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SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

BACKGROUND MATERIAL AND LECTURE NOTES ON QUANTUM HALL EFFECT

PART I

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These are preliminary lecture notes, intended only for distribution to participants.

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Background material and lecture notes on

Quantum Hall Effect

Part I

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2D Electrons in High Magnetic Field

0 Overview:

2 approaches — states (wavefunctions) / field theory

2 mappings: → bosons / → fermions

2 types of fluid — incompressible / compressible

Only pure systems will be discussed.

Plan:

① Old stuff: lowest Landau level functions & operators
Laughlin wavefunction $\nu = 1/q$, q odd
Quasiparticles, fractional statistics
Hierarchy theory

② Long range order, "composite particles".
Laughlin state as Bose condensate
Binding electrons to vortices
Fermions: Fermi liquid state $\nu = 1/q$, q even
Jain's states $\nu = n/(nq+1)$ " "

③ Equivalence of hierarchy and Jain
Numerics

④ Field Theory approach
Experiments at $\nu \simeq \frac{1}{2}$.

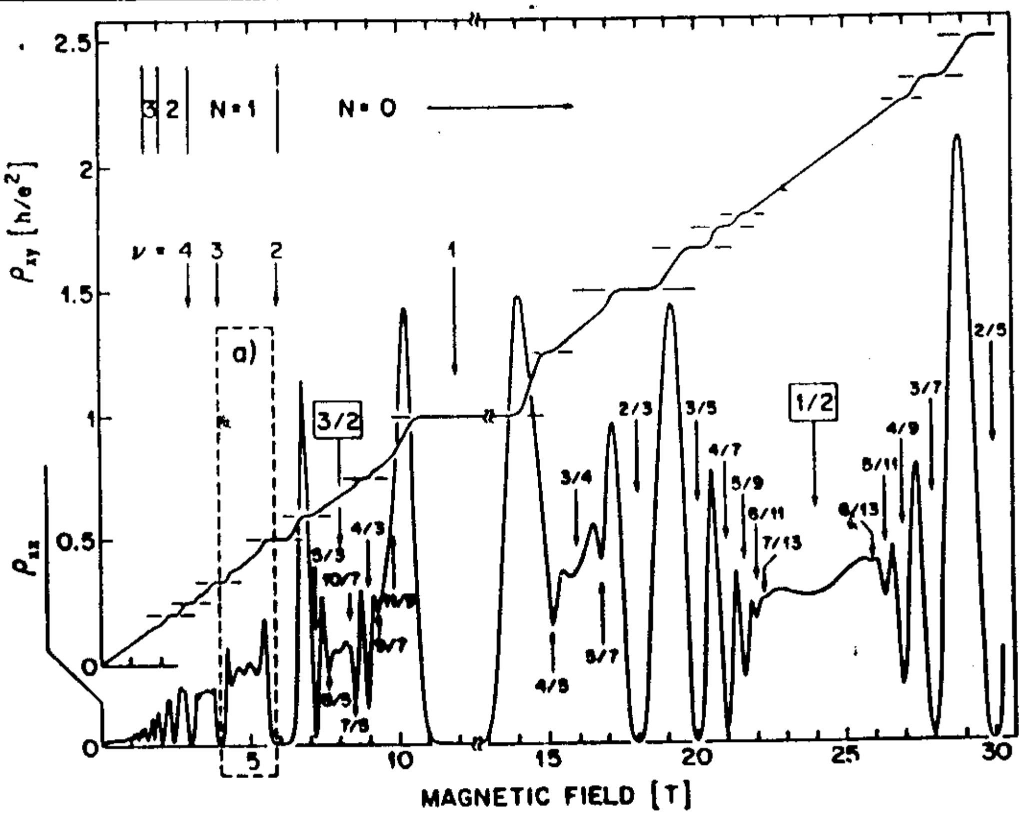


FIG. 1. Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} of sample described in text. The use of a hybrid magnet with fixed base field required composition of this figure from four different traces (breaks at ≈12 T). Temperatures were ≈150 mK except for the high-field Hall trace at T=85 mK. The high-field ρ_{xx} trace is reduced in amplitude by a factor 2.5 for clarity. Filling factor ν and Landau levels N are indicated.

Willett et al

Goal: many-body theory of electron, in lowest Landau level (limit where Coulomb interaction strength $\frac{e^2}{\varepsilon} n^{1/2} \ll \hbar\omega_c$)

① Review — 1 el states in lowest Landau level (LLL)

$$\ell^2 = \frac{\hbar c}{eB}, \quad \ell = \text{magnetic length} \sim 100\text{Å at } 10\,T$$

Set $\ell = 1$ for now

normalized LLL states in symmetric gauge have wavefunctions

$$\langle z, \bar{z} | m \rangle = u_m(z, \bar{z}) = \frac{z^m e^{-\frac{1}{4}|z|^2}}{\sqrt{2\pi\, 2^m\, m!}}, \quad z = x + iy$$

(MacDonald has $z \leftrightarrow \bar{z}$)

$m = 0, 1, 2, \ldots$

Gaussian packet in LLL:

$$\sum_m \langle z, \bar{z} | m \rangle \langle m | w, \bar{w} \rangle = \varphi(z, w) = \frac{e^{\frac{1}{4}(z\bar{w} - \bar{z}w) - \frac{1}{4}|z-w|^2}}{2\pi}$$

Note: fns of z, w from now not automatically analytic (shortens notation)

$$= \frac{e^{-\frac{1}{4}|z|^2 - \frac{1}{4}|w|^2 + \frac{1}{2} z\bar{w}}}{2\pi}$$

centred at w

z = coord of electron $\quad \int |\varphi|^2 d^2 z = \frac{1}{2\pi}$

LLL Uncertainty principle: $\Delta x \Delta y \sim 1$

$x, y \leftrightarrow x, p$

Applied uniform electric field causes w to drift ⊥ with constant velocity ↔ classical $\underline{E} \times \underline{B}/|\underline{B}|$ drift i.e. drift **current** (→ Hall effect)

(Exercise: check using time-dep Schrö eq & $V = \underline{E}.\underline{x}$ and expression for $\underline{x}$ in LLL in MacDonald's lectures)

Second Quantization in LLL

Let $\psi(z) = \sum_{m=0}^{\infty} a_m \bar{u}_m(z)$ (standard 2nd quantization using complete orthonormal set)

$$\{a_m, a_{m'}^{+}\} = \delta_{mm'}$$

↑ anticommutator

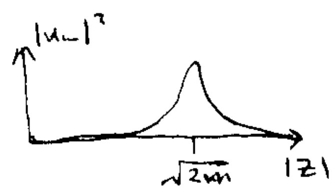
Show $\{\psi(z), \psi^{+}(z')\} = \varphi(z, z')$

φ = LLL "δ-function":

$$\int d^2 z \, f(z)\, \varphi(z, w) = f(w)$$

Important property of u_m's: 1 state in LLL per area 2π

Heuristic proof: $|u_m|^2 = \dfrac{|z|^{2m} e^{-\frac{1}{2}|z|^2}}{2\pi\, 2^m\, m!}$



Max of $|u_m|^2$ at $|z| = \sqrt{2m}$
Most of weight at $|z| \simeq \sqrt{2m}$

$\Rightarrow$ No. of states in circle radius R is $M = m_{max}$
$R = \sqrt{2 m_{max}}$, $M = \frac{1}{2} R^2$

Area $A = \pi R^2$, $M/A = \dfrac{1}{2\pi}$ //

Also no. of flux quanta inside $R = N_\phi = AB/\Phi_0 = A/2\pi = M$ //

Can now show, if ν = # els / # states = filling factor,
$= n\Phi_0/B$ $= N/N_\phi$

$j = \underline{\underline{\sigma}}\, \underline{E}$,

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

(exercise)

Laughlin (many-particle) wavefn

$$\langle \{z_i\} | L^{(q)} \rangle = \Psi_L^{(q)}(\{z_i\}, i=1\ldots N) = \prod_{i<j} (z_i - z_j)^q e^{-\frac{1}{4}\sum_i |z_i|^2}$$

$0 < q$ odd $\leftrightarrow$ antisymmetry (q even for bosons)

and is in lowest Landau level. $q=1$ is full LLL (= Vandermonde det)

Also total angular momentum $= \sum_{i=1}^{N} m_i$ in each term (in expansion in Slater dets) $= q N(N-1)$ = total degree of polynomial part.

Any one electron has $m_i \leq q(N-1) = m_{max} = N_\phi$
$\Rightarrow$ lies inside $R = \sqrt{2 m_{max}}$

Laughlin's motivation was rapid vanishing of Ψ_L as qth power as any $z_i \to$ any z_j (Jastrow form) & lowest LL property. I.e. binding zeros to particles — low Coulomb energy.

As fn of z_i, there are $q(N-1)$ zeros, q at each other electron.

Thus $N_{zeros} = N_\phi$ (= # flux inside R) (a general fact) and if density is essentially uniform inside R we get

$$\nu = \frac{N}{N_\phi} \rightarrow \frac{1}{q} \quad \text{as } N \to \infty$$

Uniform density follows from Laughlin's plasma "analogy" (actually a precise mapping to 2D one component plasma

Quasiparticles

Quasihole (Laughlin):

$$\Psi_L^{(q)} \rightarrow \Psi_{L,qh}^{(q)} = \prod_i (z_i - z)\, \Psi_L^{(q)}(\{z_i\})$$

z = location of quasihole

Els avoid z $\Rightarrow$ charge $-\frac{1}{q}$ (times charge of el) accumulated.

Proof: either by plasma mapping (MacDonald lectures)
or by counting: let $z=0$, then z_i increases angular momentum of ith el by 1, all i.
Each ang mom state has average occupation $1/q$ in Laughlin state
$\therefore$ missing charge $1/q$ at origin ($m=0$ state)

Quasielectron (Laughlin)

$$\Psi_{L,qe}^{(q)} = \prod_i \left(2\frac{\partial}{\partial z_i} - \bar{z} \right) \Psi_L^{(q)}(\{z_i\})$$

($\frac{\partial}{\partial z_i}$ not acting on $e^{-\frac{1}{4}|z_i|^2}$)

More difficult, but charge $= +\frac{1}{q}$ (by counting)

These are trial wavefunctions for ground and excited states. $\therefore$ Either the quasielectron or quasihole (or both) has finite excitation energy $\leftrightarrow$ incompressibility
$\leftrightarrow$ jump in chemical potential at $\nu = 1/q$.

Quasihole = vortex

$\prod_i (z_i - z)$ adds a zero (at z) for each electron.

$\Rightarrow$ missing charge

Also phase of wavefunction winds by 2π as z_i circles z for all i — hence "vortex" (like superfluid).

Likewise quasielectron.

Quasiparticle dynamics = vortex dynamics = dynamics of charged particle in LLL in magnetic field!

The "effective magnetic field" B_{eff} = $n_e \Phi_0$, n_e = electron density

$= \nu \Phi_0 / 2\pi$

One way to see this: can put quasihole at any z, get an exact eigenstate. Just like the Gaussian wavepackets of electron earlier!

What is B_{eff}? Count states: For N els, a basis of orthogonal quasihole states is obtained by acting with

$$\prod_{i=1}^{N} (z_i - z) = (-z)^N + (-z)^{N-1} s_1 + (-z)^{N-2} s_2 \cdots + s_N$$

$$s_1 = \sum_i z_i, \quad s_2 = \sum_{i<j} z_i z_j, \text{ etc}$$

s_i are elementary symmetric polynomials

States $s_i \Psi_L^{(2)}$ are orthogonal because they have different angular momentum.

Hence no of distinct states is $N+1$, in an area $A = \pi R^2 = 2\pi m_{max} = \pi \cdot 2q(N-1)$.

$$\Rightarrow N_\phi^{qh} = N, \quad B_{eff} = \frac{N_\phi^{qh}\,\Phi_0}{2\pi q(N-1)} = \frac{1}{2\pi q}\Phi_0 = \frac{\nu\Phi_0}{2\pi} \qquad (l=1)$$

(N " A)

Actually works for any ℓ, $B_{eff} = \nu\Phi_0/2\pi$.

Useful alternative method: adiabatic "dragging" of quasihole (not tied to $\nu = 1/q$)

Arovas, Schrieffer, Wilczek, PRL, 1984.

State with quasihole at z: $|z\rangle$ i.e. $\langle\{z_i\}|L^{(q)}, z\rangle = \Psi^{(q)}_{L,qh}(\{z_i\}, z)$

Normalize: $|\tilde{z}\rangle = \dfrac{|z\rangle}{\langle z|z\rangle^{1/2}}$

Adiabatic transport: calculate a vector potential

$$A(z) = \langle\tilde{z}|\frac{\partial}{\partial z}|\tilde{z}\rangle$$

$$= \frac{1}{2}\frac{\langle z|\frac{\partial}{\partial z}|z\rangle}{\langle z|z\rangle}$$

$$= \frac{\frac{1}{2}\int \prod_i d^2 z_i\ \overline{\Psi_L^{(q)}(\{z_i\})}\ \prod_i(\bar{z}_i - \bar{z}) \sum_j \frac{-1}{z_j - z}\prod_k (z_k - z) \times \Psi_L^{(q)}(\{z_i\})}{\int \prod d^2 z_i\ |\Psi_L^{(q)}(\{z_i\})|^2 \prod_i |z_i - z|^2}$$

$$= -\frac{1}{2}\int d^2z' \, \frac{\langle \tilde{z} \,|\, n(z') \,|\, \tilde{z}\rangle}{z' - z} \quad , \quad n(z) = \text{density operator} = \sum_i \delta^2(z_i - z)$$

Then $\underline{\nabla} \times \underline{A}(z) = 2\pi \langle n(z) \rangle$ $\left(\text{using } \frac{\partial}{\partial \bar{z}} \frac{1}{z} = -\pi \delta^2(z) \right)$

$\nabla^2 \ln|z| = -\pi \delta^2(z)$

$= 2\pi n$ in ground state

Then if we drag $|z\rangle$ around a closed curve C we get a phase in wavefunction

$$e^{i\oint_C \underline{A} \cdot \underline{dl}}$$

$$\int_{\text{area in } C} d^2z \, \nabla \times A = \oint_C \underline{dl} \cdot \underline{A} = 2\pi n \times (\text{area}) \quad \text{for simple closed curve}$$

i.e. 2π for each particle enclosed

Note: it's the electron number density that counts, not the external magnetic field, even when latter is not constant.

Quasihole inside loop $\Rightarrow$ extra phase $\pm 2\pi\nu$

$\Rightarrow$ Phase $\pm \pi\nu$ if 2 quasiparticles are exchanged adiabatically along loop not enclosing any others.

