\vec{k}\sigma}}\Big) f_{\vec{k}\sigma} + \frac{1}{2}\Big(1 - \frac{\xi_{\vec{k}\sigma}}{E_{\vec{k}\sigma}}\Big)(1 - f_{\vec{k}\sigma}) \\ &= \frac{1}{2}\Big(1 - \frac{\xi_{k\sigma}}{E_{k\sigma}} \tanh \frac{E_{k\sigma}}{2T}\Big) \\ F_{\vec{k}\sigma\sigma'} &= \frac{\Delta_{\vec{k}\sigma\sigma'}}{2E_{\vec{k}\sigma}}(1 - 2f_{\vec{k}\sigma}) \end{aligned} \tag{79}$$

The expression for $F_{\vec{k}\sigma\sigma'}$ in terms of $\Delta_{\vec{k}\sigma\sigma'}$ is used to determine $\Delta_{\vec{k}\sigma\sigma'}$, by substituting into (73). One obtains the socalled gap equation

$$\Delta_{\vec{k}\sigma\sigma'} = -\sum_{\vec{k}'} V_{\vec{k}-\vec{k}'} \frac{\Delta_{\vec{k}'\sigma\sigma'}}{2E_{\vec{k}'\sigma}} \tanh \frac{E_{\vec{k}'\sigma}}{2T} \tag{80}$$

The solution of the gap equation is most conveniently discussed in terms of the eigenfunctions of the pair potential on the Fermi surface, assuming $V_{\vec{k}-\vec{k}'}$ to vanish outside a shell

of width $2\epsilon_c$ about the Fermi energy. These eigenfunctions may be classified according to the representations of the symmetry group of the Hamiltonian. In the case of ^{3}He the symmetry group is $G = S0(3)_{\vec{L}} \times S0(3)_{\vec{S}} \times U(1)_\phi$ corresponding to the full rotation groups in orbital space and spin space and the gauge group (we neglect the small spin-orbit coupling for the present). The representations are labelled by the angular momentum L and by the spin S of the Cooper pairs, and hence are $(2L+1) \times (2S+1)$ fold degenerate. We may write the eigenvalue equation

$$V \psi^{L,S}_{m,\mu} = V_{L,S} \psi^{L,S}_{m,\mu} \tag{81}$$

where $m = -L, -L+1, \ldots, L$ and $\mu = -S, -S+1, \ldots, S$. The Pauli principle requires the gap parameter to be antisymmetric, or $\Delta_{\vec{k}\sigma\sigma'} = -\Delta_{-\vec{k}\sigma'\sigma}$, and therefore only even (odd) values of L are allowed for spin $S = 0$ ($S = 1$). The eigenfunctions are given by

$$\begin{aligned} \psi^{L,0}_{m,0}(\vec{k}; \sigma\sigma') &= i\tau^y_{\sigma\sigma'} Y_{LM}(\hat{k}) \quad , \quad L \text{ even} \\ \psi^{L,1}_{m,\mu}(\vec{k}; \sigma\sigma') &= i(\tau^y \tau^\mu)_{\sigma\sigma'} Y_{LM}(\hat{k}) \quad , \quad L \text{ odd} \end{aligned} \tag{82}$$

where the τ^M are the Pauli matrices and the Y_{LM} are spherical harmonics. For liquid ^{3}He, the $L = 1$ pair interaction component is the strongest attractive one, and hence there are $3 \times 3 = 9$ degenerate eigenfunctions $\psi^{1,1}_{m,\mu}(\vec{k} : \sigma\sigma')$ available, in which the gap function may be expanded:

$$\Delta_{\vec{k}\sigma\sigma'} = \sum_{m=1}^{3} \sum_{\mu=1}^{3} \bar{d}_{\mu m} Y_{1m}(\hat{k})(i\tau^y \tau^\mu)_{\sigma\sigma'} = \sum_{j=1}^{3} \sum_{\mu=1}^{3} d_{\mu j} \hat{k}_j (i\tau^y \tau^\mu)_{\sigma\sigma'} \tag{83}$$

where $\hat{k}_j$ are the components of $\vec{k}/\mid \vec{k} \mid$. The matrix $d_{\mu j}$ is the order parameter proper of the system. The considerable freedom associated with the large number of components gives rise to a rich variety of behavior, in particular the existence of several different equilibrium phases as a function of pressure, temperature and magnetic field, the appearance of new broken symmetries and collective modes.

