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SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

SUPERCONDUCTOR-INSULATOR TRANSITION:

PART I AND II

and

QUANTUM HALL EFFECT IN DOUBLE LAYERS

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SPRING COLLEGE IN CONDENSED MATTER
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SUPERCONDUCTOR-INSULATOR TRANSITION:

PART I

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These are preliminary lecture notes, intended only for distribution to participants.

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Superconductor-Insulator Transition in 2D

M.P.A. Fisher   A.P. Young   Min-Chul Cha
G. Grinstein   Erik Sørensen   Mats Wallin

PRL 64, 587 (1990); 65, 923 (1990).

PRB 44, 6883 (1991)

PRL 69, 828 (1992).

PRB (in press '94) Review article: Mats Wallin et al.
(Posted on Trieste Bulletin Board 11/93)

Wen + Zee Int. J. Mod. Phys. B 4, 437 (1990)

Kim + Weichman PRB 43, 13583 (1991)

Batrouni + Scalettar PRB 46, 9051 (1992), 48, 9628 (199

Krauth, Trivedi, Ceperley PRL 67, 2307 (1991)
M. Makivić, Trivedi, Ullah PRL 71, 2307 (1993)
A. Gold, Z. Phys. B 52, 1 (1983); 81, 155 (1990)

S. Chakravarty, et. al. PRB 35, 7256 (1987).
K. Runge, PRB 45, 13136 (1992).
+ many others

Claim:

Superconductor-Insulator transition is

— a quantum critical phenomenon characterized by a diverging phase correlation length ξ and diverging time ξ_τ at $T=0$.

— The conductivity at the critical point is UNIVERSAL

$$\sigma^* = \frac{(2e)^2}{h} g \; ; \quad g \sim \mathcal{O}(1)$$

g independent of microscopic details:
depends only on universality class
- Coulomb / short range
- dirty / clean
- magnetic field

Experiments

Bi, Ga, Al	Haviland, Liu, Goldman	PRL 62, 2180 (1989).
	Liu, et al.	PRL 67, 2068 (1991).
	Jaeger, et al.	PRB 40, 182 (1989).
DyBaCuO	T. Wang et al.	PRB 43, 8623 (1991). 47, 11619 (1993).
NdCeCuO	S. Tanda et al.	PRL 69, 530 (1992).
MoC	Lee + Ketterson	PRL 64, 3078 (1990).

JJ arrays:

L.J. Geerligs, et al.	PRL 63, 326 (1989).
van der Zant, et al.	PRL 69, 2971 (1992).

In-InO_x

Hebard + Paalanen	PRL 54, 2155 (1985)
	PRL 65, 927 (1990)
	Helv. Phys. Acta 65, 197 (1992)

Nota Bene: Most experiments not yet in critical scaling regime. Only Hebard + Paalanen field-tuned transition shows proper scaling.

his is that when they note that en carried ons, and it ays in the ot be inert films de- c recently layers can *in situ* us- ermit pre- The film from the deposited with the density of tal. in both Pb he Pb films a nominal nductivity, suppressed a 3.28-Å- onductivity er films of moved to nduction at f less than derlayer is process, or on is occur- ndeed, a fit e measured ut. Assum- onal to the on problem, re the con- ory.[14] The at $T=14$ K or the most were 4.08

curves with ween curves ... $T \to 0$)

unstable fixed point

$R^*(T=0) \sim \left[g \frac{(2e)^2}{h}\right]^{-1}$

universal critical amplitude

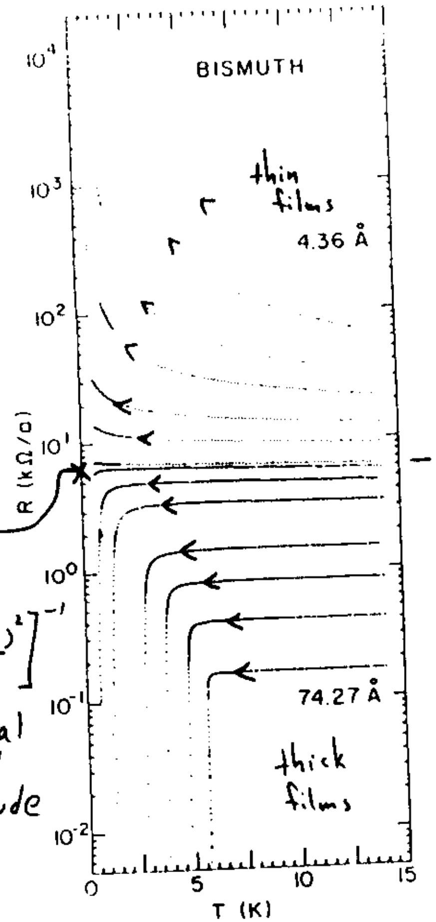


FIG. 1. Evolution of the temperature dependence of th sheet resistance $R(T)$ with thickness for a Bi film deposite onto Ge. Fewer than half of the traces actually acquired a shown. Film thicknesses shown range from 4.36 to 74.27 Å.

point at $T=0$ separating insulating and superconductir behavior. In the case of Bi the separatrix occurred ve close to $R=h/4e^2$ or 6.5 kΩ/□. In the case of Pb films occurred at somewhat greater resistance, $R=9.5$ kΩ/ but in addition, a tail in $R(T)$ could be observed. Tl connection between these observations and those ta seen in the resistive transitions of granular films, wh the downturn in $R(T)$ occurred at resistances alwa

Can we make a scaling theory

R^*_{NORMAL}

$\updownarrow\ \delta \equiv R_N - R_N^*$

<u>non</u> universal coupling constant like T_c in classical system

silver evaporation source, and pressing the bond wires to the silver pads. The sheet resistances were measured using the van der Pauw technique[10] with an estimated accuracy of about 2%. Films were cooled in a continuous-

Dy Ba Cu O 2 unit cells thick Goldman et al. (MBE grown)

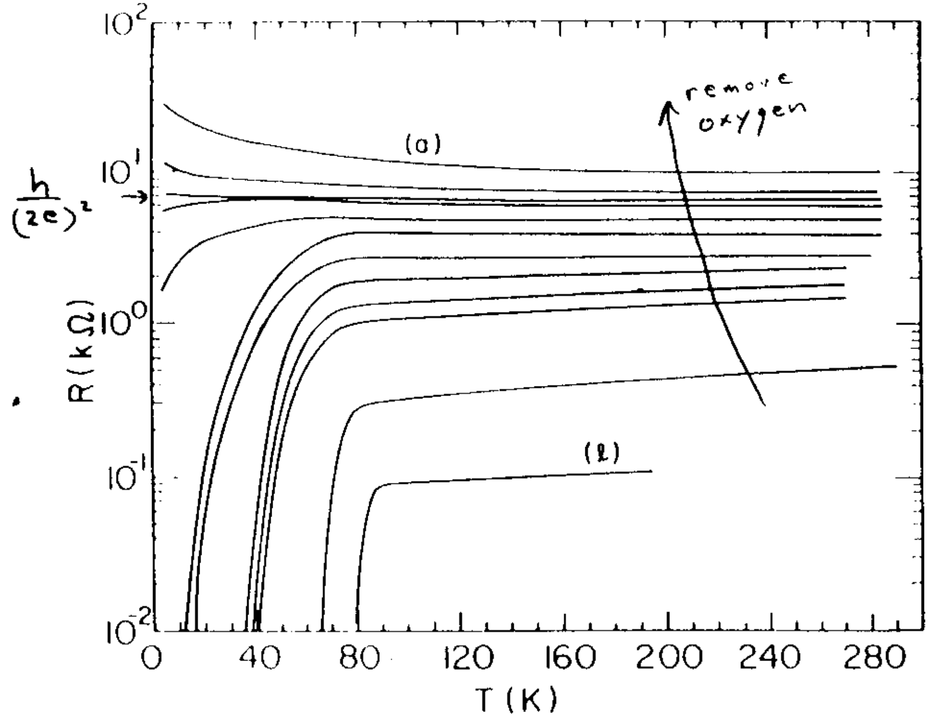


FIG. 1. Sheet resistance as a function of temperature for <u>Dy-Ba-Cu-O</u> films of various nominal thicknesses, some of which have been aged in vacuum at room temperature. Curve (e) is for a 35-Å-thick film deposited onto $SrTiO_3(100)$. Curves (b)-(d) and (a) result from successive aging steps. Curve (g) is for a 35 Å-thick film deposited onto ...

The theoretical model used to analyze these data is based upon a universal relation previously used in studies of gap-

A. Hebard, M. Paalanen In/InO

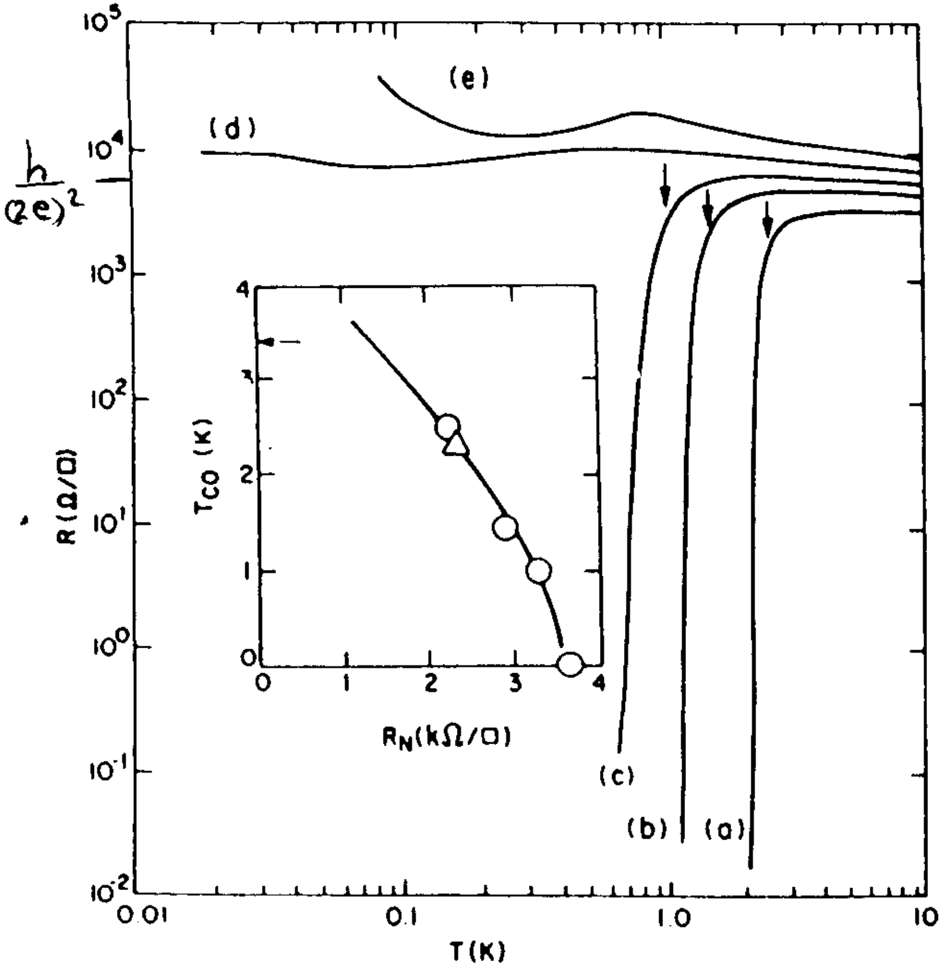


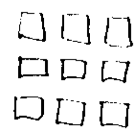
FIG. 1. Logarithmic plot of the resistance transitions of five 100-Å-thick In/InO$_x$ composite films. The transition temperatures for films (a)–(c) are indicated by arrows and the inset is discussed in the text.

tance divided by the length/width ratio 3.14 of the a
For the five arrays shown, $E_C \approx 0.84$ K is constan
R_n varies from 4.8 to 36 kΩ. Since the critical tem
ture of the aluminum was also approximately con:
$T_c = 1.37$ K, this causes x to vary from 0.53 to 3.

Hans Mooij

Josephson Junction arrays

$$\frac{(2e)^2}{C} \gtrless E_J$$



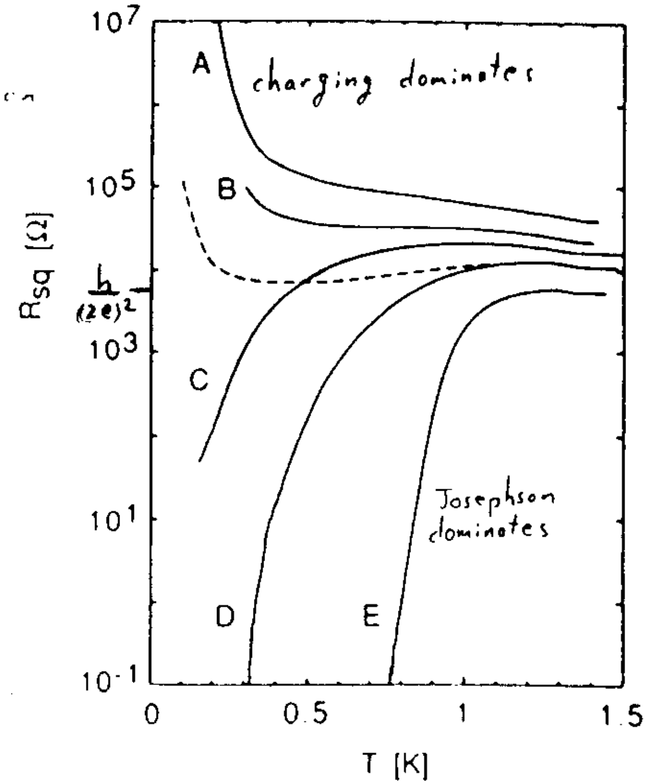


FIG. 1. $R(T)$ curves for arrays of 0.01-μm^2 junc ($E_C \approx 0.84$ K). R_{sq} is the resistance divided by length/width ratio 3.14. Each solid curve corresponds to a ray with a particular normal-state resistance R_n in zero The dashed curve is for array D with $f \approx \frac{1}{2}$. Values of kΩ, E_J/k_B in K, and $x = E_J/E_C$ are, sample A: 36, 0.22, 3. 15.3, 0.51, 1.8; C: 14.1, 0.55, 1.5; D: 9.7, 0.80, 1.0; E: 4.8

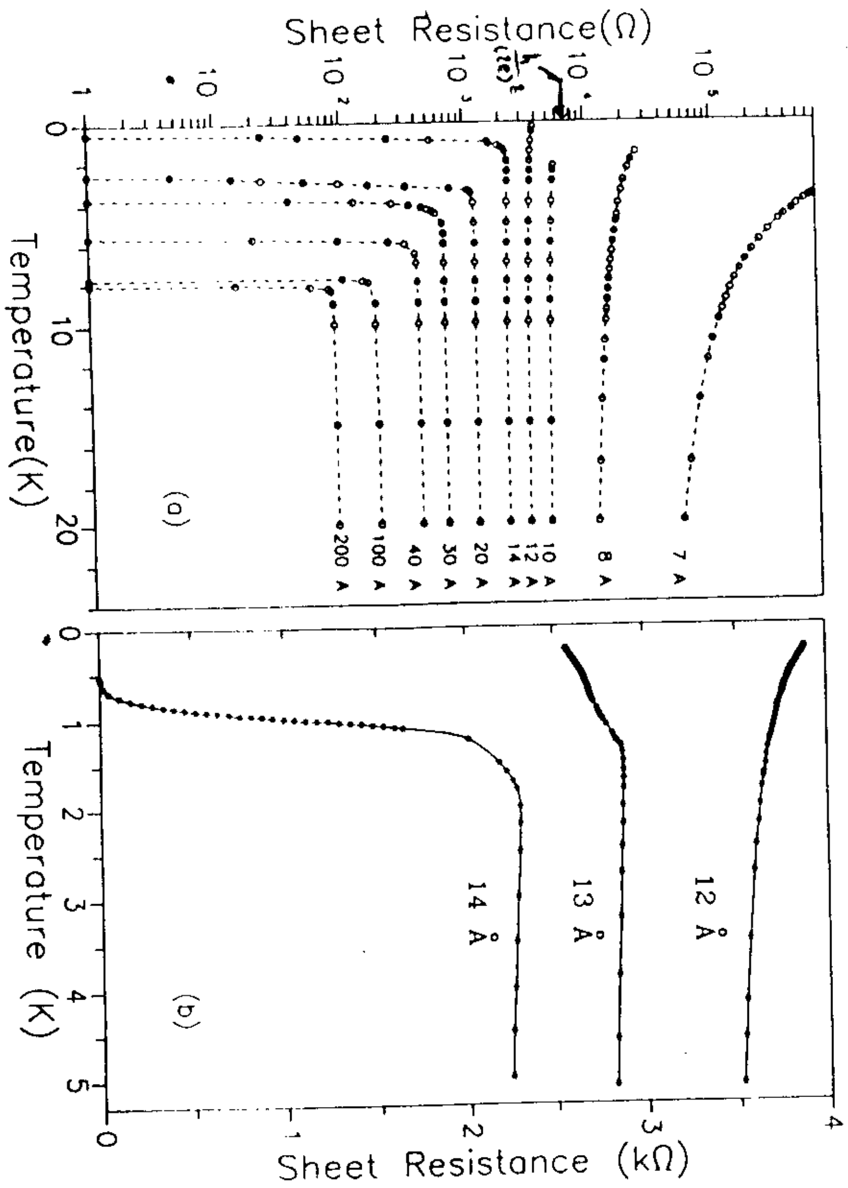


Figure 1

Lee + Ketterson MoC films (not similar R^* to others)

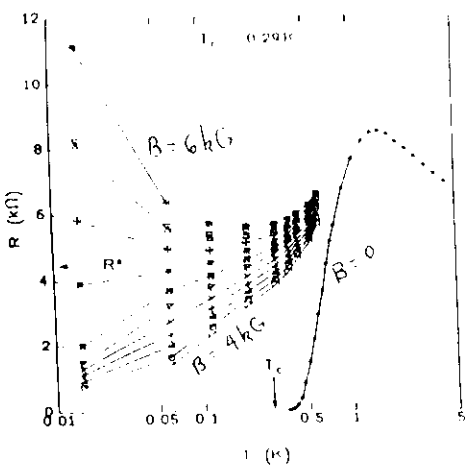


FIG. 1. Logarithmic plots of the resistance transitions in zero field (●) and nonzero field (open symbols) for a film with $T_c = 0.29$ K. The isomagnetic lines range from $B = 4$ kG (○) to $B = 6$ kG (□) in 0.2-kG steps. The horizontal and vertical arrows identify R^* and T_c, respectively.

