INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR. 758 - 25

SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

===

INTERACTING FERMIONS IN ONE DIMENSION

DISORDERED BOSONS

Thierry GIAMARCHI
Universite' de Paris XI (Paris-Sud)
Laboratoire de Physique des Solides
Batiment 510 - Centre Universitaire
F-91405 Cedex Orsay, France

===

These are preliminary lecture notes, intended only for distribution to participants.

===

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392

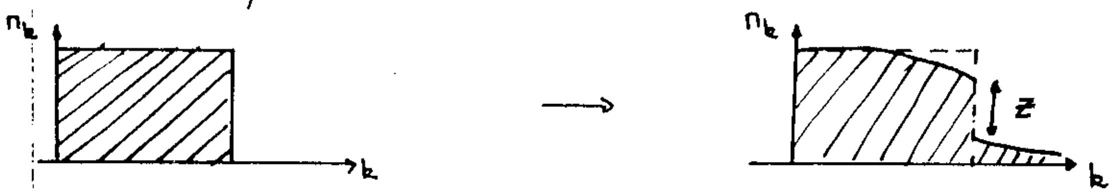
I] The goal of the game:

1) Fermi liquids (crash-course !)

$d=3$ $\quad H_{\text{free electrons}}$  k_F $\quad + \quad H_{int} = \int V(x-x')\,\rho(x)\rho(x')$

? $\rightarrow$ Fermi Liquid theory (Landau) $\Rightarrow$ "Free" electrons

$C_v = \gamma T \qquad \chi = \text{cste}$

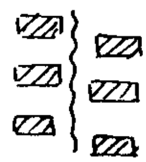


- One instability: $U>0 \rightarrow$ AF, Ferro, CDW etc
  $U<0 \rightarrow$ Supra (BCS)

- Mean-field theory: $\int V \rho\rho \rightarrow \int V \rho \langle\rho\rangle$

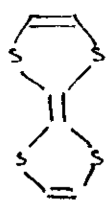
2) Why look in 1D:

?? Little [W. A. PRA 134 1416 (64)]
better for superconductivity

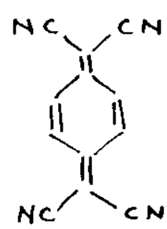


?? Simpler (!) to study theoretically
["Theorems"; Bethe Ansatz] $\rightarrow$ interacting system !!

... TTF - TCNQ



TTF



TCNQ

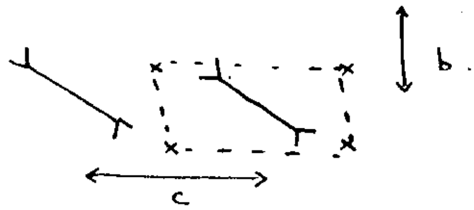
- Bechgaard Salts $(TMTSF)_2 X$

Mono cristals ≃ $0.5 * 0.5 * 5 \ mm^3$

Structure : cf transpar.

PF_6	(Å)	a = 7.297	b : 7.711	c : 10.522
ClO_4		a = 7.266	7.678	13.275

Triclinic



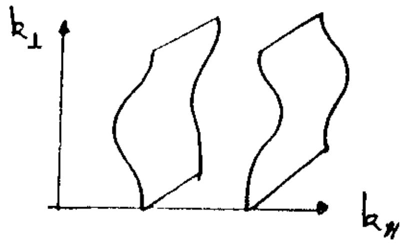
- Two families: TMTSF TMTTF

- Dimerization more important in TMTTF
  0.08% in $(TMTSF)_2 PF_6$
  0.6 % in $(TMTTF)_2 Br$

⇒ They are 1D compounds if (T big enough)

$\sigma_a / \sigma_b \simeq 200$ $\sigma_{\parallel}/\sigma_c \simeq 3000$ T petit → plus 3D

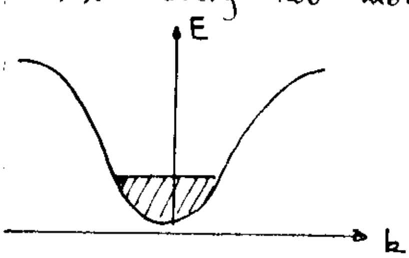
$t_{\parallel} \simeq 1000$ K $t_{\perp} = 100$ K $t_c \simeq 1$ K



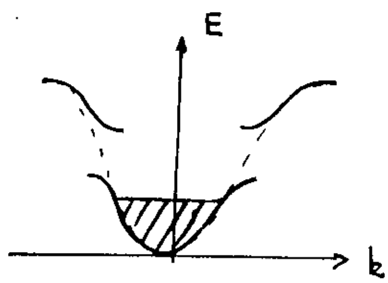
$$E = -t_{\parallel}\cos(k_{\parallel}) - t_{\perp}\cos(k_{\perp})$$

if $T \gg t_{\perp}$ ⇒ 1D (to be confirmed !)

- Filling :
  1 x every two molecules 1/4 filled.



but



- <u>Properties:</u>

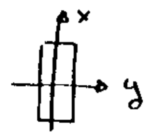
$\chi_{unif} \simeq cste$ (40% drop) SP $(TMTTF_2)PF_6$
SDW (S) PF_6

Insulator !! $\Delta \simeq 200$ K

$\rho \sim e^{\Delta/T}$ $\Rightarrow$ nothing in spins !!

$\Rightarrow$ Mott insulator ! + Spin-charge separation !

- $(TMTSF)_2\,(PF_6)$:

SDW $\uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow$ along 

+ Spin-flop along z for $H \simeq 5$ kG

Superc. under pressure : $H_{c1} = 5.0$ G ($H_\perp$ chains)
<u>extremely sensitive to disorder</u> 0.01% defects

* <u>NMR:</u>

$$T_1^{-1} \propto \sum_q |A_q|^2 \int_{-\infty}^{+\infty} dt \cos(\omega_0 t) \langle [S_q^+(t), S_q^-(0)]_+ \rangle$$

$$\propto \sum_q |A_q|^2 \frac{\chi_\perp''(q, \omega_N)}{\tanh(\omega_N/2T)}$$

if $\omega_N \ll T$

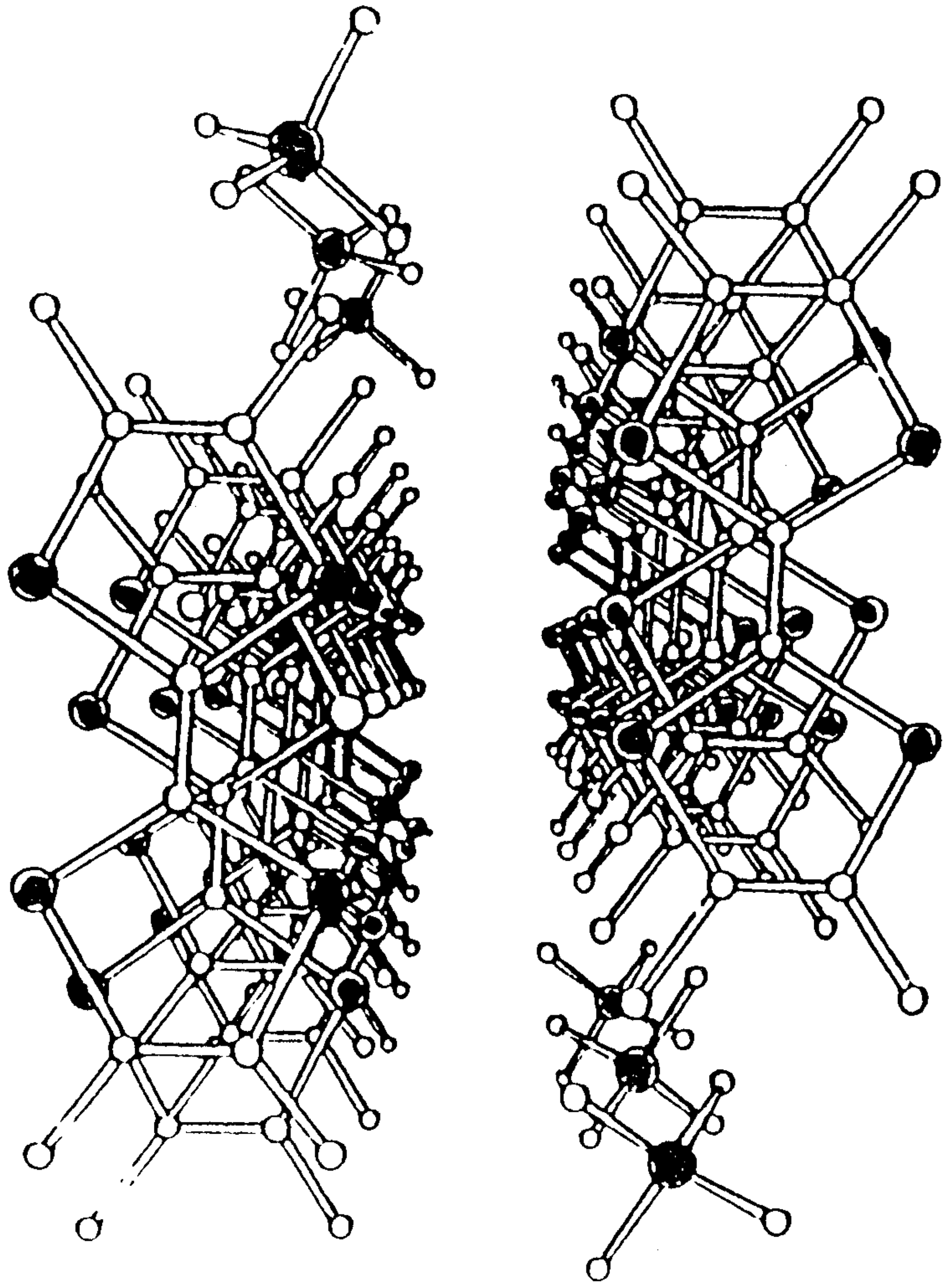
$$T_1^{-1} \simeq 2T\, \gamma_N^2 \sum_q \frac{1}{\omega_N}\, \chi_\perp''(\vec{q}, \omega_N)$$

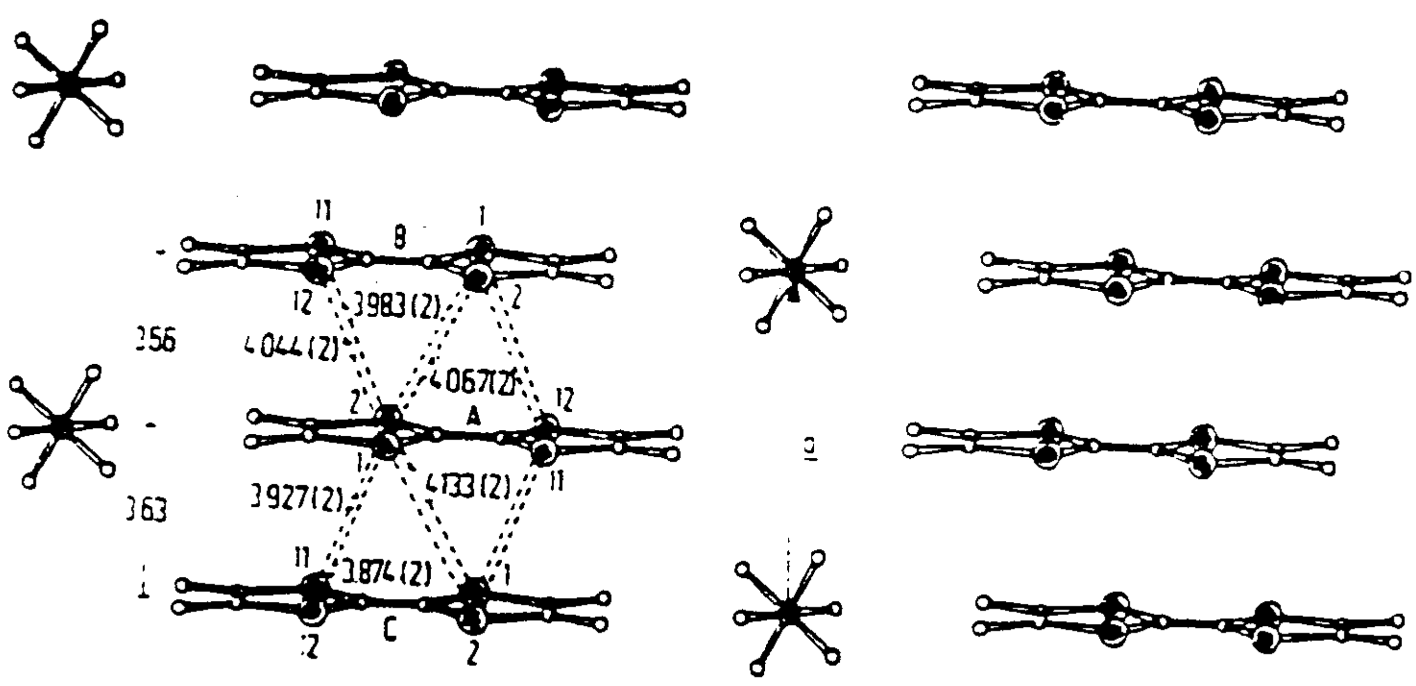
F.L : $(T_1 T)^{-1} \propto \chi_s^2 \equiv cste$

Non Fermi Liquid behavior !! $\Rightarrow$ χ_{2k_F} diverges !!