C. Thermodynamic properties

1. Transition temperature

The critical temperature T_c follows from the gap equation in the limit $\Delta \to 0$, when the different components $\psi^{L,M}$ are independent. In the L-channel one finds

$$1 = | V_L | N(0) \int_0^{\epsilon_c} d\xi \frac{\tanh(\xi/2T_c)}{\xi} \tag{84}$$

which shows that as expected all components $\psi^{L}_{m,\mu}$ emerge at the same T_c given by

$$T_c = 1.134\ \epsilon_c\ \exp\left(-\frac{1}{N(0) \mid V_l \mid}\right) \tag{85}$$

In the case of several attractive V_ℓ's, the gap parameter may have very small admixtures from these other channels, appearing at $T << T_c$, except if these V_ℓ's are accidentally very close to the leading V_L.

2. Free energy

The free energy

$$F = < H - \mu N > - TS \tag{86}$$

is obtained from (75) and the expression (9) for the entropy of an ideal Fermi gas, this time a gas of Bogoliubov quasiparticles, with $n_{\vec{k}\sigma}$ replaced by $f_{\vec{k}\sigma}$ as given by (78). One finds

$$F = \sum_{\substack{\vec{k} \\ \xi<0}} \xi_{\vec{k}} - \frac{1}{4} N(0) \sum_\sigma < (\Delta_{\vec{k}} \Delta^+_{\vec{k}})_{\sigma\sigma} >_{\hat{k}} + \frac{1}{2} \sum_{\vec{k}\sigma} \xi_k^2 \frac{\partial f_k}{\partial E_{\vec{k}}} \tag{87}$$

Of the several solutions the gap equation may have, the one with the lowest free energy is the stable one. In order to bring out more clearly the minimum property of the free energy, it is useful to have a free energy functional of $\Delta_{\vec{k}\sigma\sigma'}$, which has a minimum value given by (87) and is stationary for any $\Delta_{\vec{k}\sigma\sigma'}$ satisfying the gap equation. This is obtained by adding to (87) the expression obtained from the gap equation after multiplying by Δ^+_k and averaging over $\hat{k}$, as well as eliminating the interaction V_L in favor of the quantity $\ell n T/T_c$:

$$F\{\Delta_k, \Delta_k^+\} = F_N + \frac{1}{2} N(0) \sum_\sigma < \phi_1[(\Delta_k^+ \Delta_k)_\sigma] - \phi_2[(\Delta_k^+ \Delta_k)_\sigma] >_{\hat{k}} \tag{88}$$

where

$$\phi_1[(\Delta^+\Delta)_\sigma] = (\Delta_k^+ \Delta_k)_\sigma \int d\xi_k \left[\frac{\tanh \xi/2T}{2\xi_k} - \frac{\tanh E_k/2T}{2E_k}\right]$$
$$\phi_2[(\Delta^+\Delta)_\sigma] = (\frac{1}{2} - \ell n \frac{T}{T_c})(\Delta_k^+ \Delta_k)_\sigma + \int d\xi_k \xi_k^2 \left[\frac{\partial f(\xi)}{\partial \xi} - \frac{\partial f(E)}{\partial E}\right] \tag{89}$$

and F_N is the free energy in the normal state. The functions $\phi_1(x)$ and $\phi_2(x)$ are both monotonically increasing and concave. The difference $\phi_1 - \phi_2$ starts out proportional $-x$ at small x, goes through a minimum and increases as $x\ \ell n$ for large positive x. Thus the least anisotropic gap function will give the lowest free energy. In the case of p-wave pairing this will be the BW state to be discussed below, which actually has isotropic $\Delta_k^+ \Delta_k$.

Near the transition temperature, when the gap parameter is small, one may expand (88) to get the Ginzburg-Landau free energy functional

$$F = F_N + \frac{1}{2} N(0) \ell n \frac{T}{T_c} \sum_\sigma < (\Delta_k^+ \Delta_k)_{\sigma\sigma} >_{\hat{k}} + \frac{1}{4} \beta_0 \sum_\sigma < (\Delta_k^+ \Delta_k)^2_{\sigma\sigma} >_k \tag{90}$$

where $\beta_0 = \frac{7}{8\pi^2} \zeta(3) N(0)/T_c^2$

3. Gap parameter

For a given angular variation of the square of the gap, we may write

$$\frac{1}{2} \mathrm{tr}_\sigma (\Delta_{\vec{k}}^+ \Delta_{\vec{k}}) = \Delta^2 g(\hat{k}) \tag{91}$$

where we choose $< g(\hat{k}) >_{\hat{k}} = 1$ such that Δ^2 is the average squared gap. Minimizing (90) with respect to Δ one finds

$$\Delta^2(T) = -\frac{N(0)}{\beta_0 < g^2(\hat{k}) >_k} \left(1 - \frac{T}{T_c}\right) \tag{92}$$

and the free energy is found as

$$F = F_N - \frac{1}{2} N(0) \left(1 - \frac{T}{T_c}\right) \Delta^2(T) \tag{93}$$

Thus, $\Delta(T)$ vanishes as $(1-T/T_c)^{1/2}$ as $T \to T_c$ from below, and the gain in free energy in the condensed phase is $\sim (1-T/T_c)^2$, making the transition second order.