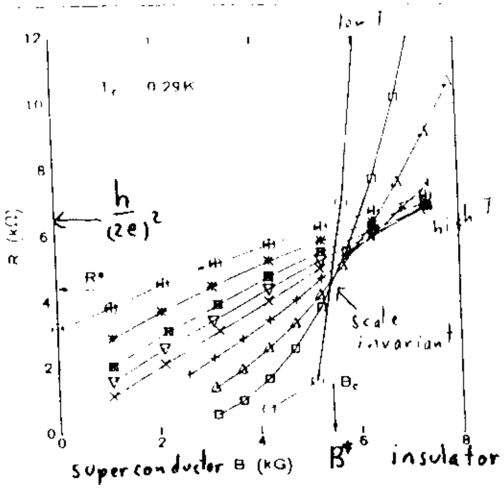


FIG. 2. Magnetic-field dependence of the resistance isotherms for the same film shown in Fig. 1. The temperatures and corresponding symbols are 0.015 K (○), 0.060 K (□), 0.114 K (△), 0.200 K (+), 0.308 K (×), 0.379 K (▽), 0.434 K (□), 0.528 K (•), and 0.599 K (⊕). The horizontal and vertical arrows identify R^* and B_c, respectively.

Five 100-Å-thick α-InO_x films[11] have been used for this study. Shown in Fig. 1 is the temperature dependence of the resistance of a film with $T_c = 0.29$ K. The voltage scales linearly with current for all resistance measurements reported here. The solid circles represent the $B = 0$ transition and the large open symbols represent isomagnetic curves for B ranging from 4 to 6 kG in 0.2-kG steps. We interpret the quasireentrant behavior near T_c as evidence for the partial formation of a superconducting condensate (above T_c and below the local maximum) which at lower temperatures evolves into the vortex phase that ultimately dominates the boson physics of the $T = 0$ superconducting-insulating transition. The transition temperature T_c has been determined from the criterion that $R(T_c) \propto B$ for low field. This criterion was originally justified by concomitant observation of a cubic power-law dependence of voltage on current at the same temperature.[11] Recent arguments based on the Kosterlitz-Thouless renormalization equations[11] and on scaling theory[15] provide additional support for such a procedure. The rapidly changing slopes of the isomagnetic curves at low temperature in Fig. 1 are consistent with the presence of a $T = 0$ field-tuned superconducting-to-insulating phase transition. The critical resistance at this transition, $R^* = 4450$ Ω, is calculated by plotting $(dR/dT)|_B$ vs R at the lowest temperature and interpolating to the resistance (horizontal arrow) where the slope is zero. We expect accuracy to be best at the lowest temperatures where quadratic corrections[6] in T/T_c are minimized.

Identification of B_c is obtained by a similar procedure, this time by plotting $(dR/dT)|_B$ vs B at the lowest temperature and interpolating to the field where the slope is zero. For the film shown in Fig. 1, this zero-slope isomagnetic curve occurring at $B_c = 5460 \pm 20$ G lies between the 5400- and 5600-G curves shown straddling the horizontal arrow at $R = R^*$. The isotherms of the R vs B plots of Fig. 2 reveal more clearly the significance of B_c (vertical arrow). As the temperature is lowered the crossover from low resistance (superconducting) to high resistance (insulating) becomes significantly more pronounced. The crossover sharpens up at $B = B_c$ and as $T \to 0$ is expected to become a sharp transition in which an *infinitesimal change of field* can, in principle, drive the film from the superconducting state, through the critical resistance R^*, to the insulating state.

From the logarithmic plot of Fig. 1, the dependence of B_c on T_c for the five films is seen to be power law with an exponent $2/z = 2.04 \pm 0.09$. This direct and unambiguous measurement of the dynamical exponent at the $B = 0$, $T = 0$ transition, i.e., $z = 0.98 \pm 0.04$, is in excellent agreement with the theoretical prediction of unity. The independent determinations of T_c and B_c to accuracies on the order of a few percent over a range in which T_c varies by more than a factor of 10 and B_c varies by more than a factor of 100 attest to the insensitivity of the

Natural Resistance Scales

1865 J. C. Maxwell

CGS) $\frac{4\pi}{c} \approx 377\ \Omega$ (SI)

Impedance of the vacuum

1915 A. Sommerfeld

$$\alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137.059\ldots} \quad \text{dimensionless}$$

1970-80 Mott, Thouless, Anderson, von Klitzing, Landauer

$$\frac{h}{e^2} \simeq 25{,}812.80\ldots\ \Omega$$

"quantum of resistance"

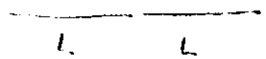
- localization transition
- QHE transition
- superconductor-insulator transition

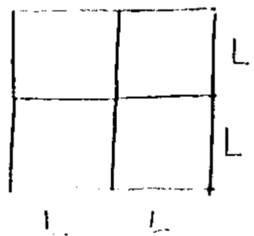
$$R_Q = \frac{h}{(2e)^2} \sim 6{,}453.20\ \Omega$$

Why is d=2 important?

Classical scaling:

$$R = \rho L^{2-d}$$

d = 1  $R(2L) = 2R(L)$

d = 2  $R(2L) = R(L)$ (scale invariant)

d = 3  $R(2L) = \frac{1}{2} R(L)$

$\frac{e^2}{h}\rho = \frac{e^2}{h} R_\square \leftarrow$ "per square" is dimensionless for d = 2

crucial for universality

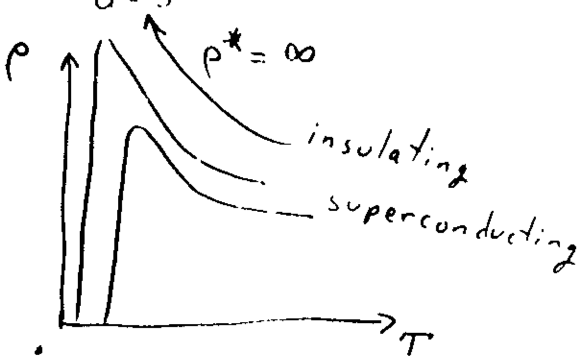
(also in QHE)

Diverging length scale $\xi \sim T^{-1/z} \to \infty$

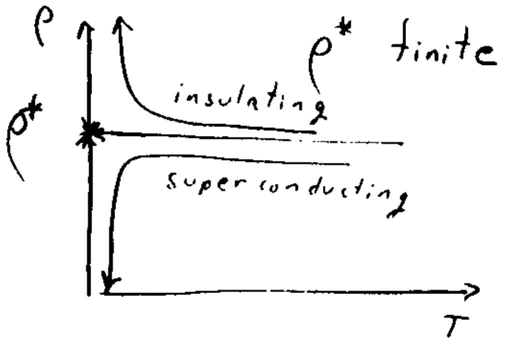
see why later (really l_φ not ξ)

$$\rho \sim \frac{h}{e^2}\, \xi^{d-2}$$

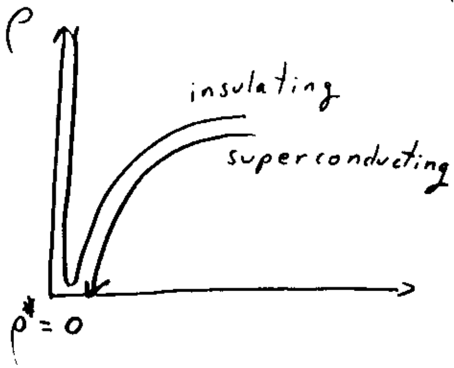
$d = 3$



$d = 2$



$d = 1$



[Possible to define universal resis<u>tance</u>, conduc<u>tance</u>

~ but hard to measure unless ξ is known]

What happens to free 3D (dirty) fermions?

- dirty 3D metal, $T = 0$, resistivity ρ of metal

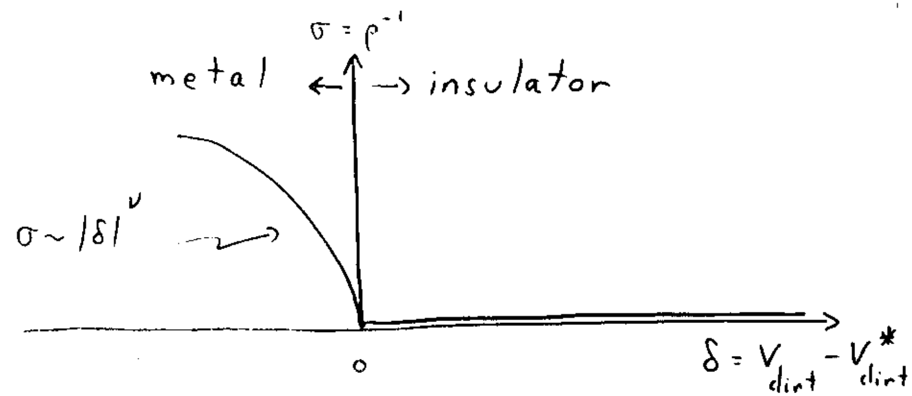
<u>Define</u> correlation length ξ' by

$$\rho = \frac{h}{e^2}\, \xi \qquad\qquad \xi \equiv \frac{e^2}{h}\, \rho$$

Boltzmann / Drude $\quad \rho = \sigma^{-1} = \dfrac{m}{ne^2}\, \tau_{TR}^{-1}$

$\tau_{TR}^{-1} \propto |V_{dirt}|^2$ scattering rate from disorder

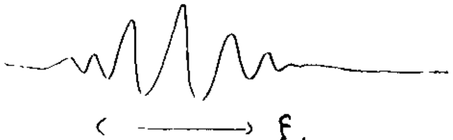
But experimentally $\rho \sim \xi \to \infty$ at <u>finite</u> V^*_{dirt}



Why?

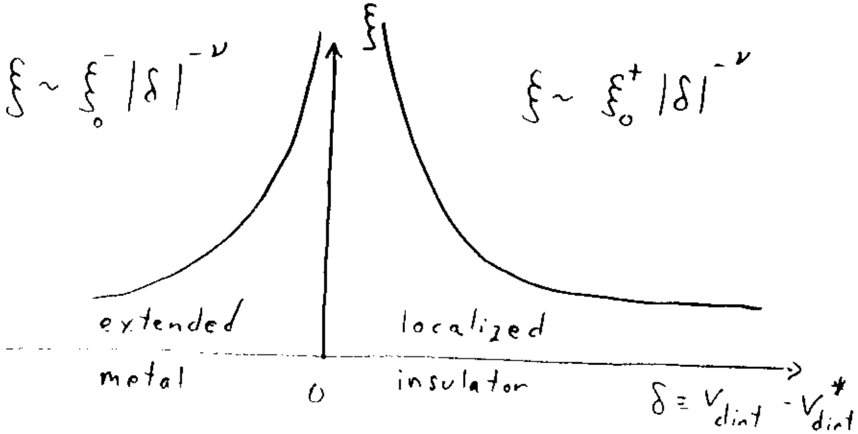
Anderson Localization Phase Transition

extended states



localized states

$\xi \sim \xi_0^- |\delta|^{-\nu}$ $\qquad$ $\xi \sim \xi_0^+ |\delta|^{-\nu}$



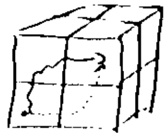
ξ_{metal}: $\quad \rho = \frac{h}{e^2}\xi \qquad \xi \equiv \frac{e^2}{h}\rho$

$\xi_{insulator}$: $\quad \xi \equiv \xi_{loc}$

analogous to correlation length in a magnet

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle - \langle \vec{S}_i \rangle \cdot \langle \vec{S}_j \rangle \sim e^{-|\vec{r}_i - \vec{r}_j|/\xi}$$

Why does Boltzmann transport theory fail?



$R(\lambda L) \neq \lambda^{2-d} R(L)$ $\quad$ classical ohm's law scaling fails

for $L < \xi$

- quantum interference corrections to semiclassical result
- $\Rightarrow \xi \to \infty$ at V_{dirt}^*, localization

3D V_{dirt}^* finite

2D $V_{dirt}^* = 0$! (weak) localization

"All states localized in 2D"

But need: $\left.\begin{array}{l} - T = 0 \\ - L = \infty \end{array}\right\}$ to have true insulator

ξ astronomically large for small V_{dirt}

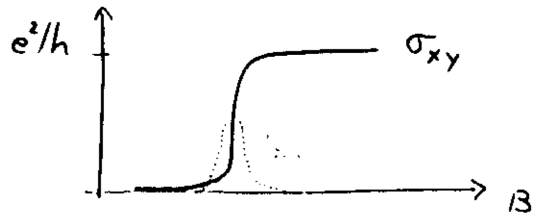
$(\xi > L$ or $\xi > \ell_\varphi \Rightarrow$ metal$)$

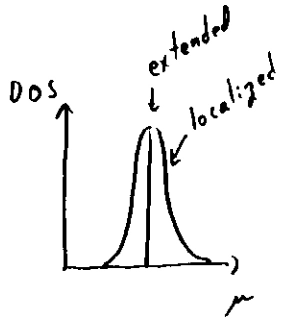
Exceptions to "all states localized" in 2D:

I. weak localization requires T reversal

- B field, IQHE

$$\sigma^*_{xx} = \sigma^*_{xy} = \frac{1}{2}\frac{e^2}{h}$$



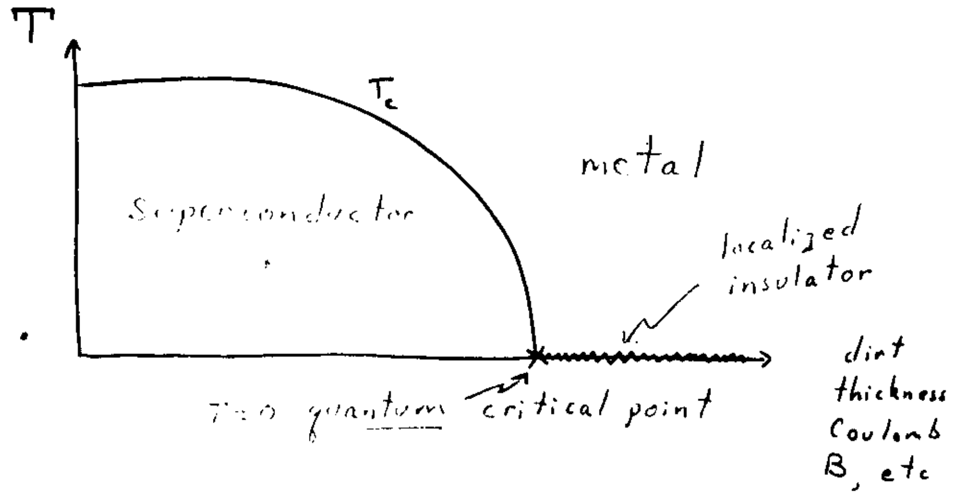


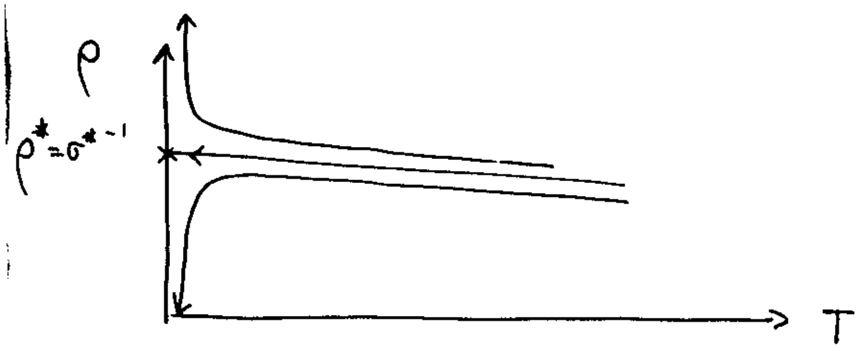
II. attractive pairing interactions produce superconductivity

V^*_{dirt} is finite (to destroy SC)

We assume direct transition at $T=0$ from superconductor to insulator, tuned by V_{dirt}, film thickness, B, etc.

