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SMR. 758 - 43

SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

QUANTUM HALL EFFECT
(NUMERICAL STUDIES)

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These are preliminary lecture notes, intended only for distribution to participants.

The Integer Quantum Hall Transition
(non-interacting electrons)

1. INTRODUCTION - LOCALIZATION IN THE LOWEST LANDAU LEVEL
2. WAVEFUNCTION TOPOLOGY & LOCALIZATION *,+
3. UNIVERSALITY OF CONDUCTANCE TENSOR *,+
4. UNIVERSAL CONDUCTANCE FLUCTUATIONS *
5. EIGENVALUE STATISTICS & LOCALIZATION *
6. CONCLUDING REMARKS

* D. Arovas	+ Y. Huo	+ M. Guo
P.B. Littlewood	* Y. Huo + R. Hetzel	
F.D.M. Haldane	* Y. Huo	
R. Rammal	* Y. Huo	

ELECTRON IN A MAGNETIC FIELD

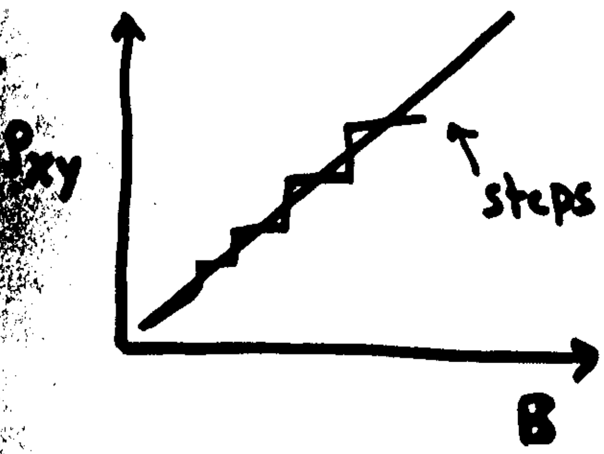
$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + V(r) \quad 2D \qquad \vec{B} = B\hat{z}$$

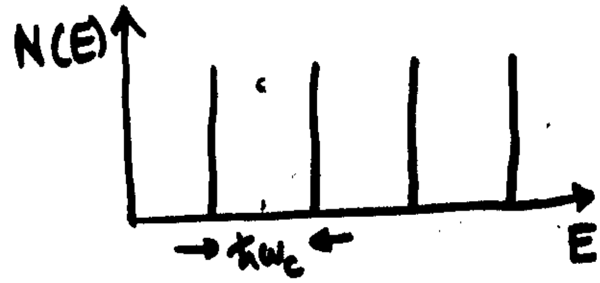
Landau levels (1930)
L&L §112

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar w_c$$

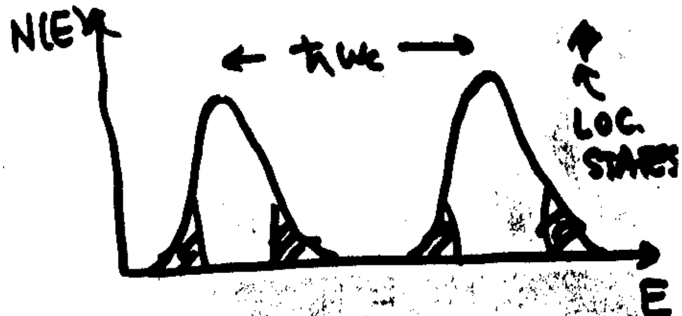
$$w_c = \frac{eB}{mc} \qquad l = \sqrt{\frac{\hbar c}{eB}}$$

Extensively degenerate.





↓ DISORDER

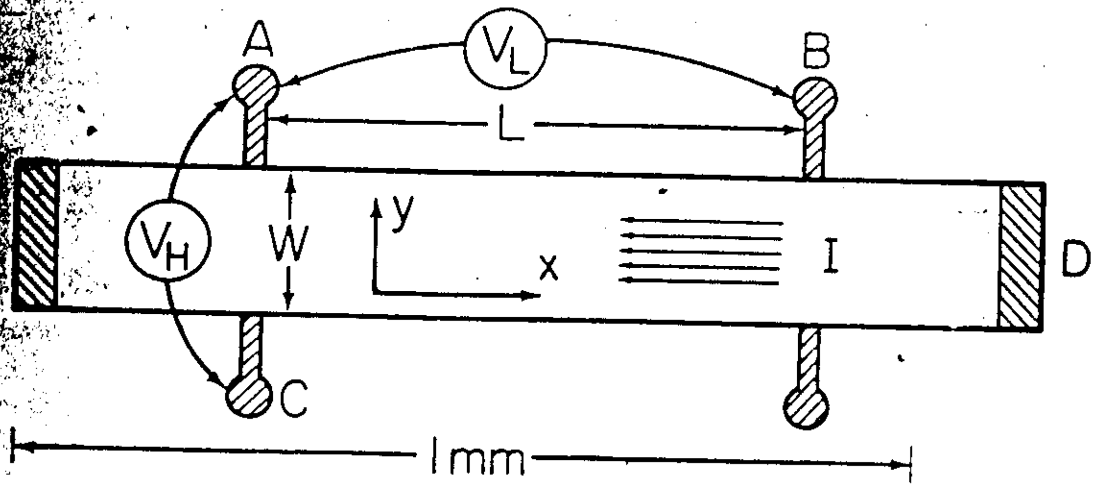


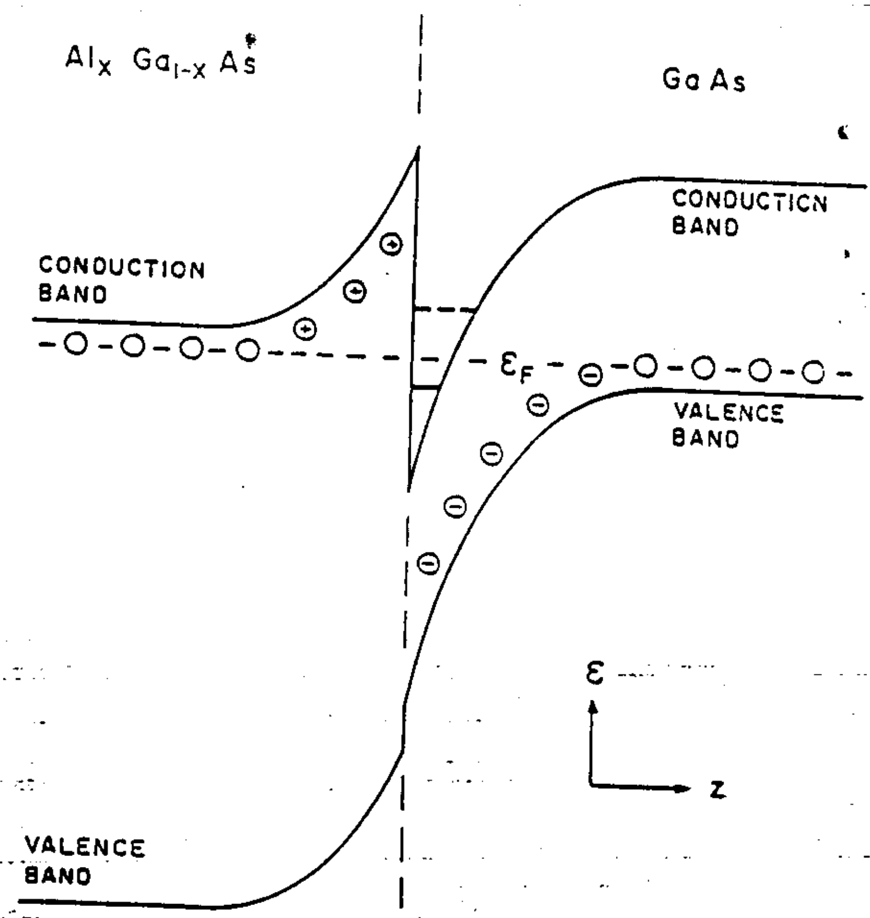
steps quantized so

$$\frac{1}{\rho_{xy}} = \sigma_{xy} = n\frac{e^2}{h}$$

n = integer

Laughlin: Gauge argument for quantization whenever E_F in localized region


A
B
V_L
L
S
V_H
W
y
x
I
D
C
1mm


$Al_x\ Ga_{1-x}\ As$
Ga As
CONDUCTION BAND
CONDUCTION BAND
ε_F
VALENCE BAND
VALENCE BAND
ε
z

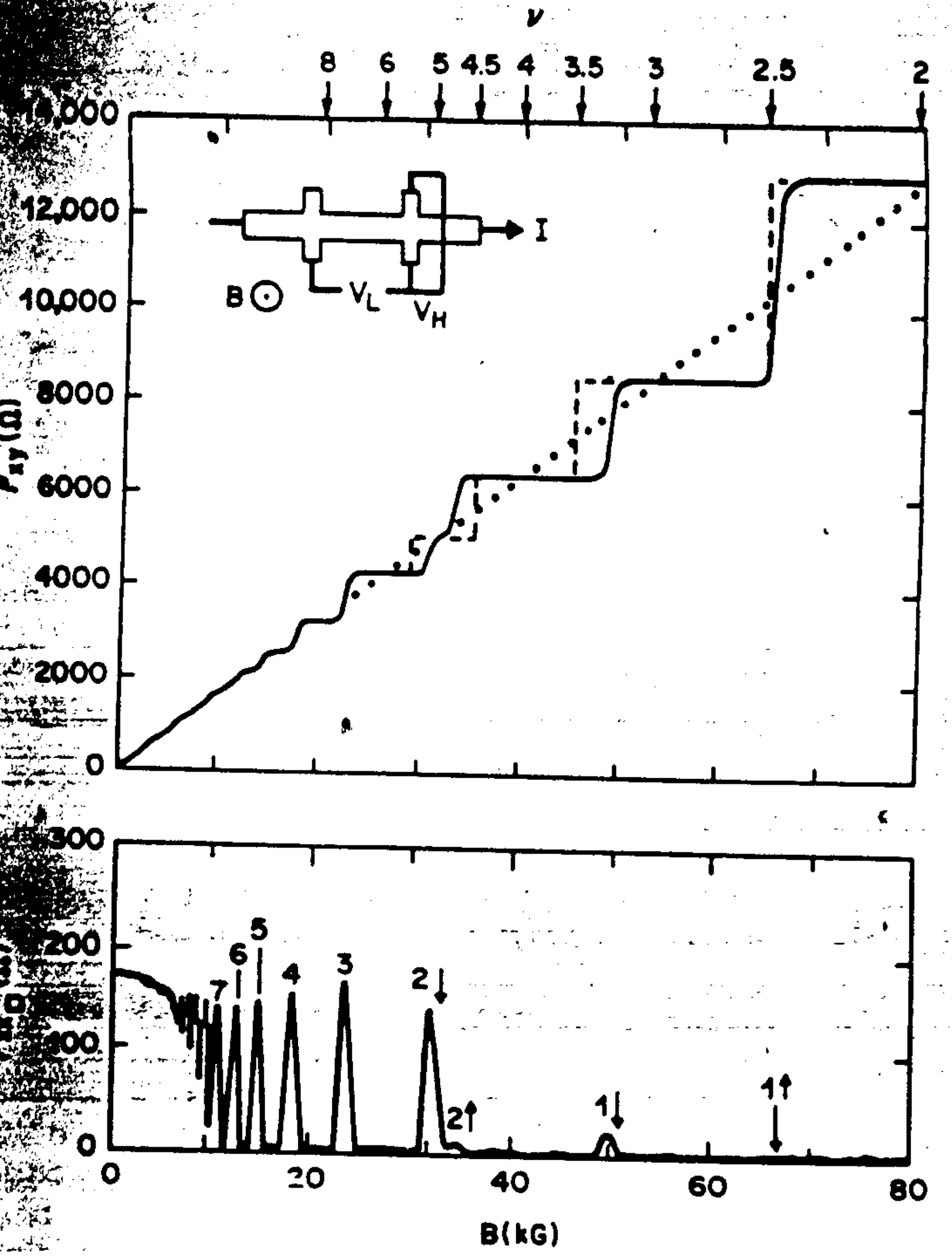
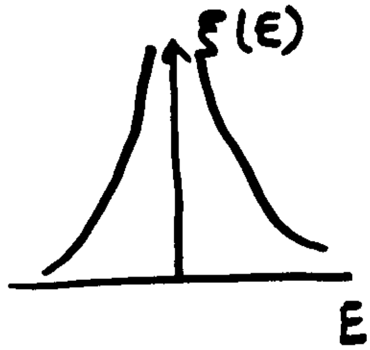
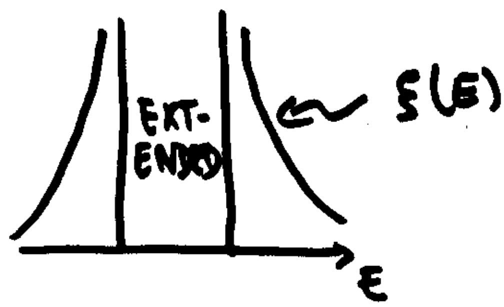


Figure 6.4.11 Experimental evidence for the energy position shift of the extended states due to disorder in the IQHE. The dashed curve indicates the ideal Hall resistance in which the transition between plateaus occurs at $\nu = i+1/2$. Note in particular the shrinkage of the $i = 5$ plateau, corresponding to the situation in Fig. 6.4.9b. The data is from Paalanen et al. (1982).

Halperin: For jump, do not need a band of extended states, enough to have one (or more) critical energies E_c, where localization length diverges.

Question: Which is the true situation (here limited to non-interacting electrons)





Nature of divergence?