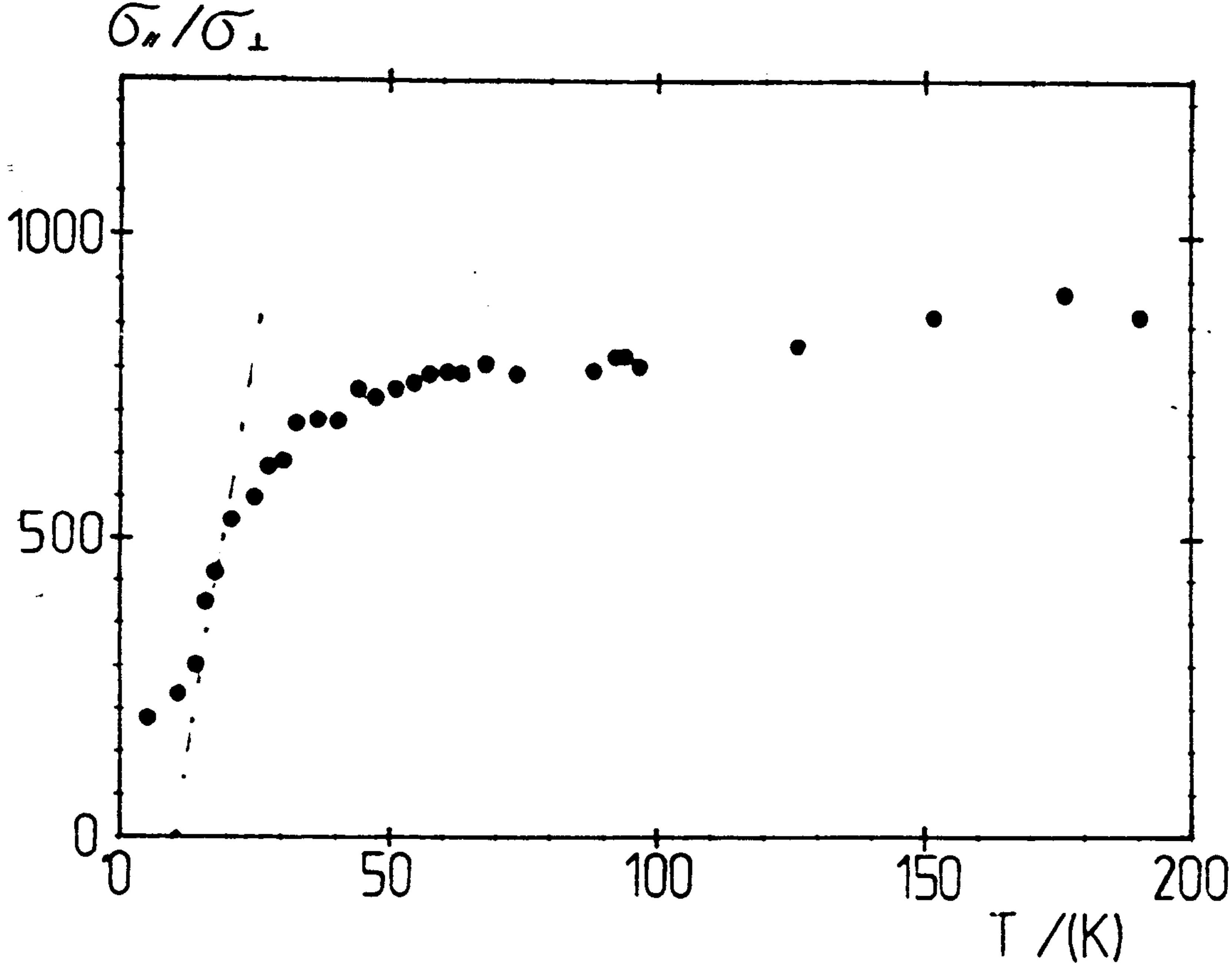
* <u>Other systems</u> :

$\rightarrow$ Quantum Wires (Coulomb interactions)
$\rightarrow$ Edge states (QHE)
$\rightarrow$ Spin Systems

