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SMR. 758 - 5

SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

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QUANTUM-COHERENCE EFFECTS IN DISORDERED CONDUCTORS: ANDERSON LOCALIZATION AND MESOSCOPIC FLUCTUATIONS

Vladimir E. KRAVTSOV
Institute for Spectroscopy
Russian Academy of Sciences
Troitsk, Moscow Region 142092
Russian Federation
present address:
ICTP, P.O. BOx 586, I-34100 Trieste, Italy

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Quantum-coherence effects in disordered conductors: Anderson localization and mesoscopic fluctuations.

Main subjects to be covered:

1. Free electrons in a random potential: impurity diagrammatic technique.
2. Diffusons and Cooperons: a perturbative approach for describing quantum-coherence effects in the limit of diffusion propagation of electrons.
3. Weak localization.
4. Anderson localization as a critical phenomenon: scaling approach.
5. Role of dimensionality and brocken symmetries.
6. Mesoscopic fluctuations of conductance and DOS.
7. Energy-level statistics and the Anderson transition

① Everything begins with the Hamiltonian!

$$H = \frac{\hat{p}^2}{2m} + U(r)$$

$U(r)$ is a random impurity potential

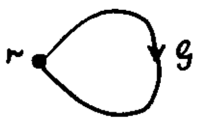
$$\langle U(r)\rangle = 0\,; \qquad \langle U(r)\,U(r')\rangle = \frac{1}{2\pi\nu\tau}\,\delta(r-r')$$

So simple and so nontrivial problem!

Main object of interest: electron Green's function

$$\mathcal{G}^{R(A)}(r,r';E) = \sum_n \frac{\psi_n(r)\psi_n^*(r')}{E-E_n \pm i0}$$

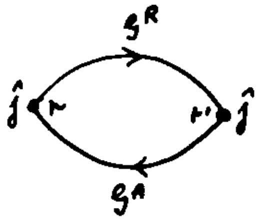
Density of states: $\nu(E) = -\frac{1}{2\pi i L^d}\int dr\,[\mathcal{G}^R(r,r;E) - \mathcal{G}^A(r,r;E)]$



Conductance: $G_\omega = \frac{1}{L^2}\int K(r,r';\omega)\,dr\,dr'$

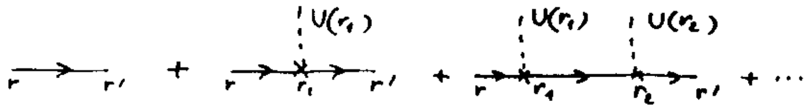
$$K(r,r';\omega) = -\frac{1}{\pi}\,\hat{j}\,\mathcal{G}^R(r,r';\omega)\,\hat{j}\,\mathcal{G}^A(r',r,0)$$

Kubo formula:



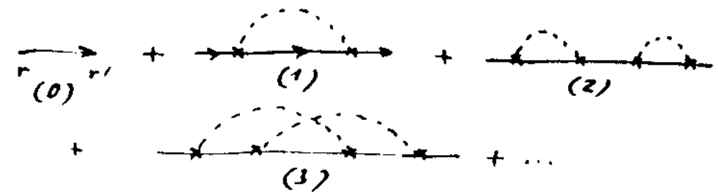
② Averaged Green's function: self-consistent Born approximation

$\mathcal{G}$ =



Averaging over a Gaussian random potential is equivalent to connecting pairs of $U(r)\,U(r')$ by all possible ways:

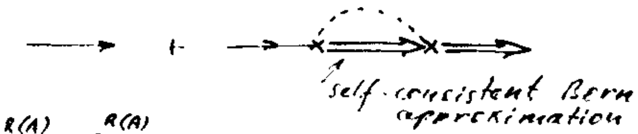
$\langle \mathcal{G} \rangle$ =



$$r\text{ - - - - - }r' = \frac{1}{2\pi\nu\tau}\,\delta(r-r')$$

Sum of all diagrams of the type (1) and (2):

$\langle \mathcal{G} \rangle_{BA}$ =



$$G^{R(A)} = \langle \mathcal{G}^{R(A)} \rangle_{BA} = (\varepsilon - \varepsilon(p) \pm i\Sigma)^{-1}$$

$$\mp i\Sigma = \frac{1}{2\pi\nu\tau}\int dp\,\frac{1}{\varepsilon - \varepsilon(p) \pm i\Sigma} = \frac{\mp i\pi\nu}{2\pi\nu\tau} = \mp\frac{i}{2\tau}$$

$$dp \to do\,d\xi\cdot\nu$$

angle — do; $\varepsilon(p)-\varepsilon$ — $d\xi$; DOS — ν

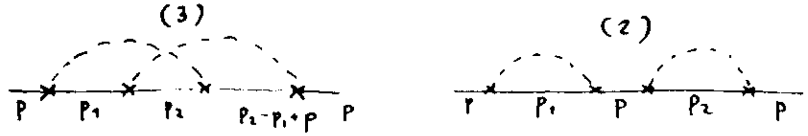
$$G^{R(A)}(p) = \left(\varepsilon - \varepsilon(p) \pm \frac{i}{2\tau}\right)^{-1}$$

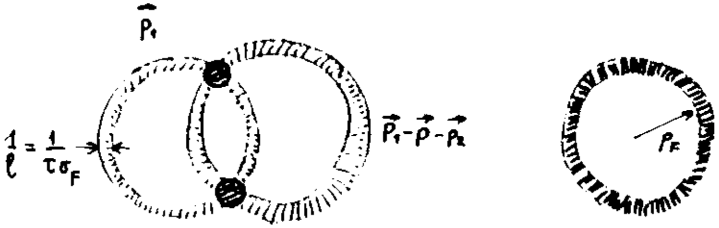
③ Averaged Green's function: effective-medium approximation

$$G^{R(A)} = \left(\varepsilon - \varepsilon(p) \pm \frac{i}{2\tau}\right)^{-1}$$

Coincides with $G_0^{R(A)}$ except for $i0 \to \frac{i}{2\tau}$ $\Rightarrow$ Effective medium result

Accuracy of the approximation:





$|p_1 - p_F| \lesssim \frac{1}{\ell}$ $\qquad$ $|p_1 - p_F| \lesssim \frac{1}{\ell}$

$|p_2 - p_F| \lesssim \frac{1}{\ell}$ $\qquad$ $|p_2 - p_F| \lesssim \frac{1}{\ell}$

extra constraint $\to$ $\left||\vec p_1 - \vec p - \vec p_2| - p_F\right| \lesssim \frac{1}{\ell}$

For $\boxed{p_F \ell \gg 1}$

The effect of averaging the Green function reduces to phase relaxation only $\left(\Sigma = \frac{i}{2\tau}\right)$

For $p_F \ell \lesssim 1$ one should take into account non-pole contributions (3)

④ Averaged Green function: phase relaxation

The main qualitative effect of averaging the Green's function:

$$\langle \psi_{sc}^{(*)} \rangle = \langle G^{R(A)} \rangle = \frac{e^{\pm i p_F r}}{4\pi r}\, \underline{e^{-r/2\ell}}$$

averaged wave function

$\langle \psi_{sc} \rangle \to 0$ at $r \gg 2\ell$

$\psi = A e^{i\phi}$ $\qquad$ $\langle \psi \rangle \to 0$

$\langle e^{i\phi} \rangle \to 0$

Randomisation of phase

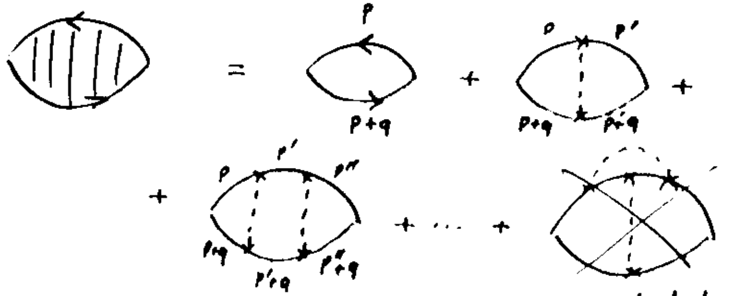
However, the intensity $\langle |\psi|^2 \rangle$ must remain finite upon averaging.