Fractional Statistics $\theta = \pi\nu$

$= \frac{\pi}{q}$ in Laughlin

For exchange of two clusters of p quasiparticles, $\theta = \pi p^2 \nu$.

Note the adiabatic definition of statistics is <u>not</u> affected by gauge transformations, singular or not. Reason is that if we change by arbitrary factor $(z-w)^{\alpha}$ for quasiholes at z, w, we must make a gauge transfor^n back after exchanging z, w. In above calc^n, the transformation was trivial. (State was $\prod_i (z_i - z) \prod_i (z_i - w) \Psi_L$

Hierarchy theory

Haldane, PRL 83
Halperin, " 84
⋮

Starting idea: Quasiholes (or quasi electrons) behave like interacting particles in LLL. Therefore try to construct a Laughlin fluid of quasiparticles.

Many - quasihole states:

$$\prod_{k=1}^{N_{qh}} \prod_{i=1}^{N} (z_i - w_k) \, \Psi_L^{(q)}$$

Prefactors commute, so qh's are formally bosons (no contradiction with above adiabatic calculation). Each boson has $N+1$ possible states it can occupy; s_m, $m = 0 \ldots N$ act as boson creation operators. Norms of these states are unknown.

For bosons, Laughlin states exist at filling factors

$$\nu_{qh} = 1/p \quad , \quad p > 0, \text{ even}.$$

Possible electron wavefunction would be

$$\int \prod_k d^2 w_k \prod_{k<l} (\bar{w}_k - \bar{w}_l)^p \prod_{k,i} (z_i - w_k) \prod_{i<j} (z_i - z_j)^q$$

$$\times\, e^{-\frac{1}{4}\sum_i |z_i|^2 - \frac{1}{4q}\sum_k |w_k|^2}$$

↑ normalizing factor for single boson states; note $l_{qh}^2 = q l^2$.

Just by counting we see that the max power of any electron zero z_i is

$$m_{max} \equiv N_\phi = q(N-1) + N_{qh} \qquad (1)$$

and also for quasihole Laughlin state

$$N = p(N_{qh} - 1) \qquad (2)$$

(This is needed to balance power of w, $\bar{w}$ in wavefunction otherwise quasiholes will not fill electron droplet appropriately)

$$\Rightarrow \quad N_\phi = q(N-1) + \frac{1}{p} N + 1$$

$$\Rightarrow \quad \nu = \frac{1}{q + \frac{1}{p}} = \frac{p}{qp+1}, \quad p \text{ even}, \ q \text{ odd}$$

as $N \to \infty$

is the filling factor of the electrons.

E.g. 1, $1/3$, $1/5$

$\frac{2}{3}$ $\frac{4}{5}$ … $\frac{2}{7}$ $\frac{4}{13}$ … $\frac{2}{11}$ ….

With quasielectrons we get instead

$$\nu = \frac{1}{q - \frac{1}{p}} \qquad \text{E.g. } \frac{1}{1} \;\; \frac{1}{2} \;\; \frac{4}{3} \ldots \qquad \frac{1}{3} \;\; \frac{2}{5} \;\; \frac{4}{11} \ldots$$

We can now repeat process using quasiparticles of the new state. What are properties of these new quasiparticles?

Include extra factor $\prod_k (\bar{w}_k - \bar{u})$ in qh case above.
Now

$$N = p(N_{qh} - 1) + 1$$

$$\Rightarrow \quad m_{max} = N_\phi = q(N-1) + \frac{1}{p}(N-1) + 1.$$

$$\Rightarrow \quad \Delta N = 1, \quad \Delta N_\phi = q$$

Increasing N_ϕ by q would leave $q \cdot \nu = \frac{pq}{pq+1}$ deficiency in centre. Adding 1 electron means we have in fact

$$-\frac{pq}{pq+1} + 1 = \frac{1}{pq+1} \quad \text{charge added}$$ //

Quasiparticles have fractional charge = 1/(denominator of ν)
What is their adiabatic statistics? (See later for answer)

When process is repeated we obtain continued fraction

$$\nu = \cfrac{1}{m \pm \cfrac{1}{p_1 \pm \cfrac{1}{p_2 \pm \ddots}}} = \frac{p}{q}$$

m = odd

All $p_k > 0$, even.

Show by induction that q = odd, all such rational $0 < \nu < \infty$ are obtained uniquely (ie from only one continued fraction of this form).

Quasiparticles at $\nu = p/q$ have charge $\pm 1/q$.

Remarks:

① Hierarchy theory predicts possibility of incompressible state at each $\nu = p/q$, q odd. Does not guarantee it is actually ground state for Coulomb interactions. Energy gaps presumably smaller in "more complex" hierarchy states, but it's hard to be quantitative. Disorder or finite T might overwhelm "weaker" states, so could proximity to a "strong" state (like 1/3) or transition to Wigner crystal.

Point-like interactions between quasiparticles were not assumed. Quasiparticles are extended objects, but have no internal degrees of freedom. Electrons are extended too, in LLL, one to $\Delta x \Delta y \geqslant l^2$ uncertainty.

② Total density of quasiparticles.

First level $\quad n_{qp1} = \frac{1}{p_1} n_e$

2nd level $\quad n_{qp2} = \frac{1}{p_2} n_{qp1} = \frac{1}{p_2 p_1} n_e$

⋮

Total $(n_{qp1} + n_{qp2} + \cdots)/n_e \quad$ all levels

$$= \frac{1}{p_1} + \frac{1}{p_1 p_2} + \cdots \frac{1}{p_1 \cdots p_k}$$

$$\leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= 1$$

Never exceeds electron density!

Not a low density theory anyway.
(Was Laughlin's?)

③ Nonappearance of even denominators does not mean some quantized Hall state can't exist at q even.
In any case, question what happens there is left open.

Order Parameter and Ginzburg-Landau Theory for the Fractional Quantum Hall Effect

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A new order parameter with a novel broken symmetry is proposed for the fractional quantum Hall effect, with the Laughlin state as the mean-field ground state. The classical Ginzburg-Landau theory of Girvin is derived microscopically from this starting point and exhibits all the phenomenology of the fractional quantum Hall effect.

PACS numbers: 73.20.Dx, 03.50.Kk, 05.30.Fk

While there is now a good understanding of the properties of the states responsible for the fractional quantum Hall effect (FQHE) in the lowest Landau level,[1] a completely general characterization of these states has not yet been given. Girvin[2] has suggested that this might be done by invoking a superfluid analogy, in which the fluid is described by a complex scalar order parameter obeying a Ginzburg-Landau equation, and the vortex excitations are identified with the fractionally charged quasiparticles of Laughlin's theory.[1] In a later Letter, Girvin and MacDonald[2] (GM) showed that a certain modified density matrix exhibits algebraic off-diagonal long-range order in the Laughlin state, providing further evidence for the superfluid analogy.

In this Letter, I construct the superfluid analogy explicitly on a microscopic basis. An order parameter that shows genuine long-range order in the Laughlin state is constructed, related to, but distinct from, that of GM. The broken symmetry is identified, and the Ginzburg-Landau action is derived at the classical, linearized level. All the phenomenology of the FQHE at filling factors $\nu = 1/q$ follows, and generalization to other filling factors can be made at least in principle. Physically, the order parameter describes the special correlations of the Laughlin state (binding of zeroes to particles).

We first exhibit a correlation function which possesses off-diagonal long-range order, indicating that the usual Laughlin state[3] is not a pure phase[4] and that a symmetry is broken. We use (i) a lowest-Landau-level-projected second-quantized field operator in the symmetric gauge,[5]

$$\psi(z) = \sum_{n=0}^{\infty} a_n u_n(z), \quad u_n = \frac{z^n e^{-|z|^2/4}}{(2\pi 2^n n!)^{1/2}}, \tag{1}$$

where a_n is a destruction operator for the nth single-particle basis state u_n, and (ii) Laughlin's quasihole operator,[3] in first quantization,

$$U(z) = \prod_{i=1}^{N} (z_i - z), \tag{2}$$

in the N-particle subspace. Note that while $\psi(z)$ removes a particle bodily from the fluid, leaving a hole of charge 1, $U(z)$ moves particles outwards by increasing their angular momentum about z, leaving a deficiency of charge $1/q$ there if the state is a *fluid state* of slowly varying density ρ close to $\rho_0 = 1/2\pi q$ with no positional long-range order. $U(z)^q$ thus leaves the same charge deficiency as $\psi(z)$, and the essence of the present approach is that these two types of hole states are physically equivalent, so that they have a nonzero overlap.[6]

In the normalized Laughlin ground state $|0_L;N\rangle$ for N particles, whose (unnormalized) coordinate representation is

$$\prod_{i<j} (z_i - z_j)^q \exp\Big[-\tfrac{1}{4}\sum_i |z_i|^2\Big],$$

we can show that

$$\langle 0_L;N|\tilde{U}^\dagger(z)^q \psi(z)\psi^\dagger(z')\tilde{U}(z')^q|0_L;N\rangle$$
$$= \rho_0^{-1}\langle 0_L;N+1|\rho(z)\rho(z')|0_L;N+1\rangle \to \rho_0 \tag{3}$$

as $|z-z'| \to \infty$ with $|z|,|z'| > N$, which is not equal to

$$|\langle 0_L;N|\tilde{U}^\dagger(z)^q\psi(z)|0_L;N\rangle|^2$$

which vanishes identically. Here and below we denote

$$\tilde{U}(z)^q|\alpha\rangle = U(z)^q|\alpha\rangle/\langle\alpha||U(z)|^2|\alpha\rangle^{1/2},$$

where $|\alpha\rangle$ is a normalized fluid state.

Equation (3) shows that the Laughlin state is not a pure state.[4] A pure state, in which $\psi^\dagger U^q$ has a nonzero expectation value, can be constructed as

$$|0_L;\theta\rangle = \sum_{N=1}^{\infty} |a_N| e^{-iN\theta}|0_L;N\rangle, \tag{4}$$

where $\{|a_N|^2\}$ is a binomial distribution function for N with mean $\bar{N} \gg 1$ and variance of order $\bar{N}$, and θ is arbitrary. For this state and arbitrary z,

$$\langle\Psi^\dagger(z)\rangle \equiv \langle\psi^\dagger(z)\tilde{U}(z)^q\rangle \to \rho_0^{1/2} e^{i\theta}$$

as $\bar{N} \to \infty$, and this defines our order parameter. From now on, fluid states $|\alpha\rangle$ will be taken to be pure states

- 16 -

with nonzero order parameter. Physical properties will be more transparent when working with pure states.

Since $\psi^\dagger$ increases N, the particle number, by 1, and U^q increases $M(z)$, the angular momentum about z, by qN, $\psi^\dagger U^q$ breaks the symmetry generated by $\frac{1}{2}N+M/qN$, while $\frac{1}{2}N-M/qN$ is unbroken. Ψ characterizes the Laughlin state, since

$$|0_L;N\rangle=\left[\int d^2z\,\psi^\dagger(z)U(z)^q e^{-|z|^2/4}\right]^N|0\rangle$$

up to a normalization factor, in exact analogy with the ground state of a Bose gas or BCS superconductor. Thus the Laughlin state as in (4) is precisely the mean-field theory of the FQHE.

Note that $\langle\Psi^\dagger(z)\rangle$ is a *local* order parameter, even though $U(z)$ acts on particles far from z, because an (in principle distinct) value can be associated with each point z; this allows it to have thermodynamic significance in a Ginzburg-Landau description, as will be shown.

While the present order parameter resembles that of GM in involving a particle bound to a flux tube (here U^q), it differs in that we find true long-range order in the Laughlin state whereas GM find only algebraic order. The algebraic order of GM is apparently an artifact of their choice of flux operator. We note that any choice of flux operator in place of U^q gives a candidate order parameter for some fluid ground state of filling factor $1/q$, since by a Berry-phase calculation[7] the flux operator will be fermionic if q is odd, and the counting of charge makes the combination, like $\psi^\dagger U^q$, a locally neutral Bose-type operator, which may Bose condense, giving a liquid state.

In constructing the Ginzburg-Landau action, we will use states

$$|\alpha;z,n\rangle=(2^n n!)^{-1/2}(2\partial/\partial z+i\mathcal{A}_-)^n\tilde{U}(z)^q|\alpha\rangle,\quad(5)$$

where $|\alpha\rangle$ is a (pure) fluid state. Equation (5) is the generalization to the normalized hole state $\tilde{U}(z')^q|\alpha\rangle$ of the expansion of the unnormalized state $U(z')^q|\alpha\rangle$ in powers of $z'-z$ about some point z. The vector potential $\mathcal{A}_-=\mathcal{A}_x-i\mathcal{A}_y$ accounts for the normalization and generalizes[8] analyticity $\partial/\partial\bar{z}\equiv0$,

$$(2\partial/\partial\bar{z}+i\mathcal{A}_+)\tilde{U}(z)^q|\alpha\rangle\equiv0,$$

giving us

$$i\mathcal{A}_-(z)=q\int\frac{d^2z'}{z'-z}\langle\tilde{U}^\dagger(z)^q\rho(z')\tilde{U}(z)^q\rangle,\quad(6)$$

where $\rho(z')=\psi^\dagger(z')\psi(z')$ is the density in the lowest Landau level. Equation (6) can equivalently be obtained from adiabatic transport of the hole.[7] $\mathcal{A}_-(z)$ can be calculated approximately by first (exactly) commuting $\rho(z')$ to the right to give

$$i\mathcal{A}_-(z)=q\int\frac{d^2z'}{z'-z}\frac{\langle\alpha|U^\dagger(z)^qU(z)^qR_q(z',z)|\alpha\rangle}{\langle\alpha|U^\dagger(z)^qU(z)^q|\alpha\rangle},\quad(7)$$

where $R_n(z',z)$ is a one-body operator. Equation (7) is now approximated by the insertion of $|\alpha\rangle\langle\alpha|$ between U^q and R_q, in which case, remarkably,

$$\tfrac{1}{2}\epsilon_{\alpha\beta}\partial_\alpha\mathcal{A}_\beta\equiv i\,\partial\mathcal{A}_-/\partial\bar{z}=-\pi q\langle\rho(z)\rangle,\quad(8)$$

where $\alpha=x,y$, $\partial_x=\partial/\partial x$, etc. For a circular droplet of density $\langle\rho\rangle=\rho_0=(2\pi q)^{-1}$, we find $\mathcal{A}_-=\frac{1}{2}i\bar{z}$, the symmetric gauge.

