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SMR. 758 - 45

SPRING COLLEGE IN CONDENSED MATTER
ON QUANTUM PHASES
(3 May - 10 June 1994)

BACKGROUND MATERIAL AND LECTURE NOTES ON

QUANTUM HALL EFFECT

PART II

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These are preliminary lecture notes, intended only for distribution to participants.

Background material and lecture notes on

Quantum Hall Effect

Part II

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Tues 11am
Wed 2pm
Thurs 9. am
Fri 9am

② Long-Range Order, "Composite Particles"

Girvin, in "The Quantum Hall Effect", argued for relation to superfluidity.
Girvin - MacDonald power-law Off-diagonal Long Range Order (PRL 58, 1252 (1987)) - MacDonald's lectures.
Return to it later.

Condensation usually means macroscopic coherence between states with different particle nos. In QHE, there are special densities where effect occurs. So should change number at fixed filling factor

$\Rightarrow$ add particles + flux in ratio $p:q$ at $\nu = p/q$.

to relate states.

To see effect in system at fixed number, look at correlation function (or density matrix).

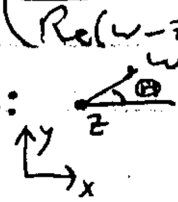
GM: consider equal time correlation function

$$\rho'(z, z') = \langle L^{(z)} | \psi^{+}(z)\, \hat{U}^{q}(z) \;\; \hat{U}^{+q}(z')\, \psi(z') \,|\, L^{(z)} \rangle$$

ψ - LLL destruction op

$$\hat{U}(z) = \prod_i e^{i\Theta(z_i, z)} \qquad \text{(first quantization)}, \qquad \Theta(w, z) = \tan^{-1}\left(\frac{\mathrm{Im}(w-z)}{\mathrm{Re}(w-z)}\right)$$

$$= \prod_i \left(\frac{z_i - z}{|z_i - z|}\right) \qquad = \text{angle:}$$



unitary operator

Note: usual density matrix

$$\rho(z, z') = \int \prod_{i=2}^{N} d^2 z_i \; \Psi_L^{*}(z, z_2, \dots z_N) \Psi_L^{(q)}(z', z_2, \dots z_N)$$
$$= \langle L^{(q)} | \psi^{+}(z)\, \psi(z') | L^{(q)} \rangle$$

Modified density matrix

$$\rho'(z, z') = \int \prod d^2 z_i \; \Psi_L^{(q)*}(z, z_2 \dots z_N) \prod_i \left(\frac{z_i - z}{|z_i - z|} \right)^{q}$$
$$\times \left(\frac{\bar{z}_i - \bar{z}'}{|z_i - z|} \right)^{q} \Psi_L^{(q)}(z', z_2 \dots z_N)$$
$$= \int \prod d^2 z_i \; |\Psi_L^{(q)}(z, z_2 \dots z_N)| \, |\Psi_L^{(q)}(z', z_2 \dots z_N)|$$
$$\sim |z - z'|^{-q/2}$$

as $|z - z'| \to \infty$ by Plasma "Analogy" after $N \to \infty$

I chose to study instead (N.R., PRL 62, 86 (1989))

$$\rho_R(z, z') = \langle L^{(q)} | U^{+q}(z)\, \psi(z) \;\; \psi^{+}(z')\, U^{q}(z') | L^{(q)} \rangle \, e^{-\frac{1}{4}|z|^2 - \frac{1}{4}|z'|^2}$$

where $U(z) = \prod_{i=1}^{N} (z_i - z)$ when acting on N ptcle system

Can evaluate by showing that

$$\langle \{z_i\}, i = 1 \dots N-1 | \psi(z) | L^{(q)}, N \rangle = \left(\frac{\nu}{2\pi}\right)^{1/2} \langle \{z_i\}, i = 1 \dots N | U^{q}(z) | L^{(q)}, N-1 \rangle$$
$$\times e^{-\frac{1}{4}|z|^2}$$

ie $\psi(z)|L^{(q)}\rangle$ and $\left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}} U^{q}(z)|L^{(q)}\rangle$ (latter with 1 ptcle less) have same wavefunction – they are the same.

ie

$$\boxed{\text{1 "real" hole} \equiv q \text{ Laughlin quasiholes at same point}}$$

(Anderson, PRB 38, 2264 (1983))

(Not nec true except for Laughlin ground state)

Hence

$$\langle L^{(q)}, N | U^{+q}\psi(z)\psi^{+}(z')U^{q}(z')|L^{(q)}, N\rangle$$

$$= \left(\frac{\nu}{2\pi}\right)^{-1}\langle L^{(q)}, N+1|\psi^{+}(z)\psi(z)\,\psi^{+}(z')\psi(z')|L^{(q)}, N+1\rangle$$

$$= \left(\frac{\nu}{2\pi}\right)^{-1}\langle L^{(q)}, N+1|\rho(z)\,\rho(z')|L^{(q)}, N+1\rangle$$

where $\rho(z) = \psi^{+}(z)\psi(z)$ is the projected density operator

$\rightarrow \left(\frac{\nu}{2\pi}\right) = \bar{\rho}$ average density, as $|z-z'|\to\infty$.

<u>True</u> long range order in $\psi^{+}(z)U^{q}(z)e^{-\frac{1}{4}|z|^{2}}$

Of course $\langle L^{(q)}, N|\psi^{+}(z)U^{q}(z)e^{-\frac{1}{4}|z|^{2}}|L^{(q)}, N\rangle \equiv 0$ since electron numbers don't match.

Like Bose condensate. If $b(\underline{x}), b^+(\underline{x})$ create, destroy bosons, $b(x) = \sum_k b_{\underline{k}} e^{i\underline{k}\underline{x}}$, groundstate (in a box)

$$|\Omega, N\rangle = \frac{(b_0^+)^N |0\rangle}{\sqrt{N!}} \quad \text{normalized}$$

(vacuum: $|0\rangle$)

$$\langle \Omega, N | b^+(\underline{x})\, b(\underline{x}') | \Omega, N\rangle = \bar{\rho} = \frac{N}{V}$$

but $\langle \Omega, N | b^+(\underline{x}) | \Omega, N\rangle \equiv 0$.

Is Laughlin state a Bose condensate?

Yes!

$$|L^{(q)}, N\rangle \propto \left(\int d^2 z\, \psi^+(z)\, U^q(z)\, e^{-\frac{1}{4}|z|^2} \right)^N |0\rangle$$

($|0\rangle$: vacuum) like $k=0$ component (even though we get a droplet, not uniform. Perfectly uniform on sphere.)

Check by calculating wavefunction, by induction.