$\langle \psi \psi^* \rangle = \langle G^R G^A \rangle \neq 0 \qquad r \gg 2\ell$

$\langle \psi \psi \rangle = \langle \psi^* \psi^* \rangle = 0$

⑤ Diffusion propagator.

$$\langle \psi(r)\,\psi^*(r)\rangle = \langle G^R(r,0)\,G^A(0,r)\rangle$$



$$\langle \psi(r)\psi^*(r)\rangle = \int e^{iqr}\,\mathcal{D}(q)\,dq \sim \frac{1}{r^{d-2}}$$

$$\mathcal{D}(q) = \frac{\Pi(q)}{1-\frac{1}{2\pi\nu\tau}\Pi(q)} \approx \frac{2\pi\nu}{Dq^2}$$

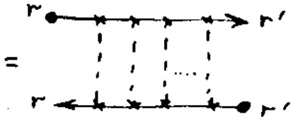
$$\Pi(q) = \int dp \; G^R(p)\,G^A(p+q) =$$

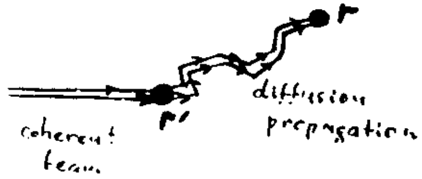
$$= \int dp \; \frac{1}{\left(\varepsilon-\varepsilon(p)+\frac{i}{2\tau}\right)\left(\varepsilon-\varepsilon(p+q)-\frac{i}{2\tau}\right)} =$$

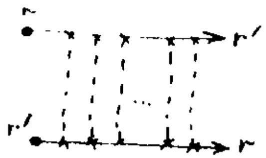
$$= \nu \int do \int_{-\infty}^{+\infty} d\xi \; \frac{1}{\left(\xi+\frac{i}{2\tau}\right)\left(\xi-\vec{v}\vec{q}-\frac{i}{2\tau}\right)} =$$

$$= 2\pi\nu\tau \int do \; \frac{1}{1-i\tau\vec{v}\vec{q}} \xrightarrow[ql\ll 1]{} 2\pi\nu\tau\left(1-\tau Dq^2\right)$$

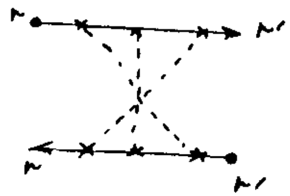
⑥ Two types of diffusion propagators: diffuson and Cooperon.

$\mathcal{D}(r,r') =$  $= \langle \psi(r|r')\,\psi^*(r|r')\rangle$

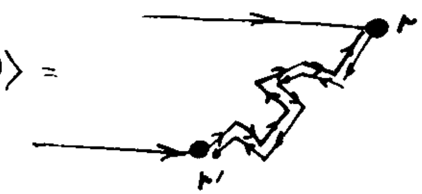


$C(r,r') =$ 

equiv

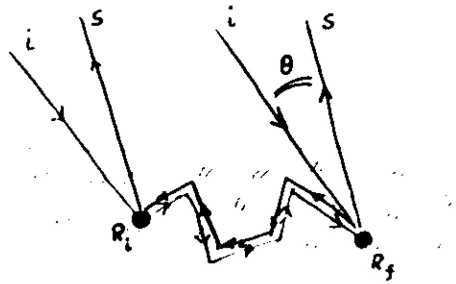


max. crossed diagrams

$C(r,r') = \langle \psi(r|r')\,\psi^*(r'|r)\rangle =$ 

where it can appear?

⑦ Enhanced backscattering



$$\varphi = \vec{k}_i \vec{R}_i + \varphi_s - \vec{k}_s \vec{R}_f$$

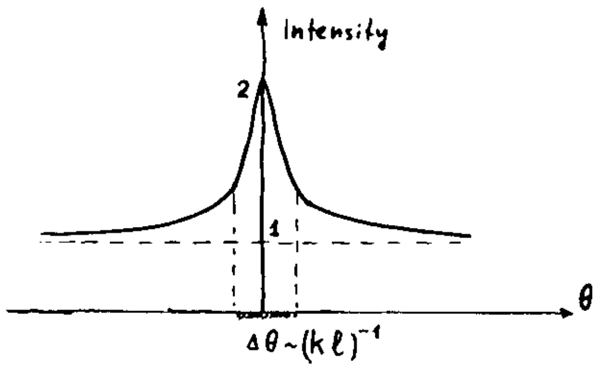
$$\bar{\varphi} = \vec{k}_i \vec{R}_f + \varphi_s - \vec{k}_s \vec{R}_i$$

$$\boxed{\Delta\varphi = \varphi - \bar{\varphi} = (\vec{k}_i + \vec{k}_s)(\vec{R}_i - \vec{R}_f)}$$

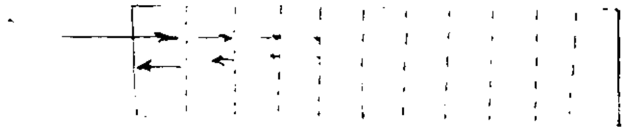
•

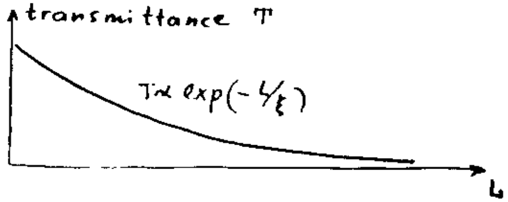
$$I = |e^{i\varphi} + e^{i\bar{\varphi}}|^2 = 2\,(1 + \underbrace{\cos(\varphi - \bar{\varphi})}_{\text{interference term}})$$

$\varphi - \bar{\varphi} = 0$ at backscattering
$\vec{k}_i + \vec{k}_s = 0$



⑧ Constructive interference, enhanced backscattering and Anderson localization.





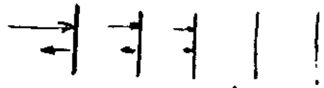
• transmittance of a (quasi) one-dimensional disordered sample decreases exponentially due to enhanced backscattering.

If interference effects are neglected (no enhanced backscattering) $\Longrightarrow$ diffusion approximation:

$$\frac{d^2 T}{d x^2} = 0 \qquad T = 1 - aL$$

linear decrease

Classical analog (1d)



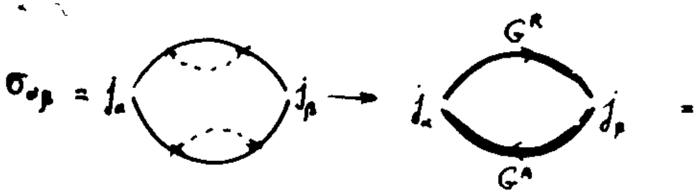
sequence of semi-transparent mirrors

Classical analog (3d)

Standing wave in a random interferometer

⑨ Weak localization corrections to conductivity

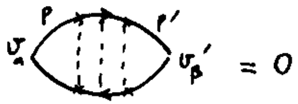
a). Drude conductivity: phase relaxation effect only



$= \frac{e^2}{\pi}\overline{v_\alpha v_\beta}\,\Pi(0) = \underline{2e^2\nu D}$

$\delta_{\alpha\beta} D = (\overline{v_\alpha v_\beta})\tau\,, \quad \Pi(0) = 2\pi\nu\tau$

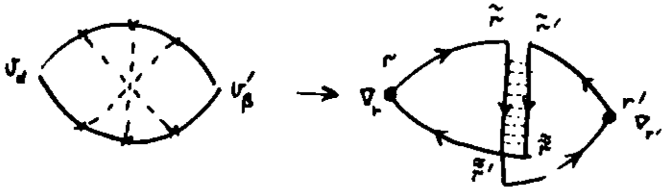
b) Simplest correlations between Green's functions



$\overline{v_\alpha v_\beta'} = 0$

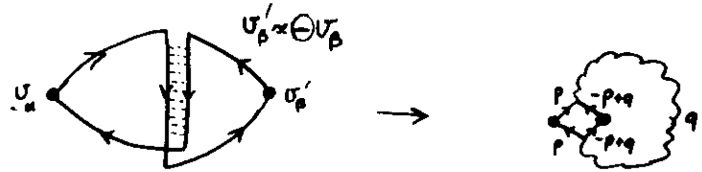
momenta p, p' are completely decoupled.

c) Cooperon weak-localization correction.