Classical: $\lambda \gg l$ $\quad \xi \sim (E-E_c)^{-\nu_{cl}}$ (TRUGMAN Hills & Valleys) $\nu_{cl} = \frac{4}{3}$

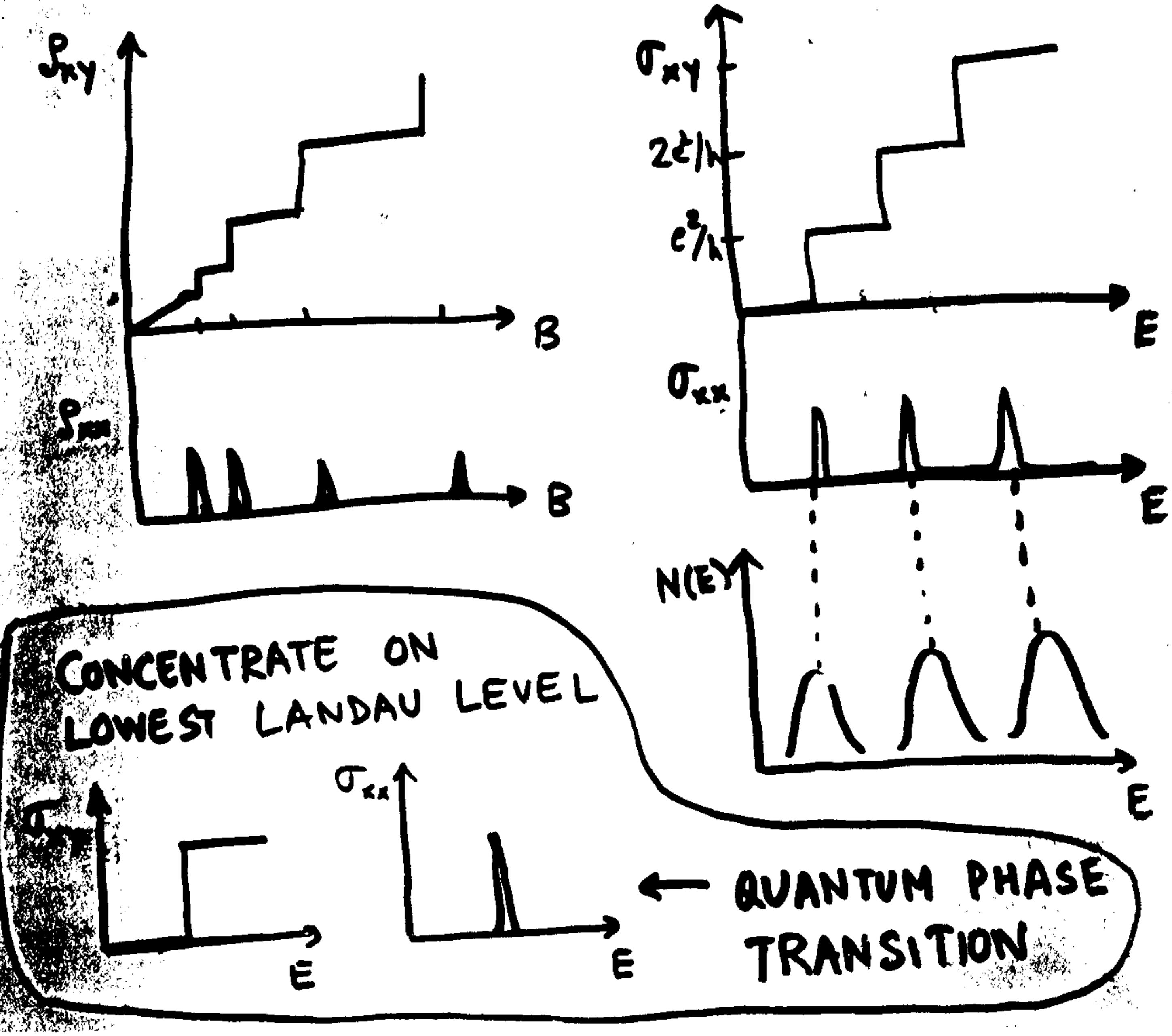
Born Approx $\quad \xi \sim e^{+\frac{1}{(E-E_c)^2}}$ (Ono, ...)

Classical + QT $\quad \xi \sim (E-E_c)^{-\nu}$ $\quad \nu = \nu_{cl} + 1$ (Milnikov Sokolov

When ρ_{xy} quantized, $\rho_{xx} \to 0$

$$\Rightarrow \sigma_{xy} = \frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \to \rho_{xy}^{-1} \; ; \; \sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} \propto \rho_{xx}$$

Equivalent pictures:



ZEROES, CHERN NUMBERS & LOCALIZATION

In the Lowest Landau Level

$$\psi \sim f(z)\, e^{-|z|^2/4\ell^2} \qquad z = x+iy$$

Zeroes of ψ specify the wavefunction.

For periodic geometry, slightly more complicated, but same holds

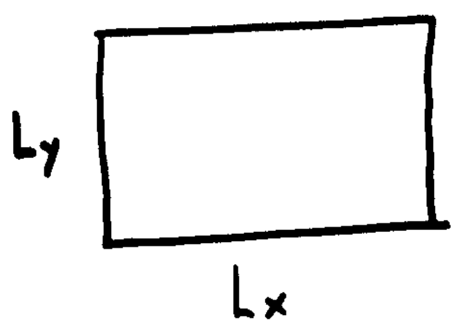
Magnetic translation operator $t(\vec{L}) = e^{i\vec{k}\cdot\vec{L}}$

$$\hbar\vec{k} = \vec{p} - \frac{e}{c}(\vec{r}\times\vec{B})$$

then $[t(\vec{L}), H] = 0$ but $t(\vec{L}_1)\,t(\vec{L}_2) = e^{-i\phi}\,t(\vec{L}_2)\,t(\vec{L}_1)$

where $\phi = (\vec{L}_1 \times \vec{L}_2)\cdot\hat{z}/\ell^2$

so $[t(\vec{L}_x), t(\vec{L}_y)] = 0$ iff $L_x L_y = 2\pi\,\ell^2 N_s$ (integral # of flux quanta)



If so, can simultaneously diagonalize H, $t(\vec{L}_x)$, $t(\vec{L}_y)$ and label states by eigenvalues

$$t(\vec{L}_x)|\psi\rangle = e^{i\theta_x}|\psi\rangle$$
$$t(\vec{L}_y)|\psi\rangle = e^{i\theta_y}|\psi\rangle$$

Analogous to crystal momentum in band theory of solids.

States satisfying "twisted" boundary condition

$$f(z) = e^{\pi N_s \xi^2/2} \, e^{2\pi i \lambda \xi} \prod_{k=1}^{N_s} \Theta\left(\pi(\xi - \xi_k) \mid i\right)$$

$\xi = z/L$

$$\Theta(w|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n+\frac{1}{2})^2} e^{(2n+1)iw}$$

is Jacobi Θ-fn. with zeroes at $w = \pi(j_1 + \tau j_2)$

$j_1 + j_2$ integers

$\lambda = n_1 + \theta_1/2\pi$ $\quad \xi_{cm} = \frac{1}{N_s}\sum_k \xi_k = \frac{1}{N_s}\left[\left(n_2 + \frac{\theta_2}{2\pi}\right) - i\lambda\right]$

$\Rightarrow$ N_s zeroes in $0 < Re(\xi) < 1$ $\quad 0 < Im(\xi) < 1$
periodically repeated.

Position completely specifies wave fn.

(3) Chern # $C_1(m)$ $\rightarrow$ gives covering of real space torus by going over regime of b.c. space.

(1) How do zeroes move with change in θ_1, θ_2

(2) Can you identify current carrying states?

(4) $\sum_m C_1(m) = 1$

MOTION OF ZEROES OF EIGENSTATES (N_s=8) UNDER CHANGE OF BOUNDARY CONDITION ANGLES (Θ_x, Θ_y).

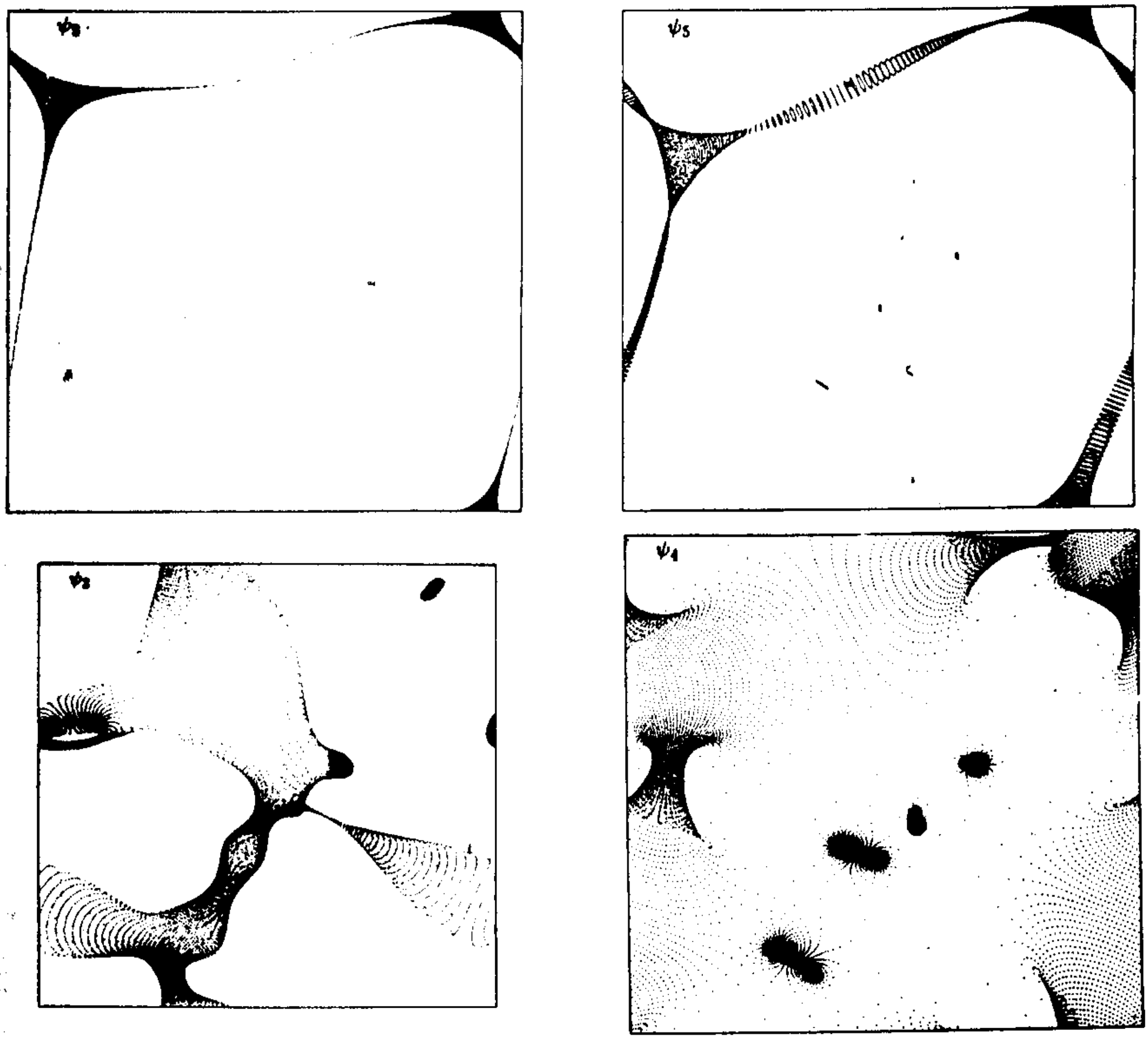


FIG. 2. Map of the nodal points of four of the $N_s(=8)$ wave functions in the potential of Fig. 1 for a fine grid of boundary conditions.

D. Arovas, R. Bhatt, D. Haldane, P. Littlewood & R. Rammal
Phys. Rev. Lett. 60, 619 (1988)

Results: [Arovas, Bhatt, Haldane, Littlewood, Rammal, Phys Rev Lett 60, 619 (88)]

Based on approach by Thouless and coworkers

- EIGENSTATES DO NOT CROSS AS FN. OF θ_x, θ_y – CAN LABEL THEM

- STATES CHARACTERIZED BY INTEGER

$$\frac{h}{e^2} \int \sigma_{xy}(i) \frac{d\theta_x}{2\pi} \frac{d\theta_y}{2\pi} = C(i)$$

First Chern character

Winding #

Topological

- STATES WITH $C(i) \neq 0$, ZEROES CAN BE MOVED TO ARBITRARY POINT IN SPACE (x, y). NOT SO FOR $C(i)=0$ STATES.

- NATURAL ASSOCIATION

$C(i) \neq 0$	FINITE σ_{xy}	ITINERANT ZEROES	"EXTENDED"
$C(i) = 0$	$\langle \sigma_{xy} \rangle = 0$	LOCALIZED ZEROES	"LOCALIZED"

$$\sigma_{xy} \sim \langle j_x \, j_y \rangle$$

$$= \frac{i e^2 \hbar}{A} \sum_n \frac{\langle 0|\hat{v}_y|n\rangle\langle n|\hat{v}_x|0\rangle - \langle 0|\hat{v}_x|n\rangle\langle n|\hat{v}_y|0\rangle}{(E_n - E_0)^2}$$

$$= \nu(E)\,\frac{e^2}{h} + \frac{i e^2}{h N_\phi} \sum_{n \neq m} \frac{\Box}{(E_m - E_n)^2}$$

$$\Box = \langle m|\frac{\partial V}{\partial x}|n\rangle\langle n|\frac{\partial V}{\partial y}|m\rangle - \langle m|\frac{\partial V}{\partial y}|n\rangle\langle n|\frac{\partial V}{\partial x}|m\rangle$$

$$\dot{x} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}[V, x] = \frac{e}{cB}\frac{\partial V}{\partial y}$$

Can use $C(i)$ to pick out states giving rise to step in σ_{xy} in thermodynamic limit (Y. Huo + R.N.B., PRL 68, 1375 (92))

Where are these states as $N_s \to \infty$?