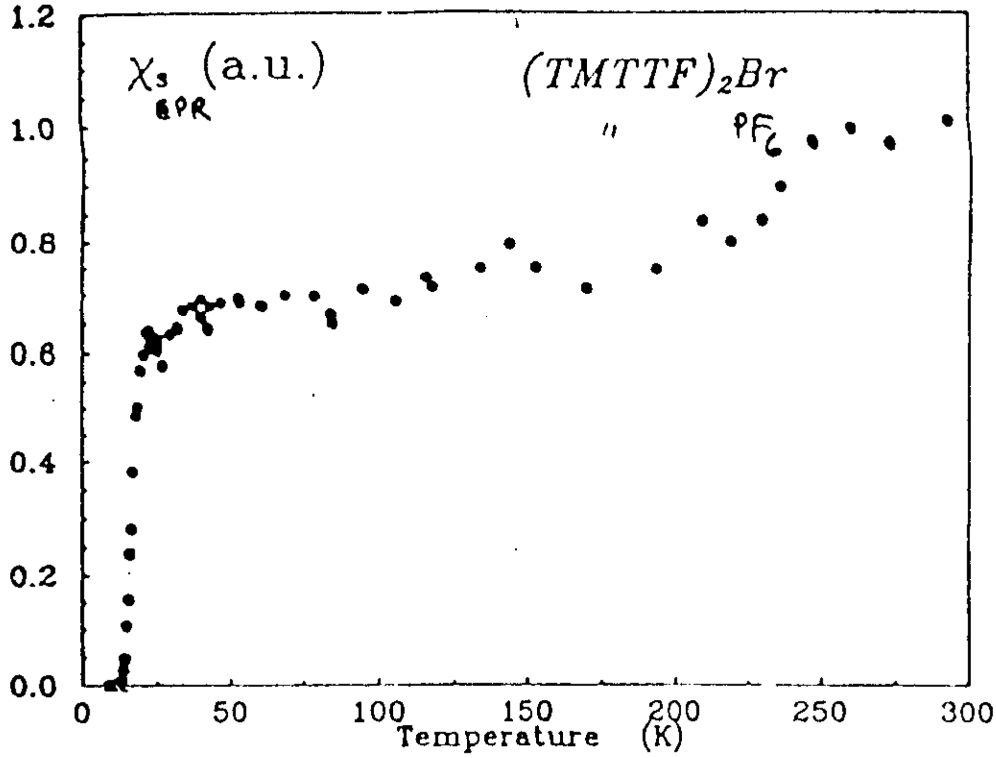



11
B
1
12
3983(2)
2
356
4044(2)
4067(2)
2
12
A
1
11
3927(2)
4133(2)
363
11
3874(2)
1
2
C
2

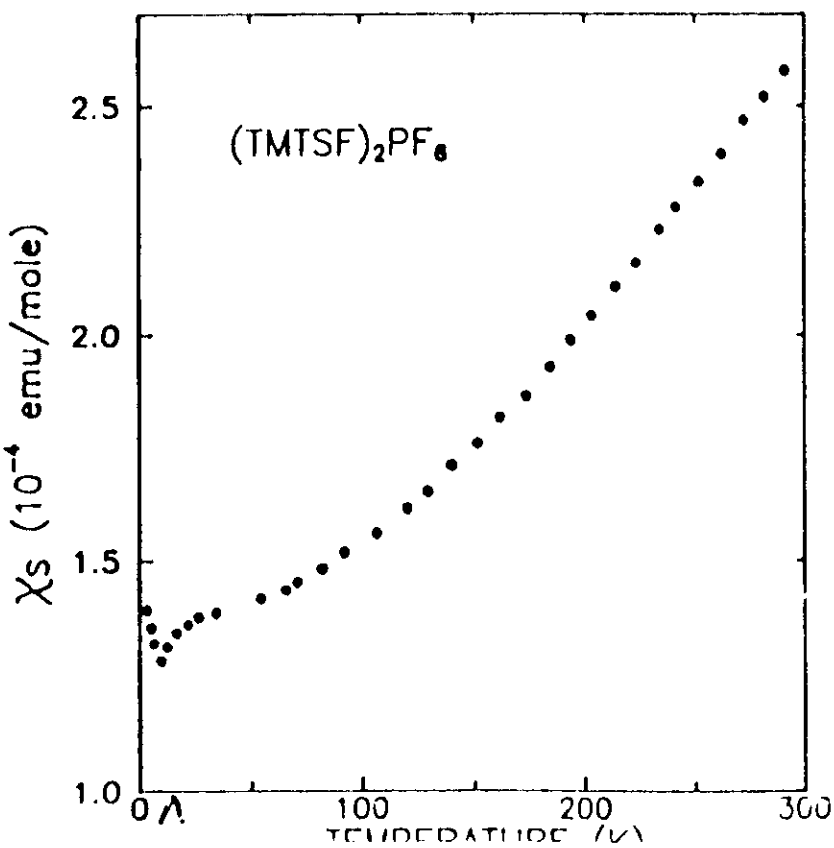
$(TMTSF)_2\ PF_6$



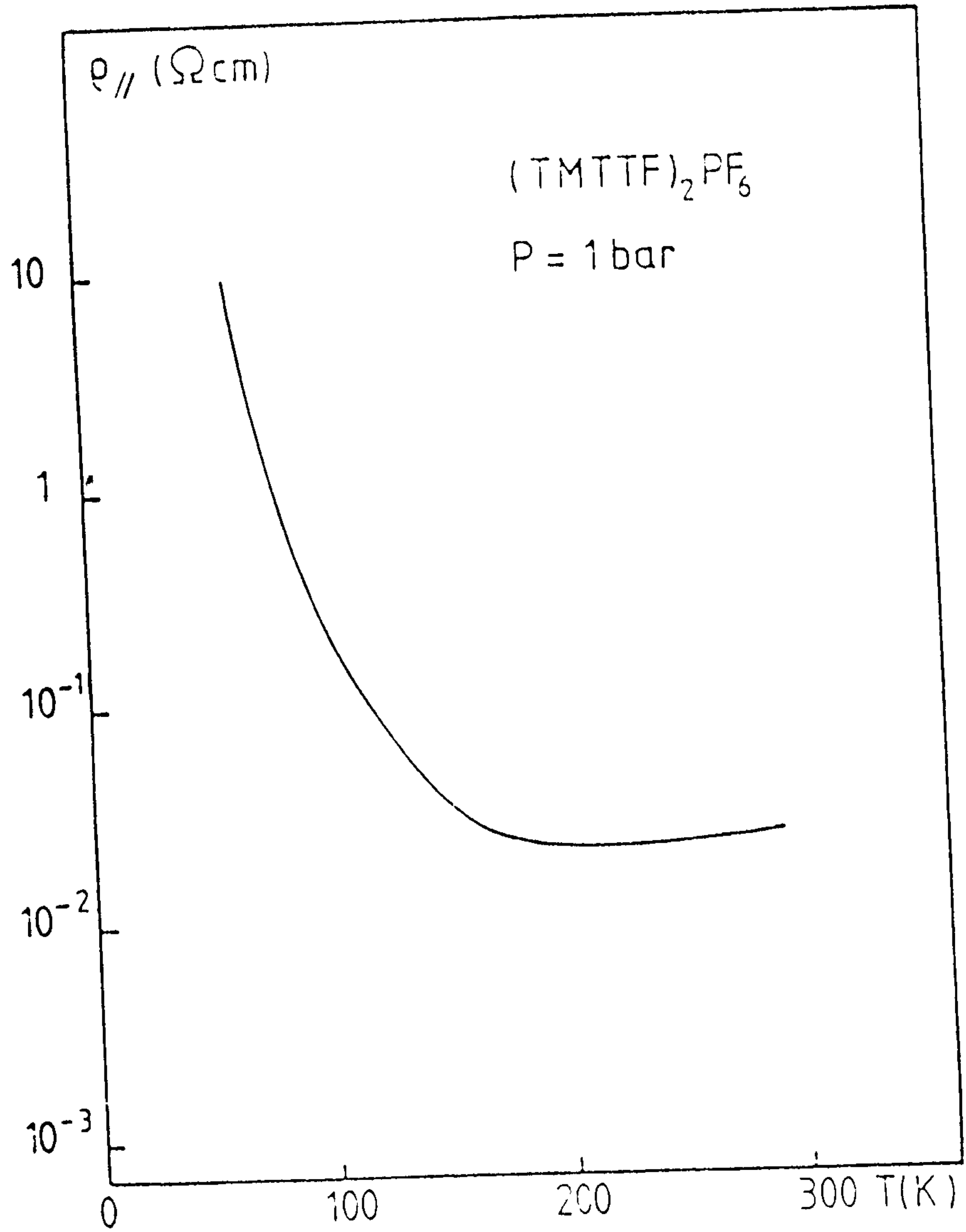
-7-



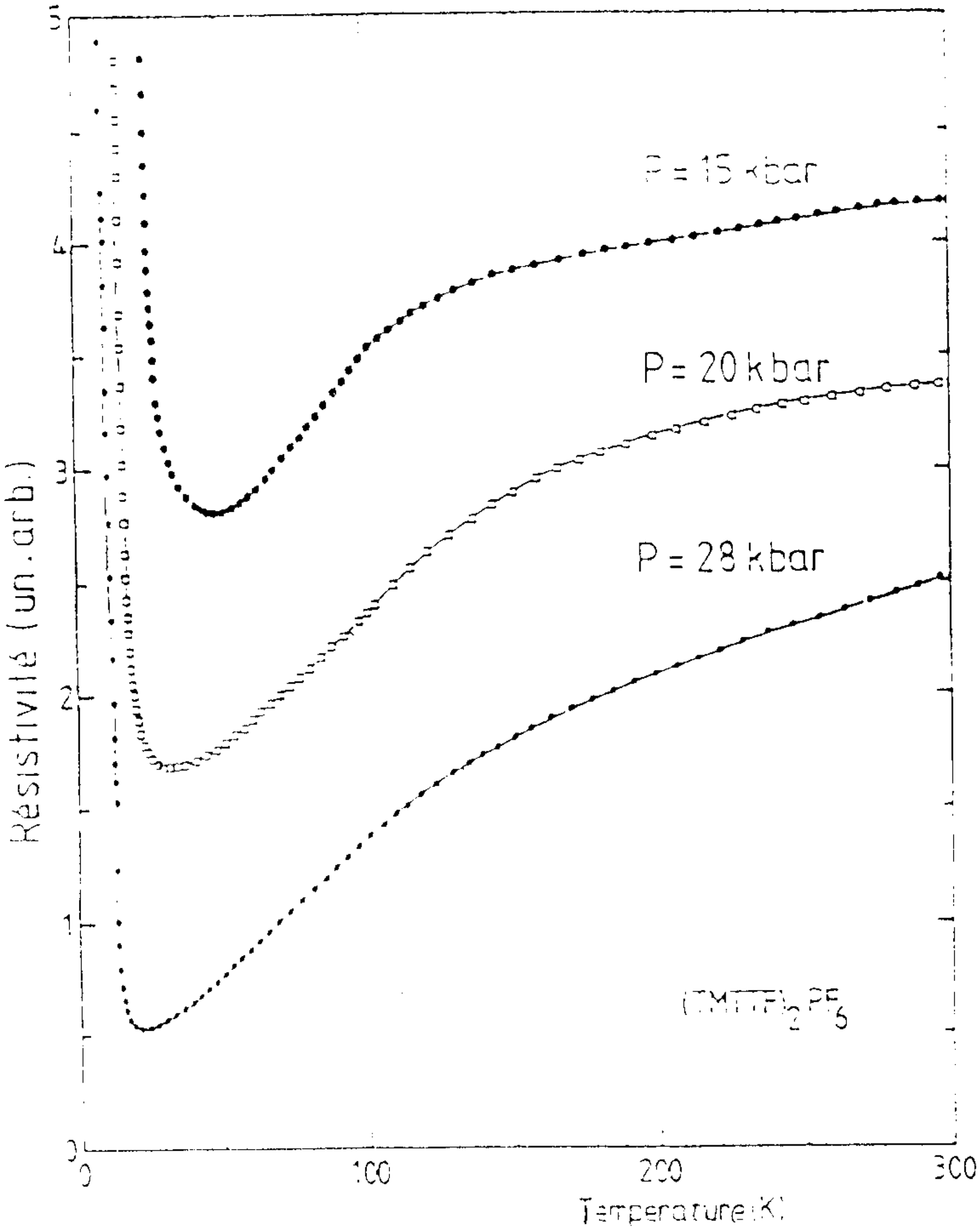
χ_s (a.u.)
EPR
$(TMTTF)_2Br$
" PF_6
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
50
100
150
200
250
300
Temperature (K)

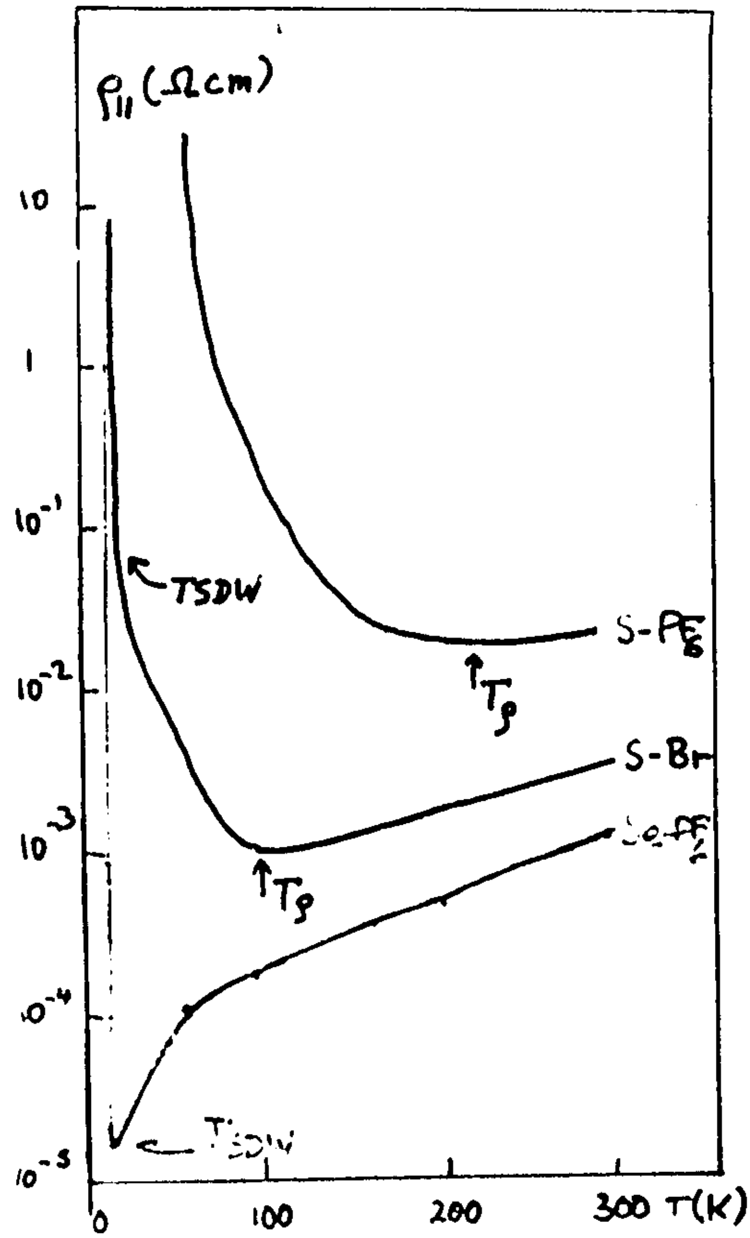


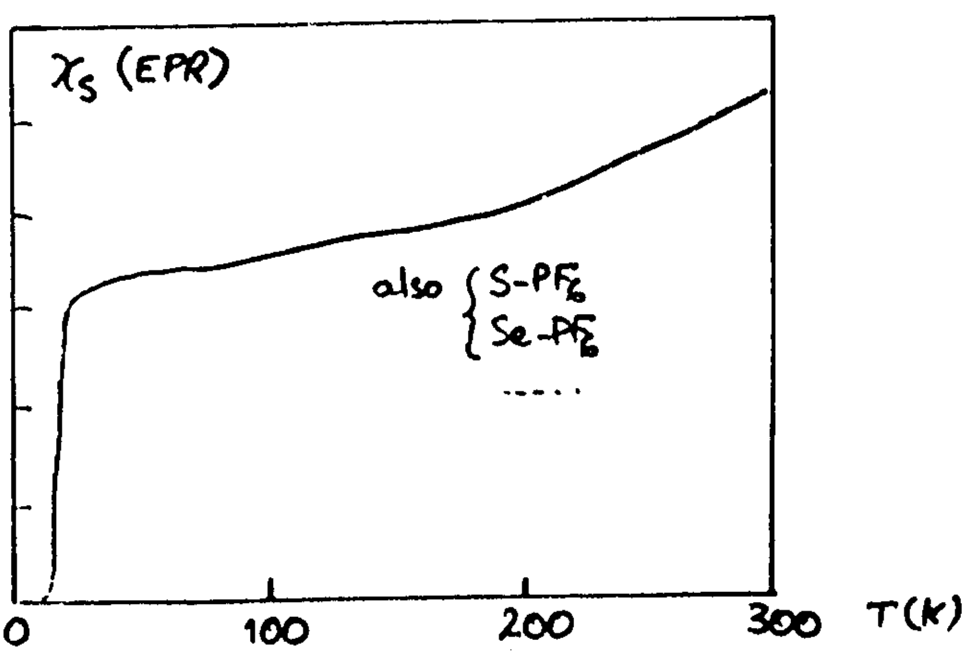
$(TMTSF)_2PF_6$
χ_s (10^{-4} emu/mole)
2.5
2.0
1.5
1.0
0
100
200
300

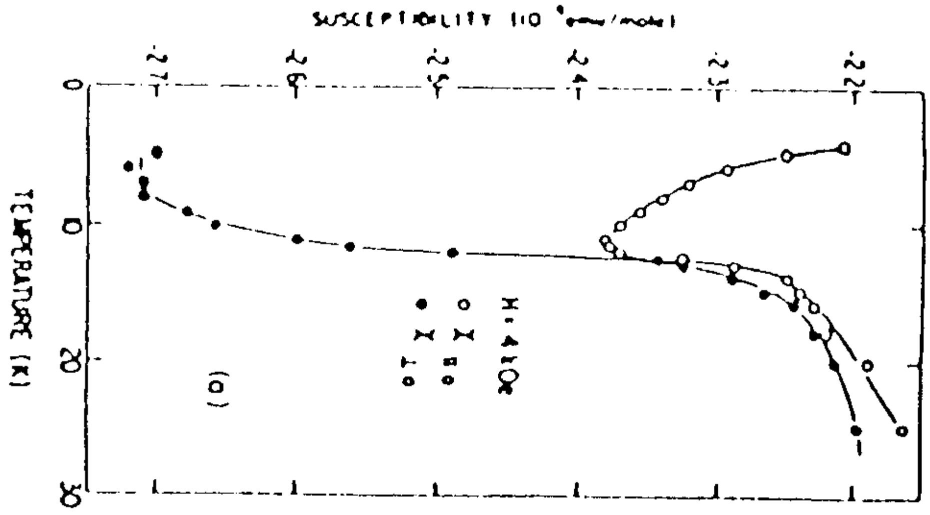

$\rho_{//}$ (Ωcm)
$(TMTTF)_2PF_6$
P = 1 bar
10
1
10^{-1}
10^{-2}
10^{-3}
0
100
200
300 T(K)