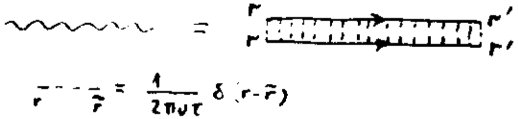


$G(r-\tilde{r}) \propto e^{-\frac{|r-\tilde{r}|}{2l}} \Rightarrow \begin{array}{l} |r-\tilde{r}| \sim |r-\tilde{\tilde{r}}| \sim l \ll |r-r'| \\ |r'-\tilde{r}'| \sim |r'-\tilde{\tilde{r}}'| \sim l \ll |r-r'| \end{array}$

⑩ Cooperon corrections to conductivity (continuation).



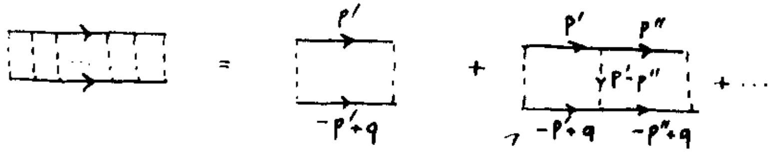
Small q make large contribution!



$= C(r-r')(2\pi\nu\tau)^{-2}$

$\frac{1}{2\pi\nu\tau}\delta(r-\tilde{r})$

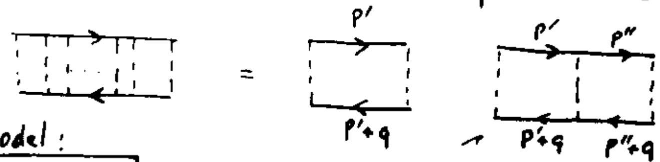
a) Cooperon in momentum representation.



The sum $p' + (-p'+q) = p'' + (-p''+q) = q = \underline{\underline{\text{const}}}$

Compare:

a') Diffuson in momentum representation



For our model:

$$\boxed{D(q) = C(q) = \frac{2\pi\nu}{Dq^2}}$$

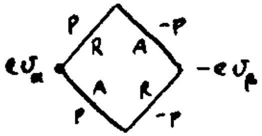
singularity at small q!

the difference

$p' - (p'+q) = p'' - (p''+q) = -q = \underline{\text{const}}$

(11) *Weak localization effects for different dimensionalities.*

$= \delta\sigma$



$$= -e^2\,\overline{v_\alpha v_\beta}\,\nu \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi+\frac{i}{2\tau})^2(\xi-\frac{i}{2\tau})^2} =$$

$$= -4\pi\nu e^2 D\tau^2$$

$$\boxed{\delta\sigma = -\frac{2e^2}{\pi\hbar}\int_{q>1/L}^{1/\ell}\frac{d\vec{q}}{(2\pi)^d}\,\frac{1}{q^2}} \qquad \delta G = L^{d-2}\,\delta\sigma$$

$d=1$:	$\delta G = -\frac{1}{\pi^2}\left(\frac{e^2}{\hbar}\right)$	$G_0 = \frac{2\ell}{L}\,\frac{e^2}{\hbar}$
$d=2$:	$\delta G = -\frac{1}{\pi^2}\ln(L/\ell)\left(\frac{e^2}{\hbar}\right)$	$G_0 = \frac{1}{2\pi}(p_F\ell)\,\frac{e^2}{\hbar}$
$d=3$:	$\delta G = -\frac{1}{\pi^3}L\left(\frac{1}{\ell}-\frac{1}{L}\right)\left(\frac{e^2}{\hbar}\right)$	$G_0 = \frac{p_F^2\,\ell L}{3\pi^2}\,\frac{e^2}{\hbar}$

In one dimension $|\delta G| \gg G_0$ (perturbative approach does not work)

(12) *The one-parameter scaling.*

$d=2$: $\quad g = g_0 - \ln(L/\ell)\;; \qquad g = \pi^2 G\frac{\hbar}{e^2}, \quad g_0 = (p_F\ell)\frac{\pi}{2}$

$d=3$: $\quad g = (g_0 - 1)\frac{L}{\ell} + 1\;; \qquad g_0 = (p_F\ell)^2\frac{\pi}{3}, \quad g = \pi^3 G\,\hbar/e^2$

$\qquad\qquad\frac{L}{\ell} = e^{\ln L}$

$$\begin{cases} d=2: & \dfrac{dg}{d\ln L} = -1 \\[2ex] d=3: & \dfrac{dg}{d\ln L} = g-1 \end{cases}$$

Generalization for arbitrary $d = 2+\epsilon$:

$$\frac{dg}{d\ln L} = \epsilon g - 1$$

$$\boxed{\frac{d\ln g}{d\ln L} \equiv \beta(g) = \epsilon - \frac{1}{g}}$$

The structure of weak localization corrections is so that

$\frac{dg}{d\ln L}$ depends only on g, not on L explicitly

$\Rightarrow$ one-parameter scaling

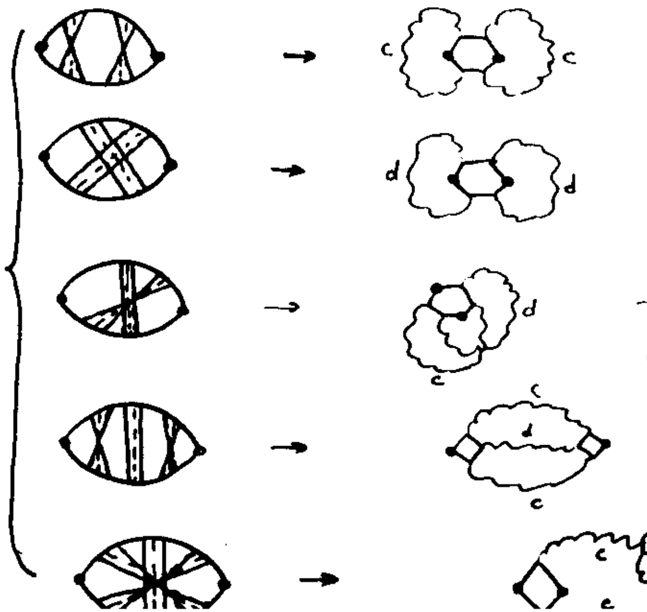
(Anderson, Abrahams, Licciardello, Ramakrishnan, 1979).

13 One-parameter scaling with a perturbative β-function and the perturbative expansion of conductance.

d=2: $$\frac{dg}{d\ln L} = \underbrace{-1 + \frac{a_1}{g} + \frac{a_2}{g^2} + \cdots}_{\text{perturbative expansion of } \beta(g)g} \equiv g\beta(g) \qquad (1)$$

$$\rightarrow \quad \frac{\delta g}{g} = -\frac{1}{g}\ln L + \underbrace{\frac{a_1}{g^2}\ln L + \frac{a_2}{g^3}\ln L}_{\text{increasing power of } \frac{1}{g} \ll 1 \text{ attached } \ln L}$$

Further corrections to conductivity (Larkin, Gorkov, Khmelnitskii)



two-loop diagrams

All of them $\sim \left(\frac{1}{g}\ln L\right)^2 \gg$

$\gg \frac{1}{g^2}\ln L$

They must cancel, if $\frac{dg}{d\ln L}$ is given by (1)

And they really DO.