<u>Result</u>:

- WIDTH OF $C(i) \neq 0$ STATES
  $\Delta E \to 0$ as $N_s \to \infty$ $\quad\quad \Delta E \sim N_s^{-x}$
  $\Rightarrow$ SINGLE CRITICAL ENERGY

- EXPECT WIDTH TO BE SCALING SUCH THAT $\xi(E - E_c = \Delta E) \sim L$
  $\Rightarrow \quad x = \frac{1}{2\nu}$ $\quad\quad \nu = 2.4 \pm 0.1$
  from $N_s = 8$ to 128
  (much smaller sizes)

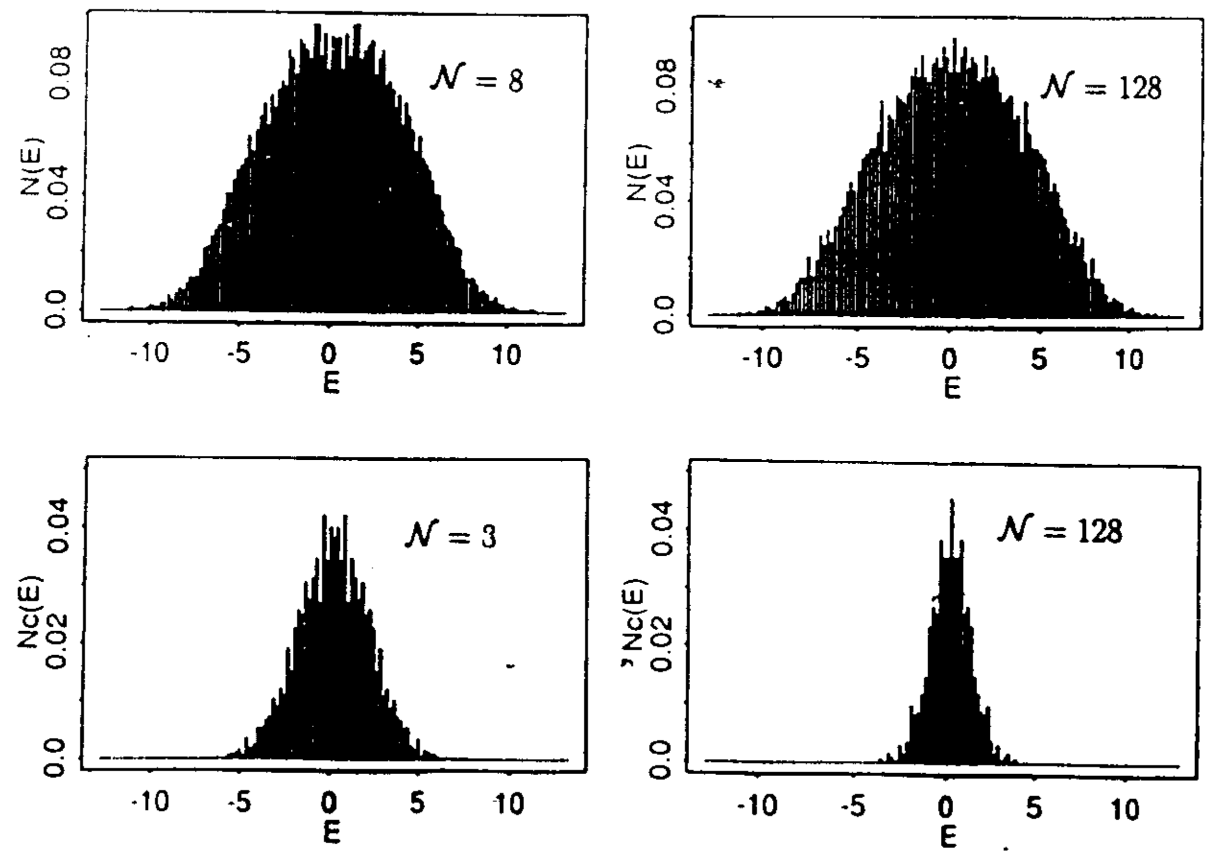
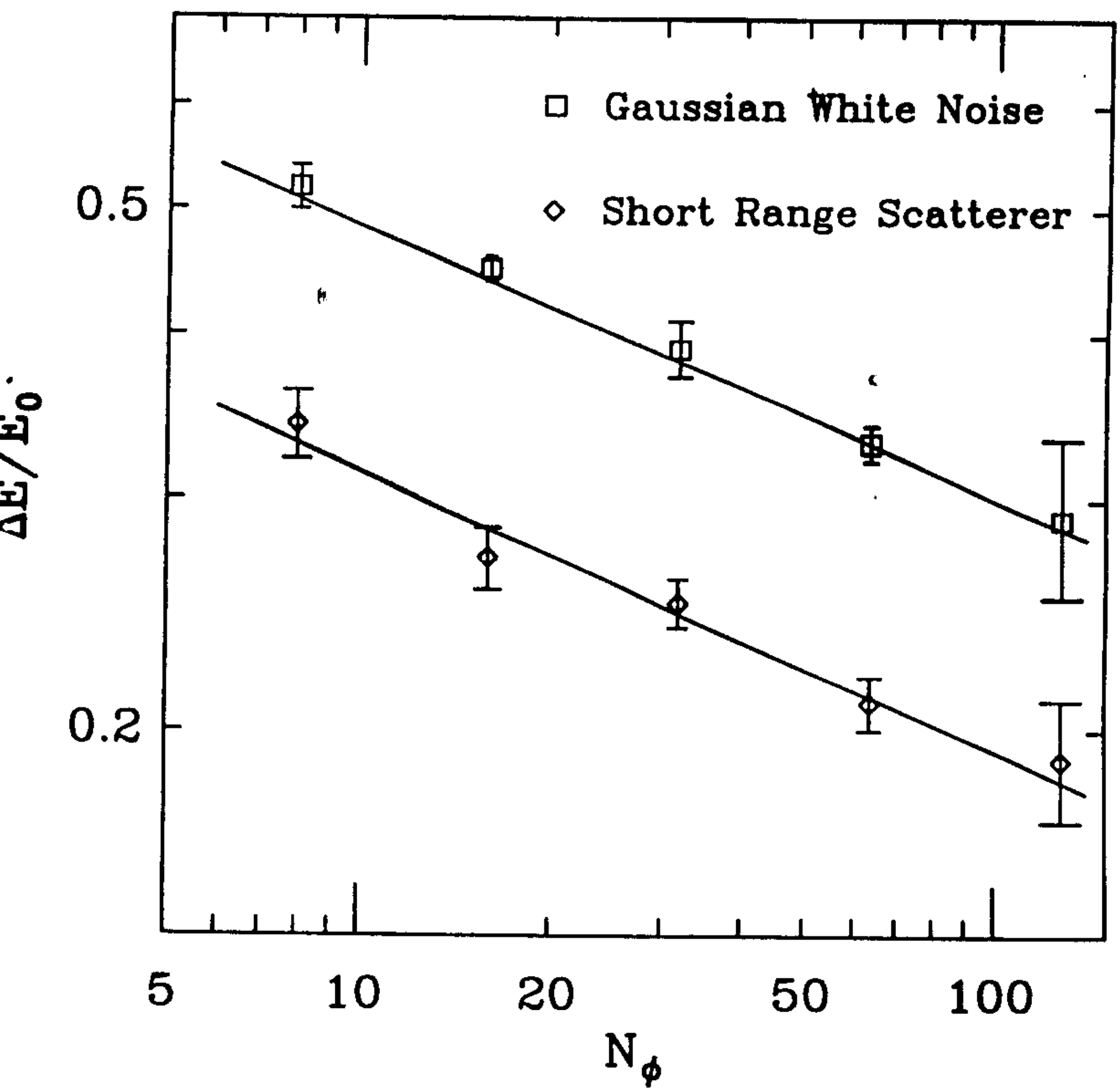


FIG. 2. Density of states of all states $N(E)$, as well as the density of nonzero Chern number states $N_c(E)$ for samples with fl quanta $\mathcal{N}$ equal to 8 and 128.



Gaussian White Noise
Short Range Scatterer
0.5
0.2
$\Delta E/E_0$
5
10
20
50
100
N_ϕ

(16) UNIVERSALITY OF σ_{xy}, σ_{xx}

- FOR SYMMETRIC POTENTIALS
  $\langle \sigma_{xy} (E = E_c = 0) \rangle = \frac{1}{2} \frac{e^2}{h}$
  HOW ABOUT ASYMMETRIC POTENTIALS?
  WHAT ABOUT DISTRIBUTION OF σ_{xy}?
- AT E_c, IS σ_{xx} FINITE? HOW DOES IT DEPEND ON POTENTIAL?

(Universality claimed by Lee, Kivelson, Zhang, PRL 68, 2386 ('92)
on basis of mapping of CS Field theory onto problem of bosons and certain assumptions)

Numerical Calculations by
Y. Huo, R. E. Hetzel and RNB
PRL 70, 481 ('93)

$$S(\vec{r}, E, \omega) = \left\langle \sum_{\alpha\beta} \delta\left(E - \frac{\omega}{2} - E_\alpha\right) \delta\left(E + \frac{\omega}{2} - E_\beta\right) \right\rangle$$

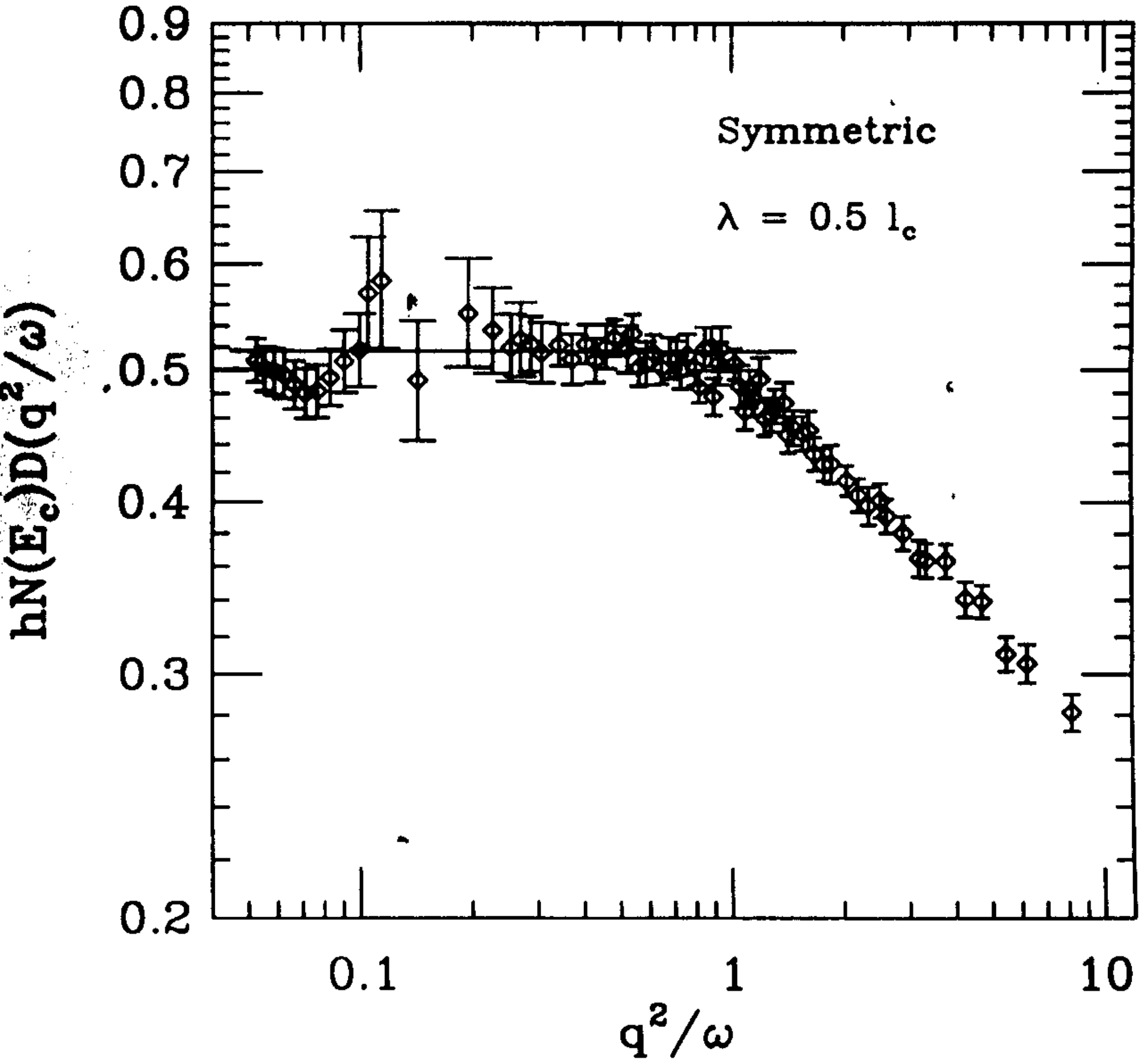
$$\psi_\alpha(0)\, \psi_\alpha^*(r)\, \psi_\beta(\vec{r})\, \psi_\beta^*(0) \rangle$$

$$\omega S(q; E, \omega) \sim \frac{\rho(E)}{\pi} \omega \frac{D(q,\omega)\, q^2}{\omega^2 + (D(q,\omega) q^2)^2}$$