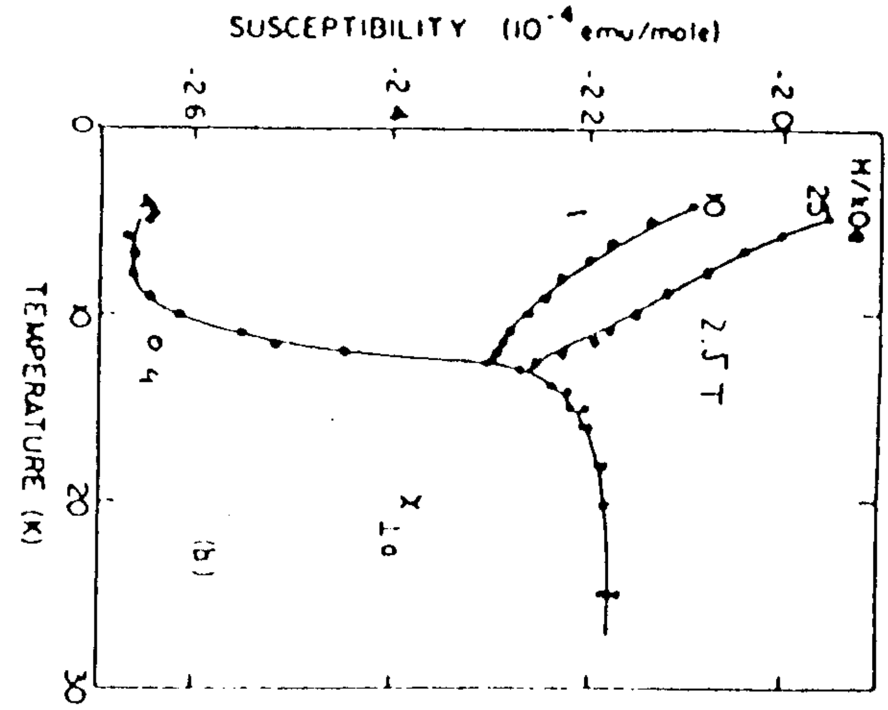
TMTTF$_2$ (PF$_6$)

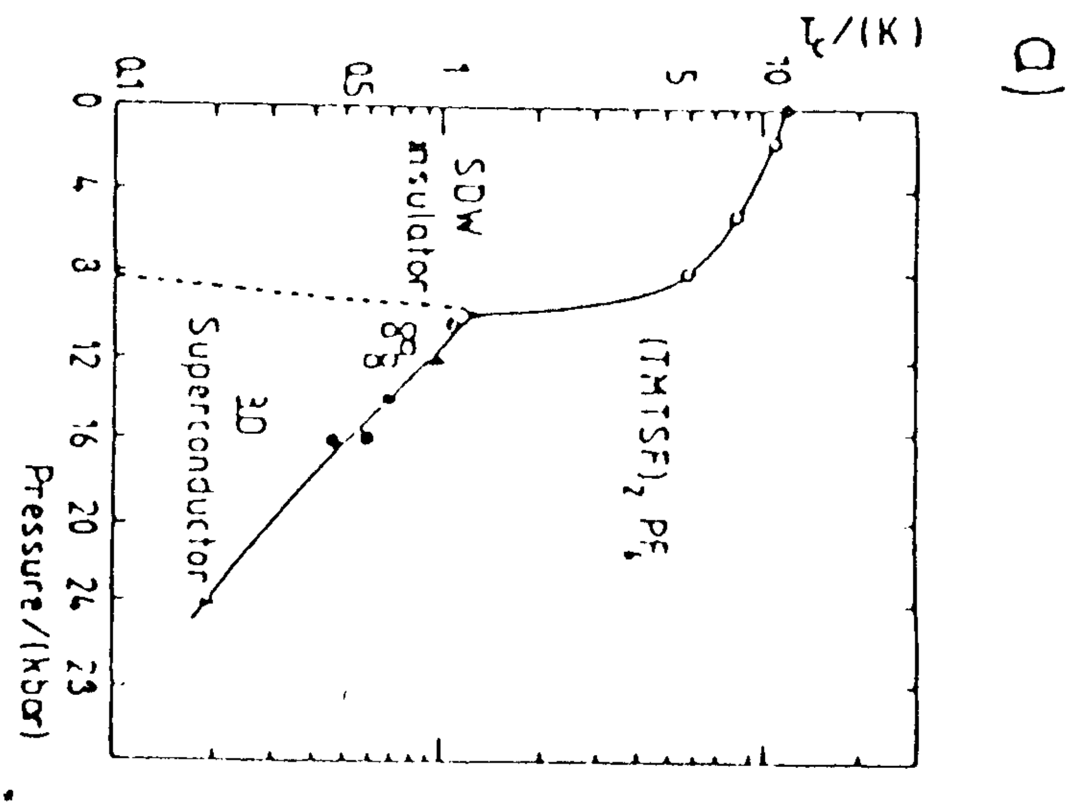

P = 15 kbar
P = 20 kbar
P = 28 kbar
(TMTTF)$_2$PF$_6$
Résistivité (un. arb.)
Temperature (K)
0
1
2
3
4
5
0
100
200
300

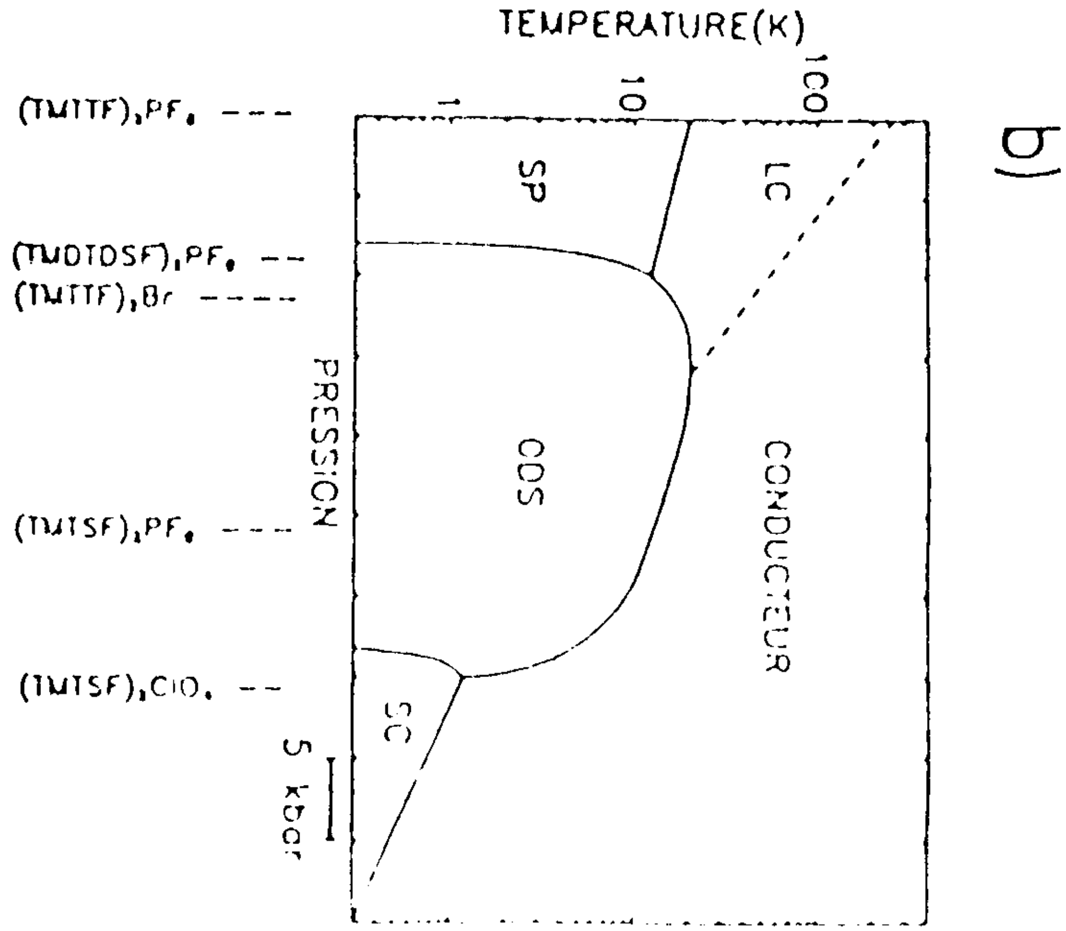

ρ∥ (Ωcm)
10
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
TSDW
S-PF6
Tρ
S-Br
Se-PF6
Tρ
TSDW
0
100
200
300 T(K)

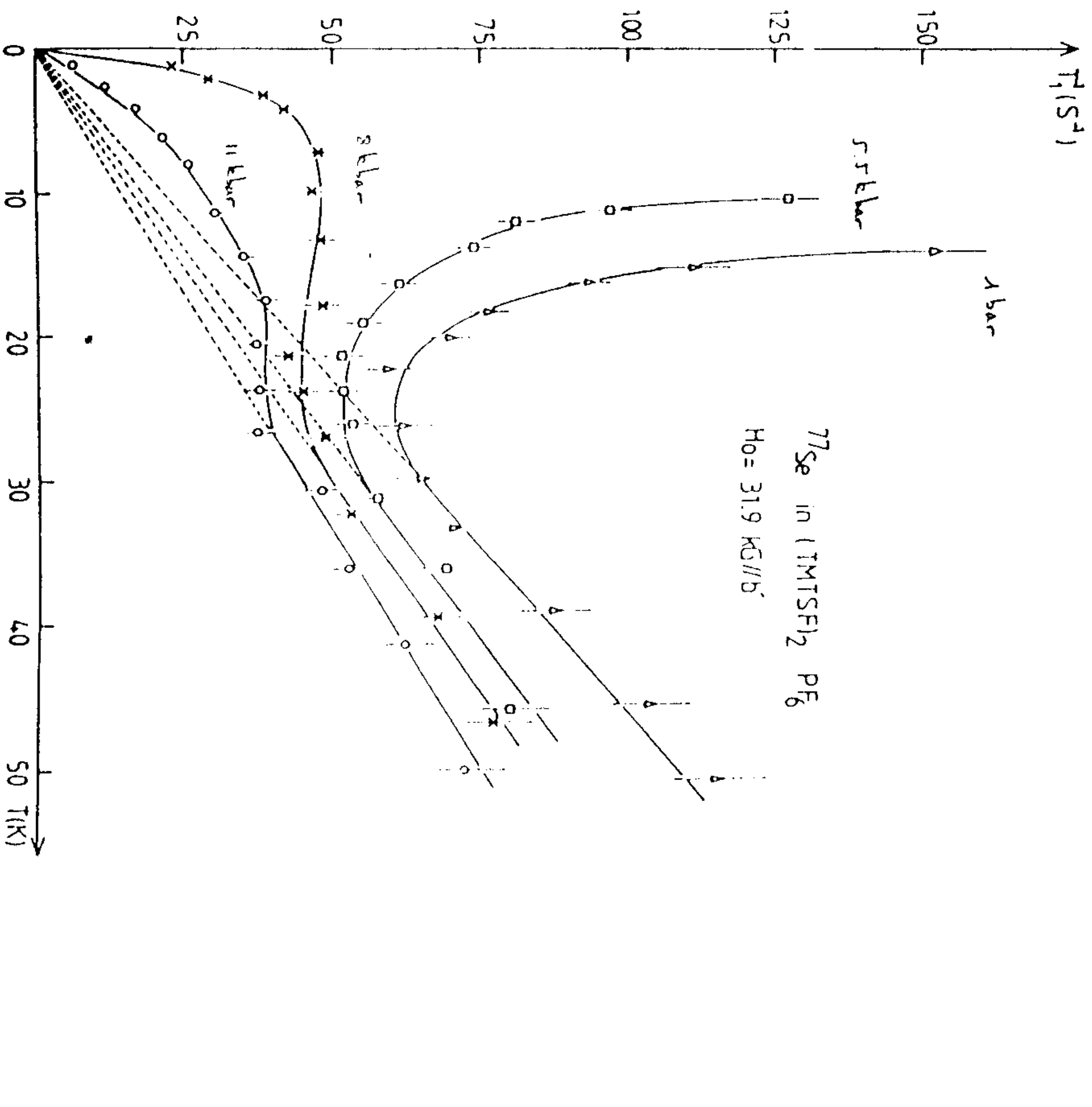

χS (EPR)
also
S-PF6
Se-PF6
0
100
200
300
T(K)