We note in passing that the ground state energy obtained from (87) in the limit $T \to 0$ is given by

$$E_0 = E_N - \frac{1}{2}N(0)\Delta^2(0) \tag{94}$$

¿From the gap equation, multiplied by Δ_k^+ and integrated over $\vec{k}$, and eliminating the coupling constant in favor of T_c, one finds the gap in the limit $T \to 0$ as

$$\Delta(0) = 1.76 k_B T_c \exp\left[-\frac{1}{2} < g(\hat{k}) \ln g(\hat{k}) >_{\hat{k}} \right] \tag{95}$$

Let us now consider two model states, thought to describe the A phase and the B phase of superfluid ^3He. The B phase occupies the low temperature part of the phase diagram at elevated pressure and is the only stable phase (in zero magnetic field) at low pressure. The corresponding model state is called the Balian-Werthheimer (BW) state [14] and has the form

$$\Delta^{BW}_{\hat{k}\sigma\sigma'} = \Delta e^{i\phi} \sum_{\mu,j} R_{\mu j} (i\tau^y\tau^\mu)_{\sigma\sigma'} \hat{k}_j \tag{96}$$

where $R_{\mu j}$ is a rotation matrix describing a relative rotation of spin and orbital space. This state has the unique feature among $L \neq 0$ pairing states of having a uniform energy gap, since

$$(\Delta_{\hat{k}}^+ \Delta_{\hat{k}})_{\sigma\sigma'} = \Delta^2 \delta_{\sigma\sigma'} \tag{97}$$

This is obviously the "least anisotropic state", which minimizes the weak coupling free energy (88). It is therefore no surprise that this state is stable over most of the superfluid region in the phase diagram.

The second observed phase, the A phase, is thought to be described by the socalled Anderson-Brinkman-Morel (ABM) state [15,5], defined as

$$\Delta^{ABM}_{\hat{k}\sigma\sigma'} = \Delta_0 \hat{d} \cdot (i\tau^y\vec{\tau})_{\sigma\sigma'} \hat{k} \cdot (\hat{m} + i\hat{n}) \ , \tag{98}$$

where the unit vectors $\hat{d}$, $\hat{m}$ and $\hat{n}$ specify preferred directions in spin space ($\hat{d}$) and orbital space ($\hat{m}$ and $\hat{n}$). The orientation of $\hat{m}$ and $\hat{n}$ is perpendicular, $\hat{n}\cdot\hat{m} = 0$, and $\hat{m}, \hat{n}$ and $\hat{\ell} = \hat{m} \times \hat{n}$ span a triad. Note that $\Delta_{\hat{k}}^{ABM} \sim Y_{11}(\hat{k})$, i.e. the Cooperpairs have a finite angular momentum projection along $\hat{\ell}$. The spins of the two partners of the pair are aligned parallel in direction perpendicular to $\hat{d}$ ("equal spin pairing").

The energy gap of the ABM state is anisotropic,

$$(\Delta_{\hat{k}}^+ \Delta_{\hat{k}})_{\sigma\sigma'} = \Delta_0^2 \left[1 - (\hat{k}\cdot\hat{\ell})^2\right] \delta_{\sigma\sigma'} \tag{99}$$

with two point nodes at $\hat{k} = \pm\hat{\ell}$. In weak-coupling theory the ABM state is unstable with respect to the BW state, as we have seen. The experimental observation of the ABM state can only be described by going beyond weak coupling theory.

4. Specific heat

The specific heat is most conventiently derived from the entropy,

$$\begin{aligned} C_V &= \sum_{\vec{k}\sigma} E_{\vec{k}\sigma} \frac{\partial f_{\vec{k}\sigma}}{\partial T} \\ &= -\frac{1}{T} \sum_{\vec{k}\sigma} \frac{\partial f_k}{\partial E_k} \left[E_k^2 - \frac{1}{2} T \frac{\partial}{\partial T} (\Delta_k^+ \Delta_k)_{\sigma\sigma} \right] \end{aligned} \tag{100}$$

The term $\sim \frac{\partial}{\partial T}(\Delta^+\Delta)$ gives rise to a discontinuity at T_c, of relative weight

$$\frac{\Delta C_V}{C_N} = \frac{12}{7\zeta(3)} \frac{1}{\kappa} \ , \tag{101}$$

where $\kappa = < (\Delta_k^+ \Delta_k)^2 >_{\hat{k}} / (< \Delta_k^+ \Delta_k >_{\hat{k}})^2$. For the isotropic BW state $\kappa = 1$ and for the ABM state $\kappa = \frac{6}{5}$. It follows from Schwarz's inequality that $\kappa \geq 1$.