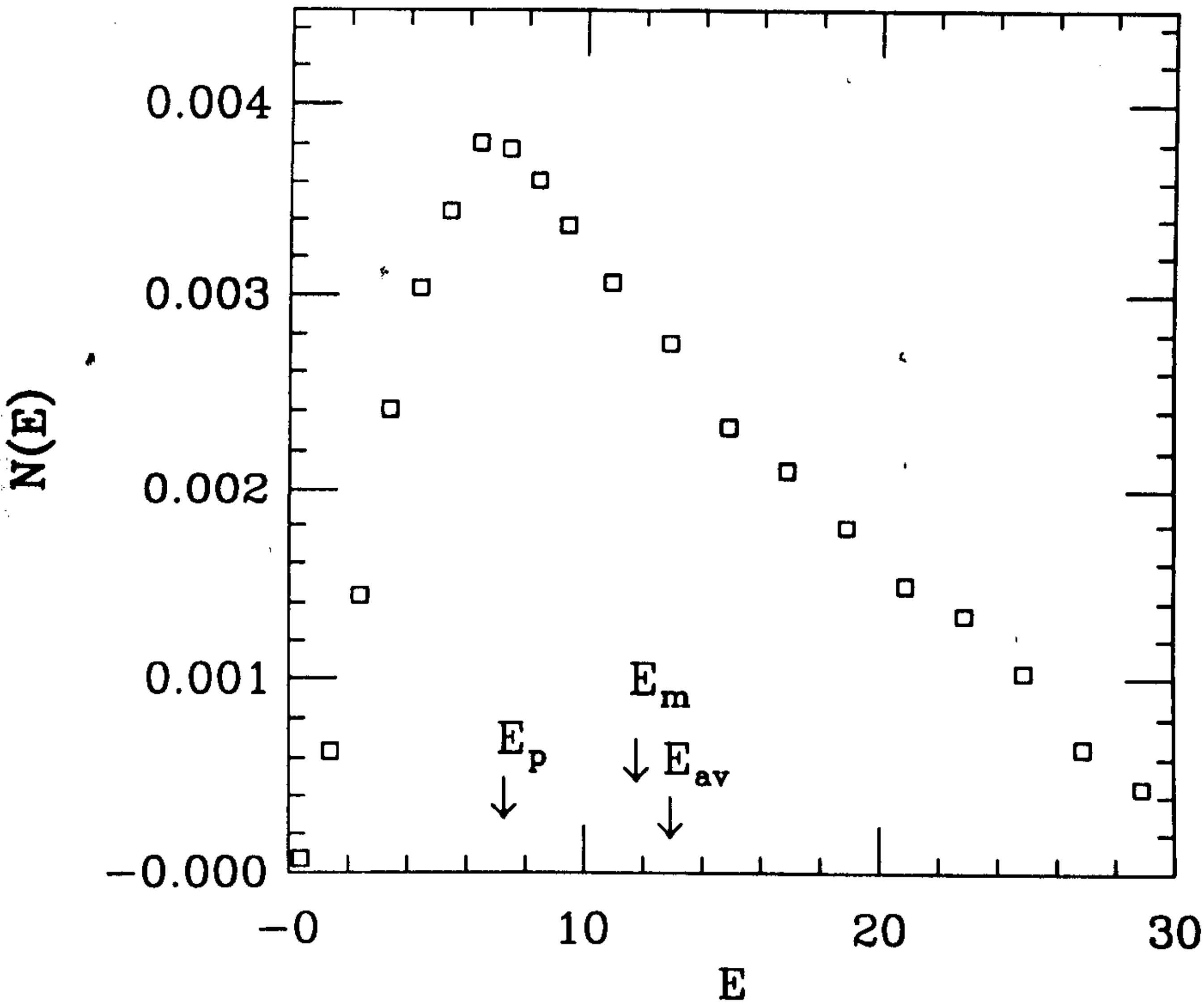
$$\omega S(q; E_c, \omega) \sim \frac{\rho(E_c)}{\pi} \frac{(\hbar q^2/\omega)\, D(q^2/\omega)}{1 + \left(\hbar \frac{q^2}{\omega}\right)^2 D^2(q^2/\omega)}$$

at critical energy.

$$\sigma_{xx}^c = e^2 \rho(E_c) \lim_{q^2/\omega \to 0} \lim_{\substack{\omega \to 0 \\ q \to 0 \\ q^2/\omega \\ \text{fixed}}} D(q^2/\omega)$$



Symmetric
$\lambda = 0.5\ l_c$
$hN(E_c)D(q^2/\omega)$
q^2/ω
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1
10



0.004
0.003
0.002
0.001
−0.000
N(E)
−0
10
20
30
E
E_p
E_m
E_av

SQUARE (TWO-DIMENSIONAL) GEOMETRY
(Arovas, Bhatt, Littlewood-86, unpublished) ($L \times L$)
(N_s flux quanta)

- RANDOM POTENTIAL - GAUSSIAN SCATTERERS (RANGE λ, EQUAL + & -)

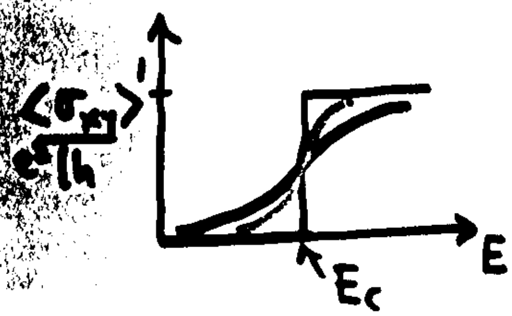
- e-h SYMMETRY GUARANTEED BY INCL. BOTH $V(\vec{r})$ and $-V(\vec{r})$ IN ENSEMBLE

- CALCULATE $\langle \sigma_{xy}(E, L) \rangle$

For $\xi, L \gg \ell, \lambda$ expect $\langle \sigma_{xy}(E,L) \rangle$ to be fn. of L via L/ξ only (finite size scaling):

$$\langle \sigma_{xy}(E,L) \rangle = f\left[L^{1/\nu}(E - E_c) \right]$$

for $\xi \sim |E - E_c|^{-\nu}$ $E_c = 0$



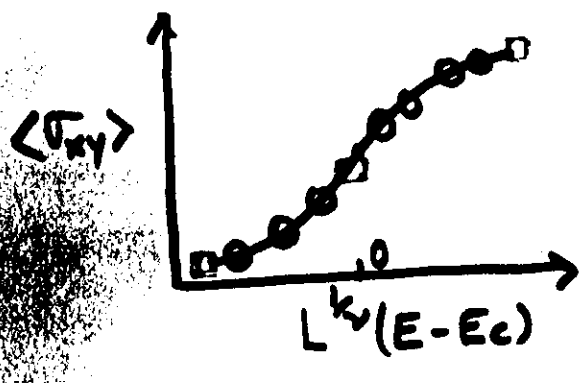
$L_1 > L_2$

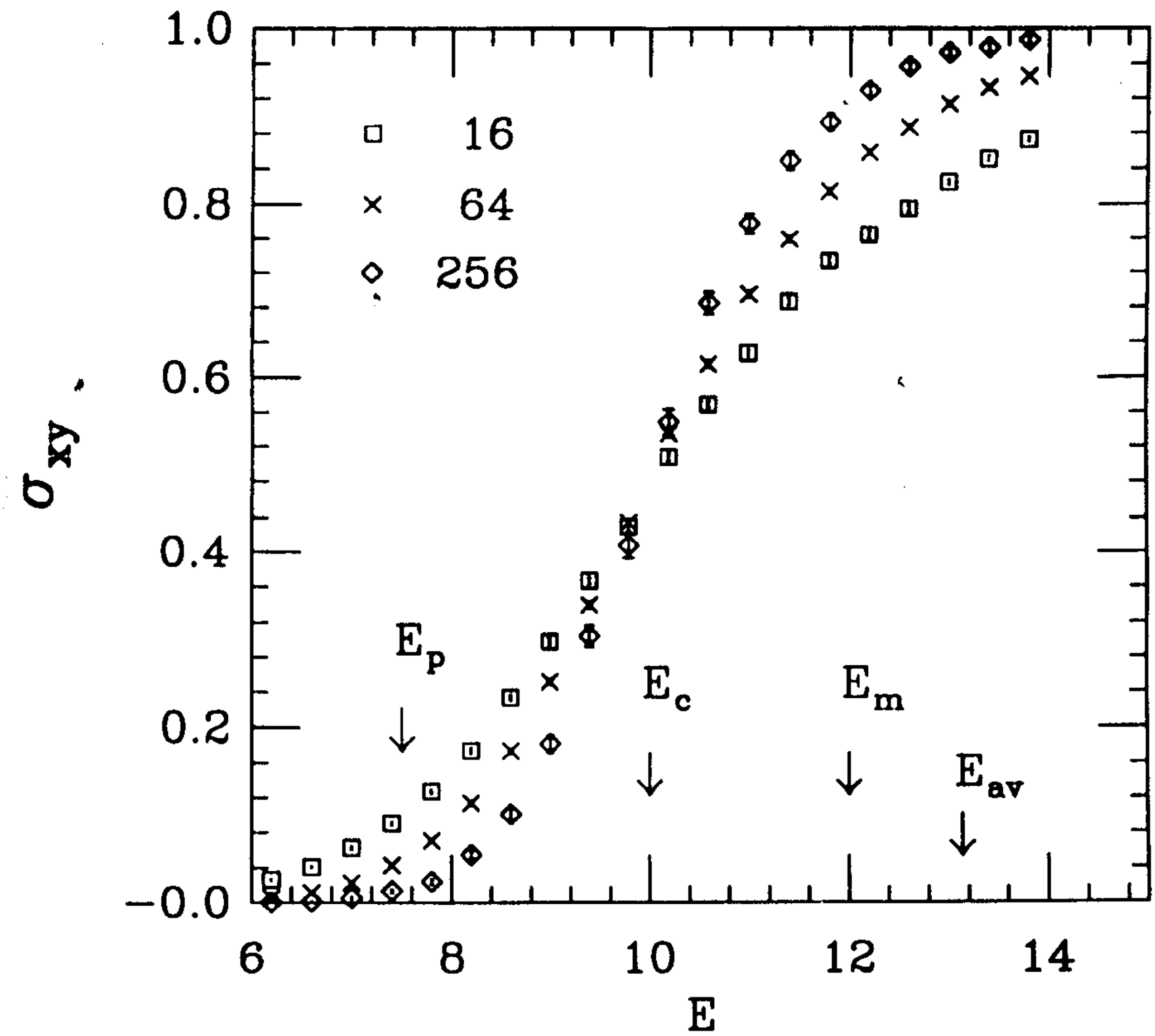
Scaling curves gives ν
Data consistent with $\xi \sim (E - E_c)^{-\nu}$

$\nu \approx 2.2$ $\lambda = .5 \ell$

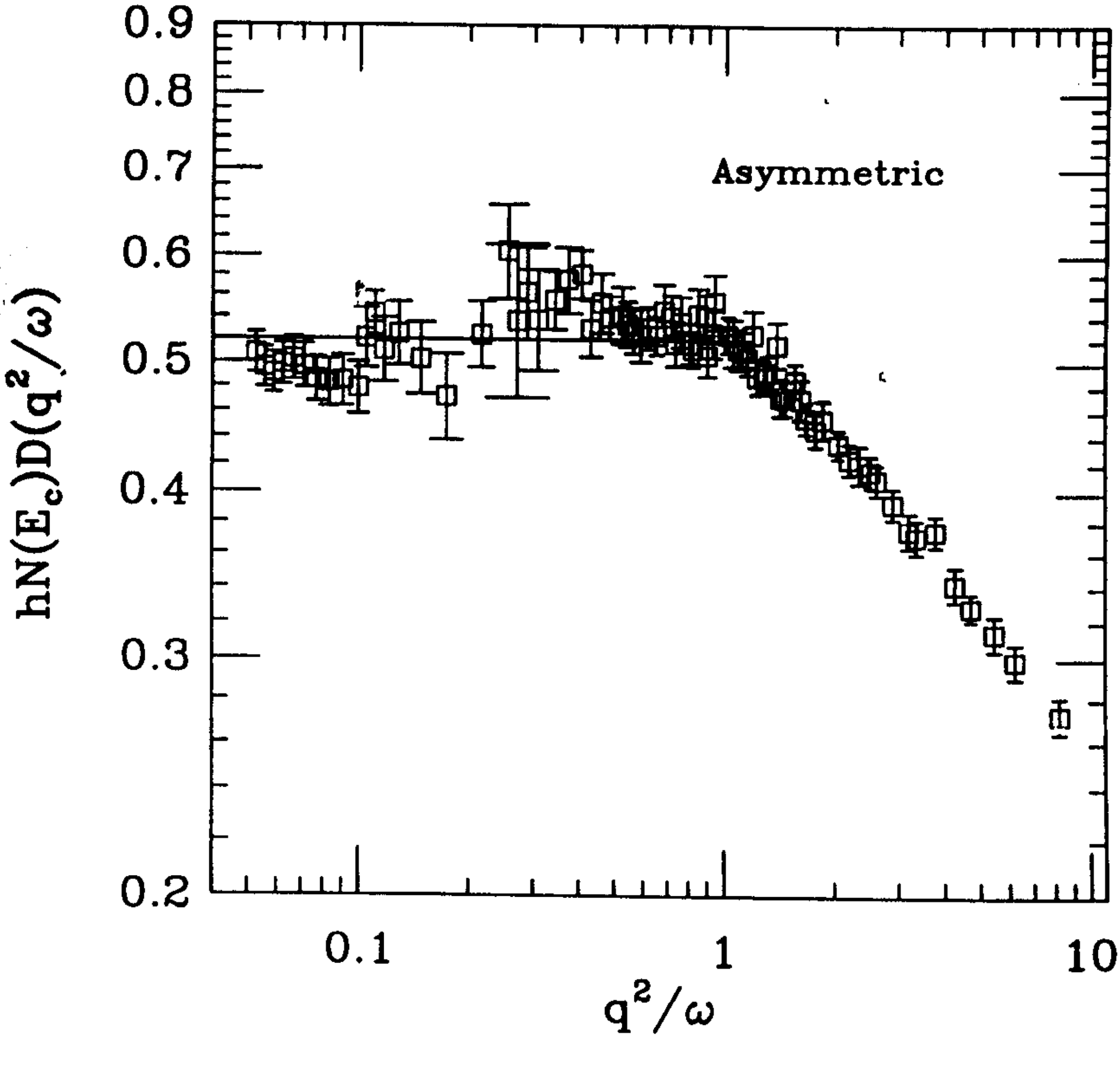
$\nu \approx 1.9$ $\lambda = 2 \ell$

using N_s upto 400
averaging ~ 1000-2000 samples per N_s





16
64
256
E_p
E_c
E_m
E_{av}
σ_{xy}
E
1.0
0.8
0.6
0.4
0.2
−0.0
6
8
10
12
14



Asymmetric
hN(E$_c$)D(q^2/ω)
q^2/ω
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1
10

TABLES

TABLE I. Results of σ^c_{xx} and σ^c_{xy} for five different potentials defined in text.