Because of the existence of the order parameter, we can relate hole states by the approximate expansion

$$\psi(z)|\alpha\rangle=\sum_{m=0}^{\infty}\beta_m|\alpha;z,m\rangle,$$
$$\beta_m=(2^m m!)^{-1/2}(2\partial/\partial\bar{z}+ia_+)^m\langle\Psi(z)\rangle,\quad(9)$$

where $\mathbf{a}=\mathbf{A}-\mathcal{A}$ satisfies[2]

$$\epsilon_{\alpha\beta}\partial_\alpha a_\beta=2\pi q(\langle\rho\rangle-\rho_0).\quad(10)$$

In Eq. (9) the $|\alpha;z,m\rangle$ for different m were treated as orthonormal; if $|\alpha\rangle$ is the ground state, this is correct; otherwise, there are overlaps between different n values because $\epsilon_{\alpha\beta}\partial_\alpha\mathcal{A}_\beta$ is not constant. These overlaps and the norms of the states may be calculated recursively; the definition of $\mathcal{A}_-$ implies that $|\alpha;z,0\rangle$ is normalized and orthogonal to $|\alpha;z,1\rangle$ in general. An orthonormal set of states can be constructed by the Gram-Schmidt method. This introduces corrections to the $|\alpha;z,n\rangle$ which, however, can be neglected in the *linearized* calculation described below, because a derivative of a slowly varying expectation value of either ρ or Ψ always appears, which is certainly already of first order in the deviation from the ground-state value. Hence, we may use (5) in (9). We see that

$$\langle\rho\rangle\simeq|\langle\Psi\rangle|^2+\cdots.$$

We now derive the Ginzburg-Landau theory for $\langle\Psi\rangle$ by first obtaining approximate equations of motion for $(id/dt)\langle\Psi^\dagger\rangle$ and then writing down an action whose variation gives these equations.

The Hamiltonian projected into the lowest Landau level contains only potential-energy terms:

$$H=-\int d^2z\,V_{\text{ext}}(z)\rho(z)+\tfrac{1}{2}\int d^2z_1d^2z_2V(z_1-z_2)\psi^\dagger(z_1)\psi^\dagger(z_2)\psi(z_2)\psi(z_1),\quad(11)$$

where $V(z)$ is a function of $|z|$ only.

Working in the Heisenberg picture, we straightforwardly obtain

$$\left\langle i\frac{d\psi^\dagger(z)}{dt}\bar U(z)^q\right\rangle = \sum_{n=0}^{\infty}(2\pi n!)^{-1}\left[\frac{\partial^n}{\partial z^n}\int d^2z'\,V_{\rm ext}(z')e^{-|z'-z|^2/2}\right]\left[2\frac{\partial}{\partial \bar z}-ia_+\right]^n\langle\Psi^\dagger(z)\rangle$$
$$-V_H(0)\langle\Psi^\dagger(z)\rangle+\nabla^2V_H(0)\left[\frac{\partial}{\partial\bar z}-\tfrac{1}{2}ia_+\right]\left[\frac{\partial}{\partial z}-\tfrac{1}{2}ia_-\right]\langle\Psi^\dagger(z)\rangle. \quad (12)$$

In the interparticle potential terms, use has been made of (9), linearized in the deviation of $\langle\Psi^\dagger\rangle$ from its ground-state value, and higher derivatives have been dropped. The "Hartree-type potential,"

$$V_H(z_1-z)=\int d^2z_2\,V(z_1-z_2)\langle\bar U^\dagger(z)^q\rho(z_2)\bar U(z)^q\rangle, \quad (13)$$

is evaluated in the ground state and is then a function of $|z_1-z|$ only. I have neglected in (12) terms arising from taking $\langle\Psi^\dagger\rangle$ to be its ground-state value, and keeping the change in V_H to linear order; these "exchangelike" terms might give mass, quartic interaction, or additional gradient-squared terms in the Ginzburg-Landau long-wavelength action. The omission of these terms, which has no effect on the physics derived here, is our main dynamical approximation.

The remainder of the equation of motion is obeyed by the projection of $(i\,d/dt)\bar U(z)^q|\alpha\rangle$ onto the basis set (5); one finds

$$i\frac{d}{dt}\bar U(z)^q|\alpha\rangle=\Phi(z)|\alpha;z,0\rangle+\sum_{n=1}^{\infty}(2^n n!)^{-1/2}\left[\left[2\frac{\partial}{\partial\bar z}\right]^n\Phi_C(z)\right]|\alpha;z,n\rangle, \quad (14)$$

$$\Phi_C(z)=\frac{\langle U^\dagger(z)^q(i\,d/dt)U(z)^q\rangle}{\langle U^\dagger(z)^qU(z)^q\rangle}\simeq\int d^2z'\,2\frac{\partial V_{\rm ext}}{\partial\bar z'}(z')\sum_{r=0}^{q-1}\frac{\langle R_{r+1}(z',z)\rangle}{z'-z}$$
$$-\int d^2z_1\,d^2z_2\,2\frac{\partial V}{\partial\bar z_1}(z_1-z_2)\sum_{r=0}^{q-1}\frac{\langle\psi^\dagger(z_2)R_{r+1}(z_1,z)\psi(z_2)\rangle}{z_1-z}, \quad (15)$$

and $\Phi=\mathrm{Re}\Phi_C$; I have approximated by decoupling as in (7) and (8) and also dropped a term in the two-body part that involves both $[\rho(z_2),U(z)^q]$ and R_{r+1}, which is a higher-order correlation. Then, by manipulations similar to those used in (8) and (12), we obtain

$$2\frac{\partial\Phi_C}{\partial\bar z}=-\int\frac{d^2z'}{2\pi q}\,2\frac{\partial V_{\rm ext}}{\partial\bar z'}(z')\sum_{r=0}^{q-1}\frac{|z'-z|^{2r}e^{-|z-z'|^2/2}}{2^r r!}-2\pi q\nabla^2V_H(0)\langle\Psi^\dagger(z)\rangle\left[\frac{\partial}{\partial\bar z}+\tfrac{1}{2}ia_+\right]\langle\Psi(z)\rangle, \quad (16)$$

up to higher gradients of $\langle\Psi\rangle$. From the exact expressions for a,Φ,

$$da_+/dt+2\,\partial\Phi/\partial\bar z\equiv2\,\partial\Phi_C/\partial\bar z, \quad (17)$$

and the right-hand side of (16) can be interpreted as the drift-motion current due to the external and interparticle potentials.

Finally,

$$i\frac{d}{dt}\langle\Psi^\dagger(z)\rangle=\sum_{n=0}^{\infty}(2\pi n!)^{-1}\left[\frac{\partial^n}{\partial z^n}\int d^2z'\,V_{\rm ext}(z')e^{-|z-z'|^2/2}\right]\left[2\frac{\partial}{\partial\bar z}-ia_+\right]^n\langle\Psi^\dagger(z)\rangle$$
$$-\sum_{n=1}^{\infty}(2\pi q n!)^{-1}\left[\frac{\partial^{n-1}}{\partial\bar z^{n-1}}\int d^2z'\,\frac{\partial V_{\rm ext}}{\partial\bar z'}(z')\sum_{r=0}^{q-1}\frac{|z'-z|^{2r}}{2^r r!}e^{-|z-z'|^2/2}\right]\left[2\frac{\partial}{\partial z}-ia_-\right]^n\langle\Psi^\dagger(z)\rangle$$
$$+[\Phi(z)-V_H(0)]\langle\Psi^\dagger(z)\rangle+\nabla^2V_H(0)\left[\frac{\partial}{\partial\bar z}-\tfrac{1}{2}ia_+\right]\left[\frac{\partial}{\partial z}-\tfrac{1}{2}ia_-\right]\langle\Psi^\dagger(z)\rangle \quad (18)$$

to linear order in deviations from the ground state. Note the similar structure of the first two terms.

Since we are working at indefinite particle number and area, we must add chemical potential and pressure terms to the Hamiltonian, which can be incorporated in $V_{\rm ext}$; this takes the form of a constant potential in the interior of the droplet, with slowly rising confining walls near the edge, and can be arranged to cancel in the ground-state case the terms on the right-hand side of (18) with no gradients of $\langle\Psi^\dagger\rangle$. Thus $(i\,d/dt)\langle\Psi^\dagger\rangle$ vanishes in the interior of the droplet in the ground-state case where $\langle\Psi^\dagger\rangle$ is uniform in space.

With the remainder of $V_{\rm ext}$ omitted for clarity, Eqs. (10) and (16)–(18) can be obtained by variation of the Lagrang-

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ian density

$$L=\Psi^\dagger i\frac{d}{dt}\Psi-\tfrac{1}{2}C\left[\left(2\frac{\partial}{\partial z}-ia_-\right)\Psi^\dagger\right]\left(2\frac{\partial}{\partial\bar z}+ia_+\right)\Psi+\Phi[|\Psi|^2-\rho_0-(2\pi q)^{-1}\epsilon_{\alpha\beta}\partial_\alpha a_\beta]-\frac{1}{4\pi q}\epsilon_{\alpha\beta}a_\alpha\frac{d}{dt}a_\beta, \quad (19)$$

where $C=\frac{1}{2}\nabla^2V_H(0)$ (as usual, L is determined by the equations of motion only up to total time derivatives).

The same linearized equations of motion can be solved for plane-wave excitations to yield a dispersion relation

$$\omega^2=C^2+\tfrac{1}{2}Ck^2(C+\tfrac{1}{2}Ck^2);$$

thus this collective mode has a gap $\omega=C$ as $k\rightarrow 0$ which is due to the long-range gauge forces in the action (19) (the Anderson-Higgs mechanism). The roton minimum[9] at a larger wave vector $k\sim\rho_0^{1/2}$ is not obtained within the present approximation. The fractional statistics[7,10] of the vortices and the quantized Hall conductance also follow from (19).

The physical meaning of the order parameter $\langle\Psi^\dagger(z)\rangle$ is that it is the amplitude for finding a particle at z at the zeroes of the many-particle wave function, and by (9) its gradients represent the amplitudes for displacements from the zeroes. A nonzero displacement leads to a higher Hartree energy (13) (which just involves the two-particle correlation function) and hence to the stiffness constant $C=\frac{1}{2}\nabla^2V_H(0)$. The long-range gauge forces are related to those in Laughlin's plasma analogy,[3] but here take on a dynamic as well as static role.

It should be possible to describe quantum fluctuations about the Laughlin state by quantizing the action (19), but this should be done with care to ensure a connection with the microscopic description. Since the Laughlin state is exact for the pseudopotential Hamiltonian[11] with $V_1,V_3,\ldots,V_{q-2}\neq 0$, $n\geq q$, quantum fluctuations will be controlled by the size of V_n, $n\geq q$.

A similar order parameter can be constructed for general filling factors $\nu=p/q$ with use of $\psi^{\dagger r}U^q$, and also extended to spin-singlet states;[12] these extensions and details of the present work will be given elsewhere.[13]

A brief report of part of this work was given previously.[14] After this work was completed, we received a preprint from Rezayi and Haldane,[15] who studied related order parameters numerically, and showed that they are nonzero in FQHE states at $\nu=\frac{1}{3},\frac{2}{5}$ but vanish in compressible states. We also learned of work by Zhang, Hansson, and Kivelson[16] on the Ginzburg-Landau action.

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Excitation Structure of the Hierarchy Scheme in the Fractional Quantum Hall Effect

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The hierarchy schemes for the fractional quantum Hall effect are reexamined and it is shown that different schemes all give the same lattice of excitations whose statistics is determined by the norm of the corresponding vector, and hence have equivalent Ginzburg-Landau theories. Similar ideas apply to the anyon liquid. The schemes can be generalized by using different lattices; many inequivalent states can be obtained at any filling factor (or value of the statistics parameter).

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Laughlin's wave functions[1] have won widespread acceptance as good model wave functions for the fractional quantum Hall effect[2] (FQHE) at filling factor $\nu = 1/q$, but the situation at most other filling factors (with the exception of those related to $1/q$ by particle-hole conjugation or by filling of lower Landau levels) is somewhat less clear. Numerous schemes for extending Laughlin's ideas have been proposed, in particular what I will call the "standard" hierarchy,[3,4] a "variant" hierarchy,[5] and recently a "new" hierarchy.[6] These hierarchies are supported by different physical arguments but are alike in producing a ground state for every fraction $\nu = p/q < 1$ such that q is odd, and in having fractionally charged excitations $\pm e^* = \pm e/q$.

Questions about the detailed structure of the excitation spectrum for filling factor ν have recently arisen because of its relevance to gapless excitations of an incompressible bulk FQHE state.[7] One may ask whether the hierarchies make equivalent predictions, whether there are other physically distinct incompressible states at the same ν, and how the Ginzburg-Landau (GL) theory, for $\nu = 1/q$,[8-10] can be properly extended to other fillings. Similar questions may be raised about the ground states of a liquid of anyons.[11,12]

The main results of this paper are as follows. Incompressible FQHE systems will be regarded as equivalent when their filling factors are equal and they possess excitations whose quantum numbers and statistics correspond. (i) The quantum numbers of the possible "charged" excitations lie on a lattice of points in r-dimensional space for r levels of the standard hierarchy. Excitations of the same physical charge all have the same (fractional) statistics. All the hierarchy schemes are equivalent in this sense; different constructions involve different bases for the same lattice. This characterizes these systems nonhierarchically. (ii) The order parameter has r components, and the GL theory also involves r gauge potentials[12] and its structure is determined by the same lattice as the excitations. (iii) The constructions can be generalized further, in a basis-independent way, by using an arbitrary lattice, subject to certain rules. This produces other inequivalent states at *any* rational ν. (iv) Similar observations apply to spin singlet and partially polarized states, and to states of an anyon liquid.

I begin by writing the standard hierarchy electron wave function[3,4] in the form

$$\Psi(\{z_{0i}\}) = \int \prod_{\alpha=1}^{r-1} \prod_{i=1}^{N_\alpha} d^2 z_{\alpha i} \exp\left(-\frac{1}{4}\sum_i |z_{0i}|^2\right) \prod_{\alpha=0}^{r-1} \left[\prod_{i<j} (z_{\alpha i} - z_{\alpha j})^{a_\alpha} \prod_{ij} (z_{\alpha+1,i} - z_{\alpha j})^{b_{\alpha,\alpha+1}}\right]. \quad (1)$$

Equation (1) describes $N = N_0$ electrons at positions $z_i = z_{0i}$ and the integrals are over coordinates of quasiparticles at level $\alpha = 1, \ldots, r-1$ in the hierarchy; the system contains N_α quasiparticles of level α at positions $z_{\alpha i}$. In the exponents, $a_0 > 0$ is odd, a_α ($\alpha > 0$) is even, $b_{\alpha,\alpha+1} = \pm 1$, and $b_{r-1,r} = 0$. Negative exponents are unconventional; quasiholes in the electron system couple with $b_{01} = 1$ as usual, but quasielectrons couple with $b_{01} = -1$. This is an acceptable alternative to the usual Laughlin quasielectron or other proposals as long as the singularity at the center is removed by projecting onto holomorphic (lowest-Landau-level) functions. Such projection only introduces a short-range interaction into the effective many-component Coulomb plasma, described below. Alternatively, the factors with negative exponents may be replaced by positive powers of the complex-conjugate factor, times additional exponential factors. This freedom of choice in the hierarchy wave functions has no influence on the following; the above form makes the structure especially clear.