At low temperatures, the thermodynamic behavior is governed by the node structure of the gap function. Thus, in the case of an isotropic gap function, as in the BW state or for s-wave superconductivity, the distribution of thermal excitations is of the Boltzmann type, $f_{k\sigma} \sim \exp -\frac{\Delta_k + \xi^2/2\Delta_k}{T}$, and

$$C_V^{BW} \sim \exp -\frac{\Delta}{T} \ , \quad T << T_c \tag{102}$$

For the case of point nodes, as for the ABM state, we can argue qualitatively that for any given temperature T, there will be Fermi surface regions around the point nodes, where $|\Delta_k| < T$ and the Bogoliubov quasiparticles behave like normal quasiparticles. For the ABM state we have $|\Delta_0 \sin\theta_k| < T$ and hence $\theta_k \tilde{<} T/\Delta_0 = \theta_c$. The specific heat is given by the normal state value times the fraction of Fermi surface contributing, $2\pi\theta_c^2/4\pi$,

$$C_V^{ABM} \sim \left(\frac{T}{\Delta_0}\right)^2 C_N(T) \propto T^3 \quad . \tag{103}$$

Obviously, a similar estimate of a state with a line of nodes in the gap function would give $C_v \sim T^2$.

5. *Normal fluid density*

At finite temperature $0 < T < T_c$, the hydrodynamic properties of a superfluid Fermisystem may be described in terms of a "two-fluid model", consisting of a superfluid component, the condensate of Cooper pairs, and a normal-fluid component, the thermal excitations, here mainly given by the Bogoliubov quasiparticles. The density of the normal-fluid component may be calculated in the following way. We assume a situation, where the superfluid component is at rest, while the normal-fluid component is flowing with uniform velocity $\vec{v}_n$. The distribution function of BQP is a shifted Fermi function

$$f_{\vec{k}}^{\vec{v}_n} = f(E_k - \vec{k}\cdot\vec{v}_n) \quad . \tag{104}$$

Since the BQP are momentum eigenstates, for which the energy change induced by a Galilean transformation is given by $-\vec{k}\cdot\vec{v}_n$. We may now calculate the momentum density in linear order in $\vec{v}_n$,

$$\vec{g}_n = \sum \vec{k} f_{\vec{k}}^{\vec{v}_n} = \sum_{\vec{k}} \vec{k}\left(-\frac{\partial f_k}{\partial E_k}\right)\vec{k}\cdot\vec{v}_n \tag{105}$$

Defining the normal-fluid density tensor $\overleftrightarrow{\rho}_n$ by

$$\vec{g}_n = \overleftrightarrow{\rho}_n \cdot \vec{v}_n \tag{106}$$

we find

$$\rho_{n,ij}^0 = \sum_{\vec{k}\sigma} k_i k_j (-\partial f_k \partial E_k) \tag{107}$$

Near T_c, ρ_{nij}^0 decreases linearly with decreasing T, $\rho_{n,ij}^0 \sim (1 - c_{ij}\frac{\Delta^2(T)}{T_c^2})$, whereas at low temperatures one finds for the largest eigenvalue of ρ_n^0

$$\rho_n^0 \sim \begin{cases} \exp(-\Delta/T) & \text{isotropic} \\ T^2 & \text{point nodes} \\ T & \text{line nodes} \end{cases} \tag{108}$$

At T_c, we should have $\rho_n^0 \to \rho$, where $\rho = mn$ is the mass density. Instead, one obtains from (107) $\rho_{nij}^0 = \frac{m^*}{m}\rho\delta_{ij}$. The discrepancy is due to Fermi liquid corrections, which we have not yet taken into account. As seen from the discussion of the effective mass relation in normal Fermi liquids, the bare velocity is screened by backflow effects, involving the Landau parameter F_1^s, and must be replaced by the effective velocity

$$\vec{v}_{n,eff} = \vec{v}_n - \frac{1}{3\rho}\frac{m}{m^*}F_1^s \vec{g}_n \tag{109}$$

[this derives from the quasiparticle energy change $\delta\epsilon_{\vec{k}} = \frac{1}{N_F}F_1^s \sum_{\vec{k}'\sigma} \hat{k}\cdot\hat{k}'\delta n_{k'\sigma} = (N_F k_F^2)F_1^s\vec{g}$, and can be shown to carry over to the superfluid state]. The correct ρ_n-tensor is then given by

$$\overleftrightarrow{\rho}_n = \frac{m^*}{m}\left(\overleftrightarrow{1} + \frac{1}{3}F_1^s\frac{\overleftrightarrow{\rho}_n^0}{\rho}\right)^{-1}\overleftrightarrow{\rho}_n^0 \tag{110}$$