Potential	σ^c_{xx}	σ^c_{xy}
(*i*) Symmetric, $\lambda = 0$	0.54 ± 0.04	$1/2^*$
(*ii*) Symmetric, $\lambda = 0.5\ell$	0.52 ± 0.04	$1/2^*$
(*iii*) Symmetric, $\lambda = \ell$	0.55 ± 0.05	$1/2^*$
(*iv*)Asymmetric, $skew[N(E)] = 0.49$	0.50 ± 0.03	0.50 ± 0.02
(*v*)Asymmetric, $skew[N(E)] = 0.91$	0.52 ± 0.04	0.48 ± 0.03

(*) from symmetry considerations.

USUALLY, FINITE SIZE SCALING DONE WITH SELF AVERAGING QUANTITIES (Q)

SO $\lim_{L \to \infty} \mathrm{VAR}(Q) / \langle Q \rangle^2 \to 0$

FOR CONDUCTANCE, (Lee, Stone, Altshuler et al)

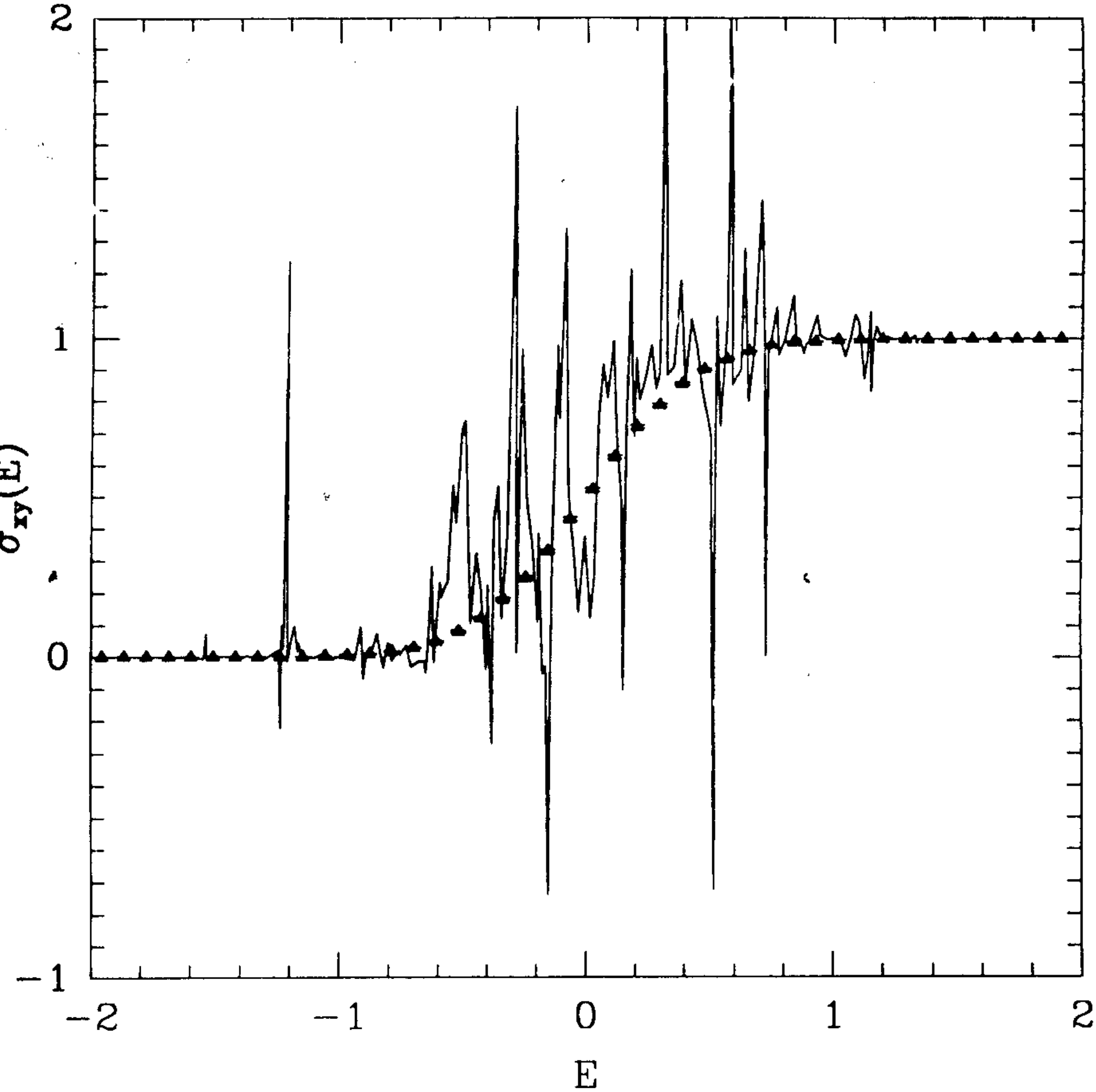
$\Rightarrow$ IN DIRTY METAL $\mathrm{VAR}(G) \sim (e^2/h)^2$

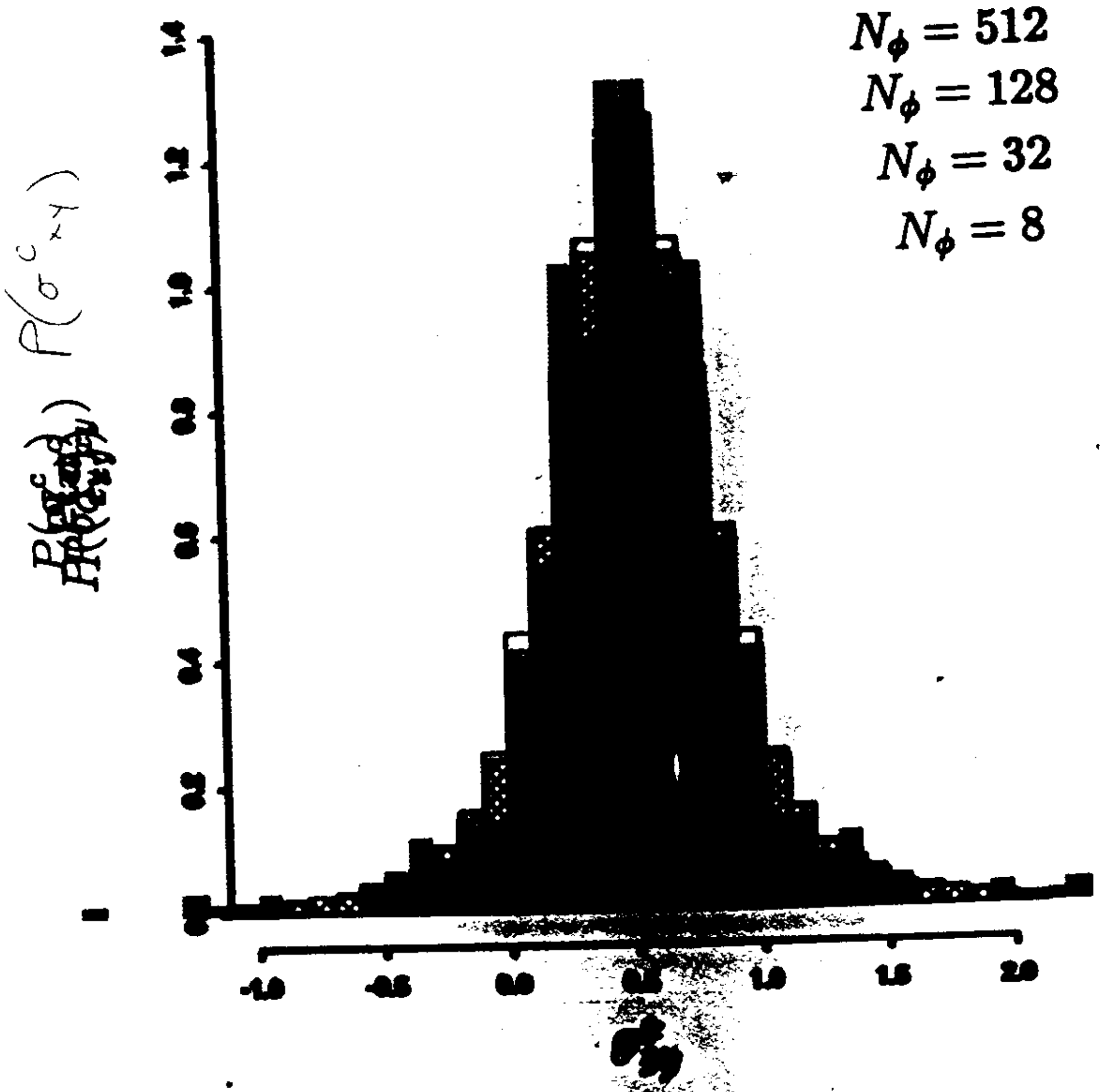
$\langle G \rangle \sim L^{d-2}$

$\Rightarrow \lim_{L \to \infty} \mathrm{VAR}(G) / \langle G \rangle^2 \to 0$

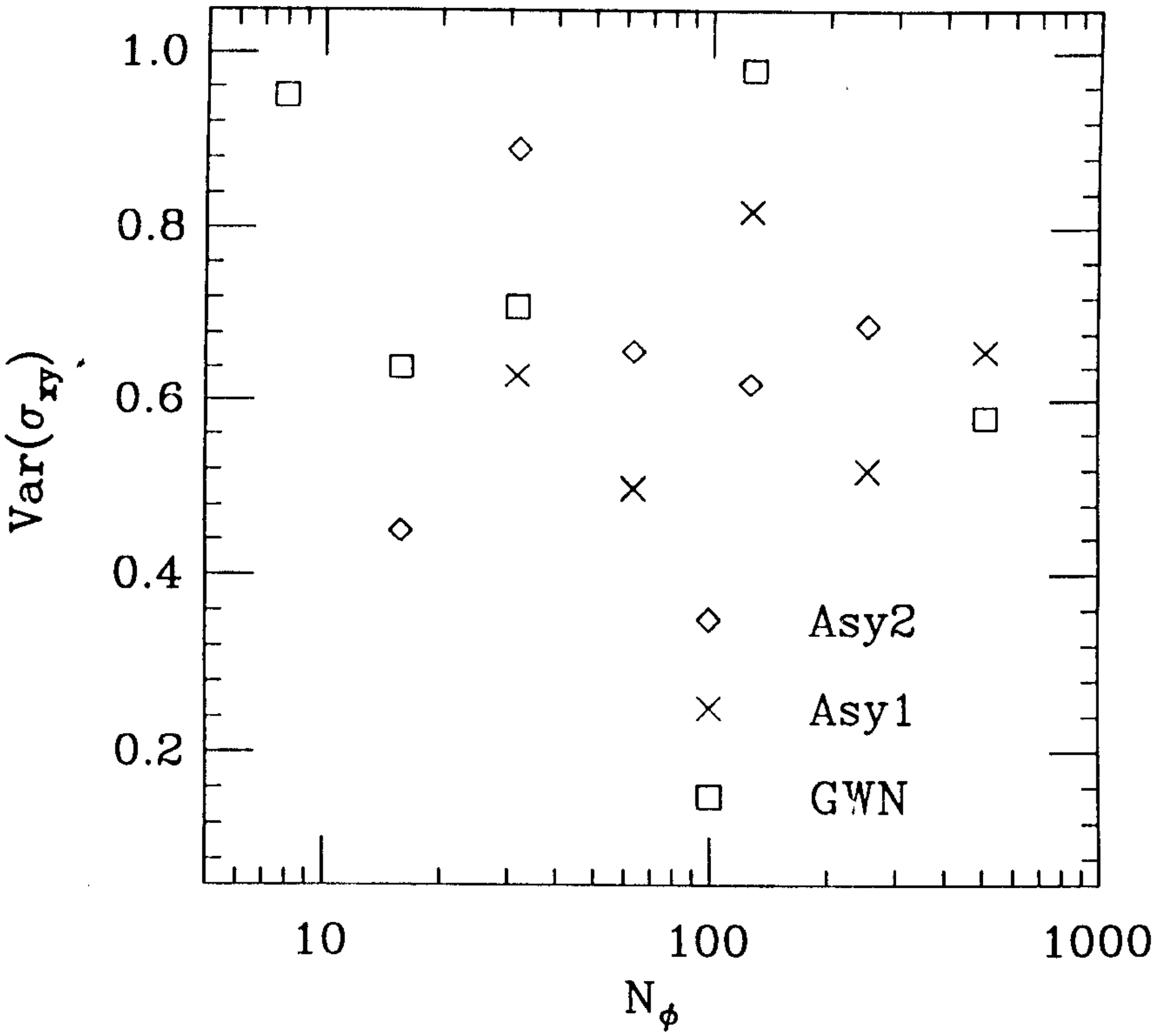
NOT SO IN DISORDERED INSULATOR WITH RESISTANCE.