Aside: Similarly, "one-boson excited states"

$$\int d^2 z\; e^{-i\underline{k}\cdot\underline{r}}\, \psi^+(z)\, U^q(z)\, e^{-\frac{1}{4}|z|^2}\, |L^{(q)}, N\rangle$$

$$\propto \rho_{\underline{k}} |L^{(q)}, N+1\rangle$$

= variational approx for one quantum of k collective mode [Girvin, MacDonald, Platzman PRB 33, 2481 (1986)]

which is very accurate at roton minimum, $k \simeq \left(\frac{\nu}{2\pi}\right)^{-\frac{1}{2}}$.

Binding electrons to vortices

Deepen physical understanding of Laughlin state and off-diagonal long-range order, then extend to other states.

Consider electron and q vortices, q = integer, in an otherwise uniform fluid at filling factor ν.

Density deficiency in center of q vortices $\Rightarrow$ electron is attracted there and since there's no kinetic energy, they can bind.

Can form the ground state itself by binding every electron to vortex in wavefunction of other $N-1$ electrons. At $\nu = 1/q$, q odd this gives Laughlin state:

$$\prod (z_i - z_j)^q \; e^{-\frac{1}{4}\sum |z_i|^2}$$

At q even, same factor $\prod (z_i - z_j)^q$ describes binding q vortices to every electron, but function is symmetric! Need something antisymmetric.

Nature of bound states 1 el + q vortices.

Using earlier calculations, charge $= 1 - q\nu$

adiabatic statistics is $\theta/\pi = \underset{\uparrow}{1} - q^2\nu \pmod{2\pi}$

Fermi statistics of el.

At $\nu = 1/q$, bound state is neutral, and has

$$\frac{\theta}{\pi} \equiv \begin{cases} 0 \pmod 2, & q \text{ odd} \leftrightarrow \text{Bose statistics} \\ 1 \pmod 2, & q \text{ even} \leftrightarrow \text{Fermi statistics} \end{cases}$$

"Neutral" means both that net electric charge is zero, and that it sees zero net effective magnetic field, when fluid density is uniform.

Bosons at zero mag field can Bose condense → Laughlin state
Fermions " " " can form Fermi sea
or (maybe) BCS paired superfluid.

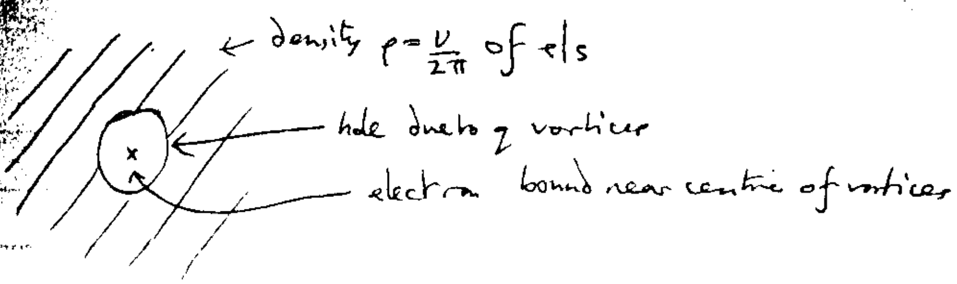
Fermi sea state is going to be a compressible fluid at even denominators $\nu = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots$

BCS paired states → incompressible fluids at $\nu = \frac{1}{2}, \ldots$
(not described further here) e.g. "Pfaffian" state, Haldane-Rezayi state

G. Moore + N.R., Nuc. Phys B360, 362 (1991) PRL 61, 1985 (1988)

Why do particles condense? To minimise their kinetic energy (neglecting interactions).

The bound states get an "effective" kinetic energy from interactions between electrons.

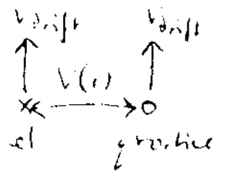


Electron experiences potential $V(r)$ attracting it to center of vortices. Same potential attracts q vortices to electron.

If electron is displaced from center of vortices, there is a potential gradient. In lowest Landau level, it tries to move perpendicular to gradient, along equipotential.

Vortices try to do same, but in opposite sense relative to gradient, and with drift velocity equal in magnitude to that of electron if $\nu = 1/q$.

Picture:



Both drift in <u>same</u> direction! Separation constant.

Velocity $\propto \dfrac{\partial V}{\partial r}$. If $V(r) \sim r^2$ for small r (as it will)

then velocity $\sim$ separation. $\propto \dfrac{\partial V}{\partial r}$

Compare group velocity $v_k = \dfrac{\partial \varepsilon_k}{\partial k}$. k plays role of r.

...ms other than $\nu = 1/q$

...archy theory — Haldane 83, Halperin 84, + others

Simple physical idea: starting from $\nu = 1/q$, change ν to get finite density of quasiparticles, make a Laughlin state of those. Repeat as often as necessary

vortex excitations in Bose condensate

Jain, using fermions filling $|p|$ LLs (Jain 89)

$$\rightarrow \quad \nu = \frac{p}{2p+1} \qquad \text{e.g. } \nu = \tfrac{1}{3},\ \underline{p=1}$$

$$\nu = \tfrac{2}{3},\ \underline{p=-2}$$

(negative B_{eff})

+ "new hierarchy" (suggestion of N.R.)

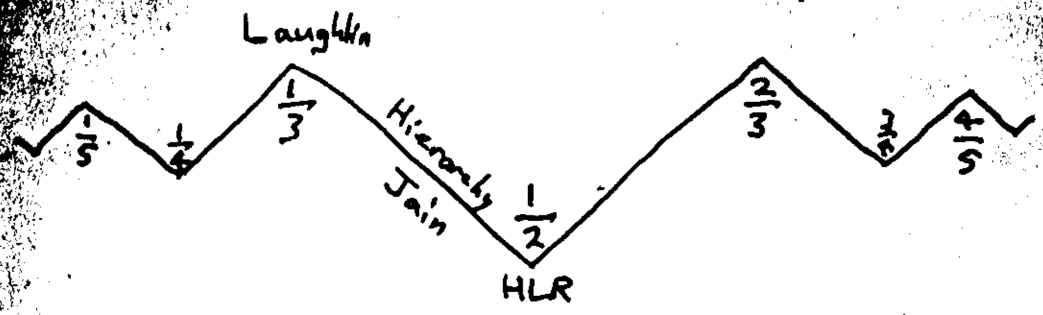
...h give:

$$\boxed{\begin{array}{l} \text{incompressible states possible at all } \nu = \frac{p}{q},\ q \text{ odd} \\ \text{excitations have charge } e^* = \pm e/q \\ \qquad\qquad \text{statistics } \frac{\theta}{\pi} = p'/q \pmod 2,\ p' = \text{odd},\ pp' = 1 \pmod q \\ \text{Two states are "same" (N.R., 1990; Blok + Wen, 1990)} \end{array}}$$

...milarly, $\nu = p/q$, q even we can have "hierarchical ...R states" — compressible

Democracy reaches FQHE

Bosons



Fermions

(Finer structure omitted)

Physics on QHE plateaus (incompressible states) can be understood in terms of <u>either</u> bosons or fermions

— but bosons more natural near $1, 1/3, 1/5, \ldots$
fermions " " " $\infty, 1/2, 1/4, \ldots$
ie $B=0$

Note "joining" of fermion sequences at $1, \frac{1}{3}, \frac{1}{5}, \frac{2}{3} \ldots$
so more than one picture is needed there.