14 The one-parameter scaling and the existance of the Anderson transition.

$$\frac{d\ln g}{d\ln L} = \beta(g); \qquad \text{Initial condition: } g(L=\ell) = g_0 = (p_F\ell)^{d-1}a_d$$

$$\beta(g) = \begin{cases} (d-2) - \frac{1}{g}, & g \gg 1 \\ \lg g, & g \ll 1 \end{cases}$$

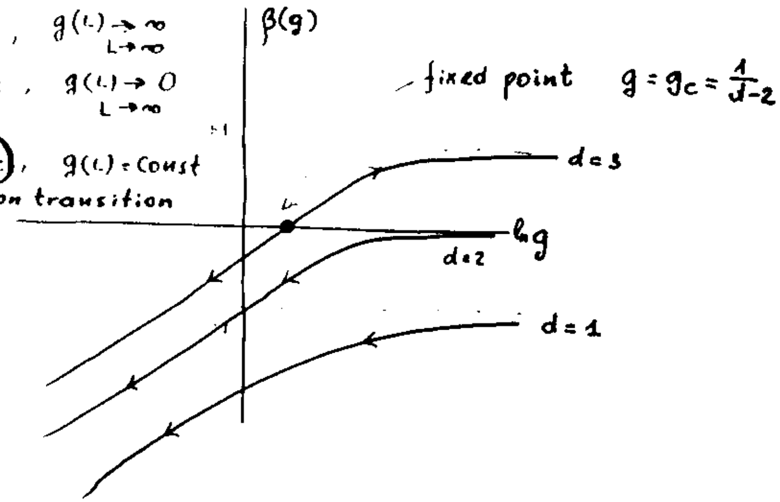
$$g \sim e^{-L/\xi}$$

$$\ln g = -L/\xi = -\frac{1}{\xi}e^{+\ln L}$$

$$\beta(g) = -\frac{1}{\xi}e^{\ln L} = \ln g$$

3-D $\begin{cases} \text{if } g_0 > g_c, & g(L) \xrightarrow[L\to\infty]{} \infty \\ \text{if } g_0 < g_c, & g(L) \xrightarrow[L\to\infty]{} 0 \\ \text{if } g_0 = g_c, & g(L) = \text{const} \end{cases}$

Anderson transition



Assumption: $\beta(g)$ is a monotoneous function: no physical reason for non-monotoneous behavior

All states are localized in 1-D and 2-D!

⑮ Critical behavior of conductance near the Anderson transition

$d=3$

$g_0 > g_c$ Metal side

$$\frac{dg}{g\beta(g)} = d\ln L$$

$$g(L=\ell) = g_0$$

$$\beta(g) \approx \frac{1}{\nu}\left[\frac{g-g_c}{g_c}\right], \quad g\to g_c$$

$$\int_{g_0}^{g(L)} \frac{dg'}{g'\beta(g')} = \ln\frac{L}{\ell}$$

- In the vicinity of $g_0 \approx g_c$: $g'\beta(g') \approx \frac{1}{\nu}(g'-g_c)$

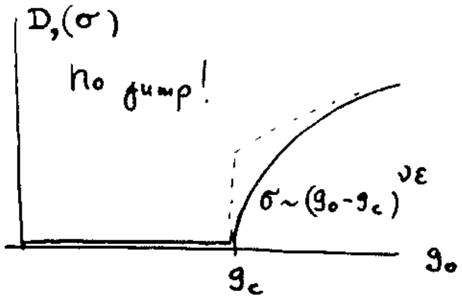
For large g': $g'\beta(g') = \underbrace{(d-2)}_{\varepsilon} g'$

choose $x_0 \sim 1$

$$\ln\frac{L}{\ell} = \nu\int_{g_0}^{x_0}\frac{dg'}{g'-g_c} + \int_{x_0}^{g(L)}\frac{dg'}{g'\varepsilon} \approx -\nu\ln(g_0-g_c) + \frac{1}{\varepsilon}\ln g$$

$$g(L) = \left(\frac{L}{\ell}\right)^{\varepsilon}(g_0-g_c)^{\nu\varepsilon}, \quad L\to\infty$$

$$\underset{L\to\infty}{D(L)} \propto \sigma(L) = \frac{g(L)}{L^{\varepsilon}} = \frac{1}{\ell^{\varepsilon}}(g_0-g_c)^{\nu\varepsilon}$$



⑯ Correlation and localization lengths.

$$g(L) = \left(\frac{L}{\ell}\right)^{\varepsilon}(g_0-g_c)^{\nu\varepsilon}, \qquad \begin{array}{c}\varepsilon = d-2>0\\ g_0>g_c\end{array}$$

$$\xi = \ell\,(g_0-g_c)^{-\nu} \Rightarrow \boxed{g(L) = \left(\frac{L}{\xi}\right)^{\varepsilon}}$$

$$D \propto \sigma \propto \frac{g(L)}{L^{\varepsilon}} = \frac{1}{\xi^{\varepsilon}}$$

- There is a divergent correlation length ξ near Anderson transition, and $\boxed{\sigma \propto \xi^{2-d}}$ $(d>2)$.

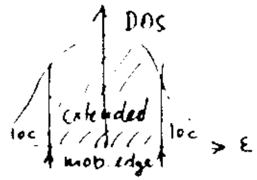
$$g_0 = a_d\,(p_F\ell)^{d-1}$$

$$\ell = (n\Sigma)^{-1},$$

n is the concentration of impurities

Σ is the scattering cross-section by impurities

$p_F = \sqrt{2m\varepsilon_F}$



$$g_0 = g_0(\varepsilon_F, n) \approx g_c + \underbrace{\frac{\partial g_0}{\partial\varepsilon_F}}_{\sim\frac{1}{\varepsilon_F}}(\varepsilon_F - E_c) + \underbrace{\frac{\partial g_0}{\partial n}}_{\sim\frac{1}{n_c}}(n-n_c)$$

$$\xi \sim \ell\left(\frac{|\varepsilon_F - E_c|}{E_c}\right)^{-\nu}, \quad \xi \sim \ell\left(\frac{|n-n_c|}{n_c}\right)^{-\nu}$$

E_c, n_c - mobility edge

$$\nu = \left[g_c \left.\frac{d\beta}{dg}\right|_{g=g_c}\right]^{-1}$$

(17) Localization length.

$g_0 < g_c$ Insulator side

$$\int_{g_0}^{g(L)} \frac{dg'}{g'\beta(g')} = \ln\left(\frac{L}{l}\right)$$

$$g'\beta(g') = \begin{cases} \dfrac{\nu}{g'-g_c}, & g' \to g_0 \\ g'\ln g', & g' \to 0 \end{cases}$$

$$\nu\int_{x_0}^{g_0} \frac{dg'}{g'-g_c} + \int_{g(L)}^{x_0} \frac{dg'}{g'\ln g'} = -\ln\left(\frac{L}{l}\right)$$

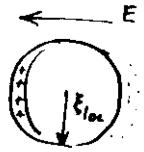
$$\nu \ln|g_0 - g_c| - \ln|\ln g| = -\ln\left(\frac{L}{l}\right)$$

$$|\ln g| = \frac{L}{l}|g_0 - g_c|^{\nu}$$

$$\boxed{\begin{aligned} g &= \exp[-L/\xi_{loc}] \\ \xi_{loc} &= l\,(g_c - g_0)^{-\nu} \end{aligned}}$$

The correlation length above Anderson transition and localization length below it are described by the same exponent ν.

(18) The catastrophe of polarizability.



$$d = \xi_{loc} \cdot q$$

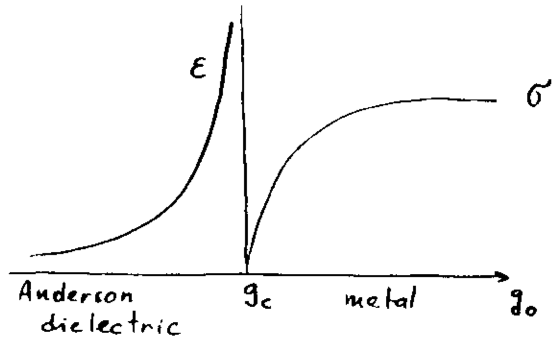
$$\frac{q}{\xi_{loc}^2} \sim E$$

$$\chi = \frac{d}{E} \sim \xi_{loc}^3 \quad (d=3)$$

$$\chi = \xi_{loc}^d \sim |g_0 - g_c|^{-\nu d}$$

Above g_c : σ vanishes as $|g_0 - g_c|^{\nu(d-2)}$

Below g_c : ε diverges as $|g_0 - g_c|^{-\nu d}$



(19) The exponent of correlation length ν.

$\nu = \left[g\beta'(g)\big|_{g=g_c} \right]^{-1}$ is the only nontrivial exponent within the one-parameter scaling.

All other exponents are functions of ν and dimensionality d.