D. General Ginzburg-Landau expansion of the free energy

1. *Uniform system*

Near a second order phase transition the order parameter (OP) is small and the change in free energy induced by the formation of order may be expanded in powers of the OP. The general form of this expansion is determined by the symmetry of the system. In the case of liquid ^{3}He, the free energy F must be invariant under rotations in orbital space and spin space (neglecting again the small spin-orbit coupling), and F must be a real quantity, of course. In terms of the normalized order parameter matrix

$$A_{\mu j} = 3^{-1/2} d_{\mu j}/\Delta(T) \quad , \tag{111}$$

the G-L free energy takes the form

$$\begin{aligned} F = & F_N + \alpha\Delta^2 + \frac{1}{2}\Delta^4\Big\{\beta_1 \mid \mathrm{tr}(AA^T) \mid^2 + \beta_2 \\ & + \beta_3 \mathrm{tr}\Big[(AA^T)(AA^T)^*\Big] + \beta_4 \mathrm{tr}\Big[(AA^*)^2\Big] + \beta_5 \mathrm{tr}\Big[(AA^*)(AA^*)^*\Big]\Big\} \end{aligned} \tag{112}$$

There are five possible fourth order invariants in this case, obtained by contracting the orbital and spin indices of AA^*AA^* in all possible ways, multiplied by phenomenological parameters β_i.

By comparison with the weak coupling result (90), one finds the values $\beta_2 = \beta_3 = \beta_4 = -\beta_5 = -2\beta_1 = \frac{6}{5}\beta_0$ in this limit.

The problem of finding the stable phase by minimizing F for a given set of β-parameters has not been solved in general so far. However, a combination of analytical and numerical studies has provided a rather complete picture: there are 13 possible order parameter structures characterized by their residual symmetry groups, which fall into two classes. The first class comprises five socalled "inert" states. These states retain a fixed structure within their respective domain of stability in β-parameter space. The BW and ABM states are of this type. The remaining "noninert" states depend continuously on the β-parameters.

The free energies of the BW state and the ABM state are given by

$$\begin{aligned} F^{BW} &= F_N - \frac{\alpha^2}{2(\beta_{12} + \frac{1}{3}\beta_{345})} \\ F^{ABM} &= F_N - \frac{\alpha^2}{2\beta_{245}} \quad . \end{aligned} \tag{113}$$

where $\beta_{12} = \beta_1 + \beta_2$, etc.. In the weak-coupling limit, $\beta^{BW} = \beta_{12} + \frac{1}{3}\beta_{345} = \beta_0$, whereas $\beta^{ABM} = \beta_{245} = \frac{6}{5}\beta_0$. Thus a 20% relative change of β^{BW} and β^{ABM} is needed to stabilize the ABM phase relative to the BW phase. There are several microscopic models involving the effect of spin fluctuations or transverse current fluctuations, and other excitations, which can

qualitatively account for the stabilization of the ABM state. The intuitively most appealing one is due to Anderson and Brinkman [5]. It emphasizes the importance of spin fluctuation exchange in producing the attractive interaction in the p-wave channel. In the superfluid state, the spin-fluctuation spectrum, and hence the pair interaction is modified. The equal spin-pairing configuration of the ABM state enhances the spin-fluctuations at higher pressure (large m^*/m) and not too low temperature, relative to the BW state.

2. *Superflow and Textures*

The most spectacular property of a pair-correlated Fermi system is of course the superfluidity. This property is related to the complex-valuedness of the order parameter, which in turn is a consequence of the broken $U(1)$ gauge symmetry. Let us consider the local pair amplitude

$$F_{\vec{k}\sigma\sigma'}(\vec{r}) = \int d^3r' e^{-i\vec{k}\cdot\vec{r}'} < \psi_\sigma(\vec{r} + \frac{1}{2}\vec{r}')\psi_{\sigma'}(\vec{r} - \frac{1}{2}\vec{r}') > \tag{114}$$

A Galilean transformation into a frame of reference moving with velocity $\vec{u}$, under which the single-particle momentum eigenstate $\varphi_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})$ transforms into $\varphi_{\vec{k}'} = \varphi_{\vec{k}} \exp(-im\vec{u}\cdot\vec{r})$ (using $\vec{k}' = \vec{k} - m\vec{u}$) causes F to change as follows

$$F_{\vec{k}\sigma\sigma'}(\vec{r}) \rightarrow F_{\vec{k}\sigma\sigma'}(\vec{r}) \exp(-2im\vec{u}\cdot\vec{r}) \quad . \tag{115}$$

The local gap parameter $\Delta_{k,\sigma\sigma'}$ transforms in the same way.