Hierarchical ideas essential at e.g. $\nu = \frac{3}{8}, \frac{4}{11}, \frac{5}{13} \ldots$
(observed (?))

i.e. $\nu = \dfrac{3p+1}{8p+3}$

quanta

$$Z \doteq \int \mathcal{D}[\psi,\psi^+]\mathcal{D}[a_i]\mathcal{D}[a_0]\, e^{-S}$$ (ψ is Grassmann field) not physical electron

Halperin, Lee + Read 93
Kalmeyer + Zhang 92
(following Zhang, Hansson + Kivelson 89 for ψ = boson)

$$S = \int_0^\beta d\tau\, d^2x\, L$$

$$L = \psi^+\left(\frac{\partial}{\partial\tau} - ia_0 - iA_0\right)\psi + \frac{1}{2m}\psi^+\left(\nabla - i\vec{a} - i\vec{A}\right)^2\psi - \mu\psi^+\psi$$

$$+ \frac{ia_0}{2\pi\tilde{\phi}}\varepsilon^{ij}\partial_i a_j + \frac{1}{2}\int d^2x'\, \psi^+(x)\psi^+(x')V(x-x')\psi(x')\psi(x)$$

in Coulomb gauge $\nabla\cdot\vec{a} = 0$

a_0 imposes constraint

$$\nabla\times\vec{a} = 2\pi\tilde{\phi}\,\psi^+\psi = 2\pi\tilde{\phi}\, n, \quad n = \text{density of electrons also}$$

In mean field at $\nu = \frac{1}{2}$ ($\tilde{\phi} = 2$) assume n = uniform

$$\nabla\times(\vec{a}+\vec{A}) = 0 \qquad A_0 = 0$$

fermions fill Fermi sea. (Fluctuations weak in limit $\tilde{\phi}\to 0$, $\nu = 1/\tilde{\phi}$, anyons in field in place of electrons)

Using $\nabla\times\vec{a} = 2\pi\tilde{\phi}\, n$, Coulomb interaction becomes

$$\frac{1}{2}\frac{1}{(2\pi\tilde{\phi})^2}\int [\nabla\times\vec{a}](r)\; V(r-r')\,[\nabla\times\vec{a}](r')\, d^2r\, d^2r'$$

$$V(r) \sim 1/r^{2-x} \Rightarrow \tilde{V}(q) \sim \frac{1}{q^x}$$

$$\rightarrow \frac{2\pi e^2}{2(2\pi\tilde{\phi})^2}\int \frac{d^2q}{(2\pi)^2}\, q^{2-x}\, a(q\,\omega)\, a(-q\,-\omega)$$

(dropping indices) N.B. $\nabla\cdot\vec{a} = 0$

$x = 0 \Leftrightarrow$ "short range"

Response functions

$$\langle j_\mu \rangle = e K_{\mu\nu}(\vec{q},\omega) A_\nu^{ext}$$

for weak perturbation A_ν^{ext}

Consider components $\parallel, \perp$ to $\vec{q}$.

$$j_\parallel = \frac{\omega}{q} j_0 = \frac{\omega}{q} n$$

so only need $j_0, j_\perp \equiv j_0, j_1$.

As a matrix,

$$K = (\Pi^{-1} + U)^{-1} = \Pi(I + U\Pi)^{-1}$$

$$U = \begin{pmatrix} v(q) & \frac{2\pi i \tilde{\phi}}{q} \\ -\frac{2\pi i \tilde{\phi}}{q} & 0 \end{pmatrix}, \quad \Pi(q) = \text{irreducible polarization matrix}$$

In RPA, $\Pi \to \Pi^0$, response of free fermi gas

$\omega = 0$	$q = 0$
$\Pi^0_{00} = \frac{m}{2\pi} + O(q^2)$ (compressibility)	$\Pi^0_{00} \simeq -\frac{n}{m}\frac{q^2}{\omega^2}$
$\Pi^0_{11} = -\frac{q^2}{12\pi m} + O(q^4)$ $= q^2 \times$ (diamagnetic susceptibility) $= q^2 \chi_d$	$\Pi^0_{11} \simeq -\frac{n}{m}$

...stivity

Above formulas lead to $\nearrow q$

$$\rho = \begin{pmatrix} \tilde{\rho}_{xx} & -\tilde{\phi} \\ \tilde{\phi} & \tilde{\rho}_{xx} \end{pmatrix}$$

$$\tilde{\rho}_{xx} = \frac{1}{\tilde{\sigma}_{xx}} \quad , \tilde{\sigma}_{xx} = \text{(irreducible) conductivity of Fermi system}$$

(includes scattering off gauge fluc^{ns} as well as impurities and $V(q)$)

Thus Drude ($B = 0$!) form for $\tilde{\sigma}$

$$\tilde{\sigma}_{xx} = \frac{\sigma_0}{1 - i\omega\tau_{tr}} \quad , \sigma_0 = \frac{ne^2\tau_{tr}}{m^*}$$

leads to

$$\rho_{xx} = \rho_{yy} = \rho_0(1 - i\omega\tau_{tr})$$

$$-\rho_{xy} = \tilde{\phi}\,\frac{h}{e^2} \quad . \quad \tau_{tr} \text{ large}$$

As $T \to 0$, we might expect weak localization ($B=0$!) effects. However CS term breaks time reversal invariance — impurities induce random magnetic field. So leading log disappears (see also Kalmeyer+Zhang) Other logs due to interactions.

Surface acoustic waves

Simple theory ($\vec{q} \parallel \hat{x}$)

$$\frac{\Delta v_s}{v_s} = \frac{\alpha^2}{2} \frac{1}{1 + (\sigma_{xx}(q)/\sigma_m)^2}$$

$$\kappa = \frac{q\alpha^2}{2} \frac{\sigma_{xx}(q)/\sigma_m}{1 + (\sigma_{xx}(q)/\sigma_m)^2}$$

α = piezoelectric coupling

$\sigma_m = \dfrac{v_s \varepsilon}{2\pi}$, σ_{xx} = long. cond.