Numerical simulations (d=3) $\boxed{\nu \approx 1.4}$

Asymptotic relationships for ν:

a). $d = 2+\varepsilon$ $\qquad \beta(g) = \varepsilon - \frac{1}{g}$; $g_c = \frac{1}{\varepsilon}$

$$\beta'(g) = \frac{1}{g^2}, \quad g\beta'(g) = \frac{1}{g}$$

$$\boxed{\nu = \frac{1}{\varepsilon}}$$

b). mean-field approximation ($d \to \infty$)

$$\boxed{\nu = \frac{1}{2}}$$

c). Harris criterion: $\boxed{d\nu > 2}$

$$\xi = \ell \left| \frac{n-n_c}{n_c} \right|^{-\nu}$$

In a finite sample of the size L n fluctuates $\left\langle \left(\frac{n-n_c}{n_c} \right)^2 \right\rangle_L \sim \frac{1}{L^d}$

Due to uncertainty in n

$$\xi < \ell \left\langle \left(\frac{n-n_c}{n_c} \right)^2 \right\rangle_L^{-\nu/2} \sim L^{d\nu/2}$$

However, for criticality to happen in $L \to \infty$ limit, one should have $\xi \sim L^{d\nu/2} \gg L$

(20) Low-dimensional systems (1-d)

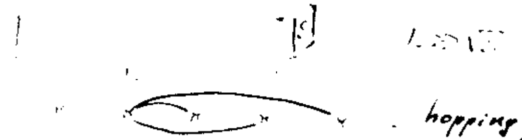
a). one-dimensional case.

$$\delta g = -\frac{1}{\pi^2}, \quad g_0 = \frac{2\ell}{L} \qquad \left| \frac{\delta g}{g_0} \right| \sim \frac{L}{\ell}$$

$\delta g \sim g_0$ already at $\ell \sim L$

$\boxed{\xi_{loc} \sim \ell}$ There is no weak-localization regime

b) quasi-one dimensional case.



hopping term connects N neighbours

$$\delta\sigma = -\frac{2}{\pi} \frac{1}{SL} \sum_{q, \vec{q}_\perp} \frac{1}{q^2 + q_\perp^2}$$

$q = \frac{2\pi}{L} n$; $(q_\perp)_\alpha = \frac{2\pi}{\sqrt{S}} n_\alpha$ $\qquad n = \pm 1, \pm 2, \dots$ $\quad (n_\alpha) = 0, \pm 1, \pm 2, \dots$

$$\delta\sigma = -\frac{2}{\pi} \frac{1}{SL} \Bigg[\underbrace{\frac{2L^2}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2}}_{\sim L^2} + \underbrace{L \sum_{q_\perp \neq 0} \int \frac{dq}{2\pi} \frac{1}{q^2 + q_\perp^2}}_{\sim \frac{LS}{\ell}} \Bigg]$$

$L \gg \frac{S}{\ell}$:

$$\delta\sigma = -\frac{1}{6\pi} \frac{L}{S}$$

$$\sigma = \nu D = \frac{p_F^2 \ell}{3\pi^2} \qquad \boxed{\frac{\delta\sigma}{\sigma} \sim \frac{L}{\ell N}}$$

$N = S p_F^2 \pi$ is the number of transverse channels

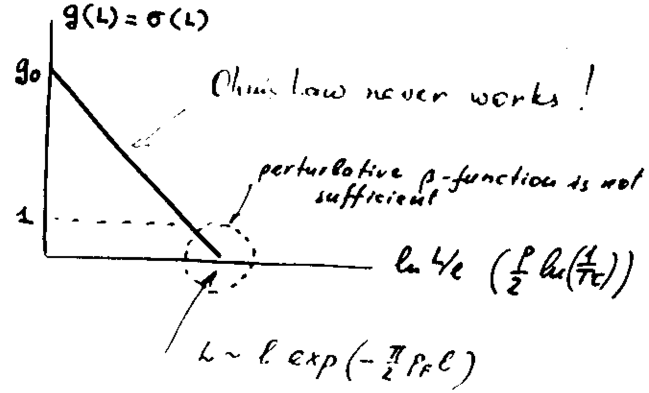
For $L \ll \xi_{loc} \sim \ell N$: weak-localization regime

For $L \gg \xi_{loc} \sim \ell N$: localized regime

(21) Low-dimensional systems (2-d).

$$\frac{1}{g}\frac{dg}{d\ln L} = -\frac{1}{g} \quad \to \quad \frac{dg}{d\ln L} = -1$$

$$\boxed{g = g_0 - \ln\left(\frac{L}{\ell}\right)} \qquad g_0 = \frac{\pi}{2} p_F \ell \gg 1$$



$$\boxed{\xi_{loc} \sim \ell \exp\left(-\frac{\pi}{2} p_F \ell\right)}$$

Perturbative estimate of the localization radius

Real experiments : $L \to L_\varphi$ - phase breaking length due to inelastic scattering (electron-electron, electron-phonon).

Landau Fermi-liquid theory : $\left(\tau_\varphi = \frac{L_\varphi^2}{D}\right)$

$$\tau_\varphi^{-1} \sim (\varepsilon - \varepsilon_F)^p \sim T^p \quad (p>1)$$

$$\ln(L/\ell) \to \frac{1}{2}\ln\left(\frac{\tau_\varphi}{\tau}\right) \propto \frac{p}{2}\ln\left(\frac{1}{T\tau}\right)$$

(22). Effect of symmetry breaking

So far we considered the system with a Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + U(r)$$

H posesses:

1). Time-reversal symmetry : $[H, T] = 0$

2). Spin symmetry: $[H, \vec{\sigma}] = 0$

$\vec{\sigma}$ is a Pauli matrix.

Magnetic interactions break the T-symmetry

a). $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$ Magnetic field

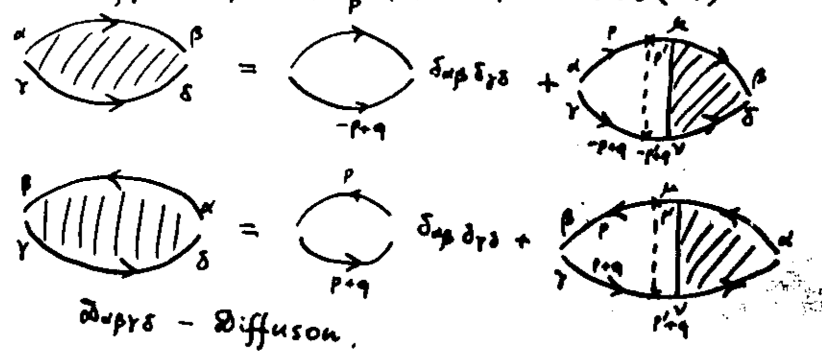
b). $H_{int} = \vec{U}_s \vec{\sigma}_{\alpha\beta}$ Magnetic impurities

Spin-orbit interaction breaks $\vec{\sigma}$-symmetry but preserves T-symmetry

c) $H_{int} = \vec{\sigma}_{\alpha\beta}[\vec{p} \times \vec{\nabla} U_{so}]$ S-O interaction

What happens with diffusons and Cooperons?

$C_{\alpha\beta\gamma\delta}$ - Cooperon, $\alpha, \beta, \gamma, \delta$ - spin indices (±1)



$\mathcal{D}_{\alpha\beta\gamma\delta}$ - Diffuson.