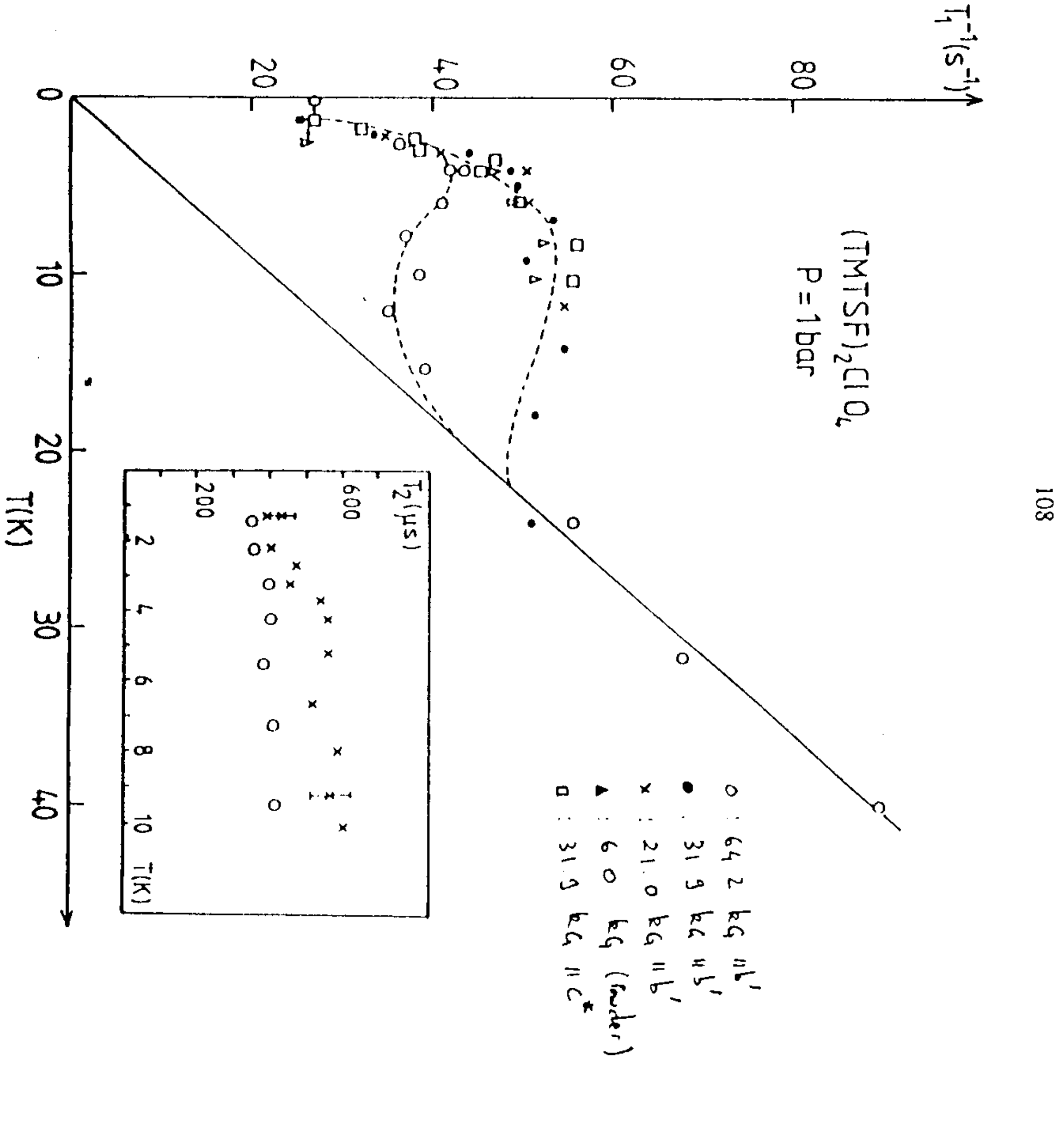

SUSCEPTIBILITY (10^{-4} emu/mole)
TEMPERATURE (K)
(a)


SUSCEPTIBILITY (10^{-4} emu/mole)
TEMPERATURE (K)
2.5 T
(b)


a)
T_c/(K)
Pressure/(kbar)
SDW insulator
3D Superconductor
(TMTSF)$_2$ PF$_6$


b)
TEMPERATURE(K)
PRESSION
(TMTTF)$_2$PF$_6$
(TMDTDSF)$_2$PF$_6$
(TMTTF)$_2$Br
(TMTSF)$_2$PF$_6$
(TMTSF)$_2$ClO$_4$
5 kbar
LC
SP
CDS
CONDUCTEUR
SC


$T_1(S^{-1})$
^{77}Se in $(TMTSF)_2\ PF_6$
Ho= 31.9 KG//b'
5.5 kbar
1 bar
8 kbar
11 kbar
T(K)


108
(TMTSF)2ClO4
P = 1 bar
T1-1(s-1)
T(K)
T2(μs)
o : 64.2 kG ∥ b'
• : 31.9 kG ∥ b'
x : 21.0 kG ∥ b'
▲ : 6.0 kG (powder)
□ : 31.9 kG ∥ c*

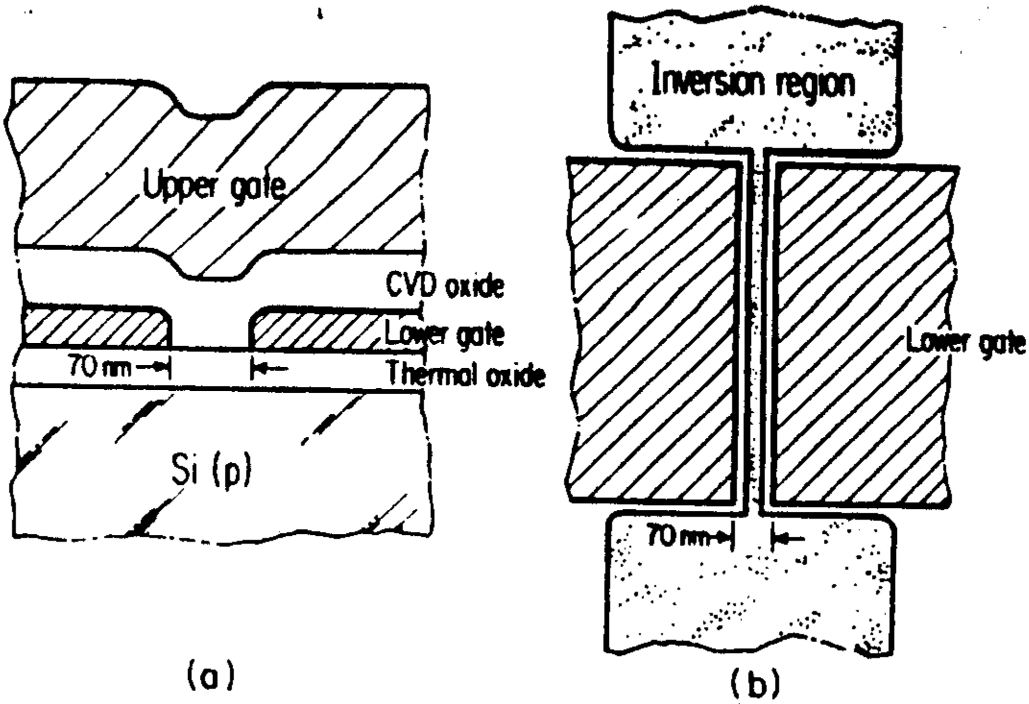


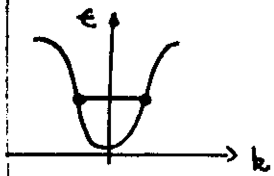
FIG. 1. Schematic (a) cross section and (b) top view of the silicon transistor with continuous upper gate and a gap in the lower gate. The electron gas, formed in the Si by the positively biased upper gate, is confined by the lower gate. The Si is *p*-type, so the surface electrons are isolated from the bulk by *p-n* junctions. That is why the electron-rich region is called an inversion layer. The cross section is roughly to scale, but the top view is not. The narrow channel is typically 20 nm wide by 1–10 μm long. Contact is made to the two wide inversion regions.

- $\chi_{q\approx 0} \sim$ cste
- T_1^{-1} diverges
- Antiferro ground state
- Superconducting " "
- 1/2 filled band : insulator
- Δ_ρ varies with P

- Spin-charge separation
- T_c ↘ with disorder
- Behavior at small T

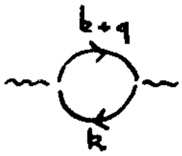
"Bad"

3) What is peculiar in 1D ?:



Fermi surface $\rightarrow$ 2 points

problem:

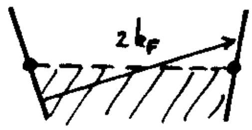


$$\chi = \sum_{k, i\nu_n} \frac{1}{i\omega_n + i\nu_n - \varepsilon_{k+q}} \frac{1}{i\nu_n - \varepsilon_k}$$

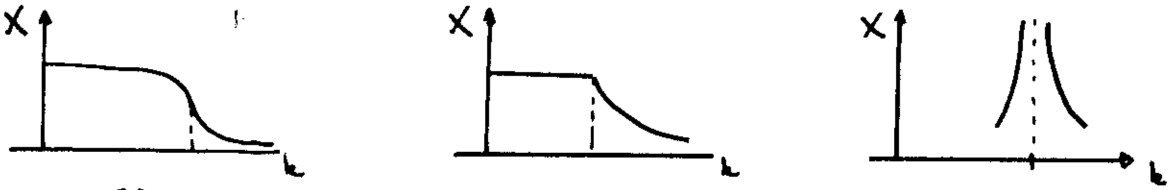
$$\chi = \sum_k \frac{f(\varepsilon_k) - f(\varepsilon_{k+q})}{i\omega_n + \varepsilon_k - \varepsilon_{k+q}} \simeq \sum_{k \sim k_F} \frac{\tanh\left(\frac{\beta \varepsilon_k}{2}\right)}{2\varepsilon_k} \rightarrow \mathrm{Log}\left(\frac{\Lambda}{T}\right)$$

$\varepsilon_k \approx v_F [k - k_F] \quad k \simeq k_F \qquad \varepsilon_k \approx +v_F[-k - k_F] \quad k \simeq -k_F$

$\forall k \qquad \varepsilon_{k+2k_F} \equiv -\varepsilon_k$

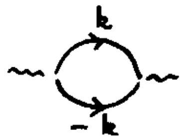


Nesting



3D 2D 1D

• BCS always Log



Competition between p.p. and p.h.

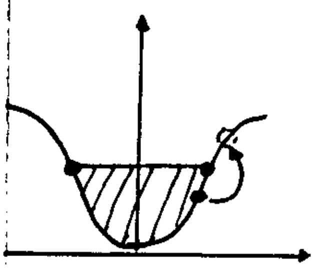
No way to do

$$\chi = \frac{\chi_0}{1 - U\chi_0}$$

(would give an ordered state at finite T $\neq$ Hermin Wagner !)

$\Rightarrow$ No mean field theory !

"Good"



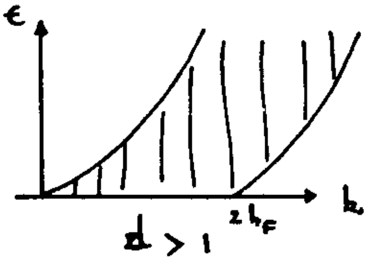
Fermi surface $\rightarrow$ 2 points

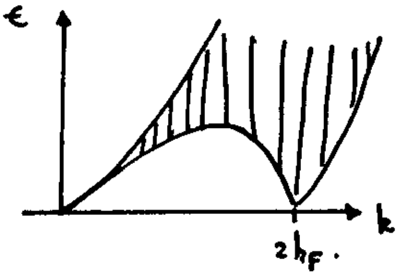
Particle hole excitations

$k \rightarrow k+q$

momentum : $P = (k+q) - k = q$
Energy : $E = \epsilon_{k+q} - \epsilon_k \approx v_f[(k+q) - k] \approx v_f q$

well defined energy <u>and</u> momentum. The p.h excitations are quasiparticles !!