WHAT ABOUT CRITICAL POINT, e.g. E_c IN IQHE ?

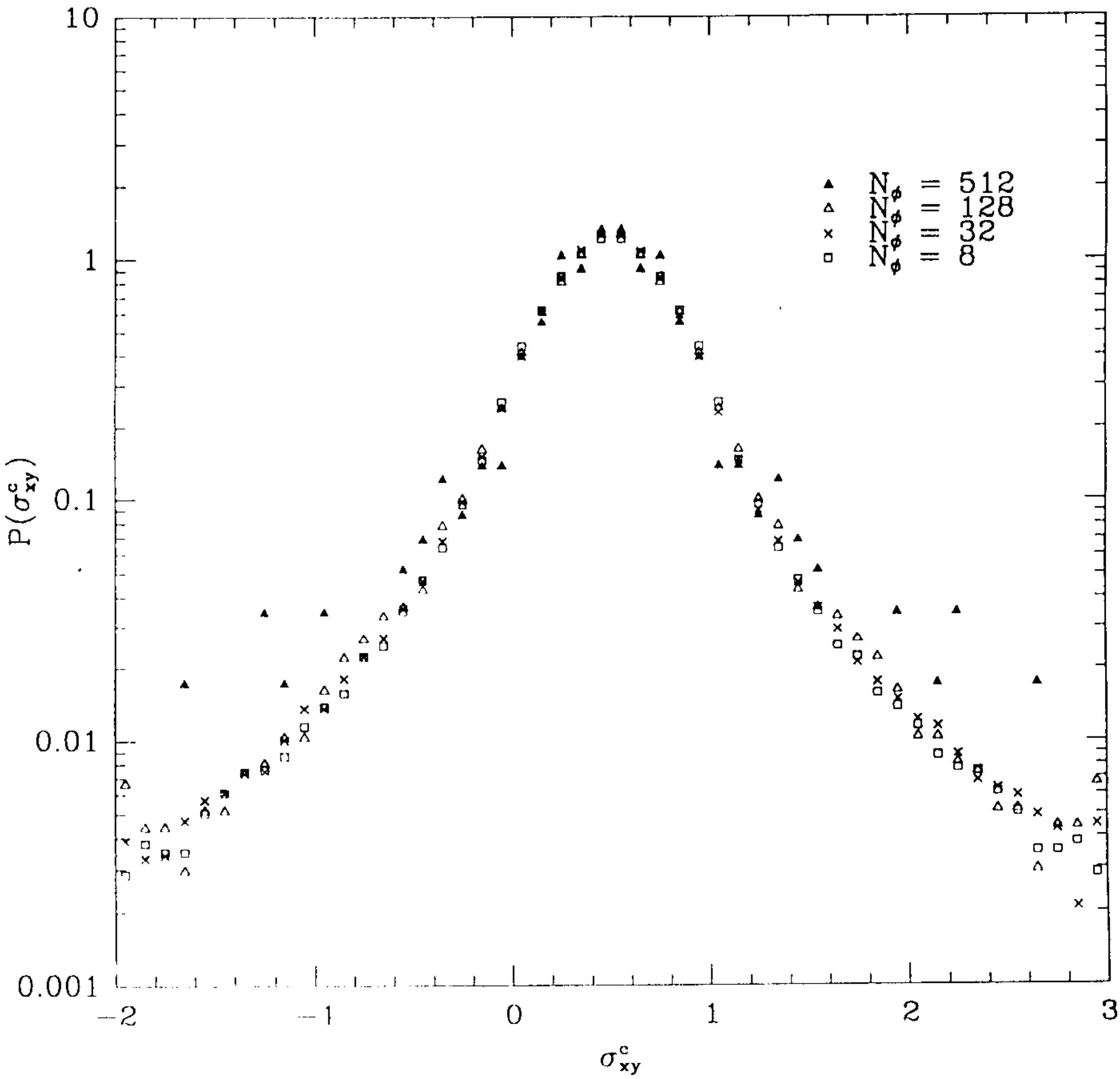

$\sigma_{xy}(E)$
E
2
1
0
−1
−2
−1
0
1
2



$P(\sigma^c_{xy})$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
σ^c_{xy}
$N_\phi = 512$
$N_\phi = 128$
$N_\phi = 32$
$N_\phi = 8$



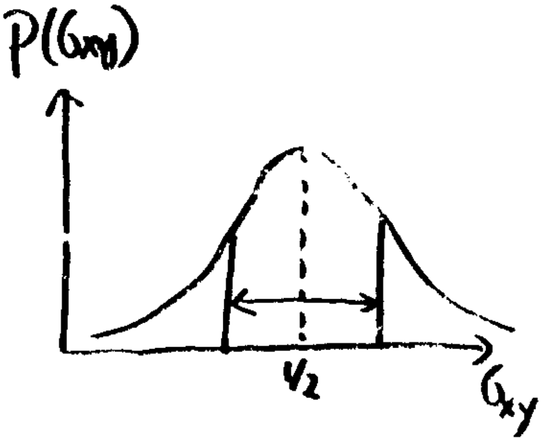
1.0
0.8
0.6
0.4
0.2
Var(σ_{xy})
10
100
1000
N_ϕ
Asy2
Asy1
GWN



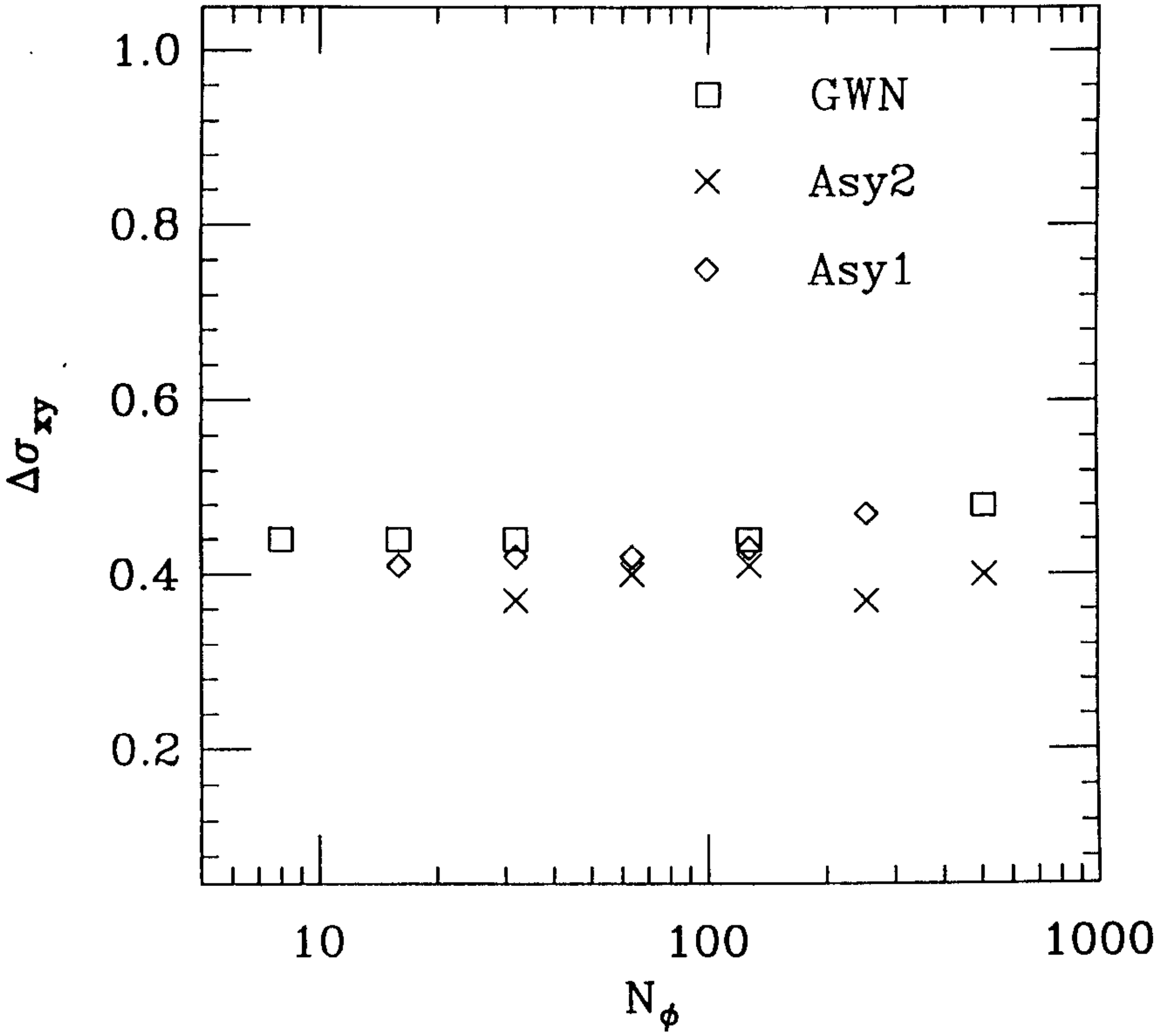
10
1
0.1
0.01
0.001
P(σ^c_{xy})
−2
−1
0
1
2
3
σ^c_{xy}
N_ϕ = 512
N_ϕ = 128
N_ϕ = 32
N_ϕ = 8

$$\int_{-\infty}^{\sigma_{xy}^{1}} P(\sigma_{xy})\, d\sigma_{xy} = \frac{1}{4}$$

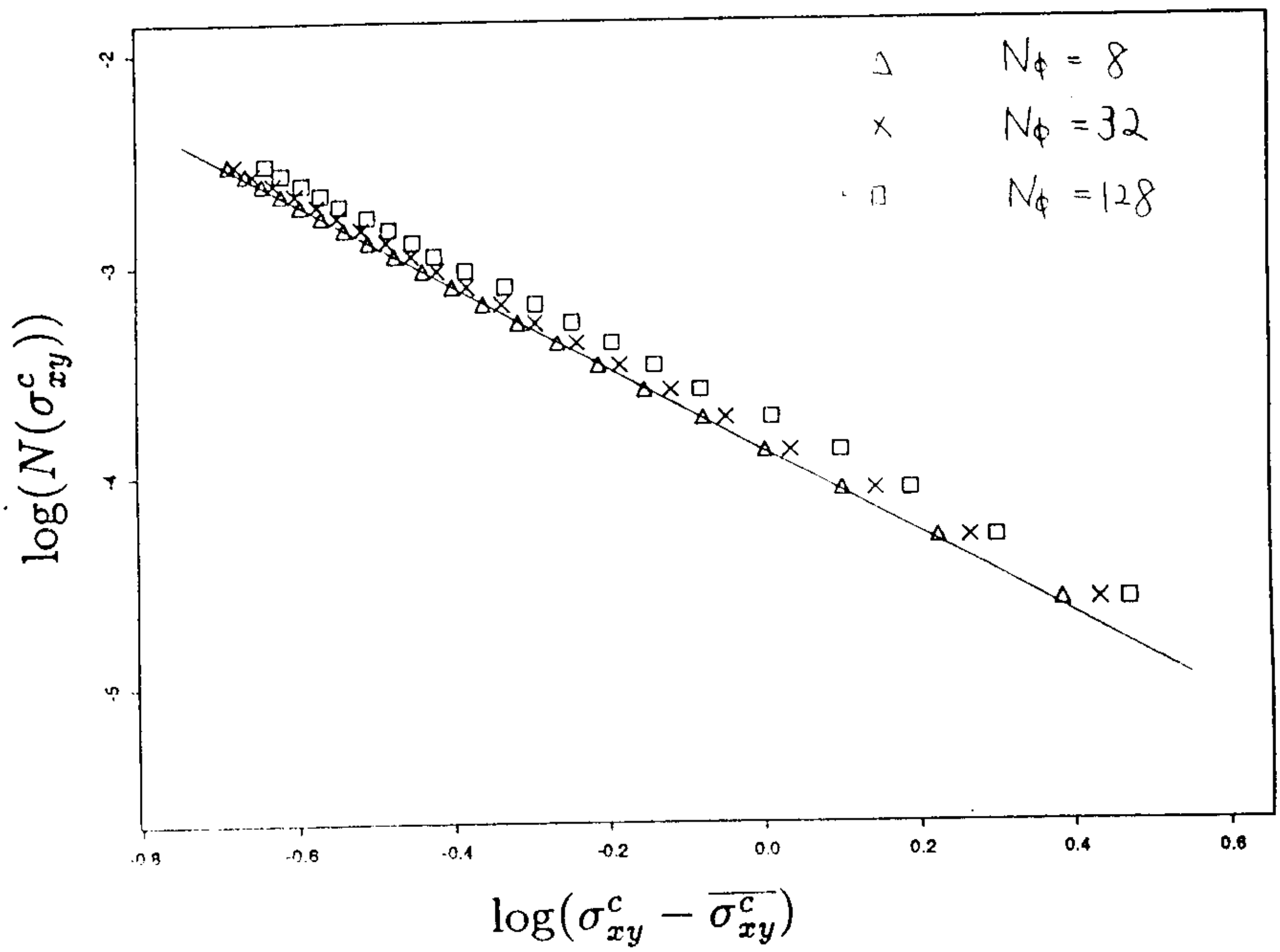
$$\int_{-\infty}^{\sigma_{xy}^{2}} P(\sigma_{xy})\, d\sigma_{xy} = \frac{3}{4}$$



$$\Delta\sigma_{xy} = \sigma_{xy}^{2} - \sigma_{xy}^{1}$$



$$N(x) = \int_x^\infty P(y)dy$$



$N(x) \sim x^{-\alpha}$ [redacted]

$P(x) \sim x^{-(\alpha+1)}$

$\alpha \lessapprox 1.9 \Rightarrow$ 2nd moment does <u>not</u> exist !