In order to work with states like (1), one needs to make an orthogonality postulate. To take overlaps of two many-quasiparticle states, one must integrate over the electron coordinates. One hopes that this makes the overlap vanish unless the positions of the $\alpha = 1$ quasiparticles in one state nearly coincides with those in the other. If so, then the integrations in (1) for each state can be reduced to a single set of integrals for $\alpha = 1$, and the process can be iterated. For a few well-separated $\alpha = 1$ quasiparticles, this can be demonstrated explicitly,[13] and so should hold for $|a_\alpha|$ large. We will assume, as is stan-

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dard, that it also holds for $|a_a|$ as small as 2. Then expectations in (1) behave like those of a multicomponent generalization of Laughlin's Coulomb plasma.

A homogeneous ground state in the shape of a disk is obtained if

$$a_a(N_a-1)+b_{a,a+1}N_{a+1}+b_{a-1,a}N_{a-1}=0, \qquad (2)$$

for $a=0,\ldots,r-1$, where $N_r=0$, $b_{-1,0}N_{-1}=-N_\phi$, and N_ϕ is the total physical flux in the area covered by the disk. Equations (2) state that charge neutrality is satisfied (including the background $-N_\phi$) in the multicomponent Coulomb gas (1). The filling factor is given by

$$\nu=\frac{N}{N_\phi}=\cfrac{1}{a_0-\cfrac{b_{01}^2}{a_1-\cfrac{b_{12}^2}{\ddots-\cfrac{b_{r-2,r-1}^2}{a_{r-1}}}}}\equiv\frac{p}{q}. \qquad (3)$$

Since a_0 is odd and positive, a_a, $a>0$, are even and of either sign, and $b_{a,a+1}=\pm1$, these give all the standard fractions; i.e., q is odd and p,q have no common factor.

Strengths of the logarithmic interactions in the Coulomb plasma resulting from (1) are given by the elements of

$$(G_{a\beta})=\begin{pmatrix} a_0 & b_{01} & 0 & \cdots & \\ b_{01} & a_1 & b_{12} & & \\ 0 & b_{12} & a_2 & & \\ \vdots & & & \ddots & \\ & & & & a_{r-1}\end{pmatrix}. \qquad (4)$$

Then (2) becomes (neglecting 1 with respect to N_a)

$$(G_{a\beta}N_\beta)=\begin{pmatrix} N_\phi \\ 0 \\ \vdots \\ 0\end{pmatrix} \qquad (5)$$

and by inversion of (5)

$$\nu=(G^{-1})_{00}=\det G'/\det G \qquad (6)$$

by Cramer's rule, where G' is the $(r-1)\times(r-1)$ matrix with elements $G_{a\beta}'=G_{a\beta}$ for $a,\beta>0$.

A quasiparticle at z may be obtained by inserting $\prod_{ai}(z_{ai}-z)^{f_a}$ with f_a integers into (1). The "fluxes" (or strictly, vorticities) f_a are screened by the generalized Coulomb plasma, producing screening "charges" δN_a *locally* around z,

$$G_{a\beta}\delta N_\beta=-f_a. \qquad (7)$$

Note that the physical electron number $\delta N=\delta N_0$ but f_0 is not the total effective physical flux because the quasiparticles δN_a constitute a backflow.

The statistics of the excitations can be found by generalizing the method of Arovas, Schrieffer, and Wilczek.[14] The phase $e^{i\theta}$ obtained by interchanging two identical excitations is

$$\theta/\pi=-f_a\delta N_a$$
$$=f_a(G^{-1})_{a\beta}f_\beta=\delta N_a G_{a\beta}\delta N_\beta. \qquad (8)$$

Here the direction of interchange is fixed for all ν by demanding that $\theta/\pi=1/q$ for a quasihole in the Laughlin state.[14] For a charge $\delta N=\pm 1/q$ excitation in the standard hierarchy

$$\frac{\theta}{\pi}=\cfrac{1}{a_{r-1}-\cfrac{b_{r-2,r-1}^2}{\ddots-\cfrac{b_{01}^2}{a_0}}}, \qquad (9)$$

which can also be obtained from Halperin's equations.[4] Some properties of this expression are given elsewhere.[15] That (9) is independent of the type of excitation will be confirmed below.

The same calculation also gives Berry's phase per unit area due to the effective background magnetic field seen by the excitation as $-f_a\bar{\rho}_a=\delta N/2\pi$, where from (5) $\bar{\rho}_a=(G^{-1})_{a0}/2\pi$ are the average densities and so only excitations with nonzero physical charge see a field, which is the physical field, as one might have expected. These excitations therefore have Landau-level-type spectra, while the neutral excitations are propagating waves.

The set of possible excitations $\{(f_a)|f_a\in\mathbf{Z}\}$ may be regarded as lying on an "excitation lattice" Λ^* in a space $\mathbf{R}^r$. The coordinates f_a are the components of each lattice point in a basis $\mathbf{e}_a^*$, $a=0,\ldots,r-1$, of Λ^* whose *Gram matrix*[16] of scalar products is $(G^{-1})_{a\beta}=\mathbf{e}_a^*\cdot\mathbf{e}_\beta^*$. Thus θ/π is just the "squared length" (norm) of a vector in the lattice (not necessarily positive since G^{-1} is not necessarily positive definite). A transformation $\mathbf{e}_a^*\to\mathbf{e}_a^{*\prime}=S_{a\beta}\mathbf{e}_\beta^*$ with S having integer matrix elements and determinant 1 changes the basis from $\mathbf{e}_a^*$ to $\mathbf{e}_a^{*\prime}$ but leaves the structure invariant.

For excitations (f_a) such that (δN_a) are all integers, one sees that the wave function is that obtained by adding or subtracting electrons or quasiparticles at z. Thus, as for Laughlin's states,[9] such combinations of fluxes are equivalent to adding or removing particles. Therefore a composite operator which adds such fluxes and compensating (quasi)particles has no net charge δN_a and exhibits long-range order; it is an *order parameter.* Pure states, with nonvanishing order-parameter expectations, are constructed[9] by taking linear combinations of states of different N_a with definite phases θ_a. Fluctuations in N_a change the quasiparticle distribution at the edge but leave the filling factor in the bulk unchanged.

The order parameters are in one-to-one correspondence with the integer-charged excitations that they contain, which form an r-dimensional sublattice Λ of Λ^*, which I call the "condensate lattice." By (8), the Gram matrix of Λ is G, so all scalar products of vectors in Λ are integers; i.e., Λ is an *integral lattice*. Λ^* is the dual lattice of Λ since it has the inverse Gram matrix,[16] and becomes an integral lattice if rescaled by $\sqrt{q}=\sqrt{\det G}$. The sublattice $\Lambda^\perp$ consisting of vectors of Λ having zero physical charge ($\delta N=\delta N_0=0$ in the original basis) has Gram matrix G'. $\Lambda^\perp$ is an *even* lattice (norms of these vectors are even because a_α are even for $\alpha>0$), and so these excitations have Bose statistics. It is easy to show from the form of (4) that they exhaust the neutral excitations, i.e., $(\Lambda^*)^\perp=\Lambda^\perp$, and hence that in the standard hierarchy the statistics of an excitation depends only on its charge δN.

These results imply that the form of the GL action[8-10] must be $S=\int d^2x\,dt\,L$, with

$$L=\tfrac{1}{2}(\partial_\mu\theta_\alpha-A_\mu\delta_{\alpha,0}-\mathcal{A}_{\mu\alpha})C^{\mu\nu}_{\alpha\beta}(\partial_\nu\theta_\beta-A_\nu\delta_{\beta,0}-\mathcal{A}_{\nu\beta})$$
$$+\bar{\rho}_\alpha(\partial_0\theta_\alpha-A_0\delta_{\alpha,0}-\mathcal{A}_{0\alpha})$$
$$+\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}\mathcal{A}_{\mu\alpha}(G^{-1})_{\alpha\beta}\partial_\nu\mathcal{A}_{\lambda\beta}, \qquad (10)$$

where $\mu,\nu,\lambda=0,1,2$ are space-time indices, the A_μ are the physical gauge potentials, $\mathcal{A}_{\mu\alpha}$ are internal gauge potentials, $C^{\mu\nu}_{\alpha\beta}=\eta^{\mu\nu}C^{(\mu)}_{\alpha\beta}$, $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1)$, and $C^{(\mu)}$ are arbitrary positive-definite matrices. The θ_α can be regarded as coordinates on a torus $\mathbf{R}^r/2\pi\Lambda^*$ and hence vortices are labeled by their flux $\int d^2x\,\nabla\times(\mathcal{A}_\alpha+A\delta_{\alpha,0})=2\pi f_\alpha$ and (7) and (8) follow from the final Chern-Simons term in (10). The second term gives the effective fields.

Jain's first construction[6] used wave functions χ_r for r filled Landau levels (LLs):

$$\chi_{(n_1+1/r_1)^{-1}}=(\chi_1)^{n_1}\chi_{r_1}, \qquad (11)$$

where $\nu=(n_1+1/r_1)^{-1}$ and n_1 is even. Even for $r_1>1$, (11) has nonzero projection to the lowest LL when $n_1>0$, the $\bar{z}$'s becoming $\partial/\partial z$'s. The resulting state may be described in terms of "fictitious LL's" or by saying that the electrons have been divided into r_1 species, each species having a different number of $\bar{z}$ factors for each electron. Thus the wave function is very close in form to a multicomponent Coulomb plasma (for χ_r itself we have r decoupled Coulomb plasmas and so the GL theory for $\nu=r$ is r copies of that for $\nu=1$). Excitations can be made by introducing holes into a single fictitious LL (or inverse powers to obtain quasielectrons). The different LL quasiholes are orthogonal in the thermodynamic limit, by a Coulomb-gas calculation, because the large number of factors of the form (z_i-z) act on distinct sets of particles. In fact, such arguments show[17] that the system exhibits a *spontaneous breakdown of permutation symmetry* and one can ignore the antisymmetrization of electrons among the species. Consequently, the $\bar{z}$ factors can be omitted and the system behaves just as an r_1 component Coulomb plasma, in which the Gram matrix G clearly has diagonal elements n_1+1 and off-diagonal n_1. These entries refer to a basis $\mathbf{e}_\alpha$ for Λ of equally charged excitations $\delta N=-1$ so the basis order parameters consist of one added electron and one of the flux combinations $\mathbf{e}_\alpha$.[18]

To make contact with the standard hierarchy, I now change basis. As the first basis vector take $\mathbf{e}_0$ which has norm n_1+1. For the remainder take $\mathbf{e}'_\alpha=\mathbf{e}_\alpha-\mathbf{e}_{\alpha-1}$, $\alpha=1,\dots,r-1$, which have $\delta N=0$ and norm 2. The off-diagonal scalar products give -1 for adjacent members of the sequence and zero otherwise. The new Gram matrix is therefore tridiagonal like (4), proving that quantum numbers and statistics of excitations are the same as those of the standard hierarchy at the same filling factor (as can also be shown by direct calculation of Λ^*). $\Lambda^\perp$ is here the root lattice A_{r-1} of SU(r),[16] and the r species behave as the fundamental representation of this group, though there is no reason why the Hamiltonian should respect all of this symmetry.

Another set of filling factors $\nu=(n_1-r_1^{-1})^{-1}$, $r_1>1$, is obtained using the conjugate of χ_{r_1} in (11), or powers n_1-1, n_1 in the Coulomb plasma, and leads in the hierarchy basis to -2 in place of 2 in G; SU(r) "symmetry" is still present. Jain has emphasized[6] that these two families include most of the experimentally observed filling factors.

Given a state χ_ν, a new filling factor is obtained[6] by adding electrons in new fictitious LLs and then attaching flux to all the particles:

$$\chi_\nu\to\chi_{\nu'}=(\chi_1)^n\chi_{r+\nu}, \qquad (12)$$

where n is even and $\nu'=[n+1/(r+\nu)]^{-1}$, giving a "new" hierarchy of states labeled by sequences $n_1,r_1,n_2,r_2,\dots,n_k,r_k$ for k steps. Once again there is a basis for Λ of $\delta N=-1$ excitations, one for each of the $r=\sum_{i=1}^k r_i$ species. Now take $\mathbf{e}_0$ to be one of the last set of r_k fluxes, and the $\mathbf{e}'_\alpha$ to be differences of the $\delta N=-1$ basis vectors, working back down the hierarchy. The resulting tridiagonal Gram matrix has diagonal n_k+1, 2 (r_k-1 times), $n_{k-1}+2$, 2 ($r_{k-1}-1$ times), $\dots$, 2, and off-diagonal elements -1, which is the standard hierarchy form (4). Including negative entries in $n_1,\dots,r_k$ gives all the standard hierarchy states.

The variant hierarchy[5] is sufficiently similar to the standard one not to require separate discussion here; it again produces the same lattices Λ^* of excitations.

The hierarchy construction can be generalized by taking an arbitrary Gram matrix G, whose matrix elements specify a ground state as in (1). In this basis, G_{00} must be odd because of Fermi statistics, and the other diagonal elements even, and so $\Lambda^\perp$ is even. Inequivalent lattices give inequivalent FQH states. Then $\nu=p/q$ where $q=\det G$ and $p=\det G'$ may have common factors. Note that ν need not have odd denominator. Equations (5), (7), (8), and (10) continue to hold. This very large set of possible states is just those having a basis of order parameters containing a single electron since a basis for Λ of $\delta N=-1$ (or a Jain-type construction) can always be obtained. An elegant example is obtained by replacing

G' by the Gram matrix of D_{r-1}, the root lattice of $SO(2(r-1))$,[16] $r>4$. Taking $G_{00}=m$, odd and depending on how G' is extended to G, one can obtain $\nu=1/(m-1)$or $\nu=1/(m-2)$, and so reproduce $\nu=1/q$ but with a lattice of dimension r.