Data can thus be understood as measuring $\sigma_{xx}(q,\omega)$

Key point:

$$\boxed{v_s \ll v_F^*}$$

ie $\omega \ll q v_F^*$. In this regime ($\vec{q} \parallel \hat{x}$) from FL theory

(transverse) $\tilde{\sigma}_{yy} \propto \dfrac{k_F}{\underbrace{q}_{=k}}$, $\tilde{\rho}_{yy} \propto \dfrac{q^{=k}}{k_F}$

$$\Rightarrow \boxed{\sigma_{xx} = \frac{\tilde{\rho}_{yy}}{\tilde{\rho}_{xy}^2} \propto \frac{q^{=k}}{k_F}}$$ which explains observed linear dependence.

No adjustable parameter (m^* cancels)

Value is about 4 times too large.

For $\Delta B \neq 0$, we find width of feature also $\propto q$.

Willett et al

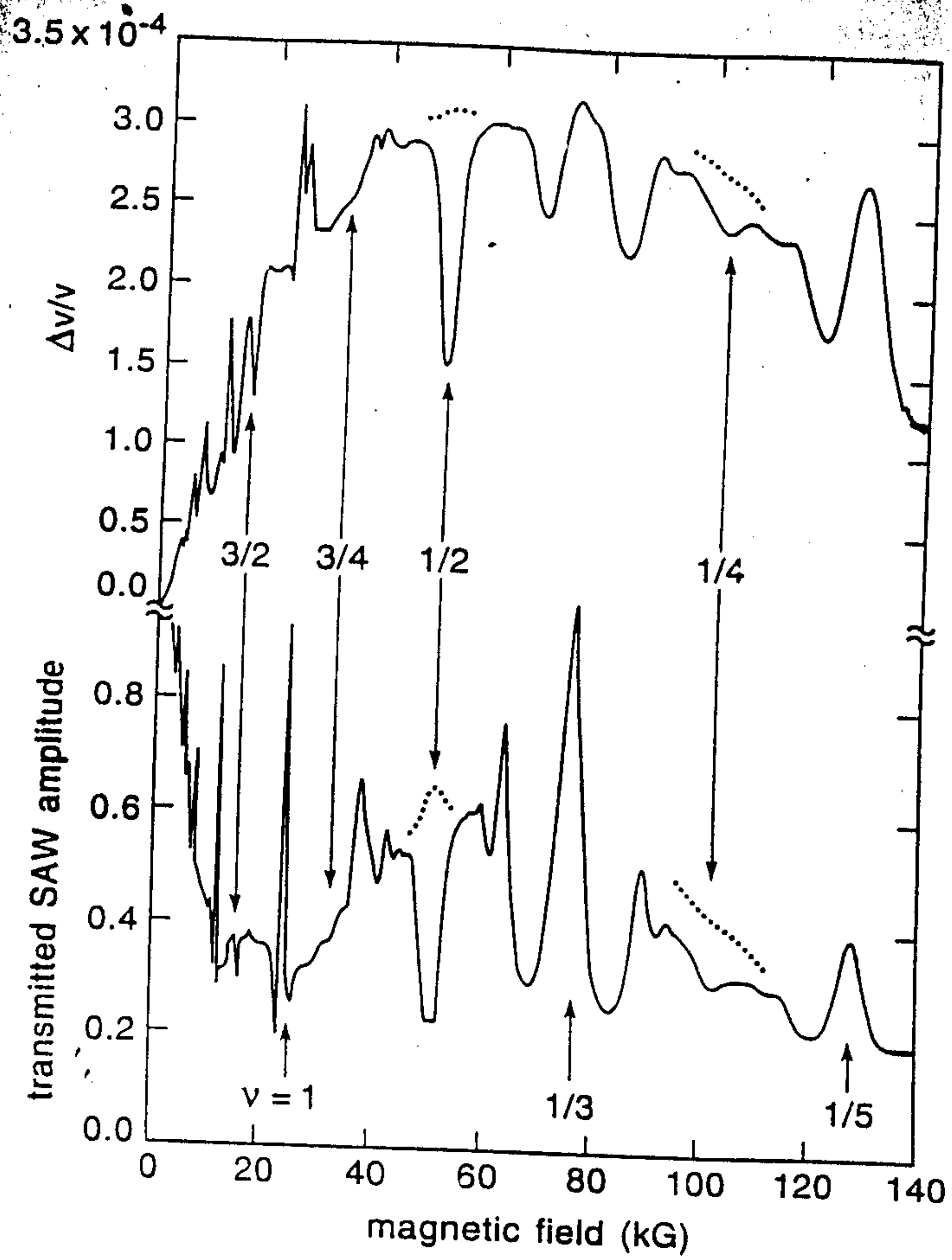


Fig. 1 $\frac{\omega}{2\pi} = 3.7$ GHz, $T = 120$ mK

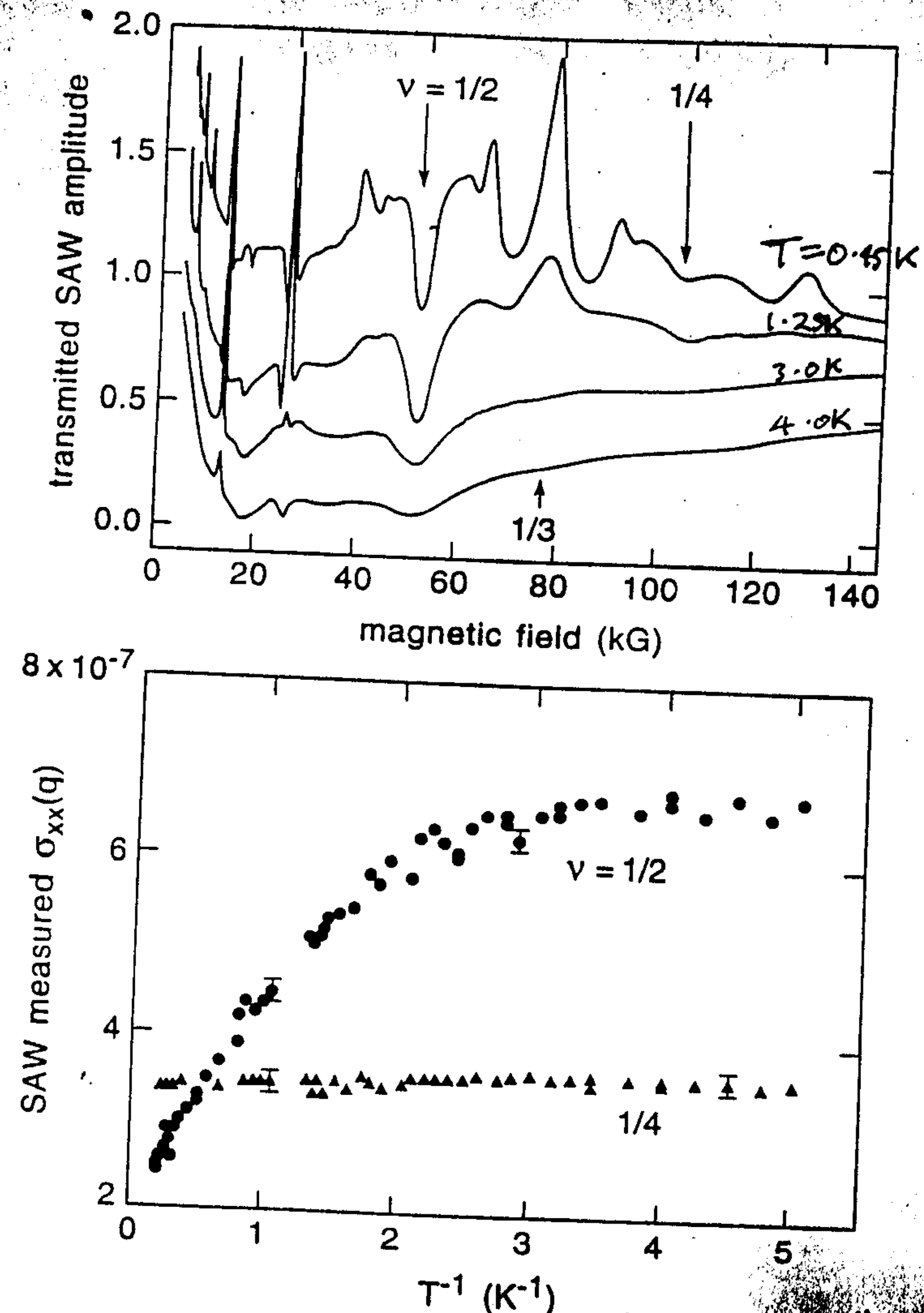
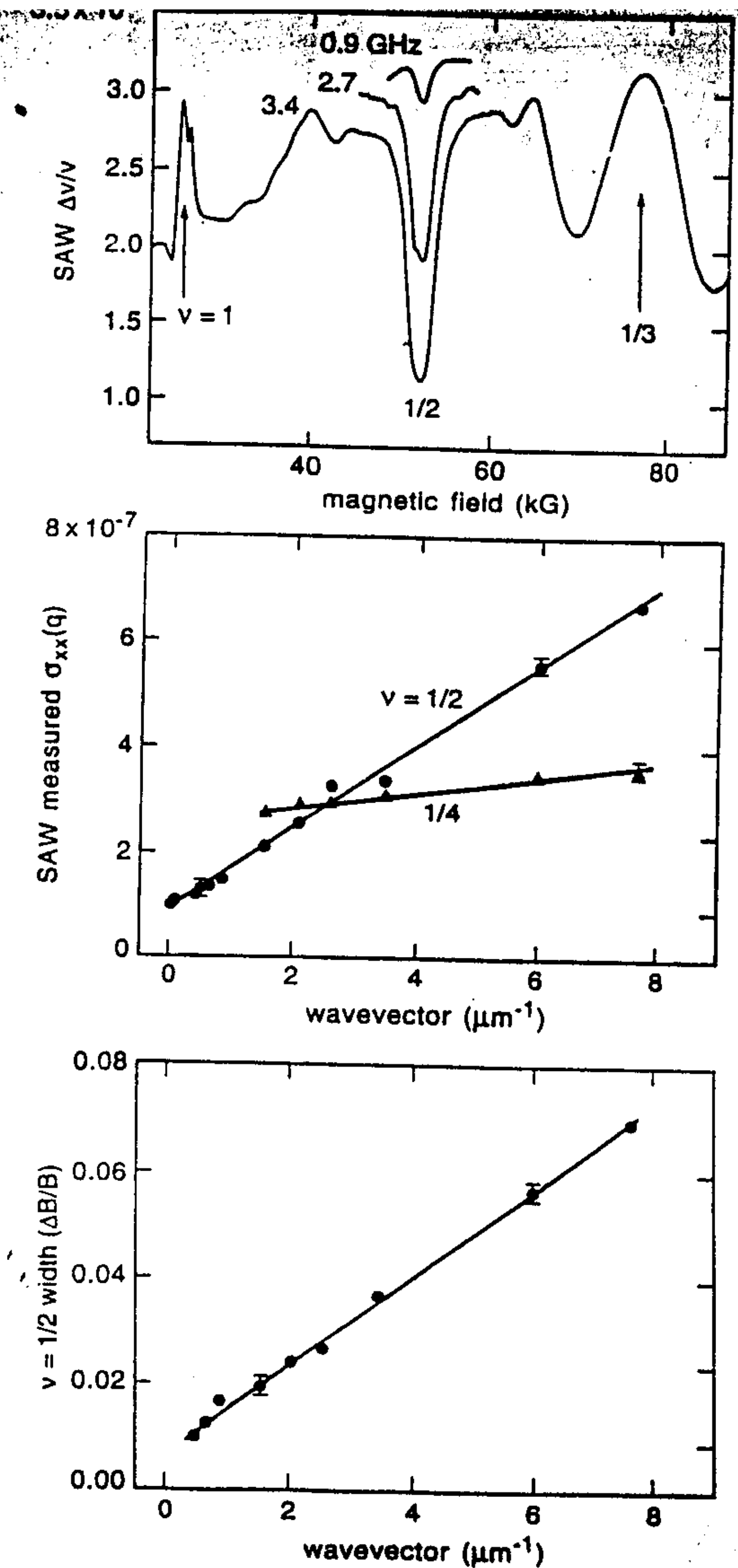