Diffuson and Cooperon for magnetic and spin-orbit scattering

$$(1+Dq^2\tau)\, C_{\alpha\beta\gamma\delta} = 2\pi\nu\tau\, [\delta_{\alpha\beta}\delta_{\gamma\delta} + \overline{\langle U_{\alpha\mu}(p,p')\, U_{\gamma\nu}(-p,-p')\rangle}\, C_{\mu\beta\nu\delta}$$

$$(1+Dq^2\tau)\, \mathcal{D}_{\alpha\beta\gamma\delta} = 2\pi\nu\tau\, [\delta_{\alpha\beta}\delta_{\gamma\delta} + \overline{\langle U_{\mu\beta}(p',p)\, U_{\gamma\nu}(p,p')\rangle}\, \mathcal{D}_{\alpha\mu\nu\delta}$$

Ladder equations for a Cooperon and a Diffuson (bar means averaging over directions of momenta p, p').

$$U_{\alpha\beta}(p,p') = \underbrace{U\delta_{\alpha\beta}}_{\text{potential scat.}} + \underbrace{\vec{U}_s\,\vec{\sigma}_{\alpha\beta}}_{\text{mag. impur.}} + \underbrace{i\,\vec{\sigma}_{\alpha\beta}\,[\vec{p}\,'\times\vec{p}\,]\,U_{so}(q)}_{\text{so scattering}}$$

$$\langle U_q U_{-q}\rangle = \frac{1}{2\pi\nu\tau_p}\,; \quad \langle U_s^{(i)}(q)\,U_s^{(j)}(-q)\rangle = \frac{1}{3}\delta_{ij}\,\frac{1}{2\pi\nu\tau_s}\,;$$

$$\overline{[\vec{p}\,'\times\vec{p}\,]^2}\,\langle U_{so}(q)\,U_{so}(-q)\rangle = \frac{1}{2\pi\nu\tau_{so}}$$

↘ Gaussian random fields

$$\frac{1}{\tau} = \frac{1}{\tau_p} + \frac{1}{\tau_s} + \frac{1}{\tau_{so}}$$ one-electron phase relaxation time.

Spin-dependent solutions for Diffuson and Cooperon:

$$C_{\alpha\beta\gamma\delta} = A_c\,\delta_{\alpha\beta}\delta_{\gamma\delta} + B_c\,\vec{\sigma}_{\alpha\beta}\,\vec{\sigma}_{\gamma\delta}$$

$$\mathcal{D}_{\alpha\beta\gamma\delta} = A_d\,\delta_{\alpha\beta}\delta_{\gamma\delta} + B_d\,\vec{\sigma}_{\alpha\beta}\,\vec{\sigma}_{\gamma\delta}$$

$$A_c = \frac{\pi\nu}{2}\left[\frac{1}{Dq^2+\frac{2}{\tau_s}} + \frac{3}{Dq^2+\frac{2}{3\tau_s}+\frac{4}{3\tau_{so}}}\right]$$

$$B_c = \frac{\pi\nu}{2}\left[\frac{1}{Dq^2+\frac{2}{3\tau_s}+\frac{4}{3\tau_{so}}} - \frac{1}{Dq^2+\frac{2}{\tau_s}}\right]$$

$$A_d = \frac{\pi\nu}{2}\left[\frac{1}{Dq^2} + \frac{3}{Dq^2+\frac{4}{3\tau_s}+\frac{4}{3\tau_{so}}}\right]$$

$$B_d = \frac{\pi\nu}{2}\left[\frac{1}{Dq^2} - \frac{1}{Dq^2+\frac{4}{3\tau_s}+\frac{4}{3\tau_{so}}}\right]$$

$\frac{1}{\tau_s}$ and $\frac{1}{\tau_{so}}$ play a role of „mass" in the diffusion propagators.

They correspond to new characteristic lengths: $L_s = \sqrt{D\tau_s}$, $L_{so} = \sqrt{D\tau_{so}}$

Even for small $\frac{1}{\tau_s}$ and $\frac{1}{\tau_{so}}$, these lengths can compete with L!

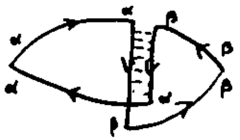
Three classes of symmetry:

1. $L_s, L_{so} \gg L$ (orthogonal) (potential scattering): $\begin{cases} A_d = A_c = \frac{2\pi\nu}{Dq^2} \\ B_d = B_c = 0 \end{cases}$

2. $L_{so} \gg L \gg L_s$ (unitary) (magnetic): $\begin{cases} A_c = B_c = 0 \\ A_d = B_d = \frac{\pi\nu}{2}\frac{1}{Dq^2} \end{cases}$

3. $L_s \gg L \gg L_{so}$ (symplectic) (SO scattering): $\begin{cases} A_c = -B_c = \frac{\pi\nu}{2}\frac{1}{Dq^2} \\ A_d = B_d = \frac{\pi\nu}{2}\,\frac{1}{\;} \end{cases}$

(25) Weak-localization corrections for a system with a broken symmetry.

$\delta\sigma$ =  $\propto \quad C_{\alpha\beta\beta\alpha} = \underset{\sim\sim\sim}{2(A_c + 3B_c)}$

$$\frac{2\pi\nu}{Dq^2} \longrightarrow \pi\nu\left[\frac{3}{Dq^2 + \frac{2}{3\tau_s} + \frac{4}{3\tau_{so}}} - \frac{1}{Dq^2 + \frac{2}{\tau_s}}\right]$$

$$\boxed{\delta\sigma = -\frac{2e^2}{\pi\hbar}\,\frac{1}{L^d}\sum_q\left\{\frac{3}{q^2 + \frac{2}{3L_s^2} + \frac{4}{3L_{so}^2}} - \frac{1}{q^2 + \frac{2}{L_s^2}}\right\}}$$

triplet contribution $I = 1$

singlet contribution $I = 0$

I is an angular momentum

Three symmetry cases

①. Symplectic case (SO int., no magnetic interactions)

Only singlet contribution survives:

$$(\delta\sigma)_{SO} = \ominus\frac{1}{2}(\delta\sigma)_{pot.}$$

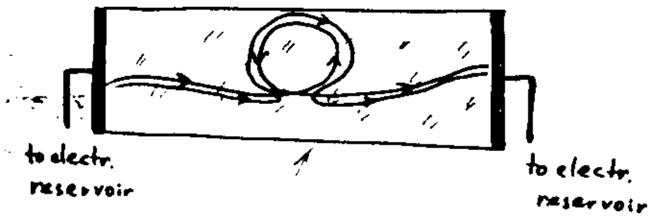
Weak delocalization instead of localization!

③ Magnetic case:

$$|(\delta\sigma)_{mag.}| \ll |(\delta\sigma)_{pot}|$$

The break-down of T-symmetry kills weak-localization!

(26) Qualitative explanation of the symmetry-breaking effects.



disordered sample

$$\boxed{g = \frac{T}{1-T}}\ , \quad T \text{ is a transmission coefficient.}$$

Landauer-Büttiker formula

$$T = \underbrace{\sum_i A_i^2 + Re\sum_i A_i^2 e^{i(\varphi_i - \bar{\varphi}_i)}}_{= A_i^2[\cos(\varphi_i - \bar{\varphi}_i) + 1]} + \underbrace{Re\sum_{i\neq j} A_i A_j\, e^{\overbrace{i(\varphi_i - \varphi_j)}^{\text{random}}}}_{\text{averages out to zero}}$$

$$\boxed{T_{int.} \propto \langle\cos(\varphi - \bar{\varphi})\rangle}$$

$\varphi - \bar{\varphi}$ is a phase difference between conjugated trajectories. $\bar{\varphi}$ is obtained from φ by T-inversion.

a). Break-down of T-invariance by magnetic field or magnetic impurities

$$\left.\begin{aligned}\varphi &= \varphi_{dyn.} + \varphi_{mag}\\ \bar{\varphi} &= \varphi_{dyn.} - \varphi_{mag}\end{aligned}\right\}\ \varphi - \bar{\varphi} = 2\varphi_{mag}$$

$$\varphi_{mag} = \underbrace{4\pi\Phi/\Phi_0}_{\text{random}}\ , \quad \Phi \text{ - random flux}, \quad \Phi_0 = hc/e$$

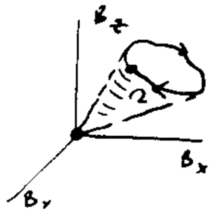
(27) Qualitative explanation of the symmetry - breaking effects.

b) effect of SO-interaction

$$H_{SO} = \vec{\sigma}\,\underbrace{[\hat{\vec{p}} \times \vec{\nabla} U_{SO}]}_{\text{effective } \vec{p}\text{-dependent magnetic field}}$$

Equivalent system: $\hat{H} = \vec{\sigma}\,\vec{B}$

$B = 0$ - degeneracy point.