POWER LAW TAILS OF $P(\sigma^c_{xy})$

$$\sigma_{xy} \sim \sum_{i,j} \frac{|M_{ij}|^2}{(E_i - E_j)^2}$$

Large value of σ_{xy} due to near degeneracy of pairs of eigenstates

If $M \sim$ constant $\quad E = E_i - E_j$

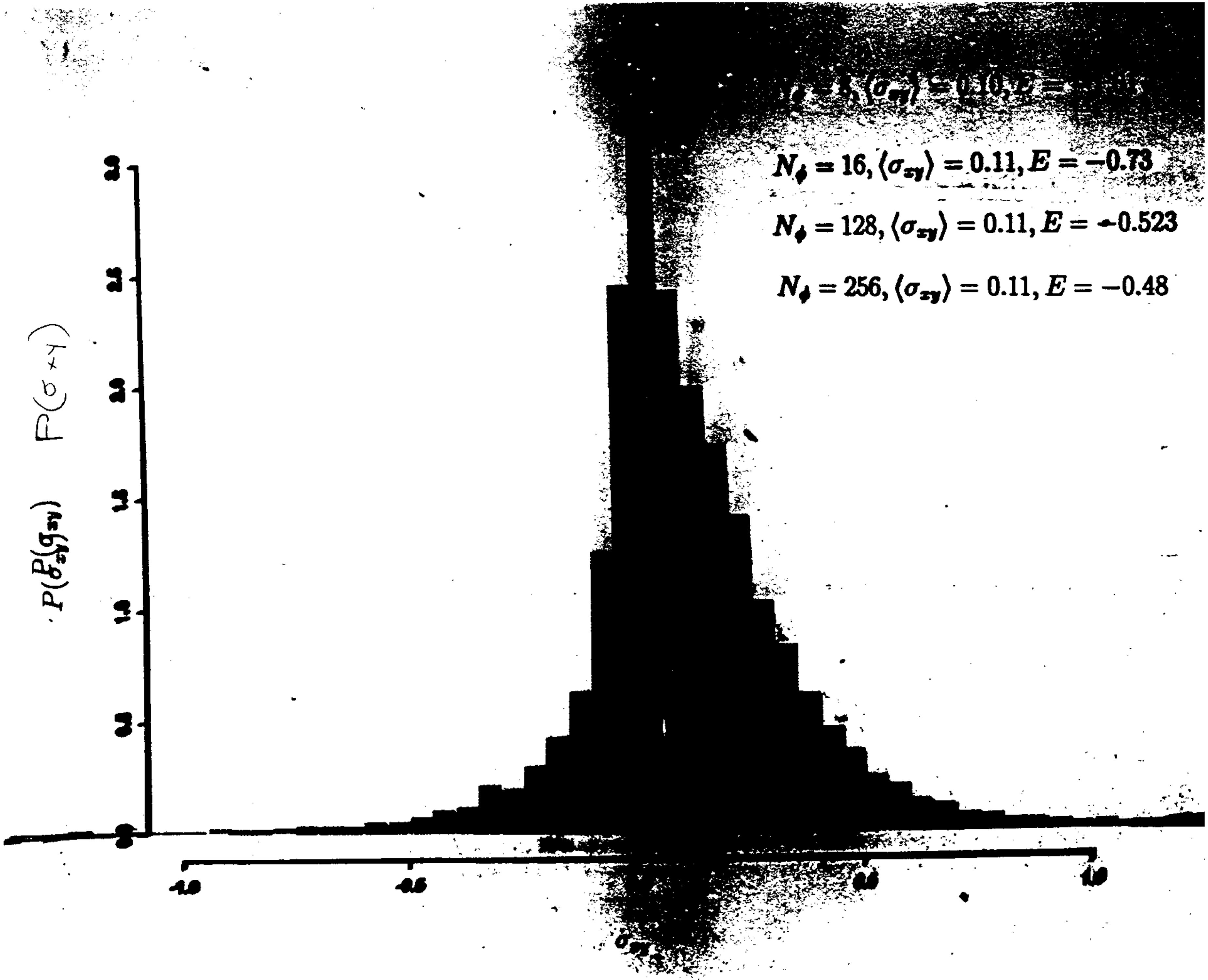
$$P(\sigma)\, d\sigma = \phi(E)\, dE$$

For GUE $\phi \sim E^2$ for small E, $\sigma \sim \frac{1}{E^2}$

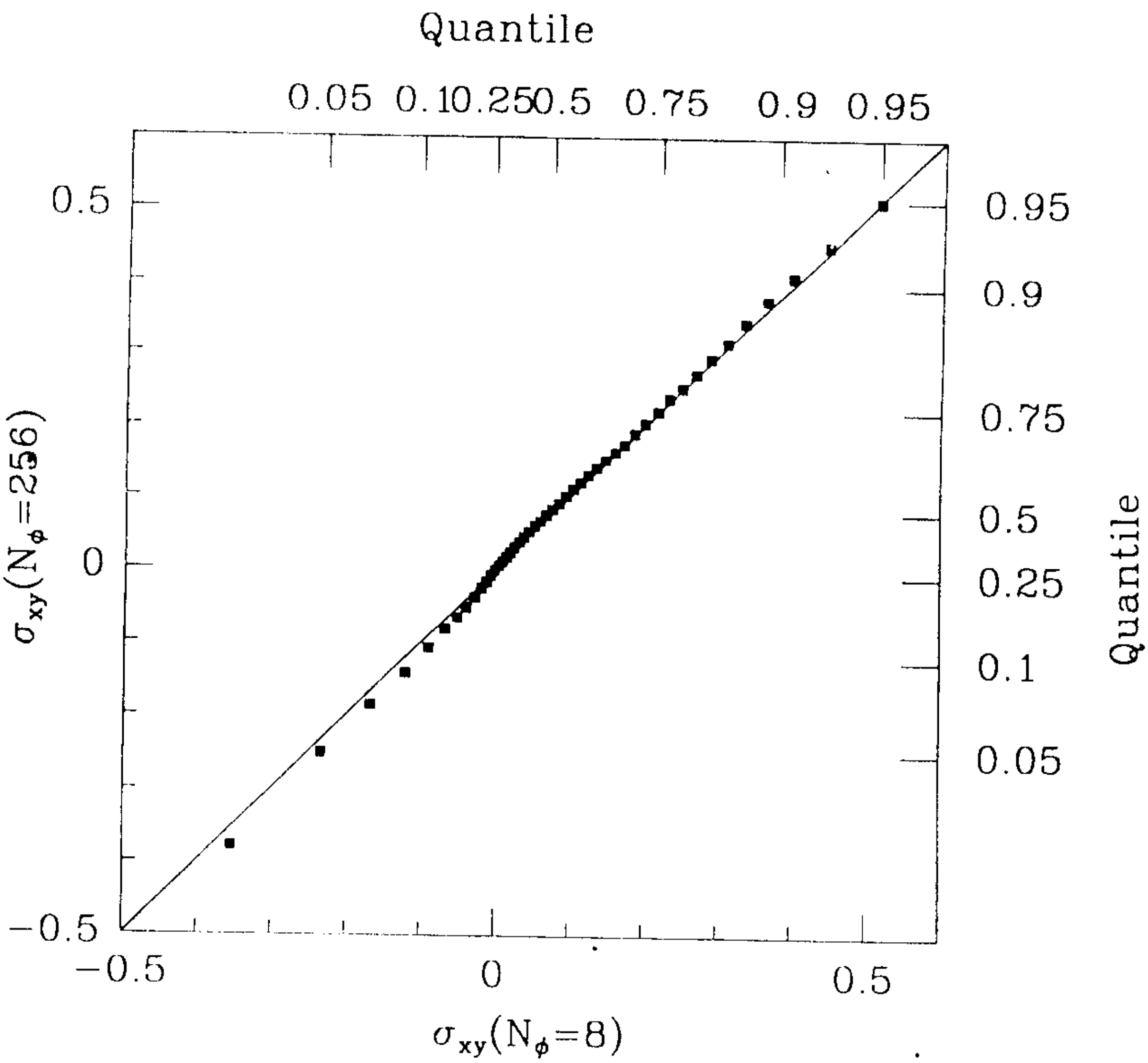
$$\therefore \quad P(\sigma) = \phi(E) \frac{dE}{d\sigma} \sim E^2 \cdot E^3 \sim E^5$$

$$\sim \frac{1}{\sigma^{2.5}}$$

Need to check what M does.

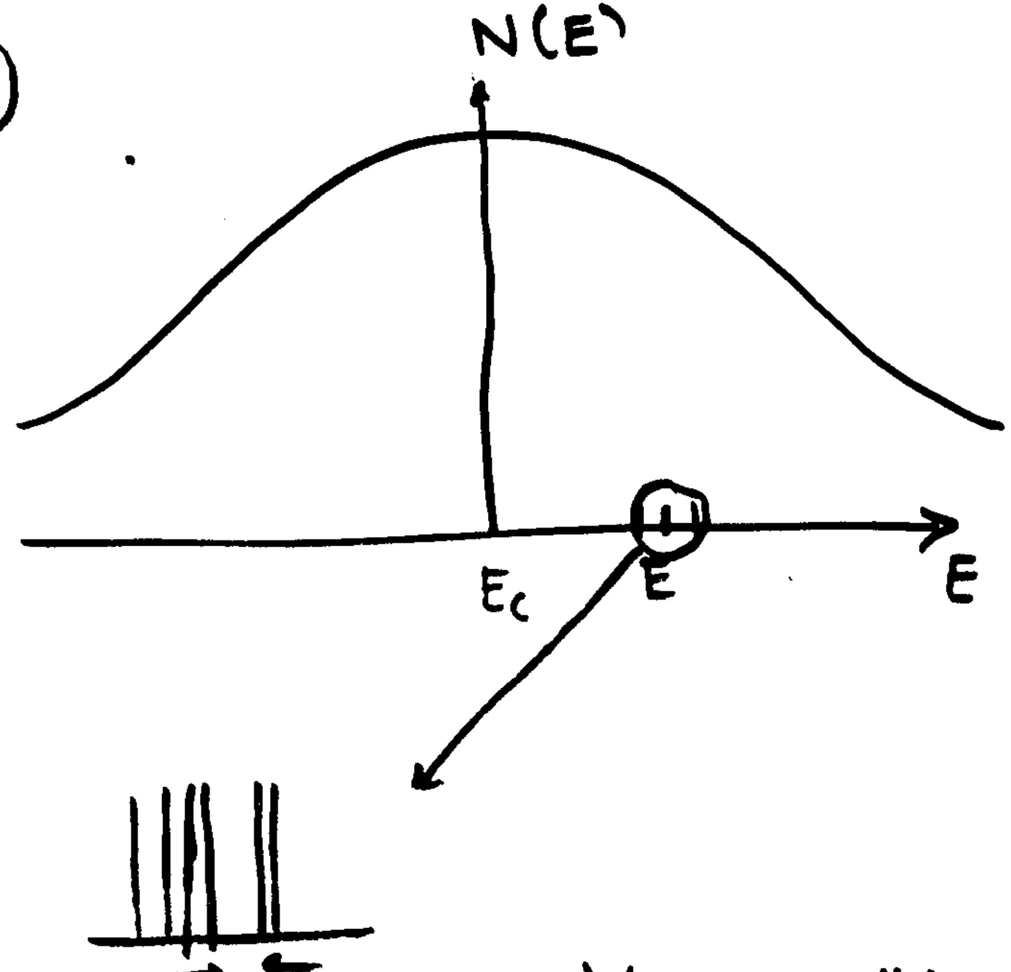


$N_\phi = 16, \langle \sigma_{xy} \rangle = 0.11, E = -0.73$
$N_\phi = 128, \langle \sigma_{xy} \rangle = 0.11, E = -0.523$
$N_\phi = 256, \langle \sigma_{xy} \rangle = 0.11, E = -0.48$
$P(\sigma_{xy})$
P(σ xy)



Quantile
0.05 0.1 0.25 0.5 0.75 0.9 0.95
0.5
0
−0.5
$\sigma_{xy}(N_\phi=256)$
0.95
0.9
0.75
0.5
0.25
0.1
0.05
Quantile
−0.5 0 0.5
$\sigma_{xy}(N_\phi=8)$

(5)