* <u>Idea</u>: $c^{\dagger}_{k+q} c_k$ quasiparticles $\rightarrow \rho(q) = \sum_k c^{\dagger}_{k+q} c_k$

quasiparticles " boson excitations ! "

$H = H_0 + H_{int}$ $\qquad H_{int} = \int V(x-x') \rho(x) \rho(x')$

Quartic in fermions $\rightarrow$ yuck !!
Quadratic in ρ !!!

if one can express everything in term of ρ the problem is solved !!

$\Rightarrow$ bosonization !

II] Bosonization (Spinless Model)

1) Model and simplifications:

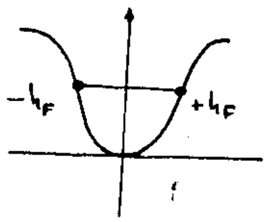
$$H = -t \sum_{\langle ij \rangle} c_i^+ c_j + V \sum_i n_i n_{i+1}$$

Analogy: $c_i^+ \leftrightarrow S_i^+$ $\quad c_i \leftrightarrow S_i$ $\quad S_z = (c_i^+ c_i - 1/2)$

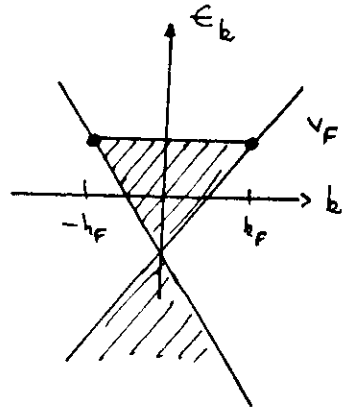
more precise : Jordan - Wigner

$$H = J_{xy} \sum_i \frac{1}{2} \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + J_z S_i^z S_{i+1}^z$$

* $V = 0$:



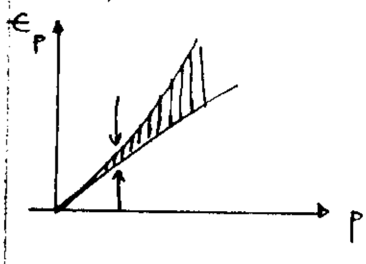
$\Rightarrow$



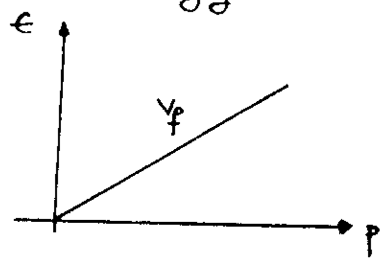
Why: $c_{k+p}^+ c_k$

$$E_{k+p} - E_k \equiv v_F p$$

All p.h. excitations have the same energy



$\longrightarrow$



[Ex: check that $\delta\varepsilon_p / \varepsilon_p \to 0$ when $p \to 0$]

Infinite negative energy states : clear later

\# Right and left movers: $(k \approx +k_F \ ; \ k \approx -k_F)$

$$H = \sum_{k,r} v_F (rk - k_F)\, c_{r,k}^+ c_{r,k}$$

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_k c_k e^{ikx} \approx e^{ik_F x} \tilde{\psi}_+(x) + e^{-ik_F x} \tilde{\psi}_-(x)$$

$$= \psi_+(x) + \psi_-(x)$$

2) Boson operators:

$$\rho_r(x) = :\psi_r^+(x)\,\psi_r(x): = \frac{1}{L}\sum_p \rho_r(p)\, e^{-ipx}$$

$$\rho_r(p) = \sum_k c^+_{r,k+p}\, c_{r,k} \qquad (p \neq 0)$$

$$= N_r = \sum_k \left(n_{r,k} - \langle 0 | n_{rk} | 0 \rangle \right)$$

Commutator:

$$[\rho_r(p), \rho_{r'}(-p')] = \sum_{k_1 k_2} [c^+_{k_1+p}\, c_{k_1},\ c^+_{k_2-p'}\, c_{k_2}]$$

$$= \sum_{k_1 k_2} \left(c^+_{k_1+p}\, c_{k_2}\, \delta_{k_1, k_2-p'} - c^+_{k_2-p'}\, c_{k_1}\, \delta_{k_1+p, k_2} \right)$$

$$= \sum_{k_2} \left(c^+_{k_2+p-p'}\, c_{k_2} - c^+_{k_2-p'}\, c_{k_2-p} \right)$$

usually change variables $\rightarrow$ 0
but here one has to worry about the ∞ $N^\#$ of state:

$$[\quad] = \sum_{k_2} :\;: - :\;: \;+ \langle \phi | c^+_{k_2+p-p'}\, c_{k_2} | \phi \rangle - \langle \phi | c^+_{k_2-p'}\, c_{k_2-p} | \phi \rangle$$

$$= \sum_{k_2} \qquad + \delta_{pp'} \left(\langle n_{k_2} \rangle - \langle n_{k_2-p} \rangle \right)$$

$$= -\,\delta_{r,r'}\,\delta_{pp'}\,\frac{rpL}{2\pi}$$

Moreover: $\rho_-(p)\,|\phi\rangle = 0 \qquad \rho_+(-p)\,|\phi\rangle = 0$

bosons:

$$b_p^+ = \left(\frac{2\pi}{L|p|}\right)^{1/2} \sum_{r=\pm} \theta(rp)\,\rho_r(p)$$

$$b_p = \left(\frac{2\pi}{L|p|}\right)^{1/2} \sum_{r=\pm} \theta(rp)\,\rho_r(-p)$$

3) <u>H° and ψ</u>

$$H^\circ = \sum_k v_F (k_r - k_F)\, c^+_{rk} c_{rk} \qquad p_0 > 0$$

$$[\, b(p_0) \,,\, H^\circ] = \sum_k \sum_r \left(\frac{2\pi}{L|p_0|}\right)^{1/2} [\, \rho_+(-p_0) \,,\, v_F (rk - k_F)\, c^+_{rk} c_{rk} \,]$$

$$= \sum_k (\quad)^{1/2} v_F k \left[c^+_{k_1 - p_0} c_k \,\delta_{k_1,k} - c^+_k c_{k_1} \,\delta_{k_1-p_0,k} \right]$$

$$= \sum_k (\quad)^{1/2} \; v_F k \; c^+_{k-p_0} c_k \; - \; v_F (k-p_0)\, c^+_{k-p_0} c_k$$

$$= v_F\, p_0\, b_{p_0}$$

$$\Rightarrow \qquad H = \sum_{p \neq 0} v_F |p|\, b^+_p b_p$$

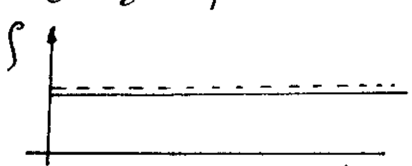
\# <u>In fact</u>: cheating !

The fermionic Hilbert space contains states with different total number of fermions. b^+_p ; b_p conserve the total number of fermions $\rightarrow$ needs operators $U_\pm$ changing the <u>total</u> number of fermions (and which commute with the bosons)

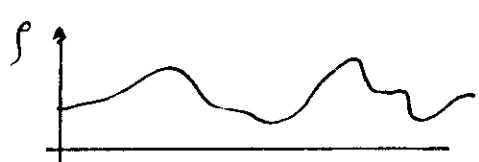
$$|\,\text{state}\,\rangle = \prod_p (b^+_p)^{n_p}\, U_+^{N_+}\, U_-^{N_-}\, |\emptyset\rangle$$

$U_\pm$ can be constructed explicitely
$\rightarrow$ one can check that the basis is <u>complete</u>

(Needs to have also Number of Left and right going particles $U_+ \leftrightarrow N_+$ $\quad U_- \leftrightarrow N_-$)



U^+ adds a charge globally



b <u>moves</u> charge

$\Rightarrow$ U not very important in thermodynamic limit

More convenient to use:

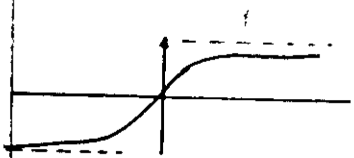
$$\phi(x)\,;\ \theta(x) = \mp \frac{i\pi}{L} \sum_p \frac{1}{p}\, e^{-\alpha|p|/2 - ipx} \left[\rho_+(p) \pm \rho_-(p) \right]$$

$$\left\{\begin{array}{l} \phi(x) = \dfrac{-i\pi}{L} \displaystyle\sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{1}{p}\, e^{-ipx} \left[b_p^+ + b_{-p} \right] \\ \theta(x) \quad \dfrac{i\pi}{L} \displaystyle\sum_p \left(\quad \right)^{1/2} \frac{1}{|p|}\, e^{-ipx} \left[b_p^+ - b_{-p} \right] \end{array}\right.$$

α is a cutoff needed to regularize the expressions
One has to take $\lim_{\alpha \to 0}$

$$[\phi(x_1), \theta(x_2)] = \sum_p \frac{\pi}{2Lp} e^{ip(x_2 - x_1)} z$$

$$= i \int_0^{+\infty} \frac{dp}{p} \sin(p(x_2 - x_1))\, e^{-\alpha|p|}$$



$$[\phi(x_1), \nabla\theta(x_2)] = i \int_0^{+\infty} dp \cos(p(x_2 - x_1))\, e^{-\alpha|p|}$$

$$\xrightarrow[\alpha = 0]{} i\pi\, \delta(x_1 - x_2)$$

thus $\pi\Pi(x) = \nabla\theta(x)$

$$\nabla\phi = -\frac{\pi}{L}\sum_p e^{-\alpha|p|/2} [\rho_+ + \rho_-]\, e^{-ipx} = -\pi[\rho_+(x) + \rho_-(x)]$$

$$\nabla\theta = \pi[\rho_+(x) - \rho_-(x)]$$

$\nabla\phi$ is related to the $q \approx 0$ part of the density
$\nabla\theta$ " part of the current

$$H_0 = \frac{1}{2\pi}\int dx\ v_F \left[(\pi\Pi)^2 + (\nabla\phi)^2 \right]$$

* What about ψ :

$$[\rho_+(p), \psi_r(x)] = \frac{1}{\sqrt{L}} \sum_k \sum_{k_1} e^{ik_1 x} [c^+_{k+p} c_k , c_{k_1}]$$

since $[AB, C] = A\{B, C\} - \{A, C\} B$

$$= \frac{-1}{\sqrt{L}} \sum_{k k_1} e^{ik_1 x} \delta_{k+p, k_1} c_k = - e^{ipx} \psi(x)$$

idem to $\left[\rho_r(p) , e^{\sum_p e^{ipx} \rho_r(-p) \frac{2\pi r}{pL}} \right]$

thus: (?)

$$\psi_r(x) = e^{irk_F x} \frac{1}{\sqrt{2\pi\alpha}} e^{-i[r\phi - \theta]}$$

[In fact: ψ changes total # of fermions $\psi = U e^{\cdots\cdots}$

$\lim_{\alpha \to 0}$: let us wait for the calculation of correlation functions

→ simple physical interpretation

4) The real stuff: interactions :

$$H = H_0 + \int dx\, dx' \, V(x-x') \rho(x) \rho(x')$$

Simplicity $V(x-x') = V \delta(x-x')$

$$\rho(x) = \psi^+(x)\psi(x) \equiv \underbrace{\psi^+_+(x)\psi_+(x) + \psi^+_-\psi_-}_{\rho\, q\approx 0} + \underbrace{\psi^+_+\psi_- + \psi^+_-\psi_+}_{\rho\, q \approx 2k_F}$$