FIG 3

$\frac{\omega}{2\pi} = 3.4$ GHz

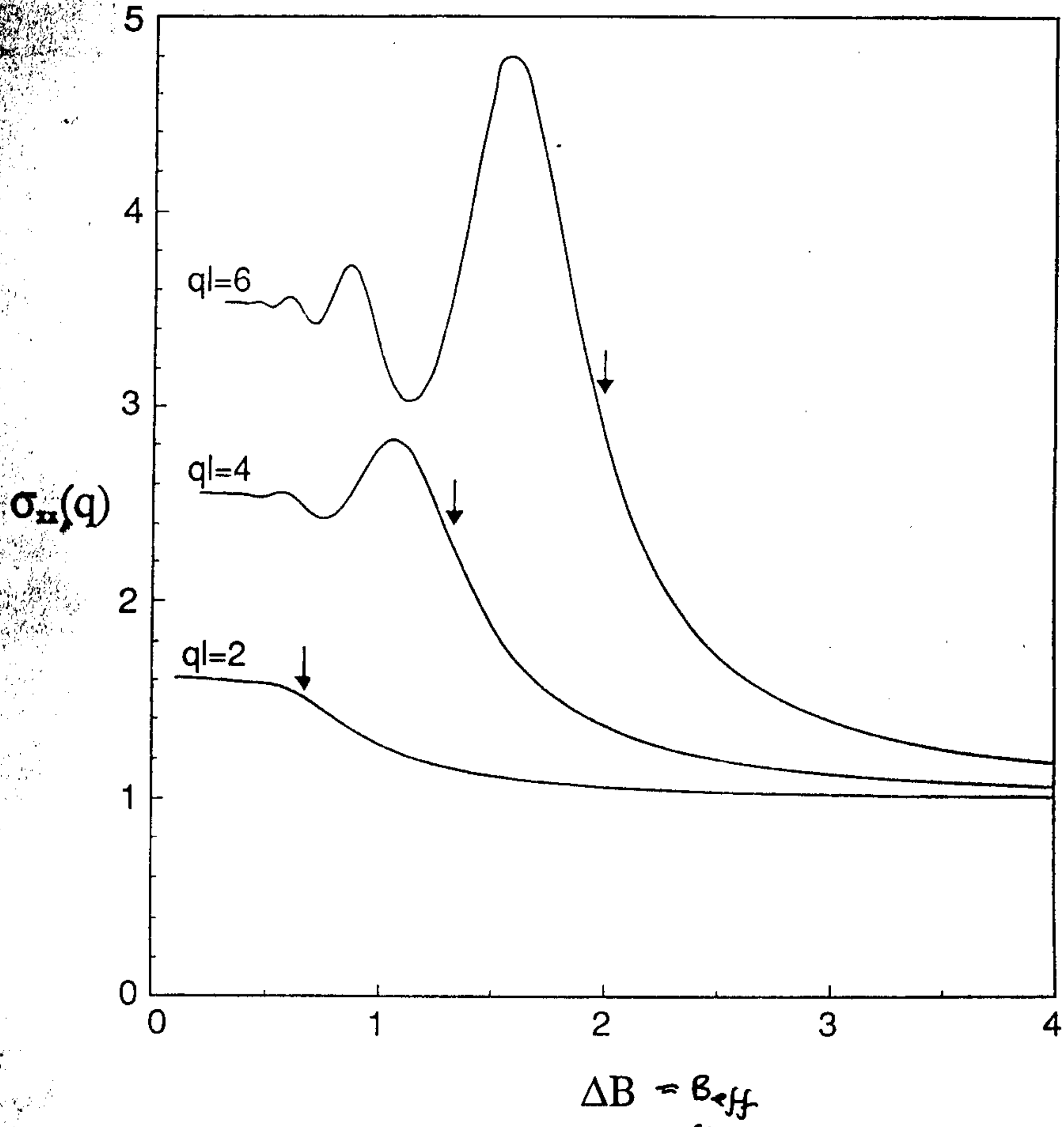


0.9 GHz
2.7
3.4
3.0
2.5
2.0
1.5
1.0
SAW Δv/v
ν = 1
1/2
1/3
40
60
80
magnetic field (kG)
8 x 10^-7
6
4
2
0
SAW measured σxx(q)
ν = 1/2
1/4
0
2
4
6
8
wavevector (μm^-1)
0.08
0.06
0.04
0.02
0.00
ν = 1/2 width (ΔB/B)
0
2
4
6
8
wavevector (μm^-1)