For the BW state where the OP is given by a real quantity times a phase factor,

$$d_{\mu j}(\vec{r}) = \mid \Delta(\vec{r}) \mid R_{\mu j}(\vec{r}) \exp[i\phi(\vec{r})] \tag{116}$$

the phase $\phi(\vec{r})$ is seen to transform as

$$\phi(\vec{r}) \rightarrow \phi(\vec{r}) - 2m\vec{u}\cdot\vec{r} \tag{117}$$

¿From this relation one concludes that the quantity

$$\vec{v}_s = \frac{1}{2m}\vec{\nabla}\phi(\vec{r}) \tag{118}$$

transforms as a velocity. It is referred to as the "superfluid velocity". The corresponding super current is given by $\vec{g}_s = \overleftrightarrow{\rho}_s \cdot \vec{v}_s$, with $\overleftrightarrow{\rho}_s$ the superfluid density tensor. By invoking the two fluid model, we can determine $\overleftrightarrow{\rho}_s$ in the following way. The total mass current is obtained by adding the superfluid and normalfluid currents

$$\vec{g} = \overleftrightarrow{\rho}_s \cdot \vec{v}_s + \overleftrightarrow{\rho}_n \cdot \vec{v}_n \tag{119}$$

By Galilean invariance, the mass current in a reference frame moving with velocity $-\vec{u}$ is $\vec{g}\,' = \vec{g} + \rho\vec{u} = \overleftrightarrow{\rho}_s \cdot (\vec{v}_s + \vec{u}) + \overleftrightarrow{\rho}_n \cdot (\vec{v}_n + \vec{u})$, and hence $\overleftrightarrow{\rho}_s + \overleftrightarrow{\rho}_n = \rho\overleftrightarrow{1}$.

For the ABM state things are more subtle, as the OP is intrinsically complex,

$$d_{\mu j}(\vec{r}) = \mid \Delta_0(\vec{r}) \mid \hat{d}_\mu(\vec{r})\left[\hat{m}_j(\vec{r}) + i\hat{m}_j(\vec{r})\right] \tag{120}$$

Multiplication of $d_{\mu j}$ by a phase factor is equivalent to a rotation of $\hat{m}$ and $\hat{n}$ in their plane by the angle ϕ, or $\hat{m}\,' + i\hat{n}\,' = e^{i\varphi}(\hat{m} + i\hat{m})$. Taking the gradient and letting $\phi \to 0$ yields

$$\vec{\nabla}\phi = -\sum_j \hat{m}_j \vec{\nabla}\hat{n}_j \tag{121}$$

Inserting (121) into (118), one can see that the superfluid velocity does no longer describe potential flow. Rather, the flow depends on the local orientation of the OP. In other words, changing the local orientation of the OP can have a major effect on the superflow. Two important consequences are (i) superflow is less stable in the ABM state: a continuous motion of the orientation of the preferred direction $\hat{\ell}$ can "unwind" the phase and dissipate the superflow. The cure for this desastrous effect is the pinning (or "locking") of the $\hat{\ell}$ vector field at boundaries or by external fields (e.g. magnetic fields). (ii) there may exist defects or "vortices" in the $\hat{\ell}$-field, which carry a finite (quantized) circulation, but do not have a normal core. These are actually energetically favored.

In general, the configuration of the preferred directions, e.g. $\hat{d}$, $\hat{\ell}$, will not be uniform, but will vary smoothly to form a socalled "texture". The textures are determined by the

interaction of the OP with boundaries and with external fields. For example, the $\hat{\ell}$-vector of the ABM state is oriented normal to a boundary, because the quasiclassical orbit of the partners of a Cooper pair obviously prefers to be in a plane parallel to the surface. The $\hat{d}$-vector of the ABM state, being perpendicular to the spin vector of the Cooper pair will orient itself perpendicular to an applied magnetic field.

The socalled "bending" of the OP costs energy, which causes a certain stiffness of the preferred directions.

This is described by the socalled "gradient free energy", which has to be added to the GL free energy in nonuniform states. It takes the form

$$F_G = \frac{1}{2}\int d^3r\Big\{K_1(\nabla_j d_{\mu\ell})(\nabla_j d^*_{\mu\ell}) + K_2(\nabla_j d_{\mu\ell})(\nabla_\ell d^*_{\mu j}) + K_3(\nabla_j d_{\mu j})(\nabla_\ell d^*_{\mu\ell})\Big\} \tag{122}$$

with coefficients K_i, which in the weak coupling limit take the values $K_1 = K_2 = K_3 = \frac{1}{5}N_F\xi_0^2$, where $\xi_0 = [7\zeta(3)/48\pi^2]^{1/2} v_F/T_c$ is the (zero temperature) coherence length characterizing the extension of a Cooper pair.

IV. UNCONVENTIONAL SUPERCONDUCTIVITY

The existence of superconductors with unconventional order parameter has not been prove beyound doubt, although clear experimental indications exist for two classes of strongly correlated electron systems, the heavy fermion compounds [7] and the cuprate superconductors. The term "unconventional" here means states with order parameter structure violating a rotation or reflection symmetry of the system in addition to gauge symmetry. As in the case of superfluid ^{3}He, the reason for pairing in states with reduced symmetry is a strongly repulsive short range interaction, which can be largely avoided if the partners of the pair are in a state with effectively finite angular momentum.