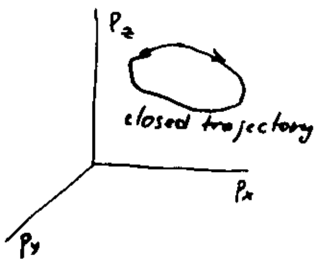


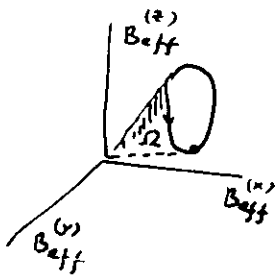
Ω - solid angle seen from the degeneracy point

On traversing the closed loop adiabatically, the wave function aquires the Berry phase:

$$\varphi_{Berry} = \frac{1}{2}\Omega$$

In our case: $\hat{H} = \vec{\sigma}\,\vec{B}_{eff}(\vec{p})$





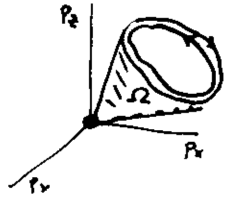
$0 < |\Omega| < 2\pi$

(28). Qualitative explanation of anti-localization.

$$\varphi = \varphi_{dyn} + \varphi_{Berry} = \varphi_{dyn} + \frac{1}{2}\Omega$$

$$\bar{\varphi} = \varphi_{dyn} - \frac{1}{2}\Omega$$

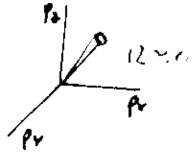
$$\boxed{\varphi - \bar{\varphi} = \Omega}$$

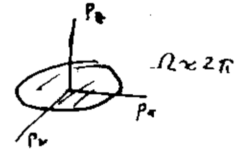


Ω is a random quantity

$0 < \Omega < 2\pi$

However, $\Omega \approx 0$ and $\Omega \approx 2\pi$ have a small statistical weight

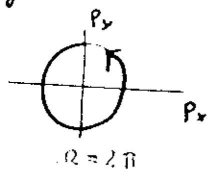


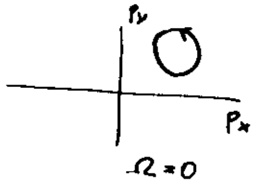


Typical trajectories correspond to $\Omega \approx \pi$ so that

$$\langle \cos\Omega \rangle = \langle \cos(\varphi - \bar{\varphi}) \rangle = -\frac{1}{2}$$

In strictly two-dimensional case





$\langle \cos\Omega \rangle = 1$ There is no anti-localization!

(29) β-function for different symmetries.

$$(\delta\sigma)_{SO} = -\tfrac{1}{2}(\delta\sigma)_{pot}$$

$$|(\delta\sigma)_{mag}| \ll |\delta\sigma|_{pot}$$

a). Potential scattering (orthogonal symmetry class)

$$\beta(g) = d-2 - \frac{1}{g} + \cdots$$

b). unitary symmetry class (magnetic)

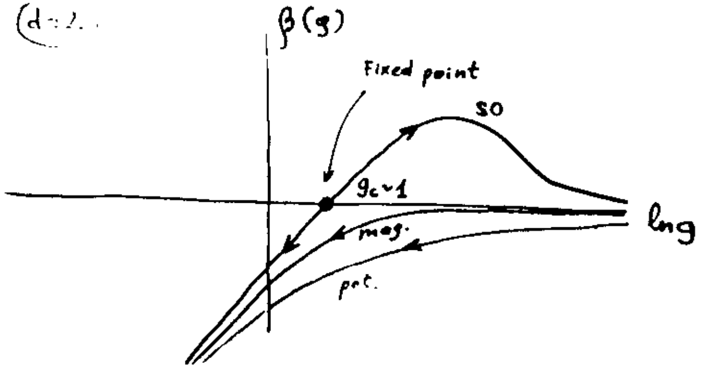
$$\beta(g) = d-2 - \frac{2}{g^2} + \cdots$$

c) symplectic symmetry class (SO)

$$\beta(g) = d-2 + \frac{1}{2g} + \cdots$$

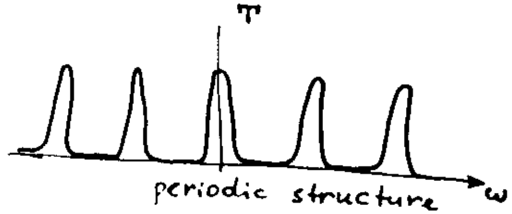
Anderson transition in (quasi)-two dimensions:

(d=2)

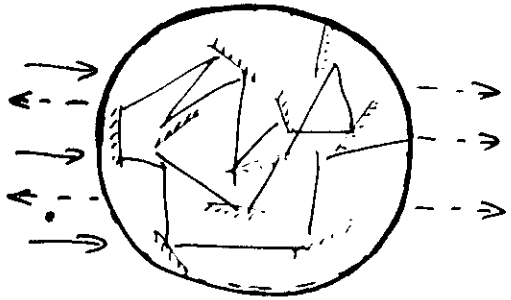


(30) Phase coherence and the lack of self-averaging.

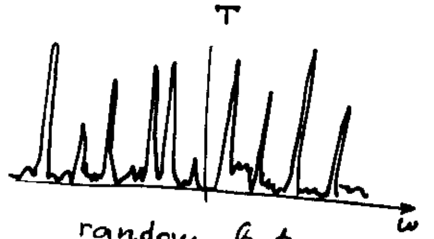
Fabri-Perot interferometer



periodic structure



Random interferometer



random but reproducible structure ("fingerprints" of the interferometer)

As long as the incident beam is coherent and decoherence does not happen in the interferometer THERE IS NO SELF-AVERAGING.

Formal basis for self-averaging: Central limiting theorem:

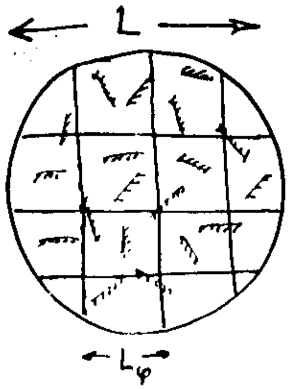
$$A = \sum_{i=1}^{N} \eta_i \; ; \qquad \eta_i, \eta_j \ (i \neq j) \text{ are statistically independent}$$

$$\bar{A} = N\langle\eta\rangle$$

$$\langle (A-\bar{A})^2 \rangle = N\langle\eta^2\rangle \qquad \frac{\delta A}{\bar{A}} \sim \frac{\sqrt{N}}{N} \sim \frac{1}{\sqrt{N}}$$

In the case of coherence . . . N = N . . . near Anderson trans

(31) The phase-breaking length.



If phase coherence is preserved only over a length $L_\varphi < L$, then the resulting intensity I is a sum of intensities I_i created by an interferometer of the size L_φ

$$\boxed{I = \sum_{i=1}^{N} I_i} \quad , \quad \boxed{N = \left(\frac{L}{L_\varphi}\right)^d N_0}$$

- For $N \gg 1$ $(L \gg L_\varphi)$,

$$\frac{\sqrt{\langle (I-\bar{I})^2 \rangle}}{\bar{I}} \sim \frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{N_0}}\left(\frac{L_\varphi}{L}\right)^{d/2} \ll 1$$

Self averaging works at $L \gg L_\varphi$.

The crucial point for mesoscopics is that $L_\varphi = \sqrt{D\tau_\varphi}$ is determined by inelastic collisions of electrons

$$\tau_\varphi = (\varepsilon - \varepsilon_F)^{-p} \sim T^{-p} \to \infty \quad (p > 1)$$

L_φ and τ_φ diverges as temperature goes to zero!

At low temperatures ($< 1°K$) L_φ is macroscopic ($L_\varphi > 1\,\mu m$), and at $T=0$ all samples display no self averaging.

(32) Coherent mesoscopics.

Originally, samples of intermediate (meso-) sizes

$$\ell \ll L \ll L_\varphi(T)$$

were called „mesoscopic samples". Later, all cases where self averaging does not take place, became a subject of mesoscopics.

-

$$T = \sum_{i,j} A_i A_j^* e^{i(\varphi_i - \varphi_j)} = \underbrace{\sum_i |A_i|^2}_{\text{all traject.}} + \underbrace{\sum_{\text{conj}} |A_c|^2}_{\text{traject. with loops}}$$

diffusion theory

localization

$$+ \boxed{\sum_{i \neq j} A_i A_j^* e^{i(\varphi_i - \varphi_j)}}$$

$T_f =$ random term. all deviations from average are there.

object of mesoscopic physics.