What happens to a "metal" film at $T=0$, poised on the brink of instability between superconductor and insulator?





$$\sigma^* = \frac{(2e)^2}{h}\, g \qquad \text{universal ?}$$

$$g \sim O(1)$$

Questions:

<u>If</u> σ^* is universal,

1. Why?

2. What is universality class of S-I transition?

3. Can we compute σ^*?
   - analytically $1/N$
   - numerically Monte Carlo, $1/N$

4. How do we tell if we are 'close enough' to the critical point $(T=0,\ V_{dirt}=V^*_{dirt})$?

   (i.e. in the 'scaling regime')

   (is the data useable?)

Basic Questions

I. what is a superconductor?

II. What is an insulator?

III. Why $(2e)^2/h$?

Really Basic Question:

I. What is condensed matter physics?

What is condensed matter physics?

A certain point of view:

a) assume all physics of <u>atoms</u>

b) gather up N atoms, $N \to \infty$ thermodynamic limit

c) seek <u>collective</u> effects

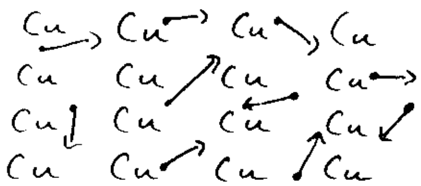
"The whole is greater than the sum of the parts"

"<u>Emergent</u> properties"

Example:

<u>1</u> copper atom does <u>not</u> conduct electricity

but Cu wire <u>does</u>



- Seek to classify <u>phases</u> of matter
- phases have <u>order</u>
    - <u>micro</u>scopic order has <u>macro</u>scopic consequences
- phase transitions
    - solid → liquid: melting
    - insulator → conductor: ('melting of electrons')

• <u>Universality</u>:

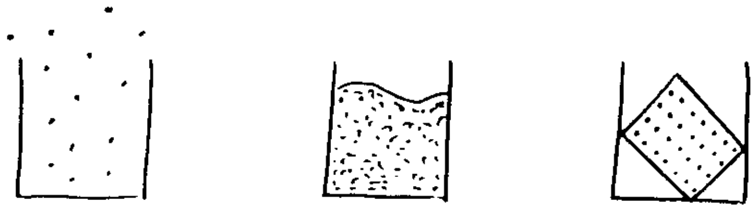
- deep similarities among many (continuous) transitions

type A order → type B order

- $\xi \to \infty$ microscopic details <u>irrelevant</u>
- scale invariance
- certain observables are scale invariant and hence <u>universal</u> (σ^* in 2d)

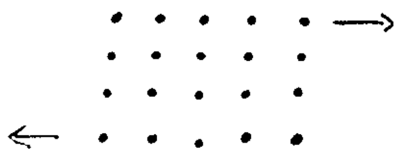
What is the <u>order</u> in a phase?

What are <u>macro</u>scopic consequences?



Gas Liquid Solid

- <u>No</u> essential difference between gas and liquid
- solid is highly <u>ordered</u>



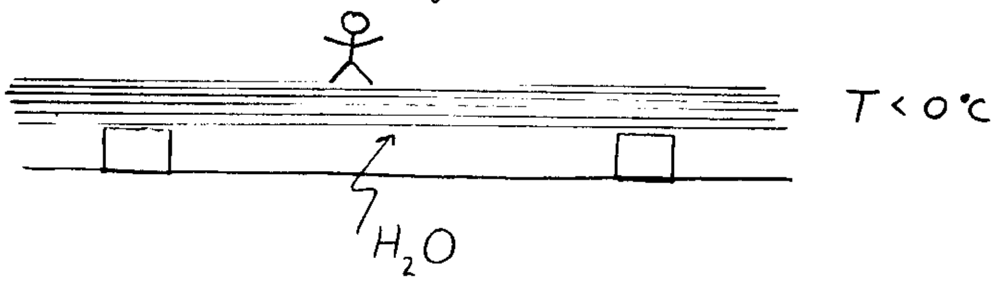
microscopic order: "broken translation symmetry"

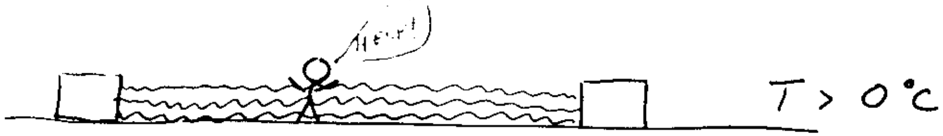
macroscopic consequence: "finite shear modulus"

Children know better:

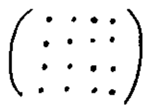
"solids are rigid"

"liquids are squishy"

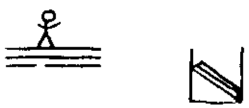




1. What is a superconductor?

2. What is the microscopic <u>order</u>? 

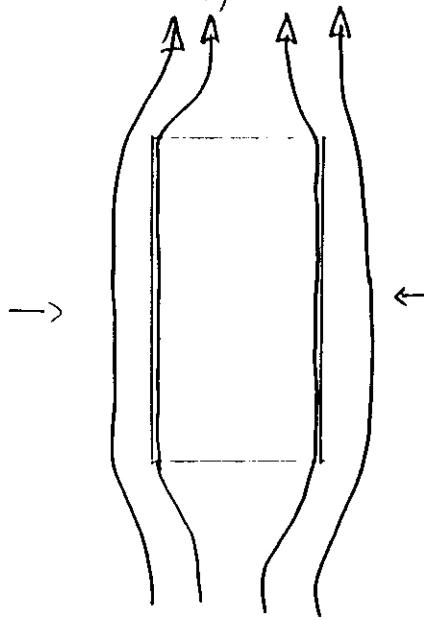
3. What is the macroscopic <u>rigidity</u>?



Rigidity in a superconductor

a) infinite conductivity

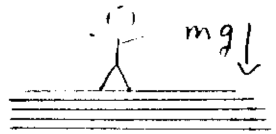
b) Meissner effect
- expulsion of (weak) B fields



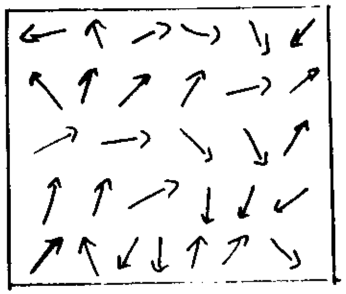
magnetic pressure

$$\frac{B^2}{8\pi}$$

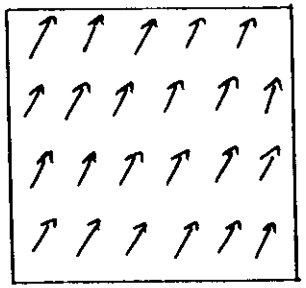
perfectly resisted by superconductor



microscopic <u>order</u> that produces <u>rigidity</u>



$T > T_c$



$T < T_c$

- analogous to XY ferromagnet

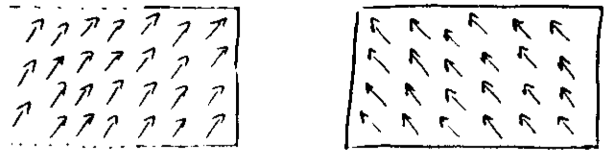
$$\cancel{k}_{ip} \qquad +J|\vec{\nabla} S^{\alpha}|^2 \quad \text{or} \quad -\sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

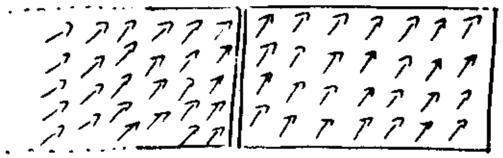
complex "wave function like" order parameter field $\Psi(\vec{r}) = |\Psi|\, e^{i\varphi(\vec{r})}$

$$S = \frac{1}{2}\rho_s |\vec{\nabla}\Psi|^2 + \beta(|\Psi|^2 - 1)^2$$

$\Psi \sim$ center of mass wave function for Cooper pairs

analog of rigidity

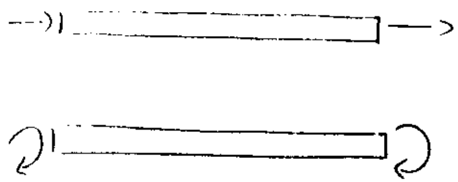




phase arrows rigidly connected

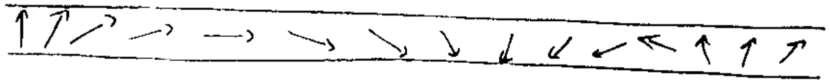
- twist of phase at one end is transmitted to the other

just as rigid rods transmit "push" and "torsion"



twist strain energy represents

energy stored in current flow

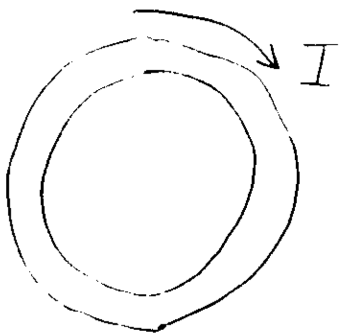


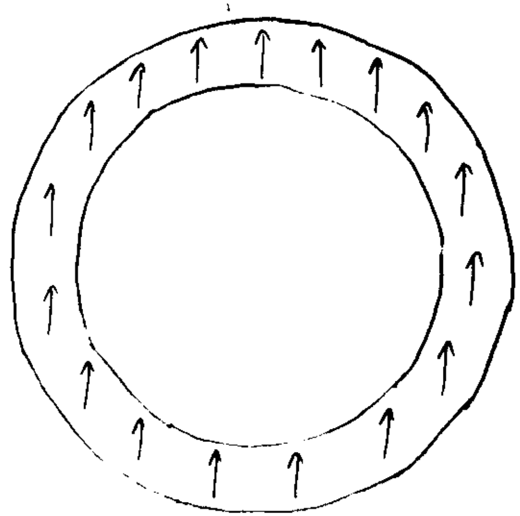
$$\Psi_{(x)} = e^{ikx}$$

$$\vec{J}\,\Psi \sim -i\vec{\nabla}\Psi \sim k\hat{x}\,\Psi$$

$$\mathcal{E} \sim |-i\vec{\nabla}\Psi|^2$$

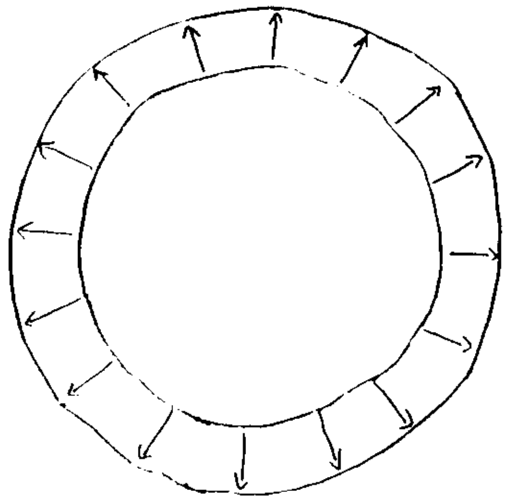
rigidity explains persistence of current in a ring





$I = 0$

large barrier to decay of current



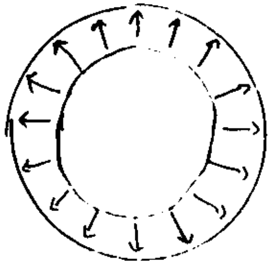
$I =$ "$+1$"

$$I \propto \oint_{loop} d\vec{r}\cdot\vec{\nabla}\varphi = 2\pi\, n_{winding}$$

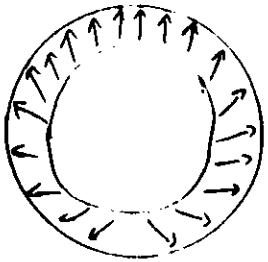
"topological stability"

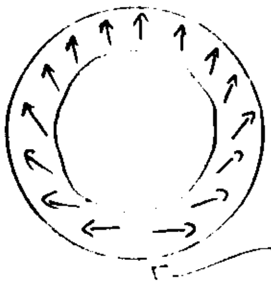
$$\boxed{\sigma \sim \infty}$$

unwinding the phase twist

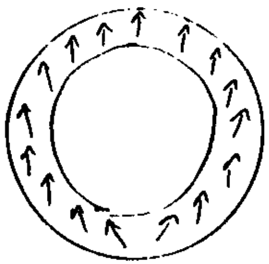


$I = $ "+1"





energetically expensive
$\langle\psi\rangle \to 0$



$I = 0$

Rigidity also explains Meissner effect

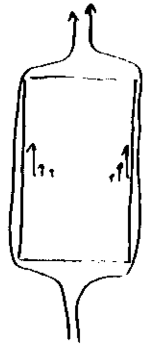
$$S = \frac{1}{2}\rho_s \left|\left(-i\vec\nabla + \frac{2e}{\hbar c}\vec A\right)\psi\right|^2 + \beta\left(|\psi|^2 - 1\right)^2$$

$$\psi \sim e^{i\varphi(\vec x)}$$

If $\vec\nabla \times \vec A \neq 0$ can not adjust $\varphi(\vec x)$ to give zero gradient energy

$\rho_s \neq 0,\ \vec\nabla\times\vec A \neq 0 \implies$ system frustrated

superconductors don't like B fields!



$$\vec\nabla \times \vec B = \frac{4\pi}{c}\vec J$$

surface currents expell $\vec B$

$B(r) \sim e^{-r/\lambda}$ London depth

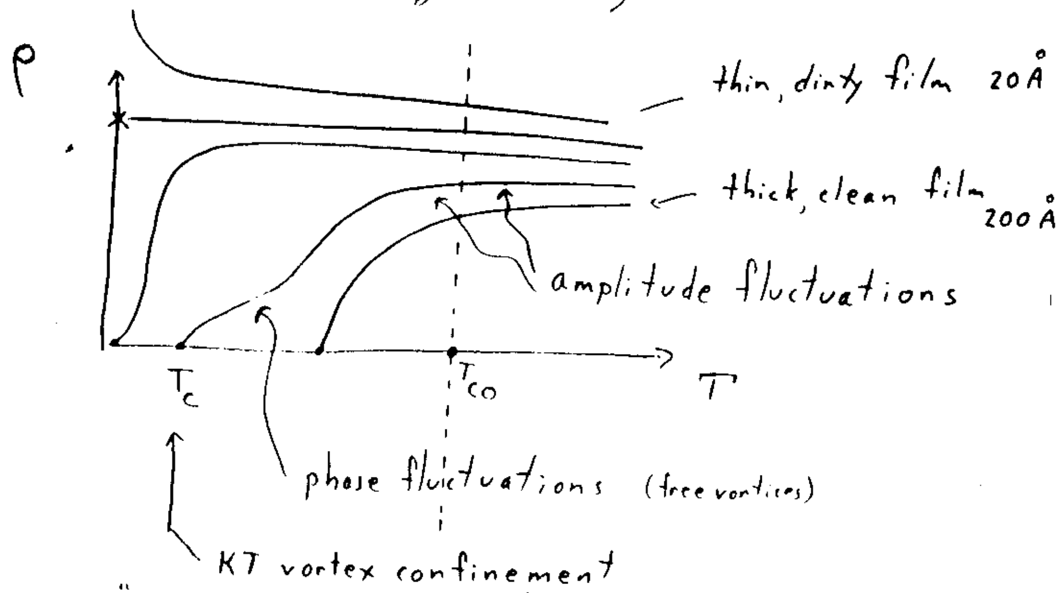
rigidity

Now we know what a superconductor is.

what is an insulator?