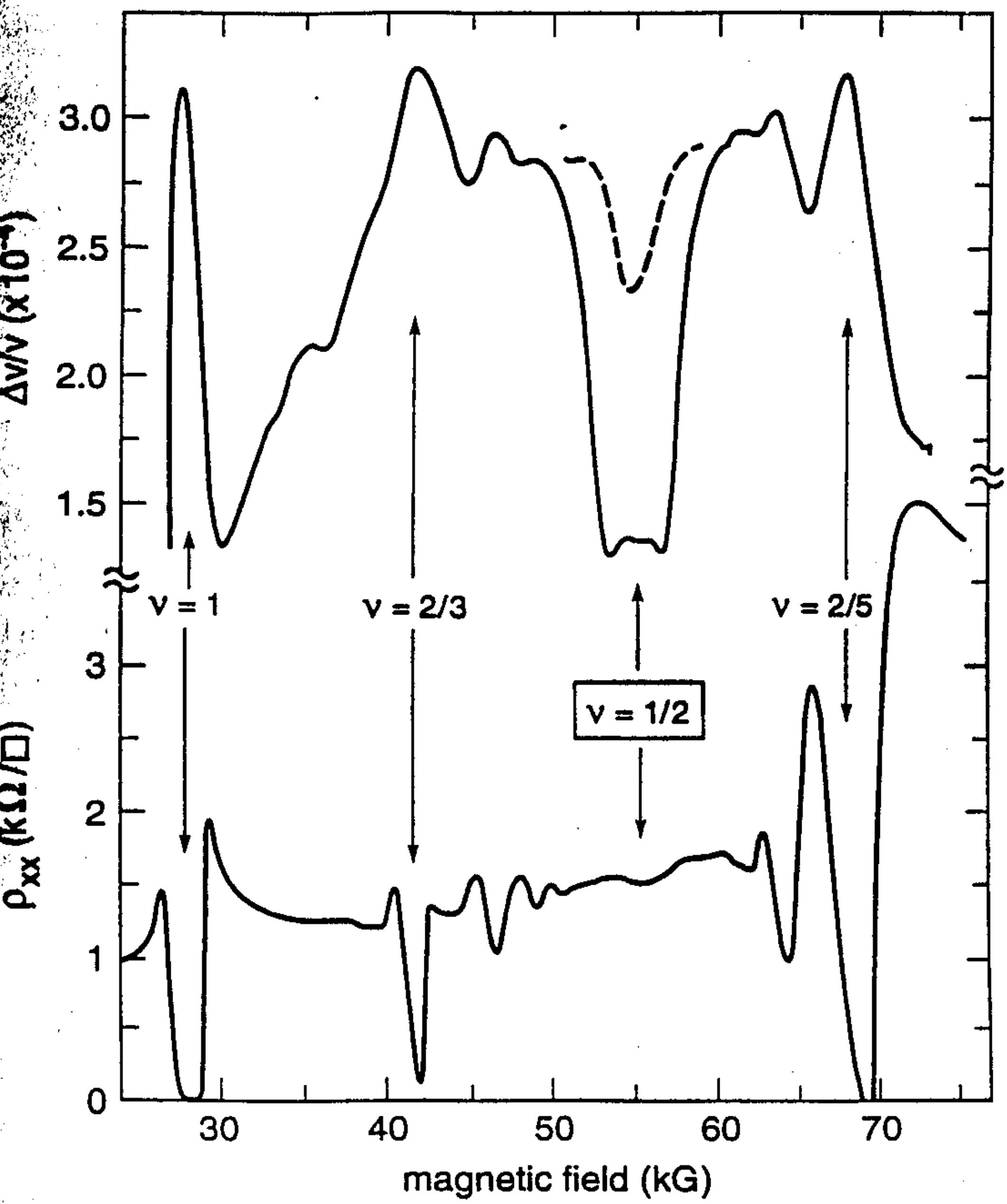
FIG 2

T = 50 mK

• Halperin, Lee, Read

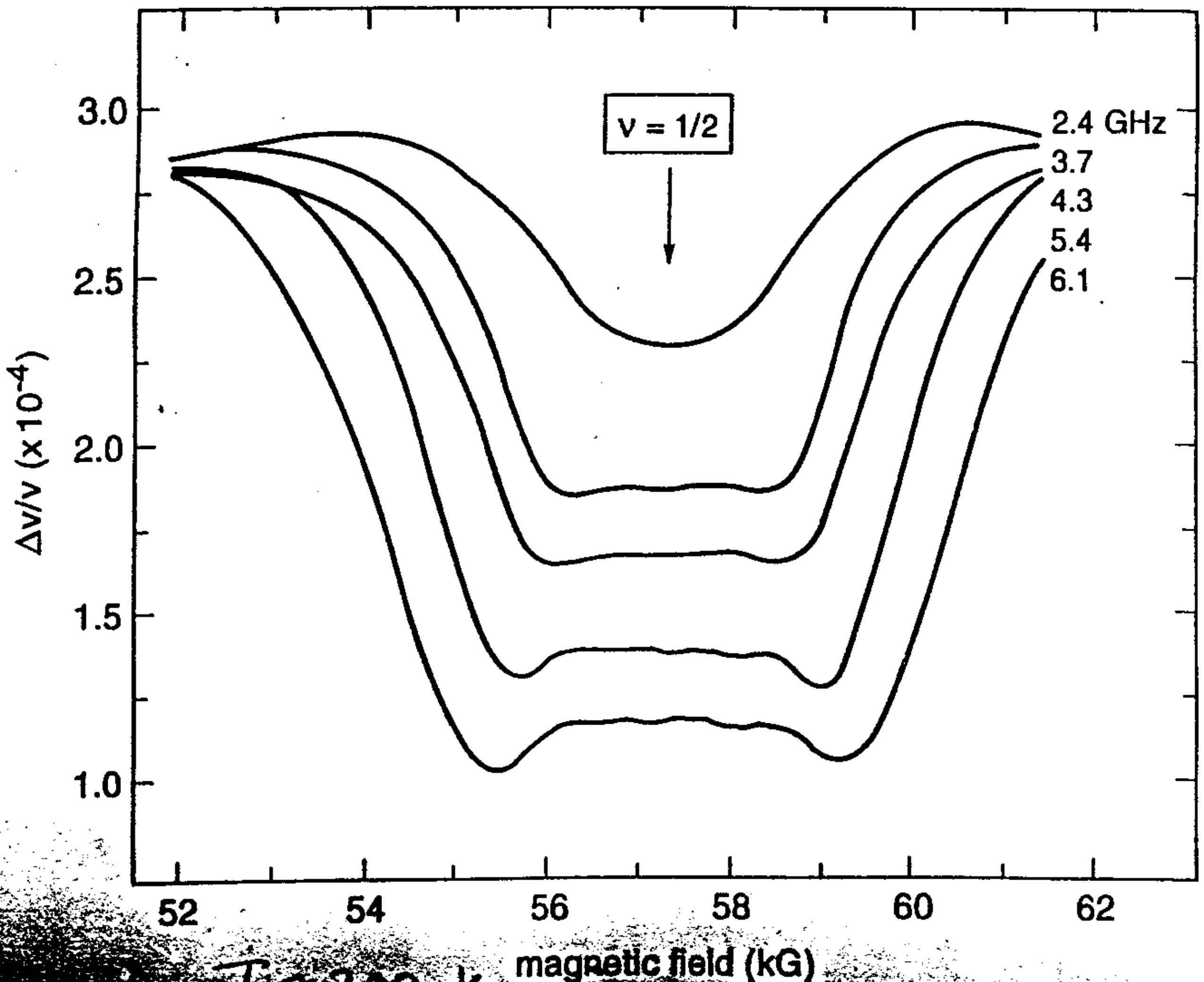


Resonances when $q R_c^{eff} = 2\pi n$.



Willett et al (1993)

FIG 1



ν = 1/2
2.4 GHz
3.7
4.3
5.4
6.1
3.0
2.5
2.0
1.5
1.0
Δv/v (x10^-4)
52
54
56
58
60
62
magnetic field (kG)