At $T=0$, DC conductance is contributed by electrons with the energy $E = E_F$.

$\varphi_i - \varphi_j$ depends on energy, and hence, on E_F. More importantly, it depends also on the external magnetic field through the magnetic phase. -> magnetic fingerprints.

(33) Averaged density-of-state and its mesoscopic fluctuations.

$$\langle \rho(E)\rangle \propto \mathrm{Im}\, G^R(r,r) = \mathrm{Im} \bigcirc G^R$$

No way to construct a diffusion propagator:

$\langle g^{R} g^{A}\rangle \Rightarrow \langle \nu(E)\rangle = \nu_0$ shows no criticality

Averaged DOS does not feel Anderson transition!

Anderson dielectric differs from the classical one: there is no gap in the energy spectrum.

$$\langle \rho \rangle \neq 0$$

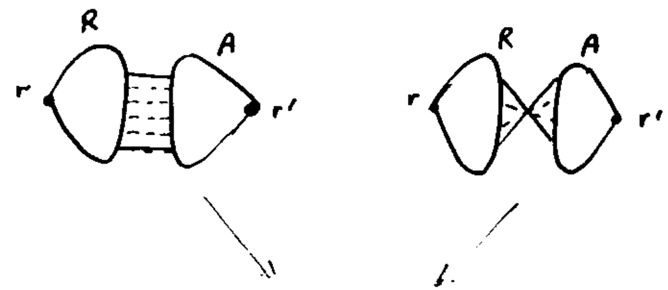
Fluctuations of DOS:

$$\rho^2(E) = \left[\bigcirc g^R - \bigcirc g^A\right]^2$$

$$\langle \rho^2(E)\rangle = \langle g^R g^R\rangle + \langle g^A g^A\rangle - \langle g^R g^A\rangle - \langle g^A g^R\rangle$$

$$\langle \rho(E)^2\rangle = \frac{1}{2\pi^2 L^d}\int dr\, dr' \langle g^R(r,r)\, g^A(r',r')\rangle$$

(34) Perturbative expression for DOS varience.



(I) $K_1(r-r') =$ 

exponentially decreases at $|r-r'| > l$ $\sim e^{-\frac{|r-r'|}{l}}$

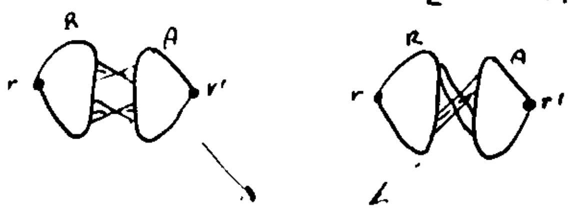
$\Rightarrow$ local DOS fluctuations

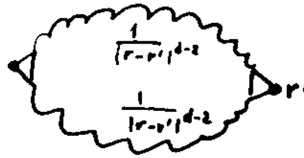
$$\langle(\rho - \langle\rho\rangle)^2\rangle = \frac{1}{2\pi^2 L^d}\int d(r-r')\, K(r-r')$$

↓ standard $\frac{1}{L^d}$ factor

$\int K_1(r-r')\, d(r-r')$ is L independent

Diagram (I) does not lead to deviations from the standard $1/L^d$ law.



(II) $K_2(r-r') =$  $\propto \frac{1}{|r-r'|^{2d-4}}$

$$\int K_2(r-r')\, d(r-r') \propto \frac{L^d}{L^{2d-4}} = (L^{4-d}),\ d<4$$

35. Dimensionless conductance and the number of effective random variables.

$$\boxed{\frac{\langle(\delta\rho)^2\rangle}{\langle\rho\rangle^2} = \left(\frac{1}{\nu_0 D L^{d-2}}\right)^2 = \frac{1}{g^2}}$$

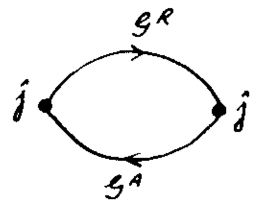
$$\frac{(\delta A)^2}{\langle A\rangle^2} \sim \frac{1}{N_0} \quad \Rightarrow \quad \boxed{N_0 = \frac{1}{g^2}}$$

Self averaging fails in mesoscopic samples, when $g \sim 1$

36. Universal conductance fluctuations.

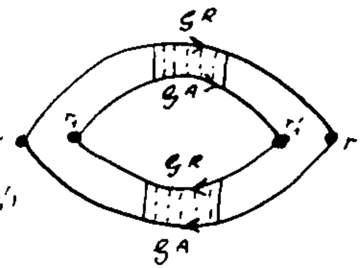
$$G = \frac{1}{L^2}\int K(r,r')\, dr\, dr'$$

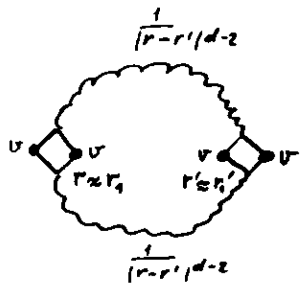
$$K(r,r') = -\frac{1}{\pi}\ \hat{j}\, G^R(r,r')\, \hat{j}\, G^A(r',r)$$



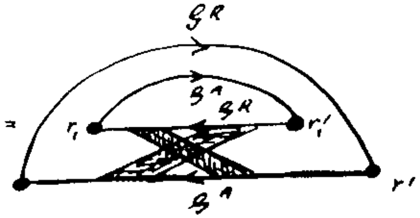
$$\langle G^2\rangle = \frac{1}{L^4}\int dr\, dr'\, dr_1\, dr_1' \langle K(r,r')\, K(r_1,r_1')\rangle$$

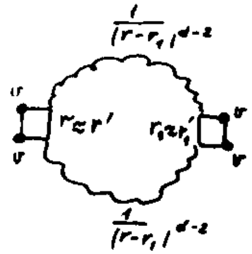
$$F_1(r-r';\ r-r_1,\ r'-r_1') = e^{-\frac{|r-r_1|}{l}}\, e^{-\frac{|r'-r_1'|}{l}}$$





$$F_2(r-r_1;\ r-r';\ r_1-r_1') = e^{-\frac{|r-r_1'|}{l}}\, e^{-\frac{|r_1-r'|}{l}}$$





$$\langle(\delta G)^2\rangle \sim \frac{1}{L^4}\, L^{2d}\, \frac{1}{L^{2(d-2)}}$$ is independent of L!

$$\boxed{\frac{\langle(\delta g)^2\rangle}{g^2} \sim \frac{1}{g^2} \sim \frac{1}{N_0}} \qquad \boxed{\langle \delta G^2\rangle \sim \left(\frac{e^2}{\hbar}\right)}$$

Universal conductance fluctuations

(37) Distribution of conductance fluctuations and the Anderson transition.

At $T \to 0$ self-averaging never happens (in its classical form).

Averaged conductance could become far from the typical one.

Only statistical description of conductance of a given sample is possible: what is the probability to find a concrete value of g.

$$\boxed{P = F(g)\,dg}$$

$F(g)$ - probability density to find a value of conductance equal to g

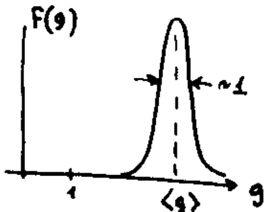
The scaling theorem is valid only for <u>averaged</u> conductance

$$\langle g \rangle = \int g\, F(g)\, dg$$

If one fixes the averaged conductance, and takes the limit $L \to \infty$ (appropriately changing g_0), one finds

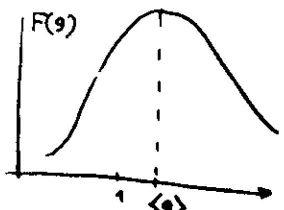
$$F(g) \equiv F(g, \langle g \rangle)$$

$\langle g \rangle \gg 1$



close to Gaussian ($N_g \gg 1$)

$\langle g \rangle \sim 1$



broad non-Gaussian ($N_g \sim 1$)