- It's a superconductor for vortices instead of charges. (details later)

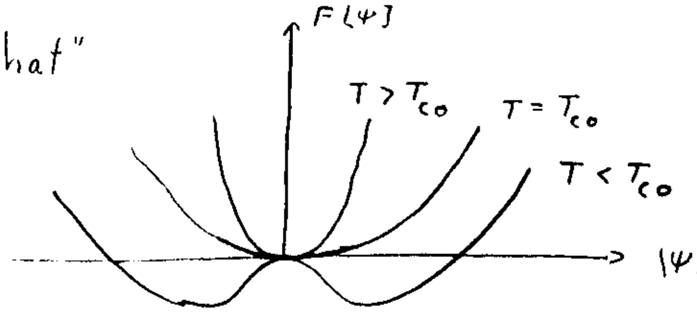
Review of finite temperature 2D normal metal → superconductor transition (Kosterlitz-Thouless)



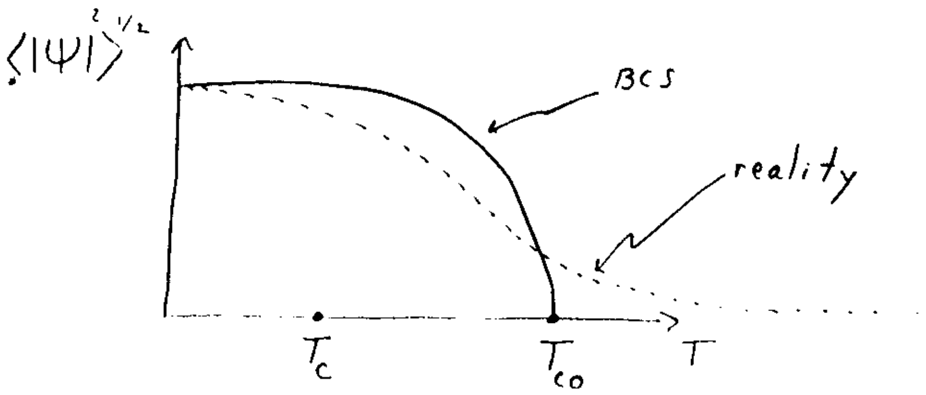
"bose physics" ⟵ ⟶ "fermi physics"

$$F = \frac{1}{2}\rho_s|\vec{\nabla}\psi|^2 - \alpha(T)|\psi|^2 + \frac{\beta}{2}|\psi|^4$$

"mexican hat"



mean field theory neglects fluctuations



nothing happens to $\langle|\psi|\rangle$ at T_c

KT theory: important physics is in soft phase fluctuations $\varphi(\vec{r})$ ("bose physics")

$$\psi(\vec{r}) \sim e^{i\varphi(\vec{r})}|\psi|$$

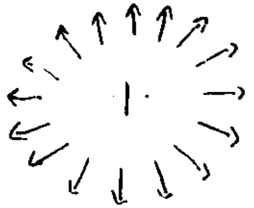
fixed at bottom of "h

Assumptions behind Kosterlitz-Thouless

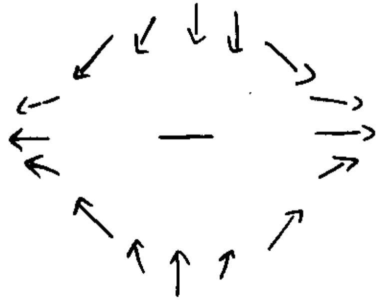
① $d=2$ $\quad (d_\perp \ll \xi)$

② low lying excitations are phase fluctuations

③ vortices are topological defects in phase field



$\oint d\vec{r}\cdot\vec{\nabla}\varphi = +2\pi$



$\oint d\vec{r}\cdot\vec{\nabla}\varphi = -2\pi$

- vortices interact logarithmically like 2D Coulomb charges

— $T < T_c = T_{KT}$ vortices "confined" in neutral pairs

— $T > T_c = T_{KT}$ free vortex flow $\Rightarrow$ dissipation

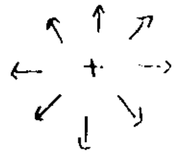
④ ~~no~~ few low-lying fermi excitations $(2\Delta > 0)$

②+③+④ $\Rightarrow$ interacting bose model

IMPORTANT: ξ_{GL} = Cooper pair size (amplitude correlation length) $\lll$ ξ phase coherence length

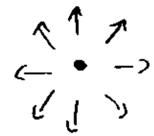
Coulomb gas analogy

$\vec{E} = \vec{\nabla}\varphi \times \hat{z}$ "electric field"



$e^{i\varphi}$

$\vec{\nabla}\varphi \sim$ current



$\vec{E}$

$\oint d\vec{r}\cdot\vec{\nabla}\varphi = 2\pi\, n_{winding} \Longrightarrow$ Poisson Eq'n

$$\vec{\nabla}\cdot\vec{E} = 2\pi \sum_j \delta^2(\vec{r}-\vec{r}_j)\, \underbrace{n_{j\,winding}}_{\pm 1 \text{ Coulomb charge}}$$

$$e^{-\beta F[\varphi]} = e^{-\beta\int d^2r\, \frac{1}{2}\rho_s |\vec{\nabla}\varphi|^2}$$

$$\rightarrow e^{-\frac{1}{T_{CG}}\int d^2r\, \frac{1}{4\pi}|\vec{E}|^2}$$

$$\frac{1}{T_{CG}} \equiv \frac{2\pi\rho_s}{k_B T} \qquad \text{"Coulomb gas temperature"}$$

$$e^{-\beta F[\varphi]} = e^{-\frac{1}{T_{CG}}\sum_{i<j} n_i n_j \left[-\log\left|\vec{r}_i-\vec{r}_j\right|\right]}$$

2D Coulomb potential

Free energy of an isolated vortex

$$\vec{E} = \frac{\hat{r}}{r}$$

$$U = \int_a^L dr\, 2\pi r\, \frac{1}{4\pi} E^2 = \frac{1}{2} \ln\left(\frac{L}{a}\right) \quad \text{energy}$$

system size

UV cutoff

$$S = \ln\left(\frac{L}{a}\right)^2 \quad \text{entropy}$$

$$F = U - T_{CG} S = \left(\frac{1}{2} - 2T_{CG}\right) \ln\left(\frac{L}{a}\right)$$

$T_{CG} < \frac{1}{4}$ $\quad F \to +\infty$ $\quad$ no isolated charges
only neutral pairs (+ −)

$T_{CG} > \frac{1}{4}$ $\quad F \to -\infty$ $\quad$ plasma forms

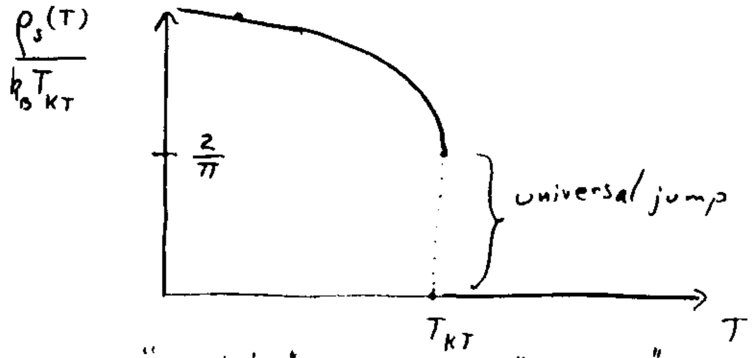
$\boxed{T_{CG} = \frac{1}{4}}$ at $T = T_{KT}^*$ $\quad$ universal dimensionless quantity

$$\boxed{\frac{\rho_s}{k_B T_{KT}} = \frac{2}{\pi}}$$

$$\frac{1}{T_{CG}} \equiv \frac{2\pi \rho_s}{k_B T}$$

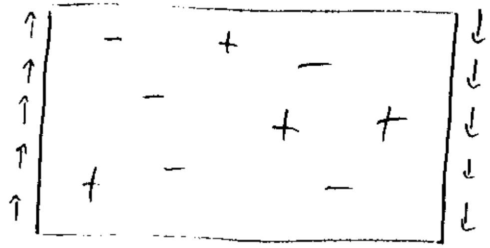
[To be precise: ρ_s is fully renormalized long distance stiffness.]

Measuring the stiffness



"insulator" ⟵ ⟶ "plasma"
(+−) $\quad$ + − + − + − +
(superfluid) $\quad$ (normal)

Twist boundary condition:

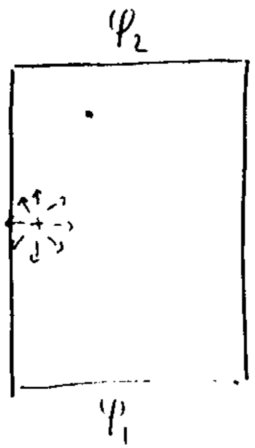


free vortices move to "screen out" b.c. twist

ρ_s renormalizes to zero for $T > T_{KT}$

$$\langle e^{-i\varphi(r)} e^{i\varphi(r')} \rangle \sim e^{-r/\xi}$$ ⟵ free vortex spacing

ξ finite $T > T_{KT}$ $\quad$ insensitive to b.c. twist for $L \gg \xi$



$\oint d\vec{r}\cdot\vec{\nabla}\varphi = \pm 2\pi \implies$ phase slip

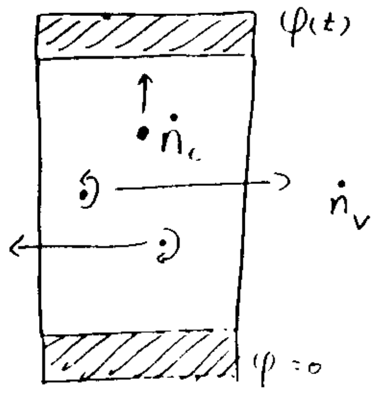
vortex moving across system winds phase by $\pm 2\pi$:

$$\varphi_2 - \varphi_1 \longrightarrow \varphi_2 - \varphi_1 \pm 2\pi$$

conversely twisting b.c. shifts vortices

- relaxes energy
- "screens out" twist

Dissipation by vortex motion



$\dot{n}_v$ vortex flux

$\dot{n}_c$ = cooper pair flu

Josephson: $V = \frac{\hbar}{2e}\dot{\varphi} = \frac{\hbar}{2e} 2\pi \dot{n}_v = \frac{h}{2e}\dot{n}_v$

$I = (2e)\,\dot{n}_c$

Ohm's Law: $R = \frac{V}{I} = \underbrace{\frac{h}{(2e)^2}}_{R_Q} \underbrace{\left(\frac{\dot{n}_v}{\dot{n}_c}\right)}_{\text{dimensionless amplitude}}$

- universal only for $T=0$ quantum critical point not KT.

- Superconductor = insulator for vortices

Insulator = superconductor for vortices

Duality $C \longleftrightarrow V$

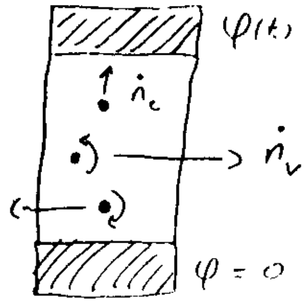
SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

SUPERCONDUCTOR-INSULATOR TRANSITION:

PART II

S-I transition: Part II

T = 0 Quantum Critical Point



$\dot{n}_v$ = vortex flux

$\dot{n}_c$ = Cooper pair flux

$$R = \frac{V}{I} = \frac{h}{(2e)^2}\left(\frac{\dot{n}_v}{\dot{n}_c}\right)$$

All finite temperature continuous transitions (even superconducting) are classical. Soft modes $\omega \to 0$

$k_B T/\hbar\omega \to \infty$

In a classical system, dynamics and statics are independent

$$Z = \frac{1}{h}\int dp \int dx \, e^{-\beta\left[\frac{p^2}{2m} + V(x)\right]}$$

$$= \frac{1}{h}\left\{\int dp \, e^{-\beta\frac{p^2}{2m}}\right\}\left\{\int e^{-\beta V(x)} dx\right\}$$

Classical system in equilibrium:
Probability distribution for $x(t)$ is independent of $p(t)$ and past history.

Not so in a quantum system
$$[\hat{p}, \hat{x}] = -i\hbar$$

- Dynamics, time evolution and equilibrium statics are all tied together.

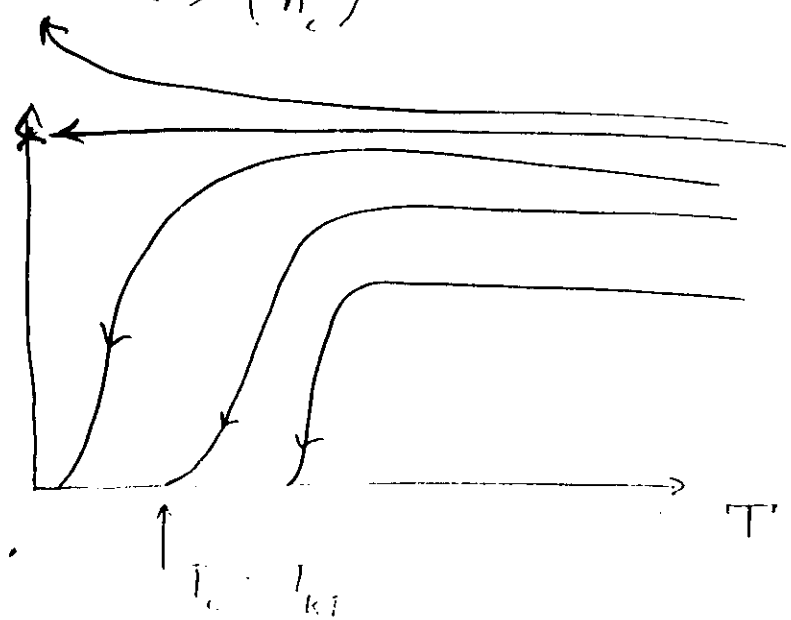
$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}}$$
$$= \sum_x \langle x | e^{-\beta \hat{H}} | x \rangle$$

$$e^{i\hat{H}t/\hbar} \qquad t \to -i\hbar\beta \qquad e^{-\beta \hat{H}}$$

evolution in imaginary time

analytic continuation gives real time dynamics

$$R = \frac{h}{(2e)^2}\left(\frac{\dot{n}_v}{\dot{n}_c}\right)$$



For finite T_c, $\left(\frac{\dot{n}_v}{\dot{n}_c}\right)$ depends on dynamics details unrelated to equilibrium stat. mech.

For $T_c \to 0$, vortices become quantum (not thermal)

$\left(\frac{\dot{n}_v}{\dot{n}_c}\right)$ dynamics part of eq. stat. me

- universal quantum amplitude

Physical picture: same assumptions as KT but int + charging energy drives $T_c \longrightarrow 0$. Bose physics.

where A_1 is a constant and z is a dynamical exponent predicted to have a value of unity. Show (solid line) is the experimental verification of this relationship for five 100 Å thick InO_x films different stages of disorder and for which T_c, B_c and T_{c0} have been independently determine slope $2/z = 2.04 \pm 0.09$ is in excellent agreement with theoretical expectations.

Hebard + Paalanen

T_{c0} by Aslamasov-Larkin

T_{KT} by linear $\rho(B)$, $V \sim I^P$

Dirt ⟶

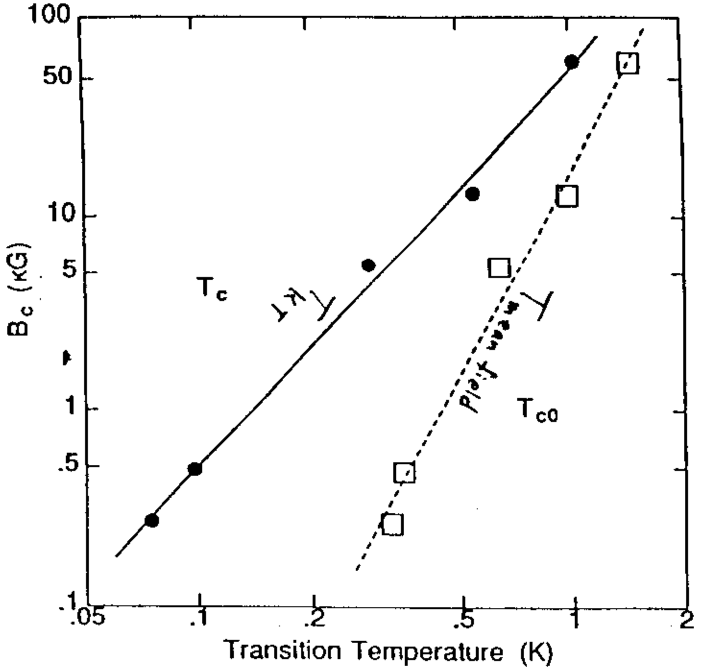


Fig. 2 Logarithmic plot of the critical field B_c versus transition temperature T_c (solid circl (open squares) for five 100-Å thick InO_x films. The regression fit solid line has a slope of 2 dashed line a slope of 3.49.