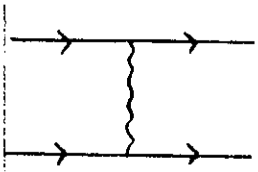
ex: $\sum_i n_i n_{i+1}$

$$= \sum_i (\rho_+ + \rho_- + \psi^+_+\psi_- + \psi^+_-\psi_+)_i (\rho_+ + \rho_- + \psi^+_+\psi_- + \psi^+_-\psi_+)_{i+1}$$

one keeps only non oscillating terms in the continuum limit

For simplicity let us take:

$$V \int dx \; (\rho_+ + \rho_-)(\rho_+ + \rho_-) = \int \frac{V}{\pi^2} (\nabla\phi)^2 \, dx$$



g_4

g_2

$$H_0 + H_{int} \Rightarrow H = \frac{1}{2\pi} \int dx \; (uK)(\pi\Pi)^2 + \left(\frac{u}{K}\right)(\nabla\phi)^2$$

with:
$$\begin{cases} uK = v_f \\ \dfrac{u}{K} = v_f + \dfrac{2V}{\pi} \end{cases}$$

$$K = \sqrt{\frac{1}{1 + \left(\frac{2V}{\pi v_f}\right)}} \qquad (< 1 \text{ if } V > 0) \qquad (> 1 \text{ if } V < 0)$$

u: new velocity of excitations;
K: can be absorbed by $\tilde{\Pi} = \sqrt{K}\,\Pi \qquad \tilde{\phi} = \frac{1}{\sqrt{K}}\phi$

ex: do it in term of bosons b^+, b

Physical properties:

• Specific heat:

$$C_v = \frac{dE}{dT} = \frac{1}{T^2} \sum_{p \neq 0} \epsilon^2 \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2} = \frac{u^2}{4T^2} \sum_{p \neq 0} \frac{p^2}{\mathrm{sh}^2\left(\frac{\beta u p}{2}\right)}$$

$$= \frac{u^2}{4T^2} \frac{L}{2\pi} \left[\frac{\pi^2}{3} \frac{8}{(\beta u)^3} \right] = \frac{T}{u} \left(\frac{L\pi}{3}\right)$$

Free fermions: $\quad E = 2 \sum_p \epsilon_p f(\varepsilon) = \frac{2}{u} \frac{L}{2\pi} \int d\varepsilon \; \varepsilon f(\varepsilon)$

$$\frac{dE}{dT} = \frac{L}{\pi u} \int_{-\infty}^{+\infty} d\varepsilon \; \varepsilon^2 \frac{1}{4\,\mathrm{ch}^2\left(\frac{\beta\varepsilon}{2}\right)} \frac{1}{T^2} = \frac{\pi}{3} L \frac{T}{u} \qquad \text{idem!}$$

$$\gamma / \gamma_0 = \frac{v_f}{u}$$

Compressibility : $\kappa / \kappa_0 = v_f \left(\frac{K_\rho}{u_\rho}\right)$

Looks like a "Fermi Liquid" for $q \approx 0$ quantities ....

Correlation functions :

$$\rho(x) \equiv \rho_+ + \rho_- + \psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+$$

$$\rho_{2k_F}(x) \approx e^{-i 2k_F x} e^{i 2\phi(x)}$$

CDW

$$\chi \equiv \frac{1}{\pi^2} \langle T_\tau (\nabla\phi)_{x\tau} (\nabla\phi)_{00} \rangle + \cos(2k_F x) \frac{1}{(2\pi\alpha)^2} \langle T_\tau \, e^{i2\phi(x,\tau)} e^{-i2\phi(0,0)} \rangle$$

Superconductivity: $\psi_+^\dagger \psi_-^\dagger \equiv \frac{1}{(2\pi\alpha)} e^{-2i\theta(x,\tau)}$

Single particle $\langle T_\tau \; \psi(x,\tau)\, \psi^\dagger(0,0) \rangle \sim \langle e^{-i[r\phi-\theta]_{x\tau}} e^{+i[\;]} \rangle$

How to get:

$$A(r) = \langle T_\tau \, e^{i2\phi(x,\tau)} e^{-i\phi(0,0)} \rangle$$

$$\mathcal{L} = \frac{1}{2\pi} \int dx \; \frac{1}{uK} (\partial_\tau \phi)^2 + \left(\frac{u}{K}\right) (\partial_x \phi)^2$$

$$\phi(x,\tau) = \frac{1}{\sqrt{L\beta}} \sum_{q\omega_n} e^{iqx+\omega\tau} \phi_{q,\omega}$$

$$S = \frac{1}{2\pi} \sum_{q\omega} \left[\frac{\omega^2}{uK} + \frac{q^2 u}{K} \right] \phi_q \phi_{-q}$$

$$A(r) = \left[\int \mathcal{D}\phi \, e^{-S} \; e^{\frac{i2}{\sqrt{L\beta}} \sum_{q\omega} (e^{i\vec{q}\cdot\vec{r}_1} - e^{i\vec{q}\cdot\vec{r}_2}) \phi_q} \right] \frac{1}{\int \mathcal{D}\phi \, e^{-S}}$$

$$= \exp\left[-2K \left(\frac{2\pi}{L\beta}\right) \sum_{q\omega} (1-\cos(qx+\omega\tau)) \frac{1}{\frac{\omega^2}{u} + q^2 u} \right.$$

$$\equiv \exp\left[-2K \int_0^\Lambda \frac{dq}{q} (1-\cos \vec{q}\cdot\vec{r}) \right. \qquad \vec{r} \equiv (x, u\tau)$$

in fact $\int dq \to \int dq\ e^{-\alpha|q|}$ $\qquad \Lambda \sim 1/\alpha$

$$A(r) \equiv \exp\left[-2K \operatorname{Log}\left(\frac{x^2+(u\tau)^2}{\alpha^2}\right)^{1/2}\right]$$

idem:

$$\langle T_\tau\, e^{i\sqrt{2}\,\theta_{x\tau}}\, e^{-i\sqrt{2}\,\theta}\rangle = e^{-K^{-1}\operatorname{Log}(r/\alpha)}$$

- How to take $\alpha \to 0$?

in fact if one has an interaction of finite range R

$$A(r) \equiv \frac{1}{(2\pi\alpha)^2}\, e^{-2\int_0^{1/\alpha}\frac{dp}{p}(1-\cos pr)\,K(p)}$$

$$\equiv \frac{1}{(2\pi\alpha)^2}\, e^{-2\int_0^{1/R}\frac{dp}{p}(1-\cos pr)\,K(p=0)\ -\ 2\operatorname{Log}(R/\alpha)}$$

$$\simeq \frac{1}{(2\pi R)^2}\, e^{-2K\operatorname{Log}(r/R)}$$

More simple to keep α finite. α can be identified with a lattice constant.

$\Rightarrow$ Power Law decay of correlation functions !

$$\begin{cases} \chi_{2k_F}(x,\tau) \propto \left(\frac{1}{r}\right)^{2K} \longrightarrow \chi(q,\omega) \sim \omega^{2K-2} \\ \chi^{SU}(x,\tau) \propto \left(\frac{1}{r}\right)^{2K^{-1}} \longrightarrow \chi^{SU}(q,\omega) \sim \omega^{2K^{-1}-2} \end{cases}$$

$V<0$: divergent SU fluctuations

$V>0$: " CDW "

$$G_{++}(r,\tau) = \langle e^{i[\phi-\theta]}\, e^{-i[\phi-\theta]}\rangle$$

$$\equiv e^{-\frac{1}{2}[K+K^{-1}]\operatorname{Log}(r)}\ e^{i\operatorname{Arg}(\tau/x)}$$

<u>Ex</u> Formule generale

$$\langle T_\tau\, e^{i\sqrt{2}\,(\alpha_i\phi_i+\beta_i\theta_i)} \cdots \rangle = e^{\sum_{i<j}\left[\alpha_i\alpha_j K + \beta_i\beta_j K^{-1}\right]F_1 + (\alpha_i\beta_j+\beta_i\alpha_j)\,F_2\ ]}$$

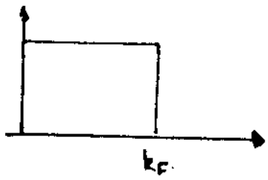
$F_1 \equiv \operatorname{Log}(r) \qquad F_2 = i\operatorname{Arg}\left(\frac{\tau}{x}\right)$

Check: $K=1$ $\quad G \equiv e^{-\mathrm{Log}[x - i\tau]} = \frac{1}{x - i\tau v} \Rightarrow \frac{1}{x - vt}$

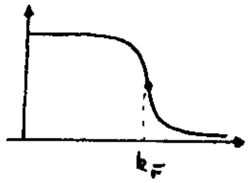
$$n(k) \equiv \int dx \; e^{i(k-k_F)x} \, G$$

$$\rightarrow \quad |k - k_F|^{\frac{1}{2}[K + K^{-1}] - 1}$$

$K=1$



$K \neq 1$

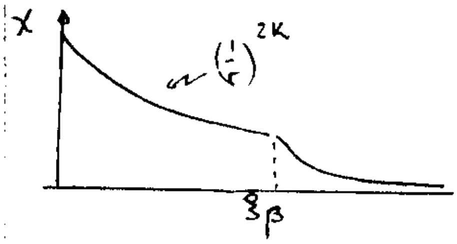


k_F not changed
no quasiparticles

Finite T :

$$F_1 = \frac{1}{2} \mathrm{Log}\left[\frac{\beta^2}{\pi^2} \left(\mathrm{sh}^2\left(\frac{\pi x}{\beta}\right) + \sin^2\left(\frac{\pi \tau}{\beta}\right) \right) \right] \qquad \Big| \begin{array}{l} \tau \equiv \tau v \\ \beta \equiv \beta v \end{array}$$

$$\chi \sim \left(\frac{1}{r}\right)^{2K} \quad \rightarrow \quad e^{-(2K)\left[\frac{\pi x}{\beta}\right]} \equiv -x/\xi_\beta$$



$$F_2 = i \; \mathrm{Arg}\left[\frac{\tan\left(\frac{\pi \tau}{\beta}\right)}{\tanh\left(\frac{\pi x}{\beta}\right)} \right]$$

5) Deviations from the ideal case: The concept of Luttinger Liquid

?? v large ?? ?? ??

in fact H describes the low energy properties provided one uses the correct u and K

e.g. compute from Bethe Ansatz

III.] Model with Spin

1) "Ze" model:

$$H = -t\sum_{\langle ij\rangle} c^{+}_{i\sigma} c_{j\sigma} + U\sum_i n_{i\uparrow} n_{i\downarrow} + \left[V \sum_i n_i n_{i+1} \dots \right]$$

"Solved" bij B.A.

$\Delta_\rho \neq 0$ 1/2 filling

$\Delta_\rho = 0$ $n \neq 1/2$

with spin: introduce d boson per spin $\phi_\uparrow$ $\phi_\downarrow$

$$H_0 = H_\uparrow + H_\downarrow = H_\rho + H_\sigma$$

$$\phi_\rho = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow) \qquad \phi_\sigma = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow)$$

\# Interaction:

$$n_{i\uparrow} n_{i\downarrow} = (\rho_{+\uparrow} + \rho_{-\uparrow} + \psi^{+}_{+\uparrow}\psi_{-\uparrow} + \psi^{+}_{-\uparrow}\psi_{-\uparrow})(\qquad)_\downarrow$$

$$\psi = \frac{1}{\sqrt{2\pi\alpha}} \exp \frac{1}{\sqrt{2}}\left[i\left(r\phi_\rho - \theta_\rho + s(r\phi_\sigma - \theta_\sigma)\right)\right]$$

$$\psi^{+}_{r\sigma}\psi_{r\sigma} \equiv \exp\left[-\sqrt{2}\, i r \phi_\rho - \sqrt{2}\, i s r \phi_\sigma\right]$$

$$H_{int} = \sum_i (\rho_{+\uparrow} + \rho_{-\uparrow})(\rho_{+\downarrow} + \rho_{-\downarrow}) + \frac{1}{(2\pi\alpha)^2}(e^{i2\sqrt{2}\phi_\sigma} + h.c.)$$

$$= \frac{1}{\pi^2}\nabla\phi_\uparrow \nabla\phi_\downarrow + \frac{2}{(2\pi\alpha)^2}\cos(2\sqrt{2}\phi_\sigma)$$

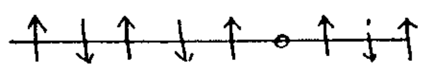
$$= \frac{1}{2\pi^2}\left[\nabla\phi_\rho^2 - \nabla\phi_\sigma^2\right] + \dots$$

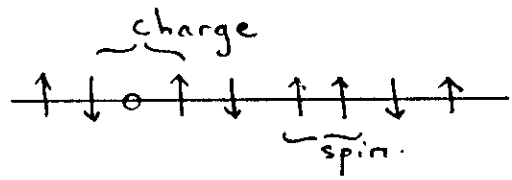
$$H = H_\rho + H_\sigma$$

$$\begin{cases} H_\rho = \dfrac{1}{2\pi} \displaystyle\int dx \; (uK)_\rho (\nabla\theta_\rho)^2 + \left(\dfrac{u}{K}\right)_\rho (\nabla\phi_\rho)^2 \\ H_\sigma = H_\nu + \dfrac{2U}{(2\pi\alpha)^2} \displaystyle\int dx \; \cos(\sqrt{8}\,\phi_\sigma) \end{cases}$$

$$\begin{cases} u_\rho K_\rho = u_\sigma K_\sigma = v_f \\ u_\rho / K_\rho = v_f + \dfrac{U}{\pi} \end{cases} \qquad \frac{u_\sigma}{K_\sigma} = v_f - \frac{U}{\pi}$$

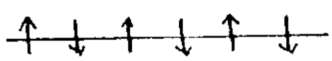
- Valid for small U. For large U see later
- Spin - charge separation

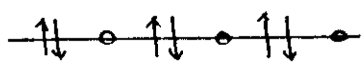




- More complicated spin H.