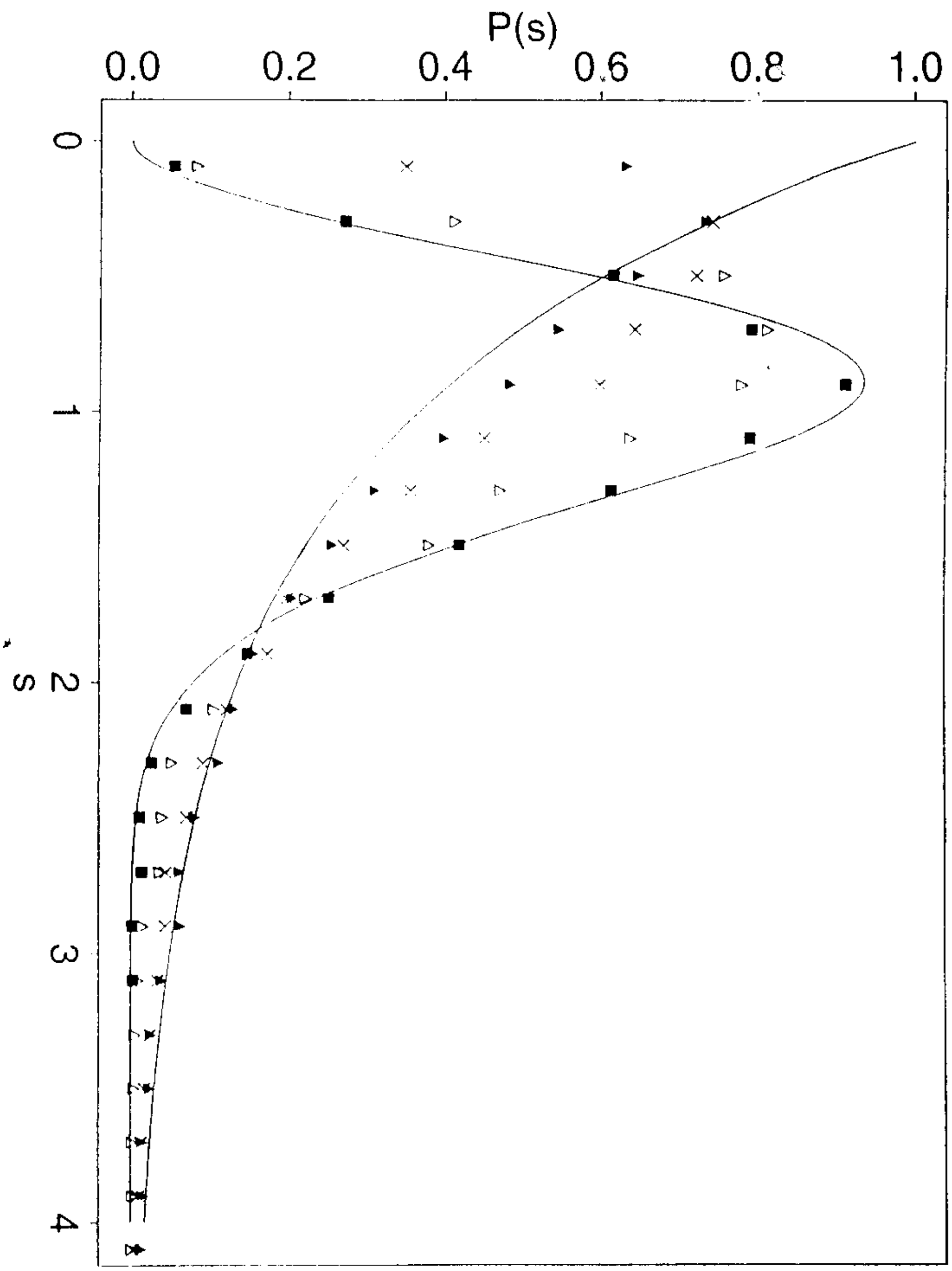


$s = (E_n - E_{n-1}) /$ mean splitting

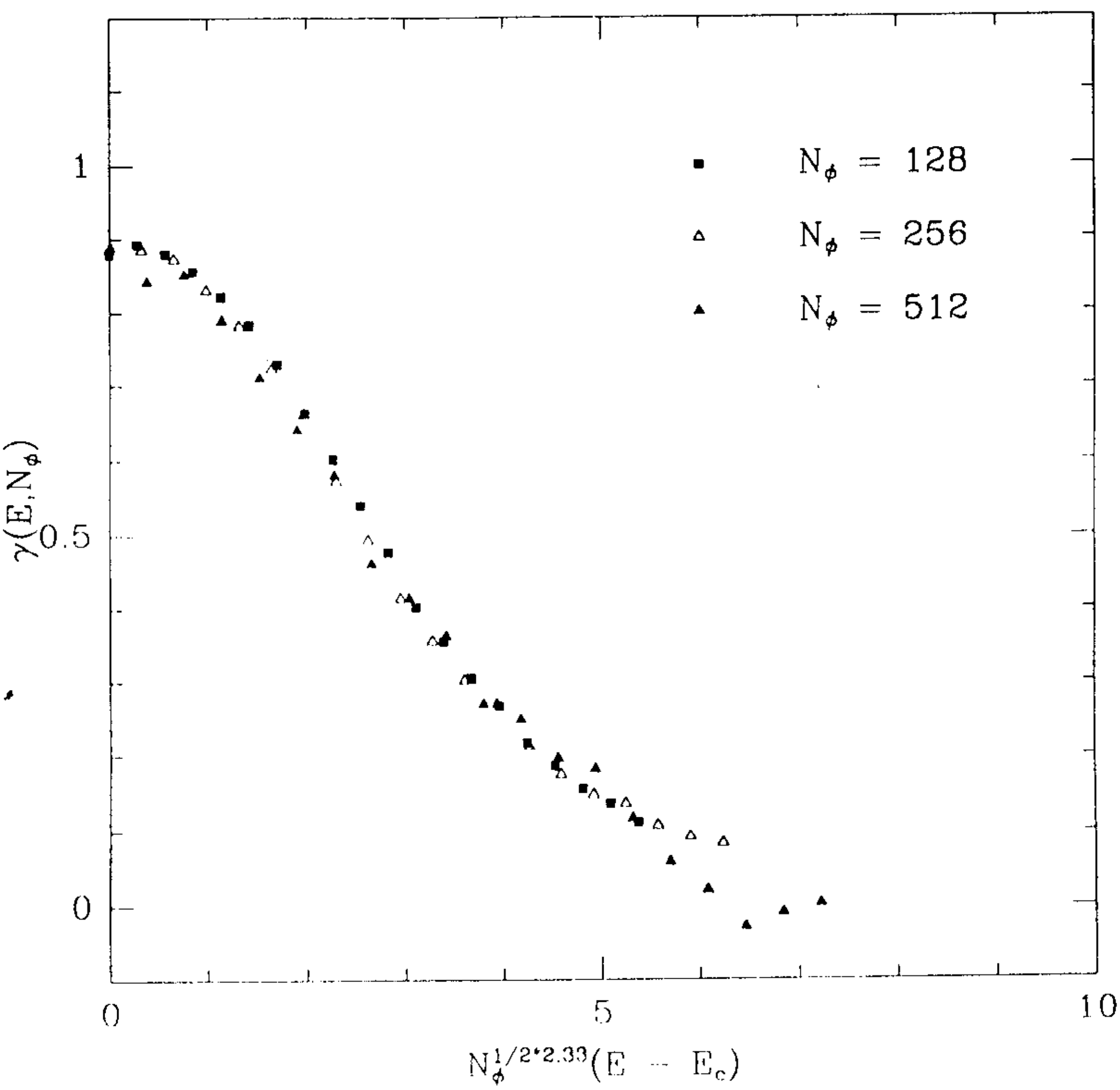
$P(s)$ → distribution of s

→ Poisson for localized states $P(s) \sim e^{-s}$

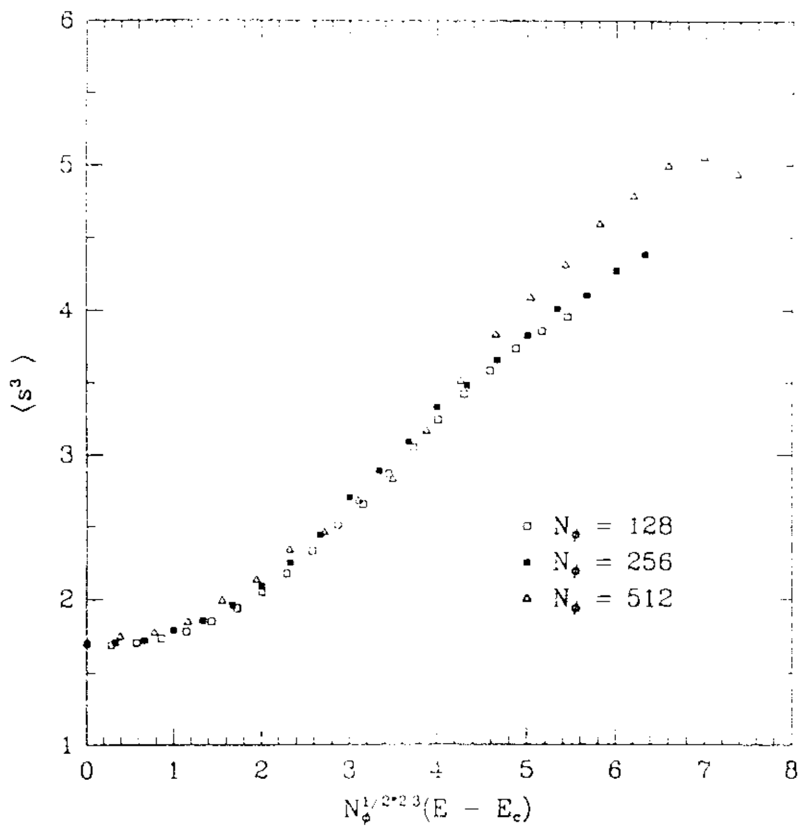
→ GUE for "extended" states
$$P(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}$$ Wigner surmise.



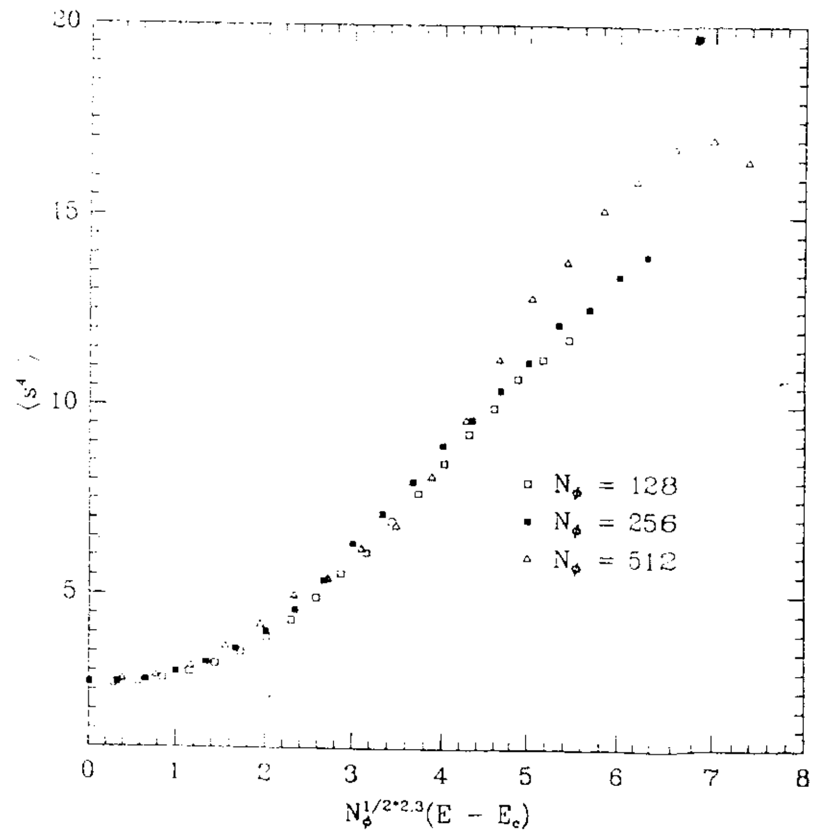
P(s)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
s



N_φ = 128
N_φ = 256
N_φ = 512
γ(E,N_φ)
1
0.5
0
0
5
10
N_φ^{1/2*2.33}(E − E_c)



$\langle s^3 \rangle$
$N_\phi = 128$
$N_\phi = 256$
$N_\phi = 512$
$N_\phi^{1/2*2.3}(E - E_c)$



$\langle s^4 \rangle$
$N_\phi = 128$
$N_\phi = 256$
$N_\phi = 512$
$N_\phi^{1/2*2.3}(E - E_c)$

CONCLUSIONS

- FOR SIMPLE SHORT RANGE POTENTIALS SINGLE CRITICAL ENERGY IN L.L.L.

  $\xi \sim |E - E_c|^{-\nu}$ $\nu \approx 2.35 \pm 0.1$

- STATES CHARACTERIZED BY TOPOLOGICAL WINDING NUMBER (CHERN CHARACTER)
  - RELATED TO EXTENSIVITY OF STATE

- $\langle \sigma_{xy} \rangle = 0.5 \frac{e^2}{h}$ at E_c

  $P(\sigma_{xy})$ SYMMETRIC & UNIVERSAL

  LONG TAILS - ? (FLUCTUATIONS)

- $\sigma_{xx} \to 0.5 \frac{e^2}{h}$ at E_c for

  $\lambda \leq l \Rightarrow$ UNIVERSALITY ?

  LONG RANGE POTENTIALS ?

- $P(\sigma_{xy}, E = E_c, L \gg l)$ is a universal distribution with fluctuations comparable to mean $(\sim \frac{e^2}{h})$

- Power law tails in $P(\sigma_{xy}^c)$, second moment likely does not exist.

- Nevertheless, bulk of distribution away from E_c described by one parameter $\langle \sigma_{xy} \rangle$

- Distribution of eigenvalue splittings crosses over from Poisson to WD-GUE
  — Scales with $L^{1/\nu}(E - E_c)$ with $\nu \sim 2.3$

QUESTIONS –

1. ν CLOSE TO $\nu_{c\ell} + 1$.
(ANY DEEP SIGNIFICANCE?)

2. $\nu_{EXPT} \approx \nu_{NUMERICAL}$
with e-e interactions — non-interacting

WHY? — cf. 2D case without H

3. EFFECT OF LONG TAILED DISTRIBUTIONS FOR MESOSCOPIC SYSTEMS

4. CROSSOVER FROM HIGH FIELD (critical Energies) TO $B=0$ (all states localized in 2D)

5. NON-UNIVERSALITY OF $\sigma_{xx}(E_c)$ IN EXPT.