Questions

1. How does quantum mechanics become important as $T_c \to 0$?

2. How do quantum zero-point fluctuations produce vortices even at $T = 0$?

3. How to compute σ^*?
   - quantum rotor model is correct universality class
   - dual transformation
   - Monte Carlo calculations
   - finite size scaling data analysis

Boson Hubbard Model (ignore fermions)

- assume granular film, short-range repulsion
- only degree of freedom on grain j is n_j, the number of Cooper pairs

$$H = H_0 + H_1$$

charging   chem. potential   random dirt

$$H_0 \equiv \frac{U}{2} \sum_j \hat{n}_j^2 - (\mu + v_j)\hat{n}_j$$

$$H_1 \equiv -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i)$$

Josephson tunneling between grains

boson (Cooper pair) creation operator

Assume $\langle \hat{n}_j \rangle \gg 1$

$$b_j^\dagger |n_j\rangle = \sqrt{n_j + 1}\, |n_j + 1\rangle$$

$$\approx \text{constant}\, |n_{j+1}\rangle$$

Suppose $\langle \hat{n}_j \rangle = n_0 + \text{small}$

large integer

$\hat{n}_j - n_0$ has $\pm$ integer eigen values.

$$(\hat{n}_j - n_0)\,|m\rangle = m\,|m\rangle$$

Quantum rotor representation

$$\langle \theta | m \rangle = e^{im\theta}$$

$$\hat{n}_j - n_0 \rightarrow -i \frac{\partial}{\partial \theta}$$

$$b^\dagger |m\rangle = \text{constant}\, |m+1\rangle$$

$$b^\dagger \rightarrow \sqrt{n_0}\, e^{i\theta}$$

$$e^{i\theta} e^{im\theta} = e^{i(m+1)\theta}$$

discrete units of angular momentum represent discrete charges (Cooper pairs)

$$H_0 = \frac{U}{2}\sum_j \left(n_0 - i\frac{\partial}{\partial\theta_j}\right)^2 - (\mu + v_j)\left(n_0 - i\frac{\partial}{\partial\theta_j}\right)$$

$$H_1 = -t\sum_{\langle ij\rangle} (\sqrt{n_0})^2 \left(e^{i\theta_i} e^{-i\theta_j} + e^{-i\theta_i} e^{+i\theta_j}\right)$$

let $2tn_0 \longrightarrow t$

$$H_1 \to -t\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)$$

let $n_0 U - \mu - v_j \longrightarrow -v_j$ (drop constant)

$$H_0 \to \frac{U}{2}\sum_j \left(-i\frac{\partial}{\partial\theta_j}\right)^2 - v_j\left(-i\frac{\partial}{\partial\theta_j}\right)$$

Quantum rotor model: 'torque' from $\cos(\theta_i - \theta_j)$ transfers quanta of angular momentum (charge) around the lattice.

rotor K.E. = boson P.E.
rotor P.E. = boson K.E. (tunneling)

Drop disorder (for now)

assume $\langle \hat{n}_j \rangle$ = integer n_0 (only case that gives S-I transition)

$$H = \frac{U}{2}\sum_j \left(-i\frac{\partial}{\partial\theta_j}\right)^2 - t\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j)$$

$U = 0$: <u>classical</u> 2D XY model $\Rightarrow$ KT

$U \neq 0$: <u>quantum</u>, $\left[-i\frac{\partial}{\partial\theta}, e^{i\theta}\right] = +1\, e^{i\theta}$

$U = 0$, $T = 0$ superconductor ($T_{KT} > 0$)

$U = \infty$, $T = 0$ Mott Hubbard insulator

$\frac{U}{2}(\hat{n}_j - n_0)^2$ kills charge fluctuations

classical: $P[\theta] = e^{-\beta[-t\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j)]}$

quantum: $P[\theta] = |\Psi[\theta]|^2$

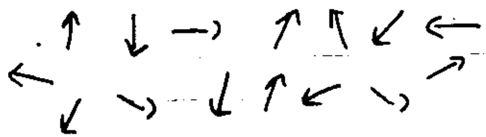
need the ground state wave function

$$H = \frac{U}{2} \sum_j \left(-i \frac{\partial}{\partial \theta_j}\right)^2 - t \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$U \to \infty$

$$\Psi(\theta_1, \theta_2, \ldots, \theta_N) = 1$$

$|\Psi|^2 = 1$ θ's totally disordered



angular momenta

Charges highly ordered

$$\Psi = e^{i \sum_j m_j \theta_j}$$

$m_j = 0$ on every site

Insulator: wild phase fluctuations localize charges

SC: wild charge fluctuations order phase

↑↑↑↑↑↑↑↑↑↑

Seek Variational wave function

Practice: harmonic oscillator

$$H = p^2 + V \qquad V = \tfrac{1}{2} k x^2$$

Trial wave function

$$\Psi_\lambda(x) = e^{-\lambda V} = e^{-\lambda \frac{1}{2} k x^2}$$

Exact!

$$\boxed{\text{Quantum rotor model} \quad \Psi(\theta_1, \ldots, \theta_N) = e^{\lambda \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \quad \text{(unnormalized)}}$$

$$H = \frac{U}{2} \sum_j \left(-i \frac{\partial}{\partial \theta_j}\right)^2 - t \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$t \gg U \qquad \lambda \to \infty$

$t \ll U \qquad \lambda \to 0$ charging causes wild phase fluctuations

$$\lambda_{\text{optimal}} \sim \left(\frac{t}{U}\right)^{1/2}$$

$$\Psi(\theta_1, \cdots, \theta_N) = e^{\lambda \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

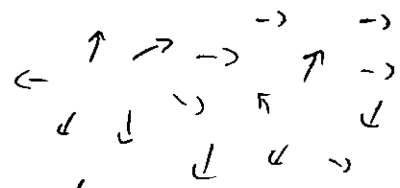
$$\lambda \sim \left(\frac{t}{U}\right)^{1/2}$$

$$P(\theta_1, \ldots, \theta_N) = |\Psi|^2 = e^{2\lambda \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

looks like $e^{-\beta\left[-J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right]}$

2D classical xy model at (fake) temperature $T_{eff}^{-1} \sim 1/2\lambda \sim \left(\frac{U}{t}\right)^{1/2}$

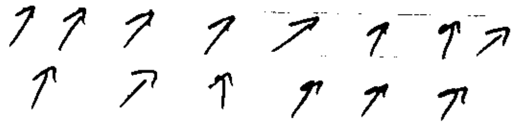
$T_{eff} > T_{KT}$



"typical" configuration

quantum disordered
(zero point motion creates vortices even at $T_{true} = 0$!) insulator

$T_{eff} < T_{KT}$ ordered (superconductor)



charging energy $\frac{U}{2}\left(-i\frac{\partial}{\partial\theta}\right)^2$
wants to make θ uncertain to minimize charge uncertainty (make charge certain)

superconductor $\rightarrow$ quantum disordered (insulator) transition

Ψ is only approximate
[see however Rana + Girvin PRB **48**, 360 (1993)]

quantum universality class is not 2D XY = KT

quantum Path integral formulation will show
$d \rightarrow d+1$: 3D XY classical xy
(dirt / Coulomb / B field will change this)

However Ψ does illustrate quantum disordering phenomenon.
(vortices in ground state)

Path integral formulation of 2D quantum XY (rotor) model

$$H = T + V$$

$$T \equiv -\frac{U}{2}\sum_j \frac{\partial^2}{\partial\theta_j^2} \qquad \text{rotor K.E. = boson P.E.}$$

$$V \equiv -t\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) \qquad \text{rotor P.E. = boson K.E.}$$

$$Z = \mathrm{Tr}\, e^{-\beta(T+V)}$$

$$[T, V] \neq 0$$

$$Z = \mathrm{Tr}\left[e^{-\frac{\beta}{M}(T+V)}\right]^M$$

$$Z = \lim_{M\to\infty}\left\{\mathrm{Tr}\; e^{-\Delta\tau T}\, e^{-\Delta\tau V}\right\}$$

$$\Delta\tau \equiv \beta/M$$

lattice constant in imaginary time direction

Insert complete sets of coherent states

$$|\{\theta(\tau)\}\rangle\langle\{\theta(\tau)\}|$$

$$Z \sim \int \mathcal{D}\theta \prod_{j=0}^{M-1} \langle\{\theta(\tau_{j+1})\}|\, e^{-\Delta\tau T}\, e^{-\Delta\tau V}\, |\{\theta(\tau_j)\}\rangle$$

$$\mathrm{Tr} \Rightarrow \{\theta(\tau_M)\} \equiv \{\theta(\tau_0)\} \qquad \text{p.b.c.}$$

$|\{\theta\}\rangle$ has def. phase so is eigenstate of $V|\theta\rangle$

$$e^{-\Delta\tau V}|\{\theta\}\rangle = e^{K_x \sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)}|\{\theta\}\rangle$$

$$K_x \equiv t\,\Delta\tau$$

Eigenstates of T

$$\langle\theta_j|m_j\rangle = e^{i m_j \theta_j} \qquad \frac{U}{2}\sum_j (m_j)^2$$

insert complete set of $|\{m\}\rangle$

$$\langle\theta|\, e^{-\Delta\tau T}|m\rangle\langle m|\theta\rangle$$

$e^{-\Delta\tau V} \longrightarrow$

$$Z \doteq \int D\theta \sum_{\{m\}} e^{K_x \sum_{(r,r')}\sum_{j=0}^{M-1}\cos[\theta_{r'}(\tau_j)-\theta_r(\tau_j)]}$$

$$e^{-\Delta\tau \frac{U}{2}\sum_r\sum_{j=0}^{M-1}[m_r(\tau_j)]^2}$$

$$e^{i\sum_r\sum_{j=0}^{M-1} m_r(\tau_j)[\theta_r(\tau_j)-\theta_r(\tau_{j+1})]}$$

$$\langle\theta(\tau_{j+1})|\, e^{-\Delta\tau T}\,|m(\tau_j)\rangle\langle m(\tau_j)|\,\theta(\tau_j)\rangle$$

$\frac{\Delta\tau U}{2} \ll 1 \Rightarrow m$ sum slowly convergent

Use Poisson summation formula

$$F(\theta)\equiv\sum_{m=-\infty}^{\infty} e^{-\Delta\tau\frac{U}{2}m^2}e^{im\theta} = \sum_{m=-\infty}^{\infty}\left(\frac{2\pi}{\Delta\tau U}\right)^{1/2} e^{-\frac{1}{2\Delta\tau U}(\theta-2\pi m)^2}$$

periodic sequence of narrow gaussians is Villain approximation to

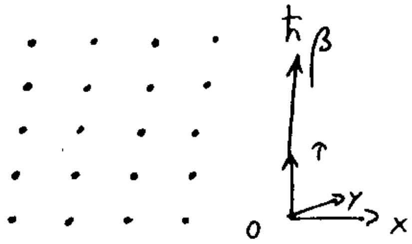
$$F(\theta)\approx e^{K_\tau\cos(\theta)}$$

$K_\tau \equiv \frac{1}{\Delta\tau U}$ $\qquad \theta = \theta_r(\tau_j)-\theta_r(\tau_{j+1})$

Finally arrive at anisotropic (2+1)-D XY Model

note

$$Z=\int D\theta\; e^{\sum_{\ell,\ell'}K_{\ell\ell'}\cos(\theta_{\ell'}-\theta_\ell)}$$



$T=0$, $\hbar\beta\to\infty$

$d\to d+1$

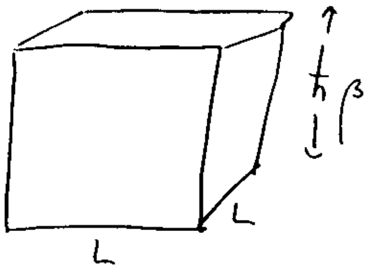
spatial bonds $K_{\ell\ell'} = K_x = t\Delta\tau \ll 1$

temporal bonds $K_{\ell\ell'}=K_\tau = \frac{1}{\Delta\tau U} \gg 1$

$K_x\to 0$, $K_\tau\to\infty$ $\qquad K\equiv(K_xK_\tau)^{1/2} = \frac{t}{U}$

Anisotropy is irrelevant. Universal properties invariant under rescaling of space and time to make $K_{\ell\ell'}=K$ isotropic. $\boxed{\text{3D XY}}$

$$Z \doteq \int \mathcal{D}\theta \; e^{K \sum_{\langle \ell, \ell' \rangle} \cos(\theta_{\ell'} - \theta_{\ell})}$$



important!

- Temperature appears not in Boltzmann factor but as finite size in time direction.

- coupling $\boxed{K \equiv \frac{t}{U}}$ controls quantum fluctuations (recall $\Psi_{variational}$)

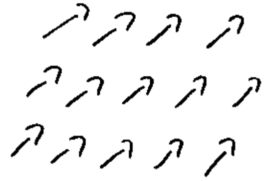
→ [exact relationship to dirt, film thickness, etc. unknown but irrelevant]

- S-I transition occurs at $K = K^*_{3DXY}$
- quantitative value of t/U non-universal (affected by Villain approx. etc)
- universal quantities like σ^* unaffected by approximations (if universality class still correct)

$K \equiv t/U > K^*$ superconducting phase

$\langle e^{i\theta} \rangle \neq 0$ (cf. Mermin-Wagner th'm)

spin wave approximation valid



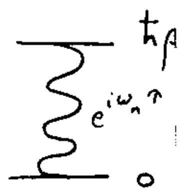
$$Z = \int \mathcal{D}\theta \, e^{K \sum_{\ell, \ell'} \cos(\theta_\ell - \theta_{\ell'})}$$

$$\rightarrow \int \mathcal{D}\theta \; e^{\frac{1}{2}\rho_s \int d^2 r \, (\partial_\mu \theta)^2}$$

$$(\partial_\mu \theta)^2 = |\vec{\nabla}\theta|^2 + (\partial_\tau \theta)^2$$ space + time gradients

$$Z = \int \mathcal{D}\theta \; e^{\frac{1}{2}\rho_s \sum_{k, i\omega_n} (k^2 + \omega_n^2) |\theta(k, i\omega_n)|^2}$$

$$\langle |\theta(k, i\omega_n)|^2 \rangle = \frac{1}{\rho_s} \frac{1}{k^2 + \omega_n^2}$$



$i\omega_n \rightarrow \omega \qquad \frac{1}{k^2 - \omega^2}$

linearly dispersing Goldstone mode

$\omega \sim k$ "Lorentz invariance"

3DXY gives correct space-time correlations

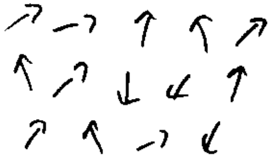
Mott-Hubbard

$K = \frac{t}{U} < K^*$ insulator phase

$$Z = \int \mathcal{D}\theta \; e^{K \sum_{\ell,\ell'} \cos(\theta_\ell - \theta_{\ell'})}$$

3DXY disordered $\Rightarrow$ short range correlations

$$\Big\langle \underbrace{e^{-i\theta(r,\tau)}}_{b} \; \underbrace{e^{+i\theta(r',\tau')}}_{b^\dagger} \Big\rangle \sim e^{-\frac{[|r-r'|^2 + |\tau-\tau'|^2]^{1/2}}{\xi}}$$



$$\langle b_r(\tau) \; b_r^\dagger(0) \rangle \sim e^{-\tau/\xi}$$

$$e^{-\tau \underbrace{(\Delta E)}_{\text{charge excitation gap}}}$$

$$\langle 0| e^{H\tau} b \, e^{-H\tau} \, b^\dagger |0\rangle$$

insert complete set energy eigenfunctions

Review

- 2+1 D classical XY model describes "relativistic" ($\omega \sim k$) boson excitations of 2D superfluid at $T = 0$
- $e^{i\theta_j}$ creates a boson on site j
- coupling K (fake T_{fake}^{-1}) $\sim \frac{t}{U}$ controls quantum fluctuations
  - small U superfluid
  - large U insulator

  medium U: quantum fluctuations drive $T_c \to 0$

Assume: (for now)
- no long range Coulomb
- no disorder
- $\langle \hat{n}_j \rangle$ must be integer for insulator to be possible

excess charges always superfluid (on site repulsion) only

Dual transform

- Problem: How to include disorder in quantum rotor model?

$$T(m) = \frac{U}{2} m^2 + v\, m$$

angular momentum

charging

dirt

- If $v \neq 0$ $\quad T(m) \neq T(-m)$

broken time reversal symmetry for quantum rotor!

$$T(m) = \frac{U}{2}(m + \tilde{v})^2 \quad + \text{constant}$$

like $(\vec{p} + \vec{A})^2$ for particle orbiting a flux tube

Path integral for particle orbiting flux tube Φ

$$e^{\,i \int_0^{\hbar\beta} d\tau\, \dot{\theta} \left(\frac{\Phi}{\Phi_0}\right)}$$

$$e^{\,2\pi \frac{\Phi}{\Phi_0} i\, n_{winding}}$$

- $T(m) = \frac{U}{2}(m + \tilde{v})^2$ leads to complex weights in path integral if $\tilde{v} \neq$ integer. Monte Carlo won't work!

Solution: <u>dual</u> <u>transform</u>

3D XY from phase rep. to charge rep.