A. Low temperature properties and node structure of the gap

As we have seen in the last lecture, the additional symmetry breaking usually leads to very anisotropic gap structures, characterized by nodes of the gap on the Fermi surface. The node structure governs the low temperture behavior, leading to temperature power laws in the thermodynamic and transport properties. A qualitative determination of the temperature power can be given following the discussion preceding (103). Essentially, the normal state result has to be multiplied by a T-dependent reduction factor accounting for the fraction of Bogoliubov quasiparticles with energy gap $|\Delta_k| < T$. In principle, experimental observation of the temperature power laws would allow the complete determination of the node structure of the gap.

Unfortunately, the power laws tend to be masked by additional effects, such as caused by impurity scattering, which can destroy the node structure at low energies.

A classification of the possible order parameter structures can be given in analogy to the earlier studies of ^{3}He [7,16]. Starting point is the symmetry group of the system, which consists of the point symmetry group of the crystal, the group of rotations in spin space, the gauge group and time reversal symmetry. In the case of heavy fermion compounds the

spin-orbit scattering is strong, and the symmetry operations of (discrete) rotation/reflection in position space and rotations in spin space are combined into a single finite group. In this case only one continuous symmetry is left, the gauge symmetry. Again a natural framework for classifying different gap structures is in terms of the eigenfunctions of the pair interaction V on the Fermi surface

$$\hat{V}\psi_{n,\nu}^{(\Gamma)} = \lambda_n^{(\Gamma)}\psi_{n,\nu}^{(\Gamma)} \tag{123}$$

The eigenstates $\psi_{n,\nu}^{(\Gamma)}$ may be grouped in multiplets labelled by the irreducible representations Γ of the point group. Different sub-states of an d_Γ-dimensional representation, labelled by $\nu = 1, 2, \ldots d_\Gamma$ have the same eigenvalue $\lambda_n^{(\Gamma)}$. For each representation Γ there is an infinite number of eigenstate multip ts labelled by n. With increasing n, the eigenstates $\psi_{n,\nu}^{(\Gamma)}$ acquire an increasing number of nodes. For given Γ, we may expect the eigenvalue $\lambda_0^{(\Gamma)}$ with the smoothest eigenfunction to be largest (in modulus).

The largest negative eigenvalue, $\lambda_0^{(\Gamma_0)}$, determines the transition temperature. Then, the equilibrium state (near T_c at least) will have the gap parameter structure

$$\Delta_{\vec{k}} = \sum_{\nu=1}^{d_\Gamma} c_\nu \psi_{0,\nu}^{(\Gamma_0)}(\vec{k}) \tag{124}$$

where the c_ν are determined by minimizing the (G-L) free energy.

Let us now consider two examples. The first will be the heavy fermion superconductor UPt_3, probably the best-studied example of its class. The crystal structure is hexagonal, the point group D_{6h}. Due to the inversion symmetry of the lattice, the single particle states are two-fold degenerate, so that inspite of the strong spin-orbit coupling there is a pseudo-spin dependence. One observes a splitting of the transition of about 10% of T_c ($T_c \simeq 0.5K$). The splitting has been shown to disappear under applied pressure, at about 4kbar. There is a weak antiferromagnetic transition at $T \simeq 5K$, which seems to be absent at pressures $\gtrsim 3k$bar. This is very suggestive of a two-dimensional representation with a weak symmetry-breaking field due to the anti-ferromagnetic order. The low-temperature behavior of the specific heat, the thermal conductivity, the ultrasound absorption indicates a line of gap nodes in the basal plane. There are two possible representations left, E_{1g} and E_{2u}, an even parity, pseudospin-singlet and an odd-parity, pseudospin-triplet state, with order parameters

$$
\begin{aligned}
E_{1g} &: \Delta_{\vec{k}\pm} = \Delta_0 \hat{k}_z(\hat{k}_x \pm i\hat{k}_y) \\
E_{2u} &: \vec{\Delta}_{\vec{k}\pm} = \Delta_0 \hat{c}\hat{k}_z(\hat{k}_x \pm i\hat{k}_y)^2
\end{aligned}
\tag{125}
$$

where $\hat{c}$ is a unit vector in pseudospin space pointing along the $\hat{c}$ axis. A general state in the two-dimensional space may be denoted by a 2-component vector $\vec{\eta} = (\eta_1, \eta_2)$, which specifies the state

$$
\Delta_{\vec{k},\vec{\eta}} = \frac{1}{2}\Big[(\eta_1 + i\eta_2)\Delta_{\vec{k}+} + (\eta_1 - i\eta_2)\Delta_{\vec{k}-}\Big] \tag{126}
$$