Instabilities:


SDW


CDW

$$O_{\substack{CDW \\ SDW}} = \sum_{\sigma\sigma'} \psi^{+}_{+\sigma} \, \sigma_{\sigma\sigma'} \, \psi_{-\sigma'}$$

$$O_{CDW} = \frac{e^{2ik_F x}}{\pi\alpha} \, e^{-i\sqrt{2}\,\phi_\rho} \cos(\sqrt{2}\,\phi_\sigma)$$

$$O_{SDW_x} = \cos(\sqrt{2}\,\theta_\sigma)$$

$$y = \sin(\sqrt{2}\,\theta_\sigma)$$

$$z = \sin(\sqrt{2}\,\phi_\sigma)$$

Superconducting : $\phi_\rho \to \theta_\rho$

2) <u>The spin part</u>:

one <u>has</u> to have cos( ) term:

$\langle O_{SDW_x} O_{SDW_x} \rangle \sim \left(\frac{1}{r}\right)^{K_\rho + K_\sigma^{-1}}$ $\langle \quad \rangle_z \sim \left(\frac{1}{r}\right)^{K_\rho + K_\sigma}$

only possible if $K_\sigma^* = 1$

<u>RG equations</u>:

$$\langle e^{i\varepsilon\sqrt{2}\phi_\sigma} e^{-i\varepsilon\sqrt{2}\phi_\sigma} \rangle \equiv e^{-\varepsilon^2 K \operatorname{Log}(r)} + \frac{1}{2}\left[\frac{g}{(2\pi\alpha)^2}\right]^2 \sum_{\alpha_1\alpha_2}$$

$$\frac{1}{u^2}\int d_2 r_1 \, d_2 r_2 \langle e^{+} e^{-} \, 2\cos(\;)_1 \, 2\cos(\;)_2 \rangle$$

$$= e^{-K\log r}\left[1 + \frac{g^2}{(2\pi\alpha)^4}\frac{1}{u^2}\int d_2 r_1 \, d_2 r_2 \left(e^{2F(r-r_1)-F(r-r_2)+F(r'-r_2)-F(r'-r_1)} - 1\right) e^{-4F(r_1-r_2)}\right]$$

$$X = \frac{r_1 + r_2}{2} \qquad \delta r = (r_1 - r_2)$$

$$1 + \frac{g^2}{(2\pi\alpha)^4}\frac{1}{u^2}\int d_2 R \, d_2 \delta r \left(e^{2[\delta r \cdot \nabla [F(r-R) - F(r'-R)]} - 1\right) e^{-4F(\delta r)}$$

$$\int d_2 R \, d_2 \delta r \quad 2\left[\delta r \cdot \nabla \left(F(r-R) - F(r'-R)\right)\right]^2 e^{-4F(\delta r)}$$

$$\int d_2 R \, d_2 \delta r \quad \delta r^2 \left[\nabla_x (F - F)\right]^2 + \left[\nabla_y (F - F)\right]^2 e^{-4F(\delta r)}$$

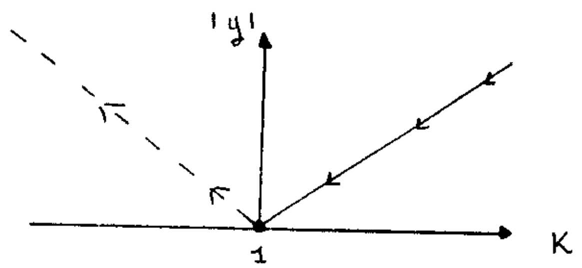
integration by part and $\Delta \operatorname{Log}(r) = +2\pi\delta(r)$

$$2\int d_2 \delta r \; \delta r^2 \, e^{-4F(\delta r)} \; K^2 \; (+2\pi) \operatorname{Log}[r - r']$$

$$\langle \quad \rangle = e^{-K\operatorname{Log}[r-r']}\left[1 + \frac{2g^2}{(2\pi)^2 u^2 \alpha^4} K^4 \int_\alpha^{+\infty} (\delta r)^3 \, d(\delta r) \; e^{-4K\operatorname{Log}(\delta r)}\right.$$

$$K_{eff} = K - \frac{1}{2} y^2 K^2 \int_\alpha \frac{dr}{\alpha} \left(\frac{r}{\alpha}\right)^{3-4K}$$

$$\Rightarrow \begin{cases} \dfrac{dK}{d\ell} = -\dfrac{1}{2} y^2 K^2 \\[2mm] \dfrac{dy}{d\ell} = (2-2K)\, y \end{cases} \qquad y = \left(\frac{g}{\pi u}\right) \qquad K = 1 + \frac{y}{2}$$



$K_\sigma^* = 1$ imposed by spin symetry.

"$K_\sigma \to 0$" $\Rightarrow$ gap in the spin sector

Luther - Emery Solution :

$$H = \frac{1}{2\pi} \int dx \, (uK)\, (\pi \Pi)^2 + \left(\frac{u}{K}\right) (\nabla\phi)^2 + \frac{2 g_1}{(2\pi\alpha)^2} \int \cos(\sqrt{8}\,\phi_\sigma)$$

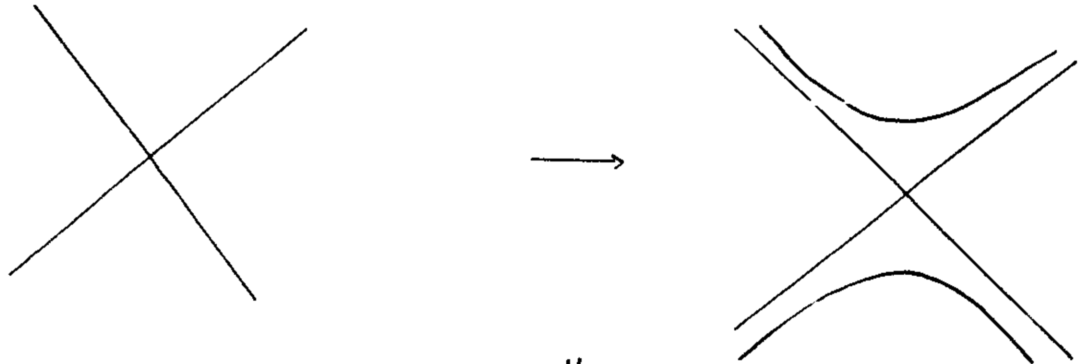
but $\psi_+^\dagger \psi_- \equiv e^{i 2 \phi}$

$$\tilde{\phi} \equiv \sqrt{2}\, \phi \quad \rightarrow \quad \int dx \, (u\, 2K) \, \ldots \left(\frac{u}{2K}\right)$$

$$+ \frac{2 g_1}{(2\pi\alpha)^2} \cos(2\tilde{\phi})$$

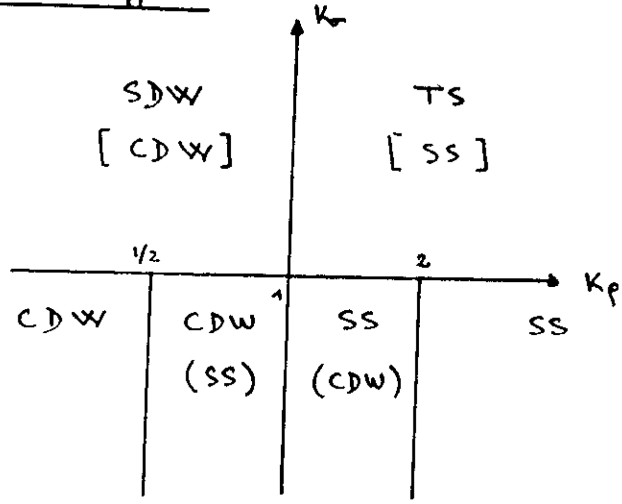
if K initial $= 1/2$

$\Rightarrow$ Free "spinless fermions" $+ \dfrac{g_1}{(2\pi\alpha)} \left[\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ \right]$



$$\Delta \sim g_1^{1/2}$$

3) Phase diagram:



$O_{CDW} \sim \omega^{K_\rho + K_\sigma - 2}$ $O_{SDW} \sim \omega^{K_\rho + K_\sigma - 2}$

$O_{SS} \sim \omega^{K_\rho^{-1} + K_\sigma - 2}$ $O_{TS} \sim \omega^{K_\rho^{-1} + K_\sigma - 2}$

- if U very large "More" SDW than CDW !

$\rightarrow$ Log corrections

$O_{CDW} \quad e^{i\sqrt{2}\phi_\rho} \cos(\sqrt{2}\phi_\sigma)$ $O_{SDW} = e^{i\sqrt{2}\phi_\rho} \sin(\sqrt{2}\phi_\sigma)$

$g_1 < O \; O \quad \cos(\sqrt{8}\phi)$ couple with opposite signs.

$R_{SDW} \sim \mathrm{Log}^{1/2}$ $R_{CDW} \sim \mathrm{Log}^{-3/2}$

- $\chi_{2k_F} \sim T^{K_\rho - 1}$ diverges if $K_\rho < 1$ RMN
- $G(r) \sim \left(\frac{1}{r}\right)^{\frac{1}{4}[K_\rho + K_\rho^{-1}] + \frac{1}{4}[K_\sigma + K_\sigma^{-1}]}$

$\Rightarrow \quad n(k) \sim |k - k_F|^{\frac{1}{4}[K_\rho + K_\rho^{-1}] - \frac{1}{2}}$

- $\langle \rho(x)\, \rho(0) \rangle = \dfrac{K_\rho}{(\pi x)^2} + A_1 \cos(2k_F x) \left(\dfrac{1}{x}\right)^{K_\rho + 1} \mathrm{Log}^{-3/2}(x)$

$\qquad + A_2 \cos(4k_F x) \left(\dfrac{1}{x}\right)^{4K_\rho} + \ldots$

4) <u>Luttinger Liquid</u>

enough to get u_ρ ; K_ρ ; u_σ $\rightarrow$ all long wavelength properties.

<u>ex</u>: $U \to \infty$ spinless fermions $K \to 1/2$ $(k_F \to 2k_F)$

$\nu = 1/8$

*References for Lect. I and II:

The list is not supposed to be exhaustive and I have tried to limit to easily accessible references.
(* : review papers)

Review 1D compounds:

* D. Jérome H.J. Schulz Adv. Phys. 31 299 (82)

• Bosonization (general):

* J. Solyom Adv. Phys 28 209 (1979)
* V. J. Emery "Highly conducting One-Dimensional Solids" ed J. T. Devreese (Plenum) (1979) p 327
F. D. M. Haldane J. Phys C. 14 2585 (81)
* I. Affleck "Field Strings and critical Phenomena" Les Houches XLIX (North Holland 1990)

• Bethe Ansatz + Luttinger Liquid

F. D. M. Haldane PRL 45 1358 (1980)
H. J. Schulz PRL 64 2831 (1990)

• More specialized (exponents, log corrections etc, ...)

A. Luther and I. Peschel PRB 9 2911 (1974)
J. Voit PRB 47 6740 (1993)
A. Luther and V.J Emery PRL 33 589 (1974)
T. Giamarchi and H.J. Schulz PRB 39 4620 (1989)
M. P. M Den Nijs PRB 23 6111 (1981)
...