3D XY (no dirt)

$$Z = \int \mathcal{D}\theta \; e^{K \sum_{i,\mu} \cos(\Delta_\mu \theta_i)}$$

$$\Delta_\mu \theta_i \equiv \theta_{i+\mu} - \theta_i \quad ; \quad \mu = x, y, \tau$$

"right derivative"

$e^{K\cos\varphi}$ is periodic in φ

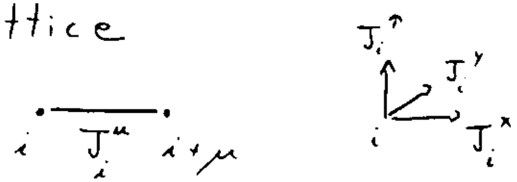
Hence

$$e^{K\cos\varphi} = \sum_{n=-\infty}^{\infty} e^{in\varphi} I_{|n|}(K)$$

$$\sim \sum_{n=-\infty}^{\infty} e^{-\tilde{K} n^2} e^{in\varphi} \quad + \text{irrelevant terms}$$

(Villain again)

Introduce this Fourier rep. on each <u>link</u> of 3D lattice



integer in Fourier series

Fourier rep.

$$e^{K \sum_{i,\mu} \cos(\Delta_\mu \theta_i)} \sim \sum_{\{J_i^\mu\}} e^{-\tilde{K} \sum_{i,\mu} J_i^\mu J_i^\mu} \; e^{i \sum_{i,\mu} J_i^\mu \Delta_\mu \theta_i}$$

"Integrate by parts"

$$\sum_{i,\mu} J_i^\mu \Delta_\mu \theta_i = -\sum_{i,\mu} \left(\Delta_\mu^\dagger J_i^\mu\right) \theta_i$$

$$\Delta_\mu^\dagger J_i^\mu \equiv \sum_\mu \left(J_i^\mu - J_{i-\mu}^\mu\right) \quad \left(\text{"} \vec{\nabla} \cdot \vec{J} \text{"}\right)$$

"left derivative"

$$Z = \int \mathcal{D}\theta \sum_{\{\vec{J}\}} e^{-\tilde{K} \sum_{i,\mu} (J_i^\mu)^2} \; e^{-i \sum_{i,\mu} (\Delta_\mu^\dagger J_i^\mu)\, \theta_i}$$

$\int \mathcal{D}\theta$ can now be done explicitly.

$$Z = \sum_{\{\vec{J}\}}{}' \; e^{-\tilde{K} \sum_i (\vec{J}_i)^2}$$

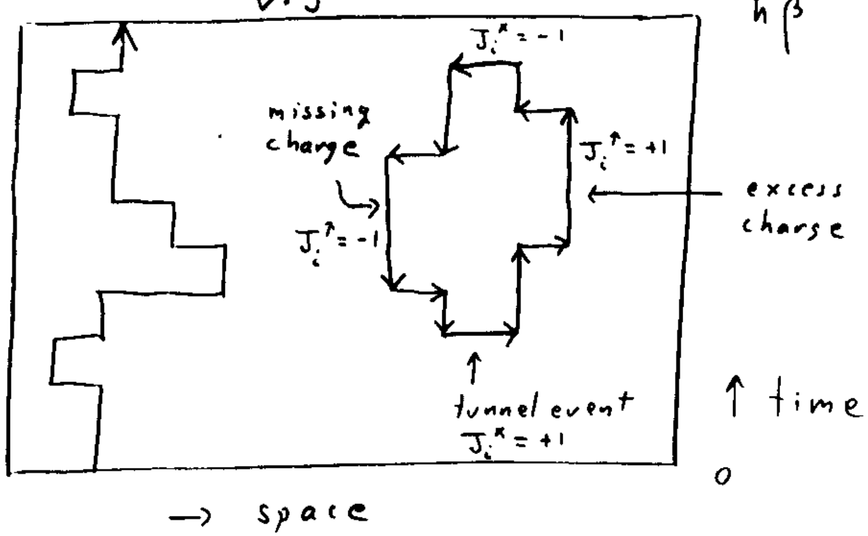
constraint $\vec{\nabla} \cdot \vec{J} = 0$

Boson world lines

$$Z = \sum_{\{J\}}{}' \; e^{-\tilde{K} \sum_{i,\mu} (J_i^{\mu})^2}$$

"relativistic"
integer valued "three current" J_i^{μ} on links

$$\vec{\nabla} \cdot \vec{J} = 0$$



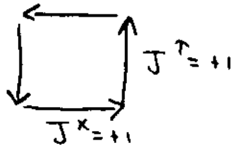
$$\partial_\mu J^\mu = 0: \quad \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

↑ J_i^τ = charge density

⟶ J_i^x, J_i^y burst of current when Cooper pair tunnels between grains

Interpretation of action cost

$$Z = \sum_{\{J\}}{}' \; e^{-\tilde{K} \sum_{i,\mu} (J_i^{\mu})^2}$$



$e^{-\tilde{K}(J_i^x)^2}$ probability (amplitude) of tunnel event

t small $\Rightarrow$ $\tilde{K}$ large

$e^{-\tilde{K}(J_i^\tau)^2}$ charging cost of excess particle

U large $\Rightarrow$ $\tilde{K}$ large

$\tilde{K} \sim \dfrac{U}{t}$ (exact relation unimportant) i.e. non-universal

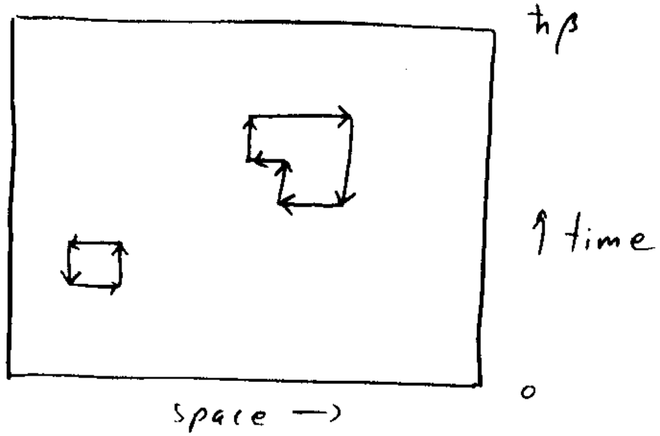
quantum fluctuation parameter
"string tension" — longer string ⇒ bigger cost

S-I Phase Transition

$$\tilde{K} \sim \frac{U}{t}$$

$$Z = \sum_{\{J\}}{}' e^{-\tilde{K} \sum_{i,\mu} (J_i^\mu)^2}$$

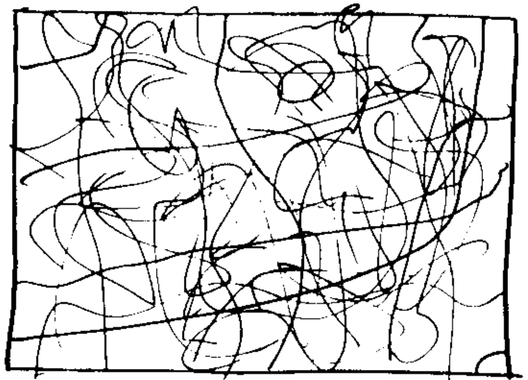
Insulator $\tilde{K} > \tilde{K}^*$



charge fluctuations expensive

Superfluid $\tilde{K} < \tilde{K}^*$

Feynman ring exchanges:



"entropy" of "loop soup" beats out energy cost

"string tension" renormalizes to zero

Introducing disorder (particle-hole symmetry breaking) now easy

J_i^τ = charge on site i

$$Z = \sum_{\{J\}}{}' e^{-\tilde{K} \sum_{i,\mu} (J_i^\mu)^2} e^{-\sum_i (\mu + v_i) J_i^\tau}$$

(no complex weights)

$$-\frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu} = \frac{1}{\beta}\left\langle \sum_i J_i^\tau \right\rangle = \langle N \rangle$$

↑ summed on all time slices

↑ number of time slices

Vector potential

$$V = -t \sum_{i,\mu} \cos(\Delta^\mu \theta_i - A_i^\mu) \qquad \text{phase rep.}$$

$$\longrightarrow e^{i \sum_{j,\mu} J_j^\mu A_j^\mu} \qquad \text{dual rep.}$$

gives correct Aharanov-Bohm phase factors for particles circling B field flux

(can't do finite B in this rep.)

$$Z = \sum_{\{J\}}{}' \; e^{-\tilde{K}\sum_{i,\mu}(J_i^\mu)^2} \; e^{-\sum_i (\mu+\nu_i) J_i^\tau} \; e^{i\sum_{j,\mu} J_j^\mu A_j^\mu}$$

$$-i\,\frac{\partial \ln Z}{\partial A_j^\mu} = \left\langle J_j^\mu \right\rangle$$ current on site j at time τ_j

$\Rightarrow$ J_j^μ is full, physical, gauge-invariant current; (not just paramagnetic piece)

- Kubo formula

$$\sigma(i\omega_n) = (2e)^2\,\frac{\rho_s(i\omega_n)}{\hbar\,\omega_n} = \frac{2\pi}{R_Q}\,\frac{\rho_s(i\omega_n)}{\omega_n}$$

$\rho_s(i\omega_n)$ = Fourier transform of current-current correlation

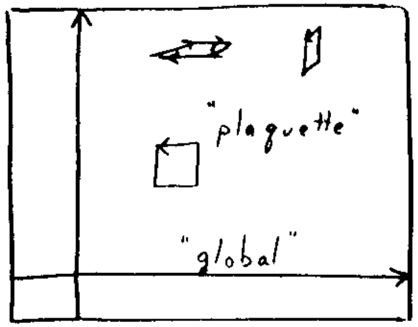
$$\rho_s(i\omega_n) = \left\langle \left| J^x(q=0, i\omega_n)\right|^2 \right\rangle$$

$$= \frac{1}{\beta}\int_0^\beta d\tau \left\langle J^x_{q=0}(\tau)\, J^x_{q=0}(0)\right\rangle e^{i\omega_n \tau}$$

J_j^μ are variables directly simulated by M.C. easy to measure!

Boson quantum path integral $\longrightarrow$ Monte Carlo for classical "loop soup"

(y direction not shown)

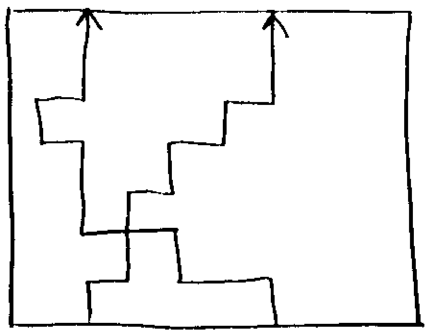


3 classes of MC "moves"

preserve $\vec{\nabla}\cdot\vec{J} = 0$ automatically

- $\{J\} \to \{J'\}$ accept/reject Metropolis

"Boltzmann factor" $e^{-\tilde{K}\sum_{i,\mu}(J_i^\mu)^2}\, e^{-\sum_i (\mu+\nu_i) J_i^\tau}$



exchange of identical bosons occurs naturally and automatically (no labels on particles!)

(no minus signs $\Rightarrow$ bosons)

no explicit permutations required (easy!)

Procedure

① bring "loop soup" to equilibrium

② adjust $\tilde{K}$ to reach critical point

③ measure $\langle J^x J^x \rangle$ correlations to get $\sigma(i\omega_n)$

④ analytically continue

$$\sigma(i\omega_n) \to \sigma_R(\omega+i\delta) + i\,\sigma_I(\omega+i\delta)$$

⑤ $\lim_{\omega \to 0} \sigma_R(\omega+i\delta) = \sigma^*$

How do we locate $\tilde{K}^*$ with great precision?

Scaling analysis shows how <u>and</u> shows σ^* universal.

SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

QUANTUM HALL EFFECT IN DOUBLE LAYERS

Quantum Hall Effect in Double Layers
Ideal Quantum Ferromagnet

- K. Yang
- K. Moon
- L. Zheng
- A. H. MacDonald
- SMG

} IU — PRL January 1994

- D. Yoshioka — Tokyo
- S. C. Zhang — Stanford

Experiments: (AT&T) PRL January 1994
S. Q. Murphy, J. P. Eisenstein, G. Boebinger

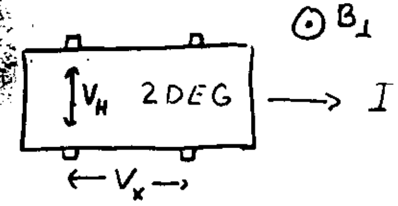
Samples: L. Pfeiffer   K. W. West

Buzz words
- skyrmions
- merons
- Pontryagin density
- spontaneous symmetry breaking

Related Theory:
- Ezawa + Iwazaki PRL, PRB 11/93
- Wen + Zee PRL, PRB $\underline{47}$, 2265 ('93)
- Sondhi et al. PRB $\underline{47}$, 16419 (1993).

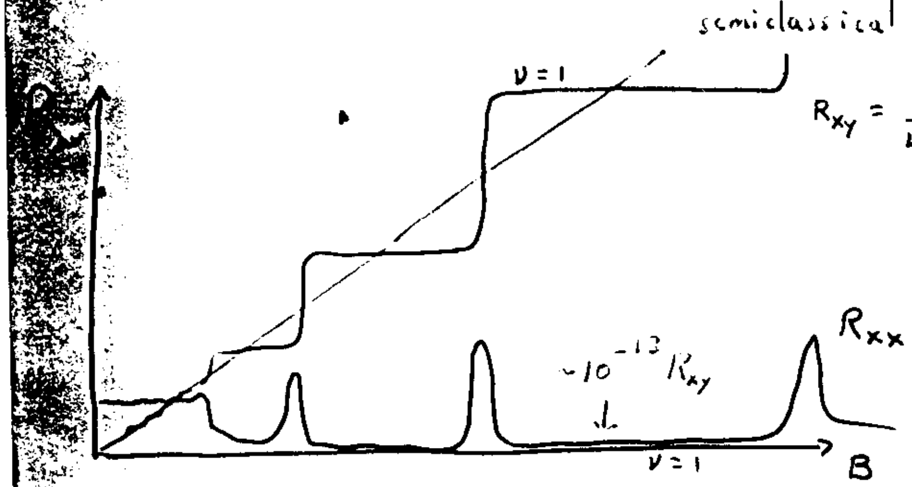
…st:

…gle Layer QHE



semi-classical
$$R_{xy} = \frac{B}{nec}$$

$$V_H = R_{xy}\, I \qquad V_x = R_{xx}\, I$$



$$R_{xy} = \frac{h}{\nu e^2}$$

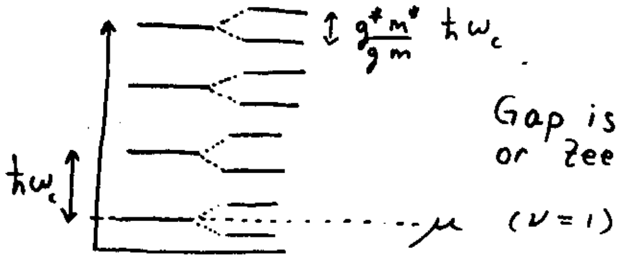
$$\frac{h}{e^2} \cong 25{,}812.80\ \Omega$$

$$\nu = 1, 2, 3, \ldots \qquad 1/3,\ 2/3,\ 3/5,\ 2/7,\ \ldots$$

focus — simplest case

Single layer in high B field (with spin)

$$H = \sum_{j=1}^{N} \frac{1}{2m^*}\left(\vec{p}_j + \frac{e}{c}\vec{A}_j\right)^2 + g^*\mu_B B \sum_{j=1}^{N} s_j^z \quad (+\,V)$$



$\hbar\omega_c \gg k_B T$

Gap is kinetic energy or Zeeman energy

Landau levels

1. ν = integer # filled levels

$\Rightarrow R_{xy} = \dfrac{h}{e^2\nu}$ plateau

$\Rightarrow$ excitation gap Δ, $R_{xx} \sim e^{-\beta\Delta} \to 0$

Focus on $\nu = 1$ $\quad \Delta \propto g^*$

Can QHE (Δ) survive $g^* \to 0$?
Yes: Coulomb interactions
$\Rightarrow$ spontaneous magnetic order
$\Rightarrow$ charge excitation gap

(pressure tune $g^* \to 0$)

Sondhi et al. PRB $\underline{47}$, 16419 (1993).

Exchange Ferromagnetism $\quad \underline{\nu = 1}$

$g^* = 0$. SU(2) symmetry

- Kinetic energy quenched in Landau level
- Hund's rule fully polarizes ground state

$z \equiv (x + iy)/\ell$

$$\Psi_\uparrow \equiv \underbrace{\prod_{i<j}^{N} (z_i - z_j)}_{\text{antisymmetric}} e^{-\frac{1}{4}\sum_k |z_k|^2} \quad \underbrace{|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle}_{\text{symmetric}}$$

single slater determinant (van der monde)

$E_X \sim 60\,\mathrm{K}\sqrt{\frac{B}{\mathrm{Tesla}}}$

- $\underline{\underline{\text{exact}}}$ ground state for $\underline{\underline{\text{any}}}$ repulsive V

$S = N/2$, $\quad S^z = \frac{N}{2}, \frac{N}{2} - 1, \ldots, -\frac{N}{2}$

$|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$, $\quad |\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rangle$, $\quad |\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$

What are excited states?