States with some or all of the spins of the electrons reversed can be handled similarly; one of the δN_a is identified as δS^z. As examples, Halperin's $\nu=2/(2n+1)$ spin-singlet states[19] have the same lattice structure[15] as Jain's construction (11) for $r_1=2$, while a singlet state proposed by Jain[6] for $\nu=\frac{1}{2}$ is equivalent to that in Ref. 20.

The present results should shed light on the fractionally charged edge excitations.[7,21] Also, on surfaces of nontrivial topology, like the torus, general principles[15] imply a ground-state degeneracy[21] in the thermodynamic limit, the degeneracy being given by a factor $|\Lambda^*/\Lambda| = \det G = q$ for each "handle." For the hierarchy states, p,q have no common factors, so this is just the minimal degeneracy q for the torus found by Haldane.[22]

The hierarchy for the anyon liquid[12] parallels that for the FQHE for bosons (for which Λ is even) with the statistics parameter $\alpha_r=\theta/\pi$ playing the role of ν; I find that the space of order parameters is $r+1$ dimensional for an r-level fraction.

In conclusion, I have shown the existence of previously unnoticed structure in the hierarchy schemes which characterizes these states completely at the GL level. This classifies all states having only single-electron condensates.

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Theory of the half-filled Landau level

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A recent theory of a compressible Fermi-liquid like state at Landau level filling factors $\nu = 1/q$ or $1 - 1/q$, q even, is reviewed, with emphasis on the basic physical concepts.

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The physics of interacting electrons in 2-dimensions in very high magnetic fields has proved to be a rich subject since the discovery of the FQHE in 1982 [1]. The initial motivation for study in this area was the expectation that a Wigner crystal should form when the (areal) density of electrons is low and the magnetic field is high, not only because, as at low magnetic field, the Coulomb energy of repulsion dominates the kinetic energy at low density, but also because if the magnetic field is high enough so that mixing of excitations to higher Landau levels can be neglected then the kinetic energy itself becomes essentially constant. In this limit the electrons would behave classically, with their dynamics given by the $\mathbf{E} \times \mathbf{B}$ drift of their guiding centre coordinates, *i.e.* drift along the equipotentials, the potential being here given by the Coulomb potential of the other electrons. The Wigner crystal is presumably the unique lowest energy, stationary state at a given density in this classical problem. The surprise was that in fact, at accessible densities, the quantum fluctuations cause this crystal to melt (as parameters are varied) into some quantum fluid, and furthermore the nature of this fluid depends on the commensuration between the density and the magnetic field, in an essentially quantum mechanical way. In terms of the Landau level filling factor $\nu \equiv n\Phi_0/B$ (where n is the density, B is the magnetic field strength and $\Phi_0 = hc/e$ is the flux quantum), this was manifested by quantum Hall effect (QHE) plateaus at $\nu = p/q$ a rational number with an odd denominator q (p and q have no common factors) in all except a handful of special cases observed more recently.

A theory that explained many features of this picture, including the "odd denominator rule" was quickly forthcoming [2, 3, 4]. Laughlin's states at $\nu = 1/q$, q odd, and the hierarchical extension to all $\nu = p/q$, q odd, are incompressible fluids capable of exhibiting a QHE plateau at $\sigma_{xy} = (p/q)e^2/h$. They also possess novel elementary excitations called quasiparticles, which carry a fraction $\pm 1/q$ of the charge on an electron, and have fractional statistics, that is the phase of the wavefunction changes by $e^{i\theta}$, $\theta/\pi = \pm p'/q$ when two quasiparticles are exchanged adiabatically [4, 5, 6, 7]. A finite energy Δ is required to create one of these excitations. The fractional QHE associated with a particular state will be destroyed if the energy scale characterizing the rms disorder exceeds about Δ.

Some questions remained unanswered, however. While the hierarchy theory predicted that states at ν with larger denominators would have smaller gaps, it would seem, at least naively, that all fractions with the same denominator would be equally strong, though it is possible that those near a stronger, smaller denominator state might be overwhelmed, for example 3/11 near 1/3. As samples improved, it could be seen that in fact most fractions observed were of the restricted form $\nu = p/(mp+1)$, $p \neq 0$, $m \geq 0$ even, $= 2, 4$. It was not clear if the hierarchy could explain this, although it could be simply a quantitative fact, without deeper explanation. Furthermore, in the region near $\nu = 1/2$, where $m = 2$, p is large in the above formula, no fractional plateaus were seen, but there was a shallow minimum in ρ_{xx} that remained $\neq 0$ as $T \to 0$. A possibly more profound theoretical question was, what is the nature of the ground state at fillings not of the hierarchy form p/q, q odd? Indeed, the construction of states at $\nu = p/q$, q odd that exhibit FQHE does not prove that FQHE cannot occur at even q. The whole question of what *does* occur received almost no attention in published work until the last few years (a notable exception is the FQHE at $\nu = 5/2$ and the Haldane-Rezayi proposal for its explanation [8, 9]). One possibility is that there is no well defined state at these fillings in the thermodynamic limit, that phase separation into domains of two nearby, stable FQHE states occurs instead. Another possibility is a well-defined pure phase constructed by taking a FQHE state at a nearby filling, and adding a low density of, say, quasiholes which at low density could form a Wigner crystal. Such a state could exist (with varying lattice constant) over a range of densities. These possibilities may actually occur near some particularly stable (large Δ) fractions such as 1/3, but seem less likely near $\nu = 1/2$ since neighbouring states would have very small energy gaps. If liquid states at non-FQHE filling factors exist, then they lay outside the then-current theoretical understanding.

With these motivating remarks, I now abandon the historical approach in order to develop the overall theoretical picture [10, 11, 12] as logically as possible. I will describe two alternative approaches. The first begins close to Laughlin's original ideas, and contains explicit trial wavefunctions, as well as other notions of the last few years. The other approach is field theoretical and eventually arrives at the same conclusions, but may seem less physical at first. However, it is much more appealing for explicit analytical calculation. I will also explain the relation with Jain's work on the hierarchy states.

Let us turn, then, to the first of these approaches. We make the usual assumption that, at $T = 0$, interaction energies $\sim \nu^{1/2}e^2/\epsilon\ell$ are weak compared with the Landau level splitting $\hbar\omega_c$, and so the electrons should all be in the lowest Landau level (LLL) with spins polarized when $\nu < 1$. We work in the plane with complex coordinate $z_j = x_j + iy_j$ for the jth electron. In the symmetric gauge [2], single particle wavefunctions in the LLL are $z^m \exp\{-\frac{1}{4}|z|^2\}$, where we set $\hbar = 1$, and the magnetic length $\ell = \sqrt{\hbar c/eB} = 1$. m is the angular momentum of the state. An N particle wavefunction has the form

$$\Psi(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N) = f(z_1, \ldots, z_N)e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{1}$$

where f is complex analytic and totally antisymmetric. If f is homogeneous of degree M in each z_i, it has definite total angular momentum and describes a (not necessarily uniform) droplet of radius $\sim \sqrt{2M}$. As a function of each z_i, f has M zeroes which is also the number of flux quanta N_ϕ enclosed by the circle of radius $\sqrt{2M}$. (Similarly, on a closed surface such as the sphere, the number of zeroes of the LLL single particle wavefunctions equals the number of flux quanta through the surface.) Thus the filling factor $\nu = N/N_\phi$ = the number of particles per flux quantum.

Now if f contains a factor

$$U(z) = \prod_i (z_i - z) \tag{2}$$

then *one* zero for each particle is located at z. ($U(z)$ can be viewed as an operator, Laughlin's quasihole operator, acting on the remainder of the wavefunction by multiplication to produce f.) This means that there are no particles in the immediate vicinity of z, there is a depletion of charge there. We describe this as a *vortex* at z, since as in a vortex in a superfluid, the phase of the wavefunction winds by 2π as any z_i makes a circuit around z.

Given a state, we can obtain a valid new state by multiplying by $U(z)$. (This increases M by 1, if $z = 0$, but otherwise by an indefinite amount. But it always increases N_ϕ by 1, if N_ϕ is defined as the flux through the region occupied by the fluid.) For reasonably uniform fluid states (such as Laughlin's, or the Fermi liquid below) the vortex can be considered as an excitation of the fluid. Now add in addition an electron. Clearly it is attracted to the centre of the vortex, due to the density deficiency there. Since there is no kinetic energy, it can certainly form a bound state. Similarly it can bind to multiple vortices $U(z)^q$.

It is natural to consider the possibility that the ground state itself contains electrons bound to vortices, since this will give a low energy. As each vortex is added, N_ϕ increases by 1. If we add 1 electron for each q vortices, we can form identical bound states, and if all zeroes are introduced in this way, we will have $N_\phi = q(N-1)$, which implies $\nu = N/N_\phi \to 1/q$ as $N \to \infty$. This case is by far the simplest to understand. In this state, there will be q -fold zeroes as one electron approaches another. It has long been realized that Laughlin's Jastrow-like ansatz

$$f_{\mathrm{L}} = \prod_{i<j} (z_i - z_j)^q \tag{3}$$

can be called a "binding of zeroes to particles".

However, the astute reader will already be aware that f_L is antisymmetric only for q odd. To solve this problem, and obtain a deeper insight into Laughlin's state, we must examine the nature of the bound objects more closely. Since we know the properties of an electron, we turn to the vortices. First, the q-fold vortex carries a charge $-\nu q$ in a fluid state at arbitrary filling factor ν (this includes our compressible Fermi liquid state below, as well as the usual hierarchy states). This can be established by Laughlin's plasma analogy [2] or indirectly through adiabatic motion of the vortex [5]; the latter will be more useful here. A crude version of this argument is that when a vortex is moved around adiabatically in some given fluid state, the wavefunction picks up a phase, since the definition (2) of $U(z)$ shows that it changes phase by 2π for each particle about which it makes a circuit. Then if it makes a circuit around a (nonselfintersecting) closed loop enclosing a region D of area A, it will pick up a phase $2\pi \int_D d^2z\, \rho(z)$ which reduces to νA if the density $\rho(z)$ is uniform $\rho = \nu/2\pi$. This is then identified as the same result as would be obtained for a particle of charge $-\nu$ in the magnetic field seen by the electrons. Similarly, a q-fold vortex has a charge deficiency of νq, which for $\nu = 1/q$ is equivalent to a real hole, so the electron–q-vortex composite at this filling has net charge zero, and behaves like a particle in *zero* magnetic field. But note that the vortex is actually sensitive to the density of electrons, which can vary in space and time, even when the external magnetic field and the average filling factor are fixed.

Now for the famous fractional statistics of vortices. If a vortex makes a circuit around a loop enclosing another vortex, with the density otherwise uniform, the missing charge around the vortex core will make a difference of $2\pi\nu$ to the phase (independent of the size and shape of the loop, as long as the vortices remain far enough apart). But a circuit is equivalent to two exchanges, up to translations. So a similar calculation gives that adiabatic exchange of two vortices produces a phase $\theta = \pi\nu$. For two q-fold vortices, the result is $\pi q^2 \nu$ which at $\nu = 1/q$ is πq. This shows that at these fillings, q-fold vortices are fermions for q odd, and bosons for q even. Hence the electron–q-vortex bound state is a boson for q odd, and a fermion for q even. For q odd, we can now argue that the bosons in zero magnetic field can Bose condense (at $T = 0$) into the zero-momentum state and that this is the interpretation of the Laughlin state [13]. If $\psi^\dagger$ creates an electron in the LLL, then $\psi^\dagger(z)U^q(z)$ creates a boson, and the condensate is obtained by letting $\int d^2z\, \psi^\dagger(z)U^q(z)\exp\{-\frac{1}{4}|z|^2\}$ act repeatedly on the vacuum. This produces exactly the Laughlin state [13]. Note that condensation arises because particles try to minimise their kinetic energy $\sim k^2$. We need to show that the bound objects do in fact have such an *effective* "kinetic" energy. The true kinetic energy of the electrons has been quenched, so this term in the effective Hamiltonian for the bound objects can only arise from electron-electron interactions. Before discussing how this arises, let us pursue the consequences. At $\nu = 1/q$, q even, the bound objects are fermions in zero net field, which cannot Bose condense, but can form a Fermi sea. This is then the proposal for a compressible state at these fillings. It will be compressible because it can be shown that incompressibility, and the quantized Hall effect, in the case of bosonic bound states is a consequence of Bose condensation (superfluidity), which does not occur in a Fermi sea unless BCS pairing occurs, in which case the state becomes a QHE state. The Haldane-Rezayi and Pfaffian states arise in this way [14].

I now return to the dynamics of the bound objects, or "quasiparticles". We have argued that there is an attraction between an electron and a q-fold vortex. In a uniform background, this will take the form of a central potential. The minimum is at the centre, meaning the electron is exactly at the zeroes of the many-particle wavefunction. Now at filling factor $1/q$, we need to consider quasiparticles in plane wave states with wavevector $\mathbf{k}$. These can be created by acting on a fluid with $\int d^2z\, e^{i\mathbf{k}\cdot\mathbf{r}}\psi^\dagger(z)U^q(z)\exp\{-\frac{1}{4}|z|^2\}$. In the wavefunction, this means a factor $e^{i\mathbf{k}\cdot\mathbf{r}_j}$ in the term where the jth electron is the bound in the state with wavevector $\mathbf{k}$. Now it turns out that this factor, acting by multiplication on some given state, displaces the jth particle by ik, where $k = k_x + ik_y$ (recall that the magnetic length ℓ is 1). This arises because in the Hilbert space of many particle states of the form (1), $\bar{z}_j$ acts on f as $2\partial/\partial z_j$ which generates displacements [15]. A quasiparticle with wavevector zero would have the electron exactly at the zeroes of the wavefunction. So a quasiparticle with wavevector $\mathbf{k}$ has the electron displaced by $|k|$ from the centre of the vortex. The electron and vortex experience a potential $V(|k|)$ due to the Coulomb repulsion of the electron by the other electrons, which are excluded from the vortex core. A good understanding is now achieved semiclassically. The electron will drift along an equipotential of $V(|k|)$. From the preceding discussion, at $\nu = 1/q$ the q-fold vortex experiences a magnetic field of the same strength as the electron, and so it will also drift with the same speed but in the opposite sense relative to the gradient of the potential, due to the opposite sign of its effective charge. This means that both components of the pair drift in the same direction, perpendicular to the vector connecting their centres, so that their separation ($= |k|$) remains constant. The picture is like that of oppositely charged particles in a magnetic field, which can drift in a straight line as a pair. The energy of our pair is $V(|k|)$ and the velocity is $\propto \partial V/\partial|k|$ as it should be. Near the bottom of the potential, it will be quadratic, and we can obtain an effective mass $\sim (\partial^2 V/\partial|k|^2)^{-1}$ due to the interactions (a similar calculation was performed in [13]). This shows clearly that the interactions favour condensation of the quasiparticles to minimize this effective kinetic energy. In the q even case, the quasiparticles are fermions and must have distinct $\mathbf{k}$'s, filling a Fermi sea. Then in the ground state not all the zeroes of the wavefunction are precisely on the electrons but some are displaced by amounts up to k_F, determined in the usual way by the density, $k_F^2/4\pi = 1/2\pi q$. For q large these displacements $\sim q^{-1/2}$ are small compared with the interparticle spacing $\sim q^{1/2}$. Thus a trial state of this form may not be much worse energetically than the Laughlin state at nearby odd q. This, I believe, is then the essential reason why this idea has a good chance of being the correct many-body ground state: it is due to the good correlations that produce a low Coulomb energy. Of course, if q is very small, this picture might break down, but in fact we have good reason to believe it holds for q as small as 2. This argument also tells us that low-lying excited states are obtained by increasing the wavevector of a quasiparticle in the Fermi sea. For q large, the Fermi velocity will be determined by the same effective mass as near the bottom of the sea; otherwise we must take the derivative at k_F.