The phenomenological G-L free energy [17] is a functional of $\vec{\eta}(\vec{r})$,

$$
\begin{aligned}
F = \int d^3r \Big\{ & \alpha(T)\vec{\eta}\cdot\vec{\eta}^* + \beta_1(\vec{\eta}\cdot\vec{\eta}^*)^2 + \beta_2 \mid \vec{\eta}\cdot\vec{\eta} \mid^2 \\
& + \kappa_1 (D_i\eta_j)(D_i\eta_j)^* + \kappa_2 (D_i\eta_i)(D_j\eta_j)^* + \kappa_3 (D_i\eta_j)(D_j\eta_i)^* \\
& + \kappa_4 (D_z\eta_j)(D_z\eta_j)^* + \frac{1}{8\pi}\vec{B}^2 \Big\}
\end{aligned}
\tag{127}
$$

Here $\alpha, \beta_1, \beta_2, \kappa_1, \ldots \kappa_4$ are material parameters, which can be calculated from microscopic theory. There are two fourth order terms and four gradient terms, due to the haxagonal symmetry of UPt_3. The D_i's are the components of the gauge invariant gardient operator $D_i = \frac{\partial}{\partial x_i} + i\frac{2e}{c}A_i$, where $\vec{A}(\vec{r})$ is the vector potential. The last term is the energy of the magnetic field in the sample.

One finds two possible homogeneous equilibrium states. For $-\beta_1 < \beta_2 < 0$, $\vec{\eta} = \eta_0(1,0)$, and for $\beta_2 > 0$, $\vec{\eta}_+ = \eta_0(1,i)$, or equivalently $\vec{\eta}_- = \vec{\eta}_+^*$.

As already mentioned, one observes a splitting of the transition, which would be caused by the symmetry breaking field due to the weak antiferromagnetic order. Assuming that the corresponding staggered field, $\vec{M}_s$, transforms as a vector in the basal plane, one obtains the additional symmetry breaking contribution to the free energy

$$
F_{SBF} = \epsilon M_s^2 \int d^3r (\mid \eta_1 \mid^2 - \mid \eta_2 \mid^2) \; ; \tag{128}
$$

This term favors an order parameter $\vec{\eta} = \eta_0(0,1)$ ($\vec{\eta} = \eta_0(1,0)$) for $\epsilon > 0$ ($\epsilon < 0$). If $\beta_2 > 0$, this state becomes unstable w.r.t. condensation of the other component at a second critical temperature T_{c2}, and develops into $\vec{\eta}_+$ for $T << T_{c2}$.

So far both, the E_{1g} and E_{2u} states would be compatible with experiment. As noted by Sauls [18], a further distinction can be derived from the anisotropy of the upper critical field H_{c2}. The H_{c2} curves for field parallel and perpendicular to the basal plane are observed to cross at low temperature. This can be understood, if the lower curve ($\vec{H} \parallel \hat{c}$), is suppressed as a consequence of "Pauli limiting". In high magnetic fields, such that the Zeeman energy is of the order of the gap energy, $\mu_o H \simeq \Delta(T=0)$, the magnetic field breaks Cooper pairs, which are (i) either in the spin-singlet state or (ii) have spin projection $S_z = 0$ in the case of triplet pairing. For the assumed E_{2u} state with orientation of the $\vec{d}$-vector $\parallel \hat{c}$-axis, the Cooper pairs have $S_z = 0$ along the quantization axis $\parallel \hat{c}$, which would explain the anisotropy of H_{c2}. In the spin-singlet E_{1g} state, on the other hand, the Pauli limiting is independent of the field direction. A still remaining problem is the existence of a tetracritical point in finite magnetic field. For a possible resolution of this problem we refer to Sauls [18].

Let us now turn to the other example, the high-T_c superconductors. Theoretical model calculations identify a spin-singlet state with d-wave symmetry

$$
\Delta_{\vec{k}} = \Delta_0(\cos k_x - \cos k_y) \tag{129}
$$

as the most stable superconducting state. This state has four lines of nodes parallel to the z-axis on the cylindrical Fermi surface. The low temperature properties may therefore be derived from the fact that the fraction of thermal excitations (BQP) near the line nodes is proportional to T.

The specific heat is varying as $C_V \sim T^2$, while the largest eigenvalue of the normal fluid density tensor $\rho_n \sim T$. Since the magnetic penetration depth λ is related to the superfluid density by $\frac{1}{\lambda^2} = \frac{8\pi e^2}{c^2 m^2}\rho_s$ and $\rho_s = \rho - \rho_n$, one finds $\lambda(T) = \text{const} + aT$. A linear dependence of λ with temperature has been observed in very clean samples [19]. This is one of the important indications of unconventional superconductivity in high-T_c superconductors.

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