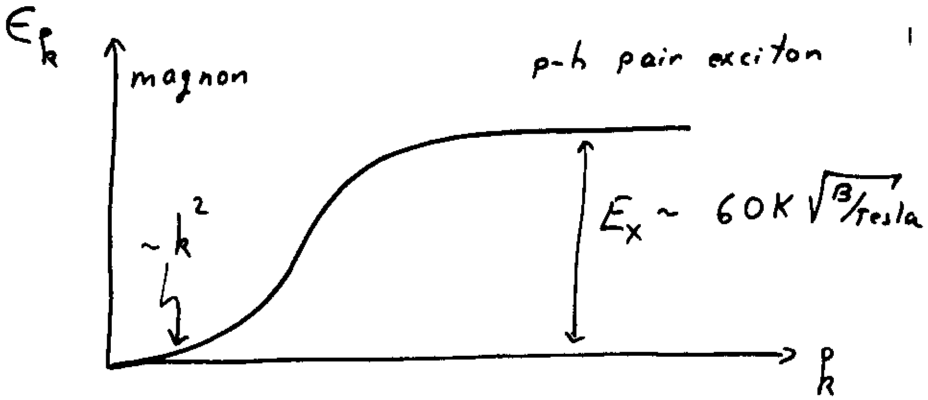
Neutral excitations: gapless spinwaves

Exact 1-magnon excited states

$$|\vec{k}\rangle \equiv \overline{S^-_{\vec{k}}}\ |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$$

$$S^-_{\vec{k}} \equiv \sum_{j=1}^{N} s_j^- \, e^{i\vec{k}\cdot\vec{r}_j}$$ $\bar{S} \equiv$ LLL projection

- Heisenberg Ferromagnet Spin Waves

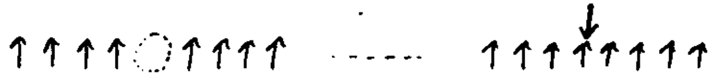


$$\epsilon_k \sim \tfrac{1}{2}\rho_s k^2 \qquad H_{eff} \sim \int d^2r\ \tfrac{1}{2}\rho_s\left(\vec{\nabla} m^\mu \cdot \vec{\nabla} m^\mu\right) + \cdots$$

$\vec{m}$ = unit vector order parameter,

Charge excitations

$\nu = 1 \Rightarrow$ no charges without spin flips

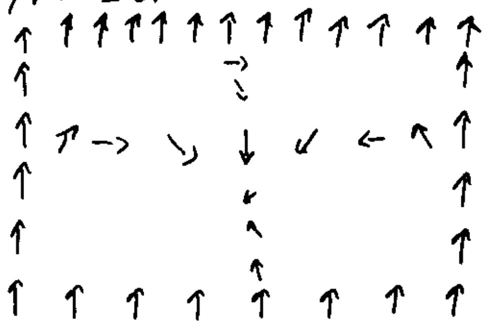


$$2\Delta = E_X = 60K\sqrt{\frac{B}{\text{Tesla}}} \sim 100K$$

Is there a cheaper charged excitation?

Spin texture (not magnon)

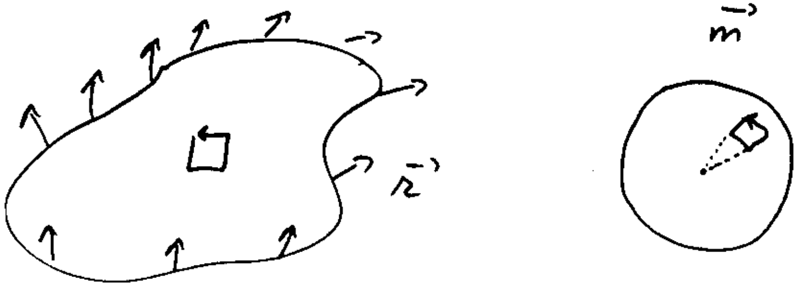
"SKYRMION"



Energy only **half** as big as spin flip

Spin textures carry charge

- adiabatic Berry's phase:



- Berry's phase looks like increased magnetic flux density δB

$\Rightarrow$ increased charge density

$$\delta\rho = \sigma^{xy} \frac{\delta B}{\Phi_0} \qquad \text{(Chern-Simons)}$$

curl of Berry's connection

$$\boxed{\delta\rho = \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{m} \cdot \partial_\mu \vec{m} \times \partial_\nu \vec{m}}$$

Pontryagin topological density

Skyrmion $Q = \pm 1$

(Static) Effective Action

Fock, Zeeman

$$H_{eff} = \int d^2r \left\{ \frac{1}{2} \rho_s |\vec{\nabla} m_\mu|^2 + g^* h_z m_z \right\}$$

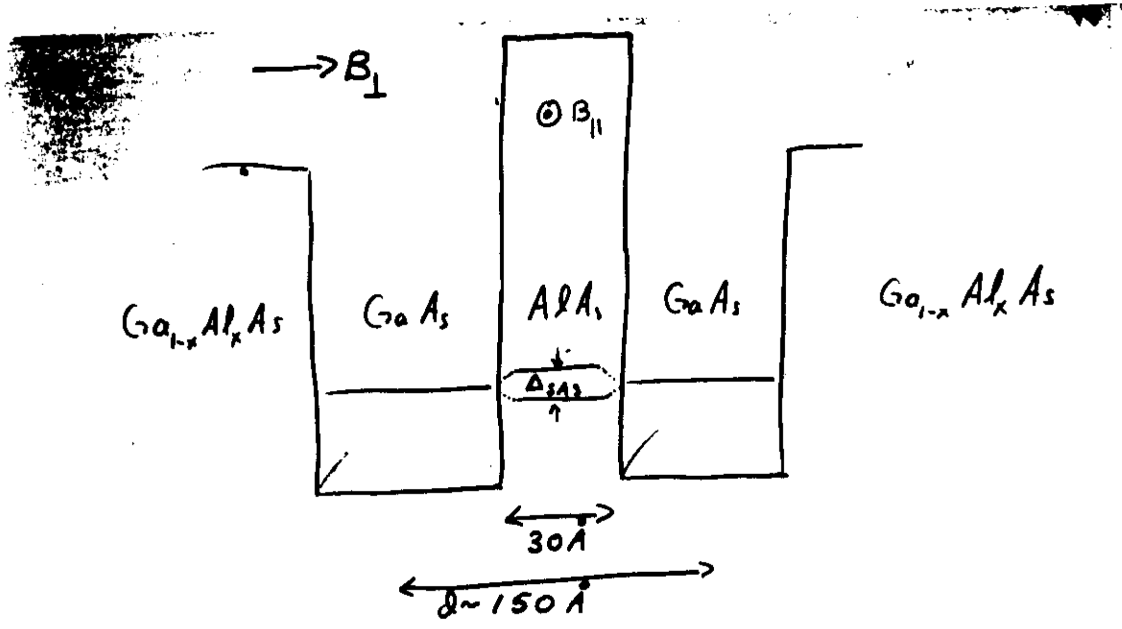
Hartree $\rightarrow$ $$+ \frac{1}{2} \int d^2r \, d^2r' \frac{e^2}{|r - r'|} \delta\rho(r) \, \delta\rho(r')$$

- $$\delta\rho(\vec{r}) = \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{m} \cdot \partial_\mu \vec{m} \times \partial_\nu \vec{m}$$

Finite spin stiffness ρ_s due to exchange

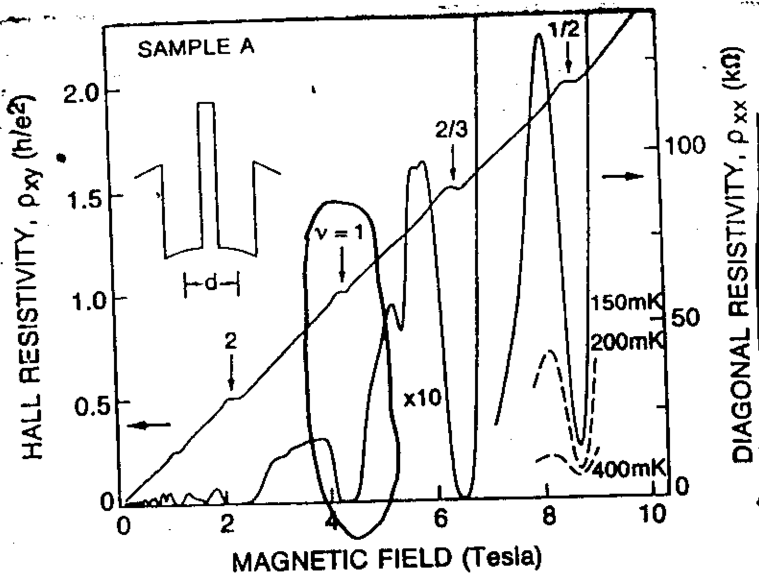
$\Rightarrow$ spontaneous magnetization

$\Rightarrow$ charge gap finite even if $g^* \rightarrow 0$



- $\mu \sim 3\times10^6$ cm²/V·s in each layer
- separate electrical contacts (for $d \gtrsim 100$ Å)
- $\Delta_{SAS} \sim$ 0-5 K (analogous to Zeeman)
- d is comparable to spacing between electrons ⇒ interlayer Coulomb correlations

In some cases: $\dfrac{e^2}{\epsilon d} > \Delta_{SAS}$



$\nu = 1$ means $\nu = 1/2$ in each lay

ignore real spin

Fig. 1: Diagonal resistivity at T=150mK and Hall resistivity at T=430mK. Note the $\nu=1/2$ fractiona quantum Hall state. Temperature dependence of ρ_{xx} near $\nu=1/2$ is also shown. The ρ_{xx} trace fo B<7T has been amplified ten-fold. Inset: Schematic conduction band diagram of the doubl quantum well.

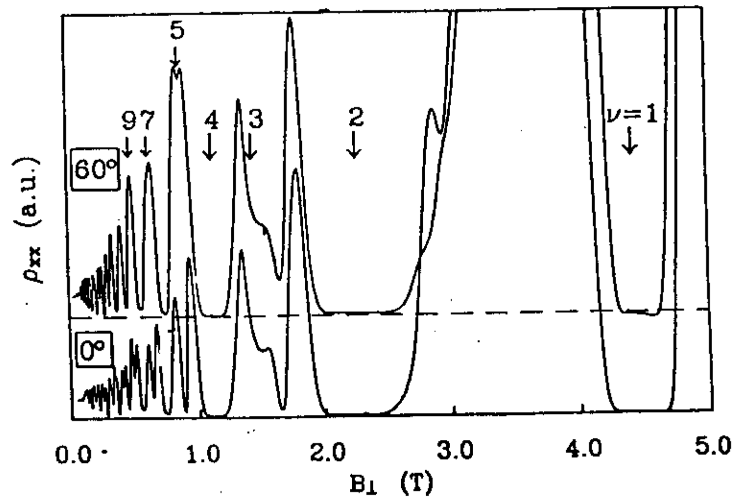
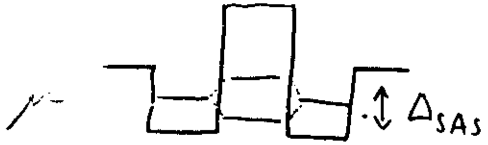


Fig. 2: Low field ρ_{xx} at T=70mK versus the perpendicular component of the magnetic field f magnetic field tilted 0° and 60° from normal to the 2DES. Note the $\nu=9,7,5$ states are destroyed b in-plane magnetic field, while the $\nu=1$ is not. This is evidence that the $\nu=1$ state arises fro inter-layer Coulomb interactions. The missing $\nu=3$ state is discussed in the text.

Boring way to get $\nu = 1$ in double layer



Δ_{SAS} large $\Rightarrow$ rapid tunneling

like $\nu = 1$ integer QHE in a <u>single</u> well

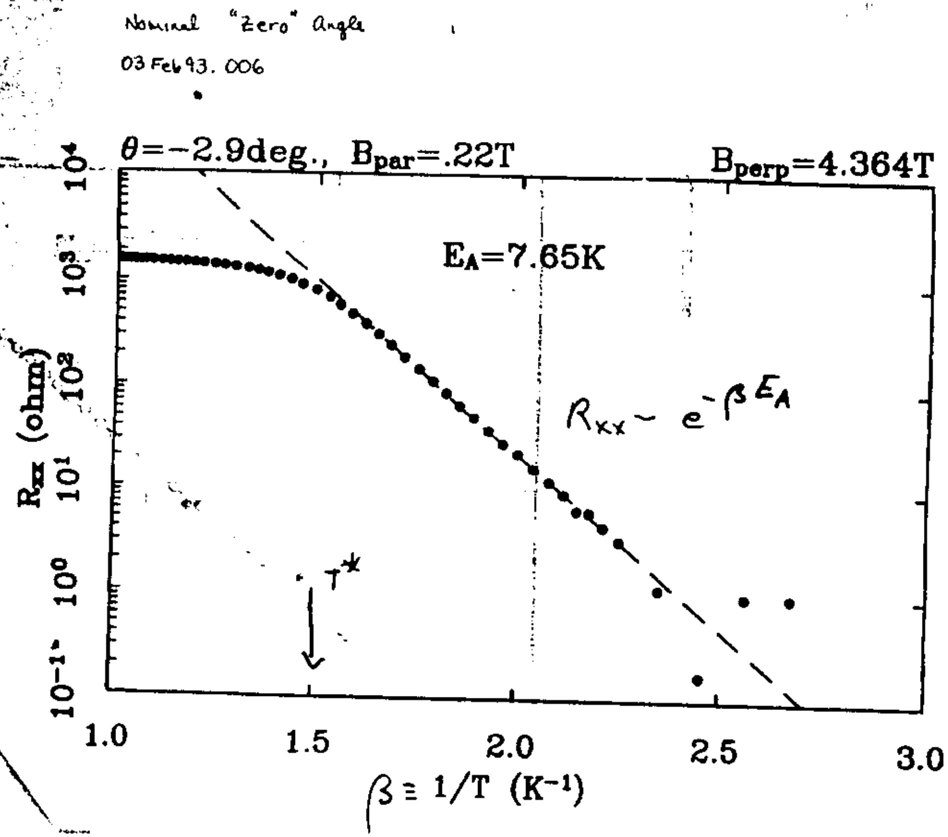
Δ_{SAS} analogous to g^* Zeeman splitting for spin

$\nu = 1$ QHE survives $\Delta_{SAS} \to 0$ due to broken symmetry

<u>Collective</u> effect due to <u>interactions</u>.

spin version:

"$\nu = 1$ is a fraction too" Sondhi, Kivelson, Karlhede, Rezayi PRB <u>47</u>, 16419 (1993)



First indication of collective nature of gap

$E_A = 7.65\,K \gg \Delta_{SAS} \sim 1\,K$

Second:

Arrhenius plot fails for $T > T^* \sim \frac{2}{3}\,K \ll \frac{E_A}{k_B} = 7.6\,K$

$B_\perp = 4.5\ T$

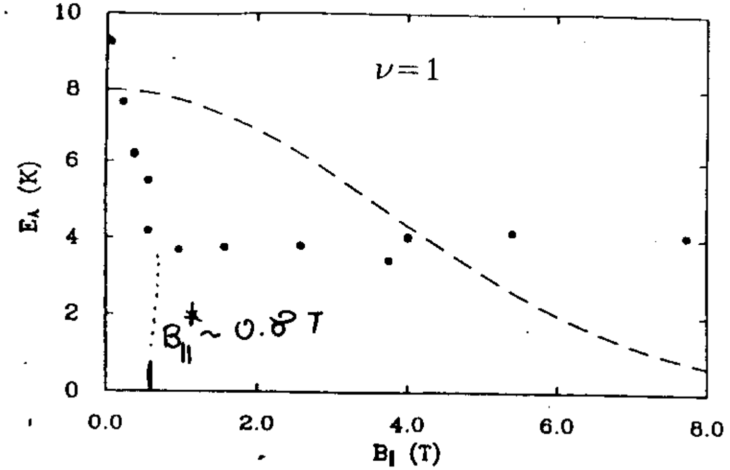
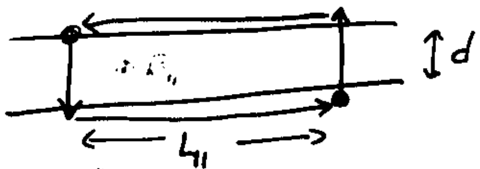


Fig. 3: Activation energy of the quantized state at $\nu = 1$ versus in-plane magnetic field, $B_\parallel$. The dashed line (normalized to the data at $B_\parallel = 0$) is the calculated dependence of a single-particle tunneling gap at $\nu = 1$. The relative independence of the activation energy over the range $1 < B_\parallel < 8\,T$ is strong evidence that the $\nu = 1$ state of Fig. 2 does not arise from single-particle tunneling.

Third indication of collective behavior:

E_A very sensitive to $B_\parallel$



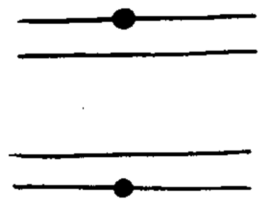
$B_\parallel^* d L_\parallel = \Phi_0$ defines characteristic length $L_\parallel$

$B_\parallel^*$ small $\Rightarrow L_\parallel$ large

$L_\parallel \sim 20\,\ell \gg$ electron spacing

Isospin Analogy

(real spin frozen out)



$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ "proton"

$|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ "neutron"

charge imbalance

$$S^z = \sum_{j=1}^{N} s_j^z = \frac{N_\uparrow - N_\downarrow}{2}$$

tunneling

$$T = -\underbrace{2t}_{\Delta_{SAS}} S^x \qquad s^x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$s_j^x \left(|\uparrow\rangle \pm |\downarrow\rangle\right) = \pm\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$$

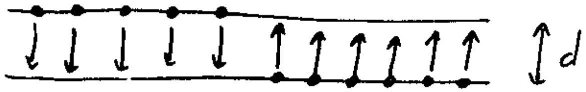
symmetric/antisymmetric tunnel eigenstate.