To consider collective effects it is necessary to go beyond an independent quasiparticle picture. There are important long range interactions that can be described as gauge fields. We will examine these in the context of the field theoretic approach later. As a test of the above ideas, we can perform numerical diagonalization of small systems at, say, $\nu = 1/2$. This has been done recently in a paper by E. Rezayi and the author [12]. Excellent agreement of trial states suggested by the above ideas is found with the exact wavefunctions of low-lying states. The trial wavefunctions were generated on the sphere, but on the plane would be roughly

$$\Psi = \mathcal{P}_{\rm LLL} \det M \prod_{i<j} (z_i - z_j)^q e^{-\frac{1}{4}\sum |z_i|^2} . \qquad (4)$$

The matrix M has elements that are essentially plane waves $M_{ij} \sim e^{i\mathbf{k}_i \cdot \mathbf{r}_j}$ for the quasiparticles, filling the Fermi sea. $\mathcal{P}_{\rm LLL}$ projects all electrons to the LLL. As explained above, the plane wave factors then act as operators within the LLL, on the Jastrow factor $\prod (z_i - z_j)^q$ which if not modified would be the Laughin state for bosons at $\nu = 1/q$. The Slater determinant makes the state totally antisymmetric. The simple product form is similar to

Jain's states off half-filling [11]; it differs a little, but inessentially, from the present author's first idea (in 1987) of building the Fermi sea by acting with Fourier components of $\psi^\dagger(z)U^q(z)\exp\{-\frac{1}{4}|z|^2\}$, which was suggested by the analogy with the Laughlin states at q odd [13]. The numerical calculations also showed that the two-particle correlation function $g(r)$ in the ground state possesses (i) a large "correlation hole" at short distances r, consistent with the argument that zeroes are bound close to the electrons, and (ii) oscillations at large r, perhaps of the form $r^{-\alpha}\sin 2k_F r$ asymptotically, as in a two-dimensional Fermi gas, which has $\alpha = 3$.

We now turn to the field theoretic approach. It begins with the observation that in 2 dimensions, particles of any (fractional) statistics can be represented by charged particles of other statistics attached to δ-function flux tubes of a certain size [16]. A charged particle dragged adiabatically around a flux tube (or *vice versa*) picks up an Aharonov-Bohm phase of 2π times the product of charge of the particle and the number of flux quanta in the tube. Thus if identical particles are attached to identical flux tubes, a circuit of one composite around the other produces a phase 4π times the charge times the flux, since each particle sees the other flux. For an exchange, we get only half this. In addition, the exchange of identical particles produces a phase $e^{i\theta}$ due to the statistics of the particles themselves. Thus fermions or bosons attached to fractional flux tubes can be used to model anyons, where the total phase obtained in an exchange is fractional [16]. In the fractional quantum Hall effect we change our terminology slightly because we view flux tubes as operators that act only on particles already present. Thus a composite is introduced by adding first a flux tube, then a particle is added at the same point. The phase produced by an exchange is then only π times the charge times the flux in each composite, plus the phase due to exchanging the particles themselves. Thus, for example, we can say that attaching two flux quanta to each boson in a system leaves the composites still as bosons, or doing the same to fermions leaves them fermions. Then in the high field situation of interest here, we may represent electrons as q (integer) flux tubes attached to some other particles. The latter must be bosons if q is odd, and fermions if q is even, in order to reproduce the fermi statistics of the electrons. The flux tubes can be represented by a "fictitious" vector potential $\mathbf{a}$, not to be confused with the physical vector poetential $\mathbf{A}$ representing the constant external field, which obeys

$$\nabla \times \mathbf{a} = 2\pi q \rho \tag{5}$$

and the density $\rho(\mathbf{r}) = \sum \delta^{(2)}(\mathbf{r}_i - \mathbf{r})$.

The next step, used in several similar problems [17, 18, 11, 10], is a mean field approximation. The new fermions (for q even) see the constant background magnetic field, and the q flux tubes attached to the other fermions. If the quantum mechanical state has a uniform density (in the quantum average), the latter becomes a constant field whose sign we can choose to be opposite to the external field. In particular, if $\nu = 1/q$, the fields cancel exactly. The fermions now see no net field, so they can form a Fermi sea, which does have a uniform density, as we assumed. We have arrived at the physical picture of the compressible Fermi liquid-like state, where we can say loosely that the fermions are electrons plus q flux quanta. But notice that this should not be taken too literally, since we have actually just made a transformation. The real flux due to the external field remains uniform, not bunched up into flux tubes attached to the electrons. It is really vorticity that is bound to the electrons, as discussed above. In mean field theory the wavefunction for the electrons is simply

$$\Psi_{\mathrm{MF}} = \det M \prod_{i<j} (z_i - z_j)^q / |z_i - z_j|^q \tag{6}$$

with M as before and no projection. This function is *not* all in the LLL. We can see that the effect of the transformation and the mean field approximation is to build the right kind of phase factors into the wavefunction, the same as possessed by q vortices on each electron. The factor $\prod |z_i - z_j|^q$ needed to recover eq. (4) can in fact be obtained from fluctuations about mean field, at least in a long-wavelength sense [19].

Excited states again involve creation of particle-hole pairs. In mean field approximation, the effective mass of fermion excitations near the Fermi surface is simply the bare electron mass m, since that is what appears in the Hamiltonian (we do not attempt to impose the LLL constraint). Consideration of collective oscillations of the system leads to the correct cyclotron frequency $\omega_c = eB/mc$ which according to Kohn's theorem cannot be renormalized. However, under the conditions stated at the beginning of this article, the low-energy fermi excitations should have an effective mass m^* determined by inter-electron interactions only. Part of the resolution of this problem is that if fluctuations renormalize the effective mass, as they usually do in Fermi liquids, then there will also be a Landau interaction paramater F_1 that obeys a relation $m^{-1} = m^{*-1} + F_1$ which guarantees that the collective mode frequency is unrenormalized (like the plasma frequency which is unrenormalized in the usual electron gas at zero magnetic field) [10]. A more serious problem is that interactions seem to play no role in the mean field theory. The prediction would be the same, even for noninteracting electrons. The Fermi liquid mean field state has finite compressibility, due to the mass m, even though in this case a partially filled Landau level should have infinite compressibility! (The same holds when this approach is applied to the FQHE states for q odd, where the compressibility vanishes.) These problems can be resolved by understanding in what limit the approach is valid. Mean field theory is good, and fluctuations in the effective magnetic field $\nabla \times (\mathbf{A} + \mathbf{a})$ are small, when q is small. But for us q is an even integer ≥ 2. However if we replace the original problem of electrons in a magnetic field by that of anyons in a magnetic field, then we can make q small. We choose to study anyons with statistics $\theta = \pi(1+q)$ (mod

2π) and filling factor $1/q$, for $q > 0$. As $q \to 0$, the magnetic field goes to zero, but for all q we can still map the problem of anyons to one of fermions in zero net average magnetic field. So at $q = 0$ we reach fermions at zero magnetic field and we attach zero flux to convert them to fermions! At this point there can be no fluctuations in the effective magnetic field, since there is no attached flux. As we increase the external field, we must attach more flux to map to fermions in zero field, so the statistics must change accordingly. When θ has made a full circle from π back to π, we recover fermions (electrons) but now at a high field and filling factor $1/q$, $q = 2$. We can continue and reach other even denominator states for electrons. This idea is an adaptation of that in [20], except that they use it to argue that certain states exist by adiabatic continuation, while I use it only for a perturbation expansion in the fluctations, *i.e.* in powers of q. Notice that in the limit $q = 0$, the system is a Fermi liquid, and the effective mass may be close to the bare mass if interactions are weak compared to the Fermi energy. A similar statement applies to the anyons for q small. It is only at $q = 2$ that we can argue that the compressibility must be infinite when interactions are set to zero, because only for fermions (such as the electrons) or charged bosons, ***and not for anyons***, can we argue that the many-particle states of a noninteracting system are (anti-)symmetrized products of Landau level states.

After that rather technical paragraph, we return to simpler discussion. If the filling factor for electrons deviates from $\nu = 1/q$, q even, then in either approach above the quasiparticles will see a nonzero net magnetic field $B_{\rm eff} = B - B_{1/2} = \nabla \times (\mathbf{A} + \mathbf{a})$. If the quasiparticles fill an integer number p of Landau levels due to this effective field, there will be a gap in the excitation spectrum (at least in mean field theory, but in fact it survives fluctuations) and we will obtain an integer QHE for the fermions, which is an FQHE for the electrons, with $\sigma = (e^2/h)p/(qp + 1)$. This of course is Jain's picture [11] of the "main sequence" of fractional QHE states as observed in experiments. Jain's approach was a hybrid of the approaches above. He begins with the transformation involving δ-function fluxes, but then simply modifies the mean field wavefunctions (6) into the form (4) with M_{ij} = Landau level wavefunctions. The argument of binding zeroes to particles to obtain a low Coulomb energy is not used. We believe the $\nu = 1/q$ case with exactly q vortices per particle gives valuable insight into the reasons for the stability of the states, even away from this filling. Also the notion of the effective mass emerges clearly in the Fermi liquid state. We predicted [10] that energy gaps in FQHE states near half-filling should scale as $1/p$ as $p \to \infty$, since the gap can be interpreted as an effective cyclotron energy $\sim B_{\rm eff}/m^*$ once we recognize the existence of the Fermi liquid, with a well-defined effective mass that controls excitation energies, as the limiting behaviour. This prediction has received experimental support [21].

The above arguments about dynamics of the quasiparticles at $\nu = 1/2$ also generalizes to this case. While the electron and the q-fold vortex still see the same potential $V(|k|)$, the drift velocities are different because that for the vortices is fixed by the electron density, not the true magnetic field. The separation still remains constant, so like the back wheels of a car turning a corner, the bound pair moves (semiclassically) on two concentric circles separated by $|k|$, giving an effective cyclotron radius for the quasiparticles. This radius is given by the usual formula $\Phi_0 k/2\pi B_{\rm eff}$ for a particle of charge 1 in a magnetic field $B_{\rm eff}$.

The reader is cautioned that away from half-filling the quasiparticles which seem to be fermions in the mean field approach in fact have their statistics modified because of screening effects of the fluctuations. They also acquire fractional charge, whereas at half filling there is perfect screening by the response of the fluid to the δ-function of flux on the fermion. This effect is in fact described by the extra amplitude factors present in the wavefunctions (4). The arguments given above for electron-q-vortex bound states at $\nu = 1/q$ extend to other fillings to predict a net charge $1 - q\nu$ and statistics $\theta = \pi(q^2\nu - 1)$. These properties of fermions excited to higher "quasiparticle Landau levels" at Jain's fractions are identical to those of the quasiparticle excitations in the hierarchy scheme, which is one of the main evidences for the equivalence of these two approaches [6, 7].

Space allows only a brief further discussion. An outstanding series of experiments has been performed by Willett and coworkers [22]. Their technique, using the propagation of surface acoustic waves across the surface of the device, probes the longitudinal conductivity of the 2 dimensional electron gas at finite wavevectors and frequencies (this and other experiments [23] will be discussed at this meeting by Dr. Stormer). Since the velocity of the waves is slow compared with the Fermi velocity of our fluid, we can consider zero frequency, finite wave vector. The theory, treating the response of the fermionic quasiparticles in an RPA-like approximation, shows that $\sigma_{xx}(k)$ increases linearly with wavevector k in the compressible state, as observed. This results from the *transverse* conductivity $\sim 1/k$ for a conventional Fermi liquid, on including the long-range effects of the gauge field $\mathbf{a}$. (Not only can $B_{\text{eff}} = 2\pi q(n - (2\pi q)^{-1}))$ fluctuate in space and time, so can $E_{\text{eff}} = 2\pi q J$, the effective electric field due to the current J in the perpendicular direction.) It also predicted resonances in the response when the wavevector is an integer multiple of the inverse cyclotron radius for the semiclassical motion of the quasiparticles close to but just off half-filling. This provides a measure of the Fermi wavevector. The recent experimental observation of these resonances confirms the (nontrivial!) existence of a Fermi surface for the charge-carrying excitations and excellent agreement of k_F with the expected value ($\sqrt{2}$ times that in zero magnetic field, because of spin) is obtained.

Open questions: a direct measure of the effective mass m^* at half-filling would be most welcome. Theoretically, a controversy remains about the possible partial breakdown of Fermi liquid theory, including behaviour of m^*, due to the fluctuations of **a** [10].

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Fermi liquid-like state in a half filled Landau level

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A system of electrons in a half-filled Landau level is investigated numerically in spherical geometry. For systems of size from $N = 1$ to 14 electrons with flux number $N_\phi = 2(N-1)$ the angular momentum of the ground state is as predicted by Hund's second rule for composite fermions of one electron and two vortices at *zero* magnetic field, as in a recent theory. Low lying excitations also fit this interpretation and trial wavefunctions give excellent overlaps with the low-lying exact states. The two-particle correlation function shows a significant correlation hole at short distances and suggests an asymptotically oscillating form at long distances, as in a Fermi liquid.