T ≃ 200 mK

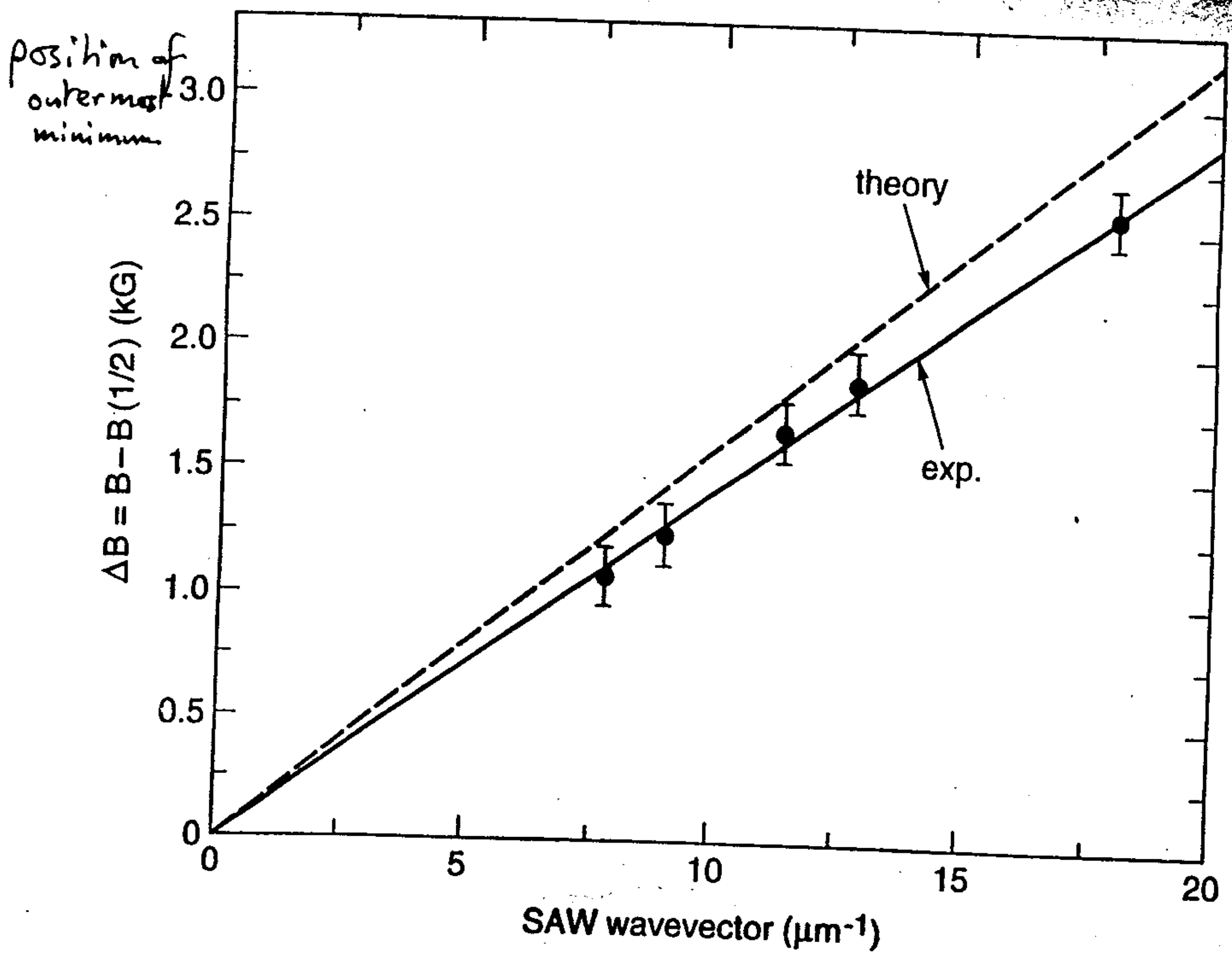
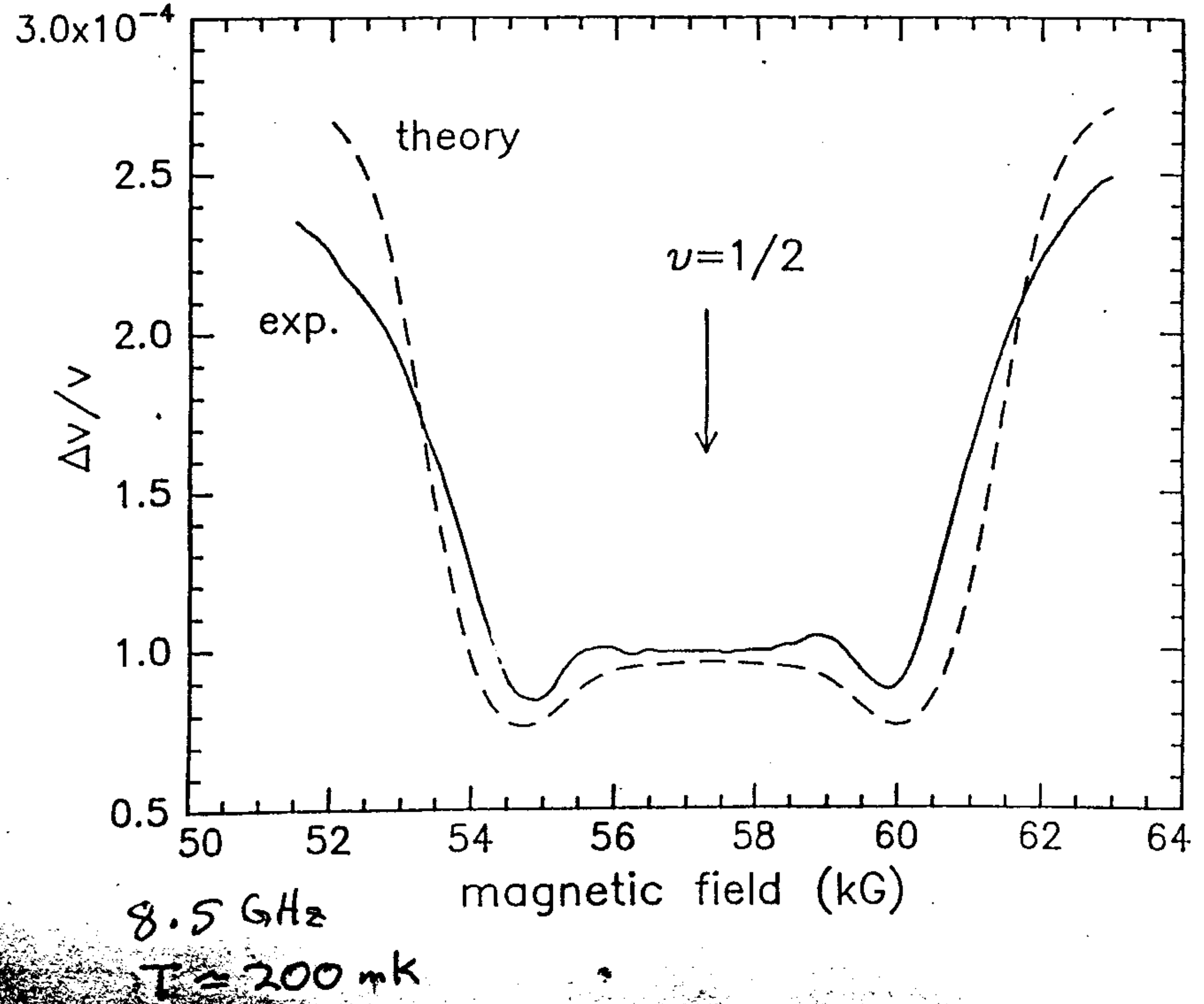

position of outermost minimum
3.0
2.5
2.0
1.5
1.0
0.5
0
ΔB = B−B(1/2) (kG)
theory
exp.
0
5
10
15
20
SAW wavevector (μm-1)

FIG 3



3.0x10^{-4}
2.5
2.0
1.5
1.0
0.5
$\Delta v/v$
theory
exp.
ν=1/2
50
52
54
56
58
60
62
64
magnetic field (kG)
8.5 GHz
T ≈ 200 mK

FIG. 4