(Spontaneous) phase coherence between layers

$$\vec{m} : e^{i\varphi/2}\cos\theta\,|\uparrow\rangle + e^{-i\varphi/2}\sin\theta\,|\downarrow\rangle$$

$$H_{eff} = \int d^2r \left\{ \tfrac{1}{2}\underbrace{\rho_s |\vec\nabla m_\mu|^2}_{\text{Fock}} - \tfrac{1}{2}\underbrace{\Delta_{SAS}\, m_x}_{\text{tunneling}} + \underbrace{U m_z^2}_{\text{charging}} + \cdots \right\}$$

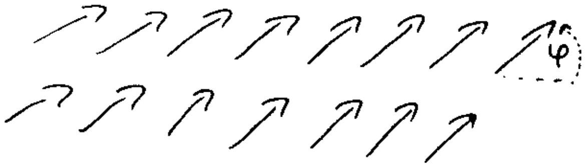
Charging energy $U = \dfrac{de^2}{2\pi\epsilon\ell^2} + U_x$



$U m_z^2$ "easy plane anisotropy"

Even if $\Delta_{SAS} = 0$ isospin spontaneously polarizes in XY plane

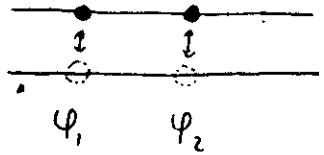
broken $U(1)$ symmetry



$$|\varphi\rangle = |\uparrow\rangle e^{i\varphi/2} + |\downarrow\rangle e^{-i\varphi/2}$$

If there is no tunneling how can energy depend on φ?

It only depends on $|\vec\nabla\varphi|^2$.



hole in one layer bound to particle in other. but layer index uncertain

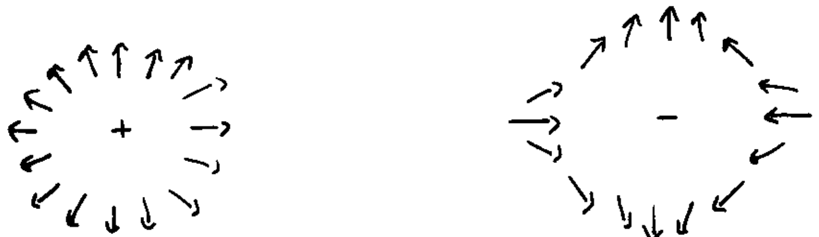
If $\varphi_1 = \varphi_2$ spatial wave function must vanish as $\vec r_1 \to \vec r_2$. $\vec\nabla\varphi$ costs exchange.

$|\varphi\rangle$ has definite phase but indefinite "charge" S_z like BCS

capacitive energy causes "zero-point" fluctuations in φ which limit S_z fluctuations

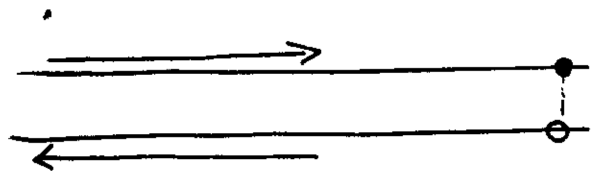
Kosterlitz-Thouless XY $\quad T_{KT} \sim \rho_s \sim 0.5\,K$

$T > T_{KT}$, unbound 'isospin vortices' $\Rightarrow$ dissipation



Vortices are "Merons": $\int d^2r \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{m} \cdot \partial_\mu \vec{m} \times \partial_\nu \vec{m} = \pm 1/2$

$T < T_{KT}$ "isospin superflow" $\vec{J}_{spin} \sim \rho_s \vec{\nabla}\varphi$



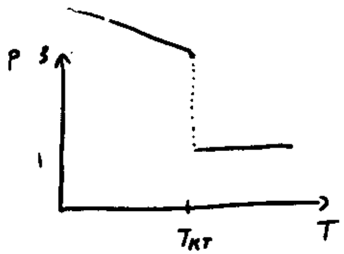
electron bound to correlation hole in opposite layer

condensate (gauge neutral!)

linear $R_{xx} \to 0 \quad T < T_{KT}$

$V_{xx} \sim I^p \quad p = 1 \quad T > T_{KT}$

$\qquad\qquad p = 3 \quad T = T_{KT}$

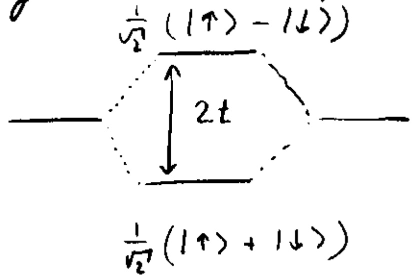


$T_{KT} \lesssim 0.5\,K \qquad \boxed{d > 0,\ t = 0}$

Finite tunneling t

$$T = -\int d^2r \, \vec{h} \cdot \vec{m} \qquad \vec{h} = t\hat{x}$$

analog of Zeeman splitting



$\Delta_{SAS} = 2t$

XY picture

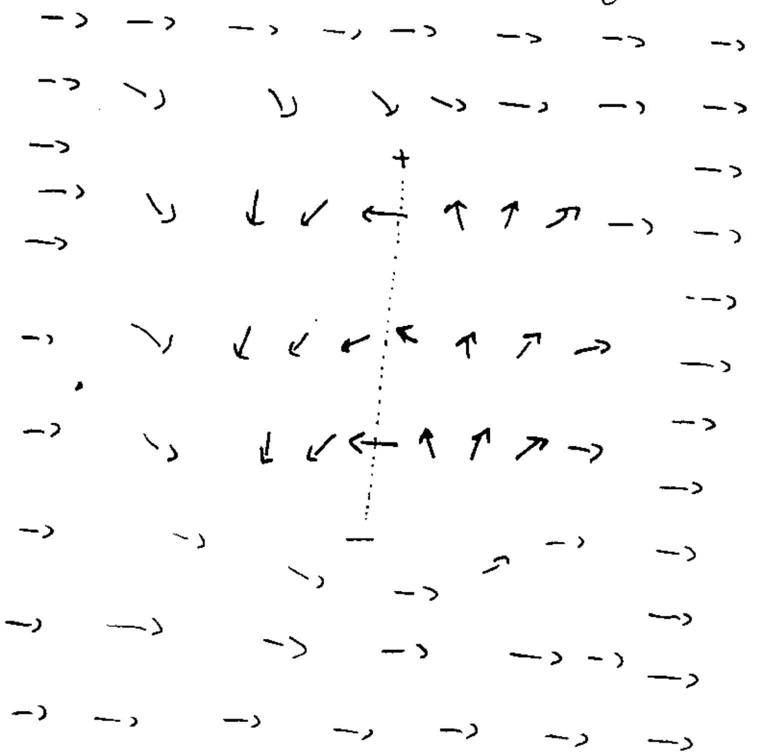
$$H = \int d^2r \, \frac{1}{2} \rho_s |\vec{\nabla}\varphi|^2 - t\cos\varphi$$

t destroys $U(1)$ symmetry and hence KT

collective mode gap $\quad \omega \sim \sqrt{t + c^2 k^2}$

t enhances magnetic order and hence charge gap

(but charge gap exists even if $t = 0$ if spontaneous magnetic order) linear vortex confinement in presence of tunneling

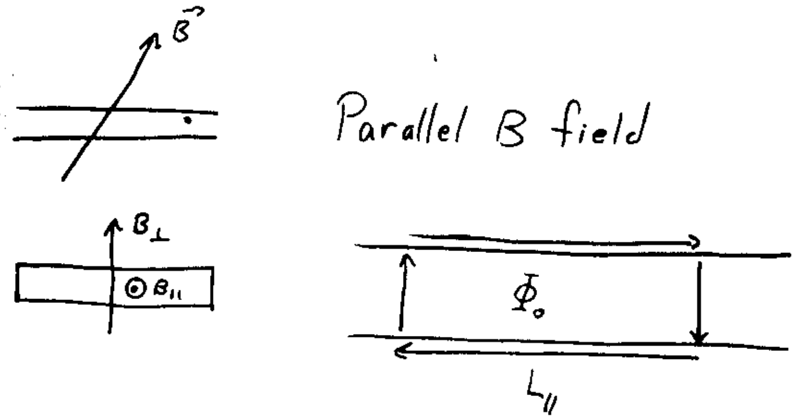


"domain wall"

$U \sim W\,L \quad (\text{not } \log L)$

$+\,2\,\Delta_{core} \pm \frac{(e/2)^2}{L}$

(KT destroyed)



Parallel B field

$t(x) = t\, e^{i Q x}$ $\qquad Q \equiv \frac{2\pi}{L_{\parallel}} = \frac{2\pi\, d B_{\parallel}}{\Phi_0}$

Pokrovsky - Talapov model

$$H_{eff.} = \int d^2 r \left\{ \frac{1}{2} \rho_s |\vec{\nabla}\varphi|^2 - t \cos(\varphi - Q x) \right\}$$

preferred phase for tunneling "tumbles"

small $B_{\parallel}$ "commensurate phase"

$\varphi \approx Qx$ optimizes tunneling costs exchange energy $\quad \mathcal{E} \sim \frac{1}{2} \rho_s Q^2$

large $B_{\parallel}$ "incommensurate phase"

$\varphi \approx \varphi_0$ give up tunneling, save exchange

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

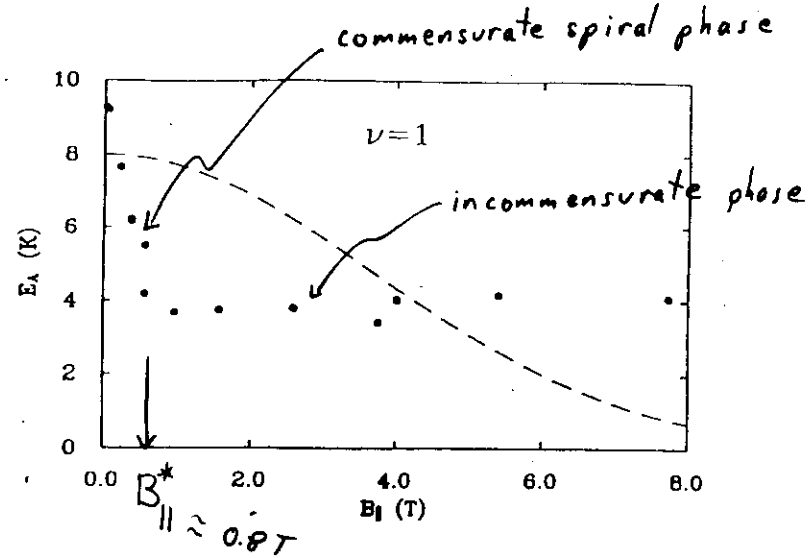


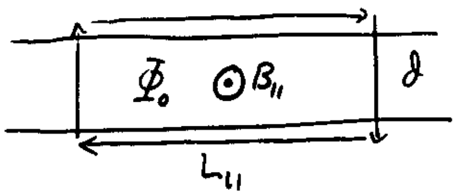
Fig. 3: **Activation energy of the quantized state at $\nu=1$ versus in-plane magnetic field, $B_{\parallel}$. The dashed line (normalized to the data at $B_{\parallel}=0$) is the calculated dependence of a single-particle tunneling gap at $\nu=1$. The relative independence of the activation energy over the range $1 < B_{\parallel} < 8T$ is strong evidence that the $\nu=1$ state of Fig. 2 does not arise from single-particle tunneling.**

Numerical exact diagonalization calculations confirm picture. E_A drops rapidly.

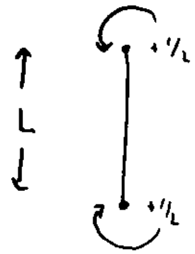
$$H = \int d^2r \ \ \tfrac{1}{2}\rho_s |\vec{\nabla}\varphi|^2 - t \cos(\varphi - Qx)$$

$$B^*_{\parallel} = B_{\perp} \frac{\ell}{d} \frac{2}{\pi} \sqrt{\frac{2t}{\pi\rho_s}} \sim 1.6\ T$$

$B^*_{\parallel} = 0.8T \Rightarrow L_{\parallel} \sim \underline{\underline{20\,\ell}}$ highly collective



Why does charge gap drop as C-I transition at $B^*_{\parallel}$ is approached?

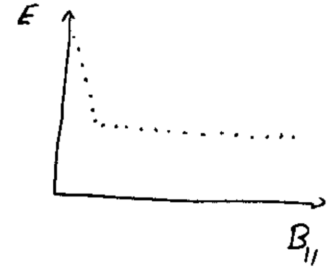


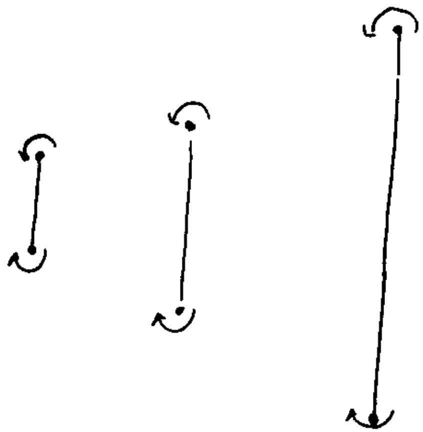
string tension   Coulomb

$$E \sim W L + 2\Delta_{core} + \frac{(e/2)^2}{L}$$

$B_{\parallel}$ encourages spin tumbling — reduces domain wall cost

$$W \sim W_0 \left(1 - \frac{B_{\parallel}}{B^*_{\parallel}}\right)$$

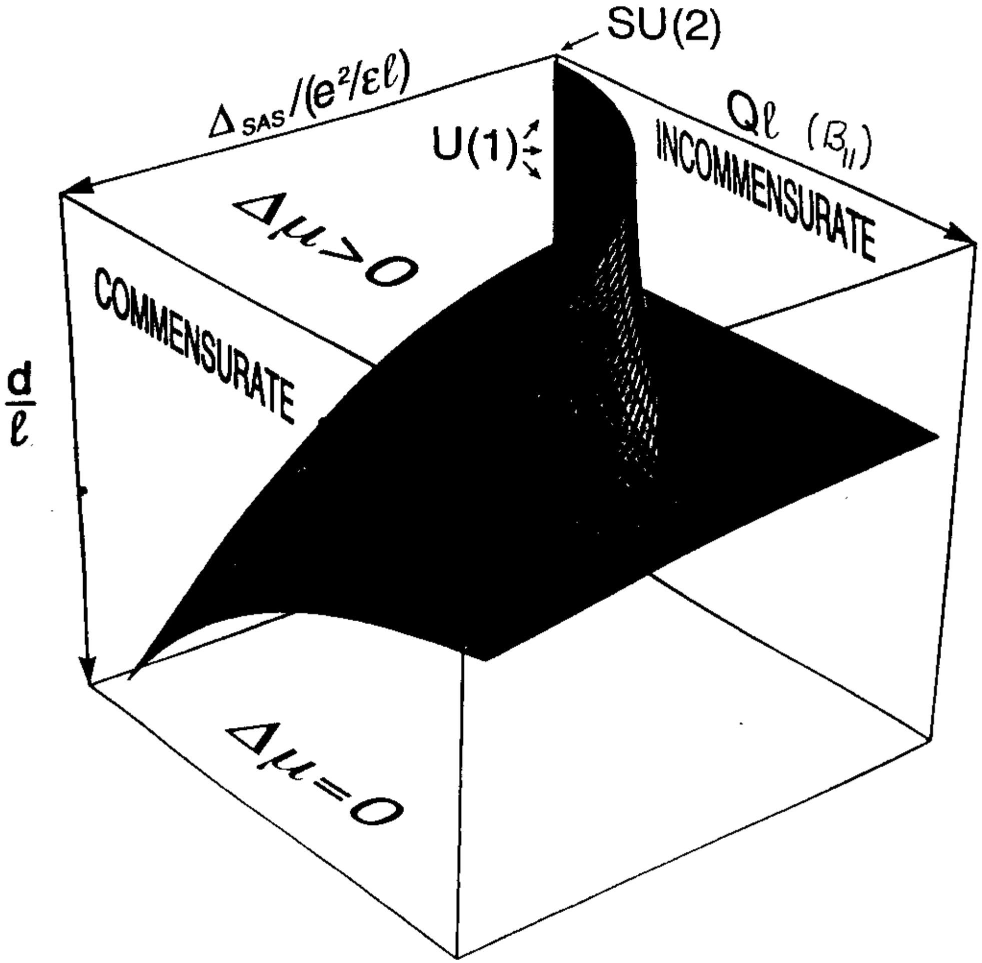




optimal length diverges as $B_{\parallel} \to B^*_{\parallel}$

Coulomb cost reduced

Zero-temperature Phase Diagram



Summary

Two layer QHE

- ideal quantum ferromagnet (isospin)
- finite layer separation

$\Rightarrow$ easy plane anisotropy

XY model $\quad T_{KT} \sim 0.5\,K$

- finite tunneling with $B_{\parallel}$

$$H = \int d^2r \; \frac{1}{2}\rho_s |\vec{\nabla}\varphi|^2 - t\cos(\varphi - Qx)$$

- commensurate-incommensurate transition

- appears to explain experiments