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The fractional quantum Hall effect [1] results from a strongly correlated incompressible fluid state [2] formed at special densities n of a two-dimensional electron layer subject to a perpendicular magnetic field B. For fully spin-polarized electrons the most dominant series occurs at filling factors $\nu \equiv n\Phi_0/B$ (Φ_0 is the flux quantum hc/e) of the form $\nu = p/(2p+1)$, where p is an integer, $\neq 0$. In contrast to these odd denominator fillings, the nature of the ground state at even denominators has long been an intriguing unsolved problem. In particular, $\nu = 1/2$ is the accumulation point of the sequence above and this state has been a subject of considerable recent interest.

Although the striking features (quantized $\sigma_{xy} = \nu e^2/h$ and vanishing σ_{xx}) seen in conductivity measurements at quantum Hall states are absent at $1/2$ filling, it does show a broad minimum [3] in ρ_{xx} and exhibits, additionally, anomalous behavior in surface acoustic wave propagation [4], indicating an entirely different type of correlations. Although it has been known for some time from numerical work [5] that $\nu = 1/2$ is compressible the exact nature of this state was unknown until now.

Recently a theory of a compressible Fermi liquid-like state at fillings $\nu = 1/q$ or $1 - 1/q$, q even, was proposed by Halperin, Lee and one of the present authors [6] (HLR). The approach was a transformation that represents each electron as a fermion attached to a δ-function flux of size $q\Phi_0$ (with q even). The attached flux can be represented as a coupling of the fermions to a gauge field whose action is the (abelian) Chern-Simons term. In a mean field approximation where the Chern-Simons gauge field is replaced by its spatial average (on the assumption that the fermions form a state of uniform density) the fermions see a net magnetic field of zero, if the filling factor for the electrons is $1/q$ and the sign of the attached flux is chosen appropriately. The fermions may then form a Fermi sea, which is a compressible state. If the filling factor differs from $1/q$, the fermions see a net field, and at filling factors $\nu = p/(qp+1)$ they may fill $|p|$ Landau levels, which is Jain's construction [7] of the incompressible quantized Hall states. The role of fluctuations in the gauge field in the compressible state has been discussed in HLR, and may lead to behaviour similar to a "marginal Fermi liquid" or to a "Luttinger liquid" [8], but we need not concern ourselves with these fine distinctions from a conventional Fermi liquid here; the essential properties of the proposed state are in any case that it is compressible and has a Fermi surface visible in its excitation spectrum.

In this paper we perform finite-size calculations for N spin-polarized electrons confined to the lowest Landau level on a spherical surface [9], and compare the numerically obtained states for Coulomb interactions with analytic forms for trial states based on the HLR picture. To aid in interpretation we therefore now give a description of the theory on a sphere.

The electrons experience a spherically symmetric magnetic field of a fixed strength B, the total flux being $N_\phi \Phi_0$, $N_\phi > 0$ integer. The sphere therefore has radius $R = \ell\sqrt{N_\phi/2}$ where the magnetic length $\ell = \sqrt{\hbar c/eB}$ ($\hbar = 1$ hereafter). The single electron wavefunctions for the lowest Landau level are monopole harmonics of angular momentum $S = N_\phi/2$ [9]. The transformed fermions (called quasiparticles hereafter) experience in addition $-q(N-1)$ flux quanta due to each other. The total flux vanishes if

$$N_\phi = q(N-1) \tag{1}$$

which is therefore the number of flux required for the HLR state. In the thermodynamic limit $N \to \infty$, we obtain $\nu = N/N_\phi = 1/q$. q must be even and we consider $q = 2$ from now on.

Now consider the mean field theory for the quasiparticles. If we approximate the effective statistical gauge field as uniform (we discuss below when this will be correct), then the net magnetic field is zero, so the single quasiparticle wavefunctions are simply spherical harmonics, of angular momentum $L = 0, 1, 2, 3, \ldots$, denoted s, p, d, f, ... Assuming that they obtain an effective kinetic energy due to electron-electron interactions proportional to, say, $\mathbf{L}^2$, and neglect residual interactions between them, then they will simply fill the lowest angular momentum shells. For $N = n^2$, $n = 1, 2, \ldots$, they will completely fill n angular momentum shells, the highest having angular momentum $L_F = n - 1$, and the total angular momentum will vanish. For other values of N, a shell will be partially filled and we expect a nonzero angular momentum in the ground state; in these cases residual interactions between quasiparticles will play a role. Also in these cases, the density in the system will not be uniform and strictly we should take this into account in finding the mean field state, but in the best tradition of the shell model in atomic and nuclear physics, we will not do this in our zeroth approximation. We will also discuss low-lying excited states as quasiparticle-quasihole pairs in the same picture of quasiparticles in zero magnetic field with weak residual interactions.

We now turn to a systematic finite-size study of systems at $N_\phi = 2(N-1)$, $N \leq 14$ and begin by discussing ground state quantum numbers. Fig. 1 shows the total orbital angular momentum of the ground states obtained by exact diagonalization for the Coulomb interaction, (*i.e.* inverse chord distance on the sphere) (filled symbols) and expected results for the next 2 sizes (open symbols). There are two interesting features here. First, the uniform, $L_{tot} = 0$, states occur for $N = 1, 4, 9, \ldots$ electrons as expected from the mean field theory picture above. Second, the angular momentum in other cases is the maximum value that can be obtained by combining the angular momenta of the quasiparticles in the partially filled shell using Fermi statistics. In other words they obey the second of Hund's rules familiar from atomic physics. Large total angular momentum just as in the case of atoms means the quasiparticles avoid one another as much as possible and thus optimize their repulsive residual-interaction energy.

The low-lying spectra for $N \neq n^2$ reflect the partially filled shell level structure. Fig. 2 shows the spectra for 7–13 particles. For example for 8 (or 10) particles there is a single quasihole (respectively, quasiparticle) in the d (f) shell thus we should obtain a single multiplet with $L_{tot} = 2$ (3). On the other hand, for 11 (7) electrons we have two quasiparticles (quasiholes) in the f (d) shell with $L_{tot} = 5, 3, 1$ (3, 1) which are the only allowed values for a pair of fermions. Finally, for 12 and 13 (which are particle-hole conjugates within the f shell, so they have the same count of low lying states) the Hilbert space is $L_{tot} = 6, 4, 3, 2, 0$. These low-lying states are clearly seen in the spectra. The ordering of these energy levels also follows the trend expected from Hund's second rule, with a slight exception at $L_{tot} = 0, 2$ for $N = 13$ (see Fig. 2).

For the 9 electron system shown in Fig. 3, the s, p, d shells are full. The low-lying excited states form a series of well separated bands. The lowest band would be expected to correspond to the lowest effective "kinetic" energy single particle-hole excitation: a particle in the f shell and a hole in the d shell. The expected values of L_{tot} would be $L_{tot} = 1, 2, 3, 4, 5$; however $L_{tot} = 1$ is missing from this band in Fig. 3. A similar phenomenon has been observed [10] for the incompressible states. We shall shortly explain this using explicit variational states in the lowest Landau level. In the noninteracting quasiparticle model, the next band would contain higher energy single quasiparticle-quasihole pairs and a set of two quasiparticle-quasihole pair states, namely two particles in f and two holes in d. The latter produce the highest L_{tot}, nondegenerate $L_{tot} = 7, 8$ multiplets, as observed in this band in Fig. 3. However the identification of states at lower L_{tot} is less unambiguous, and configuration mixing may be important here.

To go a step further, we compare with trial wavefunctions constructed as follows. Consider states of the form

$$\Psi = \mathcal{P}_{\rm LLL} \det M \prod_{i<j} (u_i v_j - v_i u_j)^2 \tag{2}$$

where $(u_i, v_i) = (\cos\theta_i/2 \exp i\phi_i/2,\ \sin\theta_i/2 \exp -i\phi_i/2)$ are the spinor coordinates on the sphere [9] corresponding to spherical polars (θ_i, ϕ_i) for the ith particle, the matrix M has elements $M_{ij} = Y_{L_i}^{M_i}(\theta_j, \phi_j)$ and $\mathcal{P}_{\rm LLL}$ projects all electrons to the lowest Landau level (LLL). Note that the Jastrow factor $\prod_{i<j}(u_i v_j - v_i u_j)^2$ is totally symmetric and alone would describe the Laughlin state for *bosons* at $\nu = 1/2$, which in the plane geometry [2] would become $\prod_{i,j}(z_i - z_j)^2$. The determinant renders the states totally antisymmetric, and the projection ensures they are entirely in the LLL. The projection makes the factors in the determinant act as operators within the LLL. If we choose the L_i, M_i to fill the lowest levels, and if the projection were omitted and the Jastrow factor replaced by $\prod_{i,j}(z_i - z_j)^2/|z_i - z_j|^2$, then the state would be just the singular gauge transformation of a Slater determinant representing a Fermi sea on a sphere, which is the basic mean field *ansatz* for the ground state. The extra amplitude factors and projection that we have included are an attempt to improve the trial state, in particular by giving it better short distance correlations and by removing the higher Landau levels to lower the electron kinetic energy. These improvements, which in the HLR approach would be due to fluctuations, make it similar to Laughlin's state [2] and to Jain's states [7]. Conceptually, it is a Fermi sea of quasiparticles that consist of one electron and 2 vortices, and the construction was originally motivated by the parallel with that in which the Laughlin state is regarded as precisely a Bose condensate (*i.e.* all particles in $\mathbf{L} = 0$ state) of bosonic quasiparticles each containing one electron and q vortices at filling $1/q$, q odd [11]. In general, we regard the set of pairs (L_i, M_i) as the set of quasiparticle angular momenta, in spite of the nonorthogonality of states with distinct sets $\{(L_i, M_i), i = 1, \ldots, N\}$ but the same $L_{\rm tot}$, $M_{\rm tot}$ which is due to the amplitude of the Jastrow factor and to the LLL

projection.

We have already described above the interpretation of the low-lying states at various sizes in terms of quasiparticles occupying different angular momentum orbitals. This specifies a trial state of the form (2) for each. The overlaps of the trial wavefunctions for the ground states at $N = 9$, 8, 7 and for the lowest excited states at several $L_{\rm tot}$ at $N = 9$, 8 with the true Coulomb potential are listed in Table 1, together with the dimensions of the Hilbert space at that N, $L_{\rm tot}$. As can be seen this construction is essentially exact for these states.

We believe that the following facts explain the absence of the $L_{\rm tot} = 1$ low-lying state (a similar argument explains the observations in [10], and also goes through for Bose quasiparticles [11]). First observe that the angular momentum components of the projected density operators $\bar{\rho}(\theta, \phi) = \mathcal{P}_{\rm LLL}\rho(\theta, \phi)\mathcal{P}_{\rm LLL}$, where ρ is the electron density operator, form one multiplet of one electron operators for each angular momentum 0–$2S$; in particular the $L = 0$ part is the total number operator and the $L = 1$ part is the total angular momentum. The latter annihilates the $L_{\rm tot} = 0$ filled shell ground states. On the other hand, these operators act on the trial states (2) by changing the angular momentum of *one quasiparticle*, just as the usual density would on ordinary Slater determinants, as can be seen using a Clebsch-Gordan series for the products of spherical harmonics involved, viewed as LLL operators. In particular the $L_{\rm tot} = 1$ trial state multiplet formed from one quasihole in the topmost filled shell, one quasiparticle in the lowest empty shell would be obtained uniquely from the filled shell trial ground state by acting with the $L = 1$ components of the projected density, *i.e.* the angular momentum, so the trial $L_{\rm tot} = 1$ state vanishes identically. The absence of the $L_{\rm tot} = 1$ state in the numerical spectra is itself evidence that the physics of our approach is correct—the absence of the trial state makes the evolution of an exact eigenstate from it impossible.

Fig. 4 shows the pair correlation functions $g(r)$ in the $N = 9$ Coulomb ground state and for the above model state which are indistinguishable in the figure over the whole range of distance (measured along a great circle). A correlation "hole" is visible at short separation; the part of this due to Fermi statistics alone can be seen by comparison with the $\nu = 1$ result, also plotted; the difference shows the "hole" due to interaction-induced correlations. The result for the Laughlin state for *bosons* at $\nu = 1/2$ is also plotted for comparison. This suggests that the Fermi liquid trial state is a good variational state because the electrons avoid one another fairly well, and we expect that the state is robust enough to produce observable effects up to temperatures of at least a few degrees, as seen in experiments [3, 4]. The difference $g_F - g_B$ oscillates with r and suggests a form $r^{-\alpha} \sin 2k_F r$ (with $\alpha > 0$ some constant) asymptotically for the HLR state, similar to the free Fermi gas, which has $\alpha = 3$.

In summary, the systematic size dependence of the ground state and excited state properties of the system, together with the agreement of trial wavefunctions with the numerically obtained states, provide convincing evidence for the correctness of the HLR theory of a compressible Fermi liquid-like state of electrons in a half-filled Landau level.

Similar results found for $\nu = 1/4$ will be presented elsewhere, along with details of this work.

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Figure Captions

Figure 1. The angular momentum of the Coulomb interaction ground state at $N_\phi = 2(N-1)$ as a function of particle number N. Solid symbols are calculated, open symbols are the prediction for the next few sizes.

Figure 2. The low lying excitation spectra at $N_\phi = 2(N-1)$ for various sizes N near the $N = 9$ filled shell configuration. Energies in all Figures are in units of $e^2/4\pi\epsilon\ell$. The lowest bands, discussed in the text, are emphasized.

Figure 3. The low lying excitation spectrum for $N = 9$ (s, p, d quasiparticle shells are completely filled); energies ΔE are relative to the ground state at $L = 0$. See text for discussion of the low-lying band at $L = 2 - 5$.

Figure 4. The pair correlation function $g(r)$ (labelled "Fermi") as a function of great-circle distance r/ℓ, for both the exact $N = 9$ ground state and the model state of eq. (2) with s, p, d, shells filled; these curves are indistinguishable. For comparison, we have also plotted $g(r)$ for the filled Landau level at the same number of flux (labelled "$\nu = 1$"), for the Laughlin wavefunction for *bosons* at $\nu = 1/2$ (labelled "Bose") and the difference of the Fermi and Bose cases.

Table Caption

Table 1. The overlaps between the exact ground state and the lowest band of excited states and variational states constructed from Fermi-Liquid states for $N = 9$ and 8 (see text), together with the ground state for $N = 7$. Also the Hilbert space dimensions (in parentheses) and angular momentum quantum number.

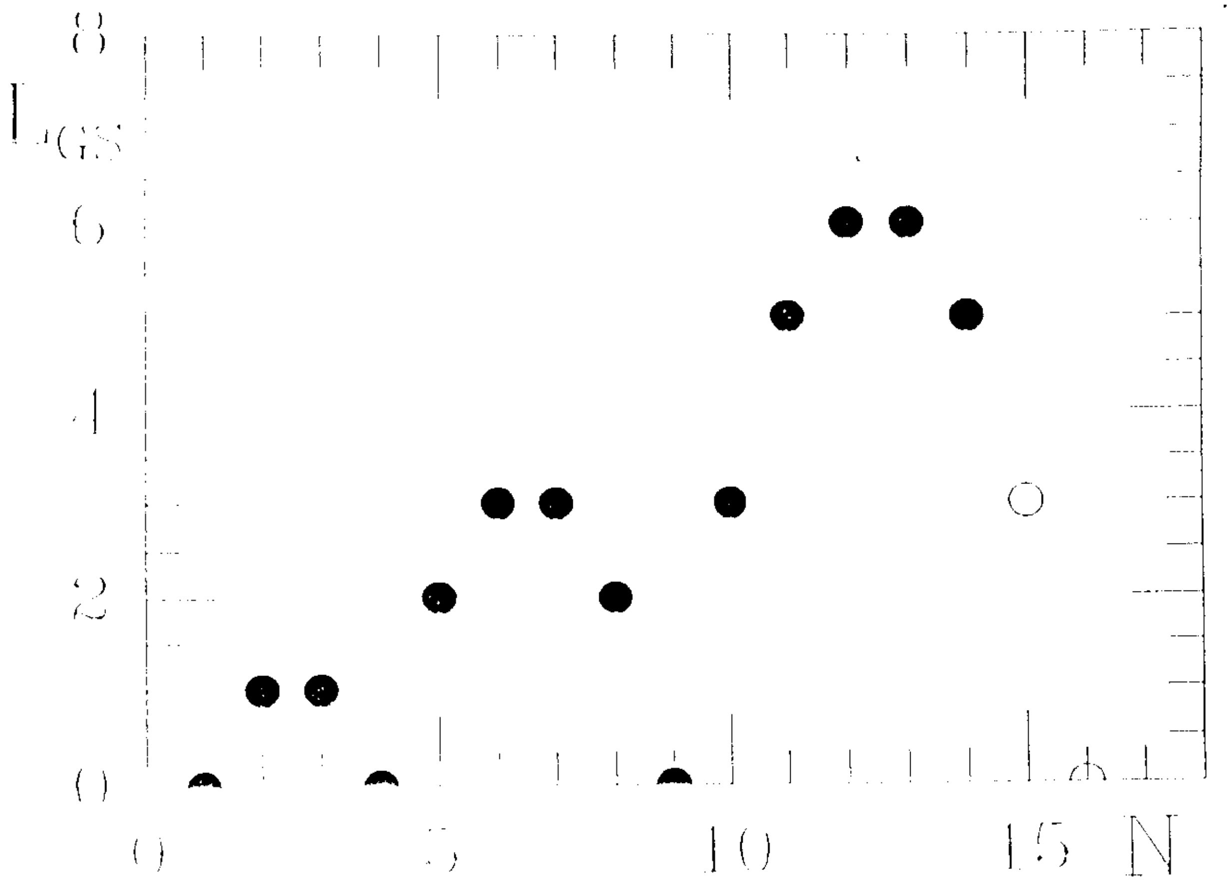

Figure 1

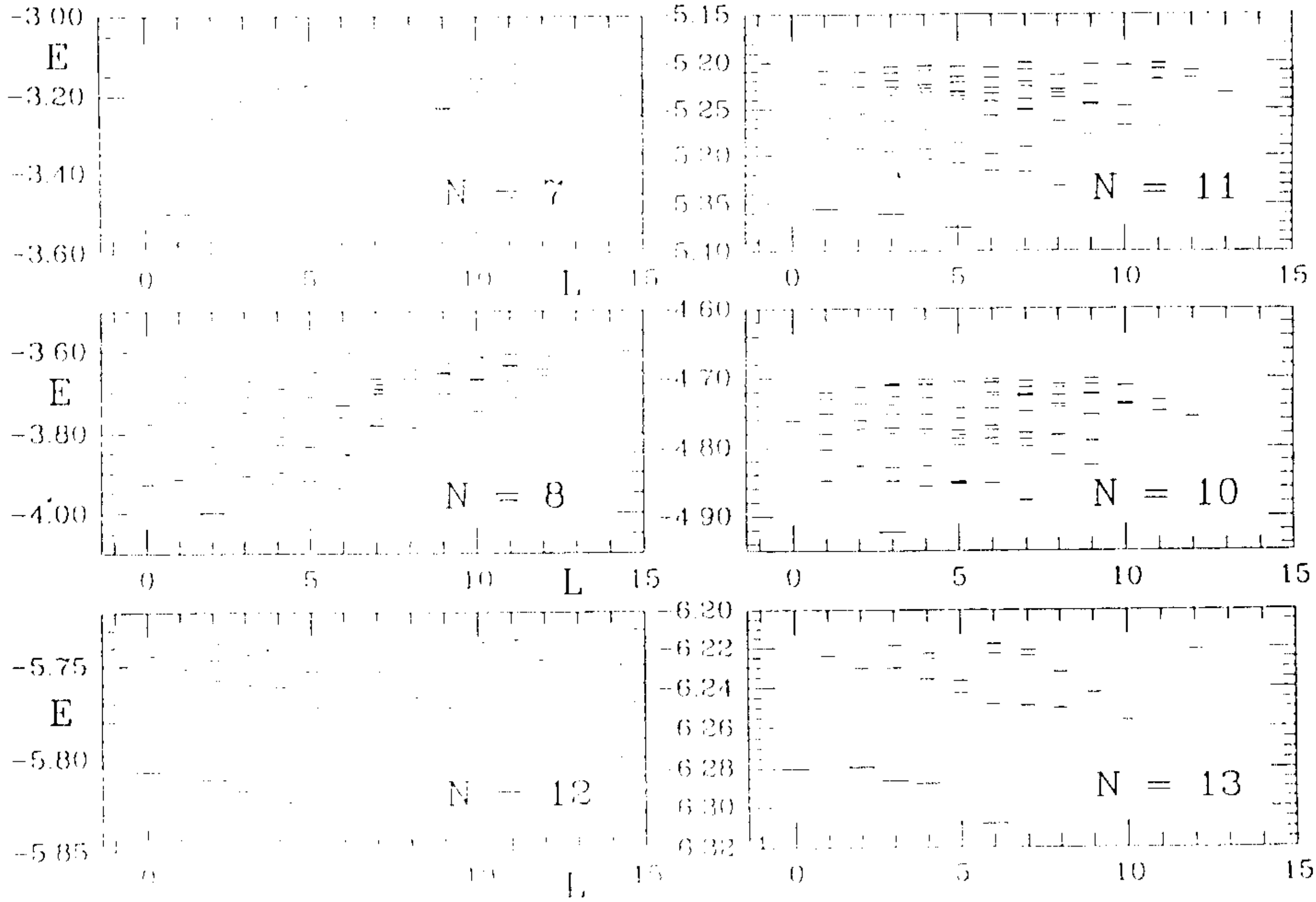

Figure 2

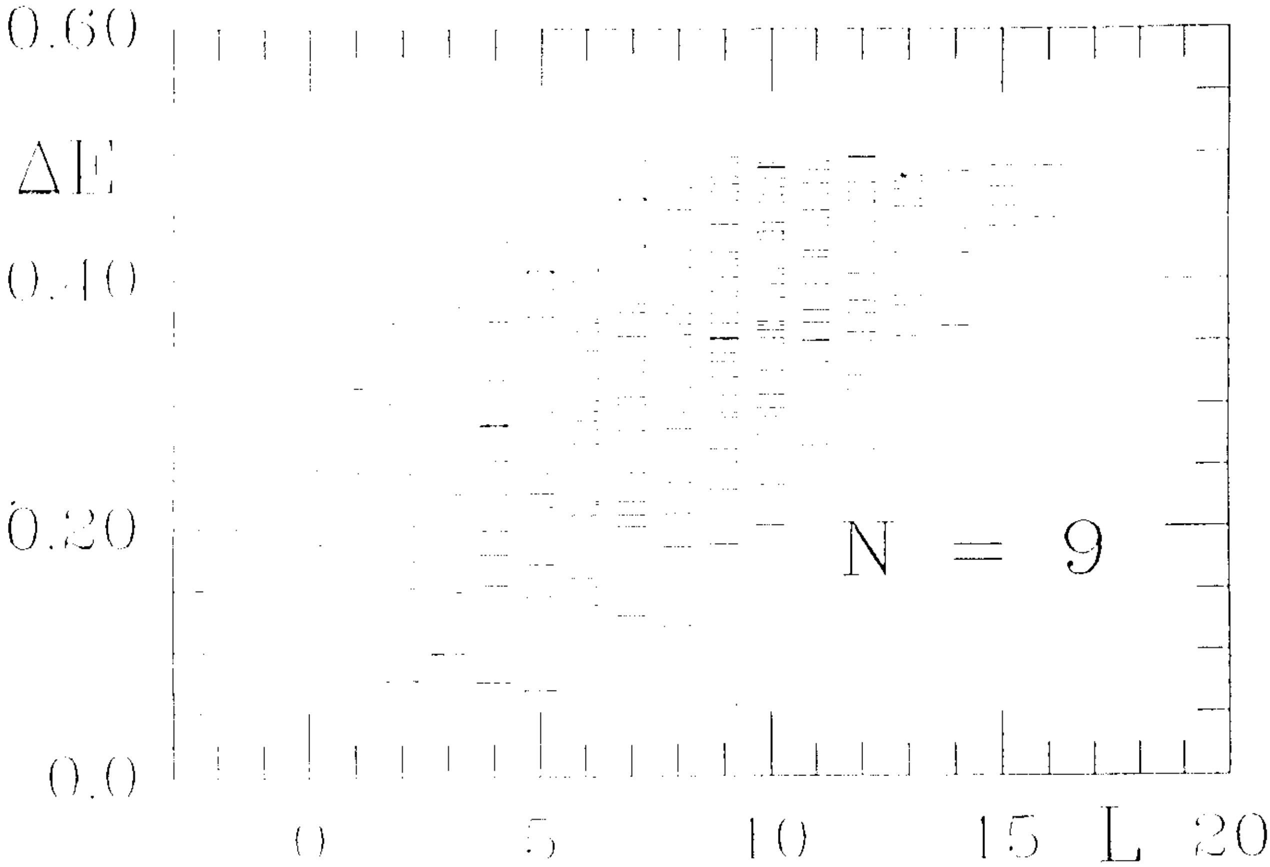

Figure 3

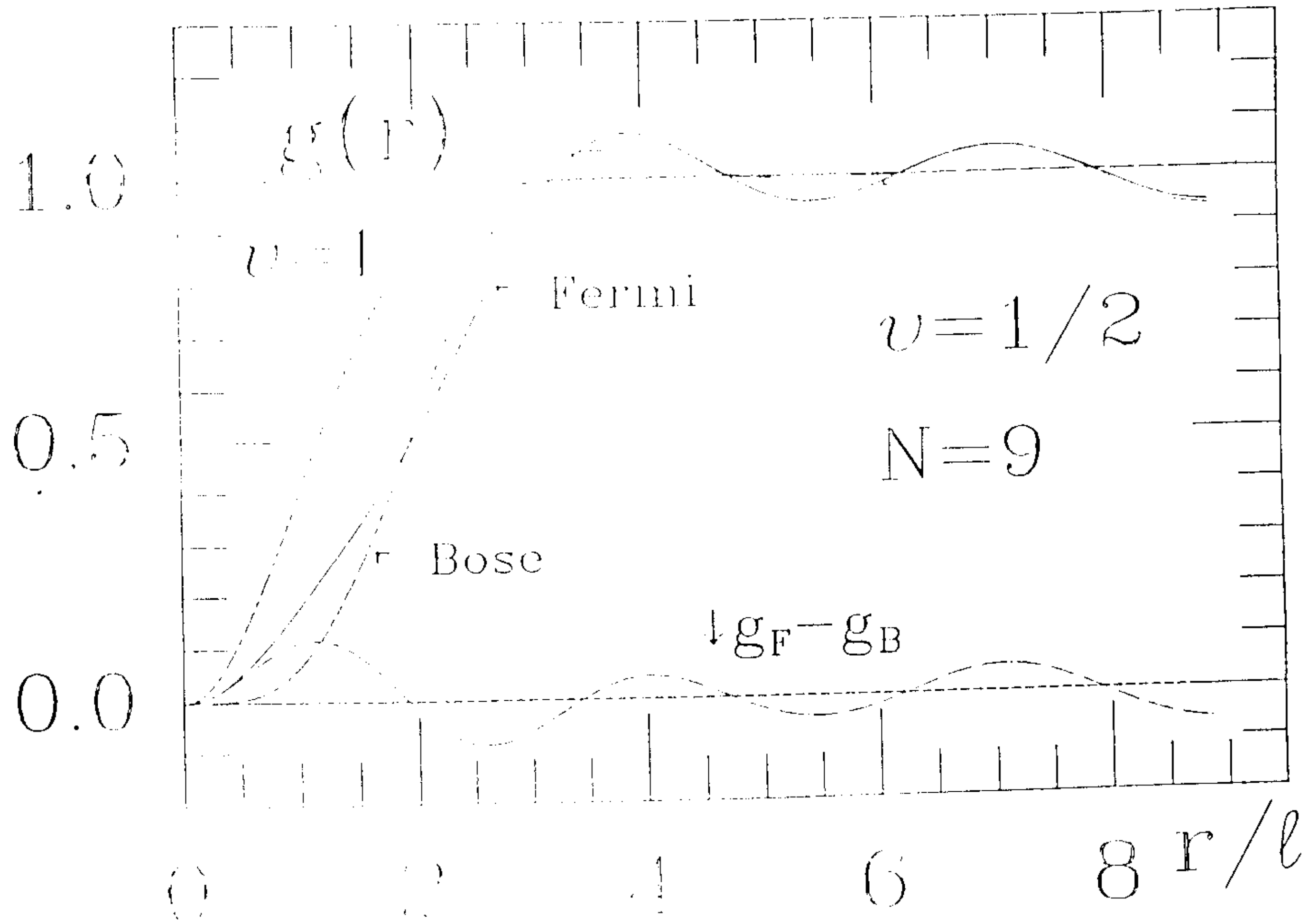

Figure 4

N	Ground State	Single P-H Excitations				Two P-H Excitations	
9	0.998779(8), L = 0	[illegible](21), 2	0.95[illegible]57(22), 3	0.948791(35), 4	0.993719(33), 5	0.977806(12), 7	0.977845(51), 8
8	0.990228(10), 2	0.9[illegible](11), 5	0.984475(19), 6				
7	0.990845(7), 3						

Table 1