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(WITH APPLICATIONS TO THE FRACTIONAL HALL EFFECT)"

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Low-Energy Properties of Two-Dimensional Fermions with Long-Range Current-Current Interactions

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We calculate the one-particle Green function of 2D fermions interacting via a long-range transverse gauge field. Its asymptotic low-energy behavior is found within the eikonal expansion which consistently sums the infrared divergent terms given by "maximally crossed diagrams." Instead of power law corrections to Fermi liquid theory we observe a much stronger singularity which implies a more radical breakdown of Fermi liquid theory than the usual orthogonality catastrophe.

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Considerable interest has focused recently on the idea that many fermionic systems may possess "singular" effective interactions between the quasiparticles. In one development, Anderson has postulated a singular form for the fully renormalized interaction function between fermions in two dimensions [1]. However, the problem of singular interactions more commonly arises in a rather different way; one finds that some process or sum of diagrams leads to a singular form for the effective fermion-fermion interaction, and one would like to go beyond such an approximation in a nonperturbative way. It is important to realize that it is possible to do this, and to give a treatment of theories with singular interactions in an entirely consistent way [2]. It is also important to find examples of singular effective interactions for which the microscopic basis is quite unambiguous.

In the present paper we consider just such an example, in which the singular interaction is generated by the transverse gauge field. In the case of relativistic electromagnetic interactions in ordinary 3D metals this problem was first considered by Holstein, Norton, and Pincus [3] (see also [4]). In the framework of 3D QCD a similar discussion was given recently by Pethick and coworkers [5]. The general problem can be discussed in a standard model of spinless nonrelativistic fermions, at zero temperature, interacting via the Abelian gauge field $\mathbf{A}(\mathbf{r},t)$ which is described by the Hamiltonian written in the gauge $A_0=0$:

$$H=\int d\mathbf{r}\frac{1}{2m}\psi^{\dagger}(-i\nabla-g\mathbf{A})^2\psi-\mu\psi^{\dagger}\psi+\frac{1}{2}\int d\mathbf{r}\left[\frac{\partial\mathbf{A}}{\partial t}\right]^2+(\nabla\times\mathbf{A})^2. \quad (1)$$

Here we concentrate on the 2D counterpart of this problem which was first recognized in the context of gauge models of doped Mott insulators [6].

We shall be primarily interested in the effects of the transverse gauge field $\mathbf{A}_{\perp}(k)=[\mathbf{k}\times\mathbf{A}(k)]/k$, discarding the longitudinal one (it was recently shown to lead to less singular contributions [7]). As a general fact, in a metallic fermion state, with gapless charge excitations, transverse gauge fluctuations are described by the (retarded) propagator

$$D_{ij}(\omega,\mathbf{q})=\Pi_{ij}^{-1}(\omega,\mathbf{q})=\frac{\delta_{ij}-q_iq_j/q^2}{i\gamma\omega/q+\chi q^2}, \quad (2)$$

which is governed by RPA corrections of Fig. 1(a). The coefficients γ and χ can be approximately found from the one-loop fermion polarization. However, intending to proceed beyond perturbation theory, we will use the expression (2) with arbitrary coefficients.

All previous attempts to analyze the effects of the transverse gauge interaction relied on the calculation of the first self-energy correction shown in Fig. 1(b) which behaves as $\Sigma_{2D}(\epsilon)\approx-g^2(p_F/m\chi^{2/3}\gamma^{1/3})(i\epsilon)^{2/3}$ and $\Sigma_{3D}(\epsilon)\approx-g^2\epsilon\ln(\mu/\epsilon)$ in 2D and 3D, respectively [3–6], where in RPA $\gamma\sim g$ and χ depends weakly on g for g

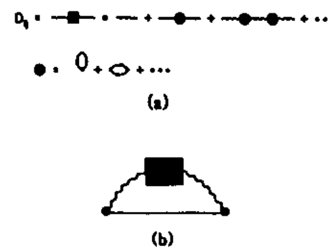

FIG. 1. (a) The RPA-screened transverse gauge field propagator; (b) the lowest-order contribution to the fermion self-energy.

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small. In fact, however, this contribution is only the first one in an infinite series of infrared divergent terms, which must all be dealt with on a consistent footing. In this paper we shall perform a summation of these terms by means of the eikonal expansion, which correctly accounts for the dominant processes of small angle scattering between the states close to the Fermi surface [2]. This will be done here using functional methods.

To proceed with this eikonal expansion we first obtain a formal solution for the one-particle Green function $G(\mathbf{r},\mathbf{r}';t,t';\mathbf{A})$ in a given external field $\mathbf{A}(\mathbf{r},t)$:

$$\left[i\frac{\partial}{\partial t}-\frac{1}{2m}(-i\nabla-g\mathbf{A})^2+\mu\right]G(\mathbf{r},\mathbf{r}';t,t';\mathbf{A})=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \quad (3)$$

This can be done in a standard way [8]; we Fourier transform in $\mathbf{r}-\mathbf{r}'$ and $t-t'$ and obtain $[\xi(p)=p^2/2m-\mu]$:

$$\left[\epsilon-\xi(p)+\frac{1}{m}\mathbf{p}\cdot(-i\nabla-g\mathbf{A})+i\frac{\partial}{\partial t}-\frac{1}{2m}(-i\nabla-g\mathbf{A})^2\right]G(\epsilon,\mathbf{p};\mathbf{A}(\mathbf{r},t))=1. \quad (4)$$

The solution of (4) can be represented in the integral form

$$G(\epsilon,\mathbf{p};\mathbf{A}(\mathbf{r},t))=i\int_0^{\infty}d\alpha\exp\{i\alpha[\epsilon-\xi(p)+i\delta_p]\}\exp\Psi(\epsilon,\mathbf{p};\mathbf{r},t;\alpha), \quad (5)$$

where $\delta_p=\delta\,\mathrm{sign}[\xi(p)]$, and $\delta=0^+$. Expanding $\Psi(\epsilon,\mathbf{p};\mathbf{r},t;\alpha)$ as a series in the coupling constant $\Psi=\sum_{n=1}^{\infty}g^n\Psi_n$ we have a set of equations for Ψ_n:

$$i\frac{\partial\Psi_n}{\partial\alpha}=\left[i\frac{\partial}{\partial t}+\mathbf{p}\cdot\frac{\nabla}{im}+\frac{1}{2m}\nabla^2\right]\Psi_n+\mathbf{A}\cdot\frac{\nabla}{im}\Psi_{n-1}+\delta_{n,1}\frac{1}{m}\mathbf{p}\cdot\mathbf{A}-\delta_{n,2}\frac{1}{2m}\mathbf{A}^2-\frac{1}{2m}\sum_{s=1}^{n-1}\nabla\Psi_s\cdot\nabla\Psi_{n-s}. \quad (6)$$

This expansion provides an efficient way to find recursively all higher order corrections to the leading eikonal approximation. The latter requires only the lowest order term Ψ_1 given by the expression

$$\Psi_1(\epsilon,\mathbf{p};\mathbf{r},t;\alpha)=\int d^2\mathbf{q}\,d\omega\exp[i(t\omega-\mathbf{r}\cdot\mathbf{q})]\mathbf{p}\cdot\mathbf{A}(\omega,\mathbf{q})\frac{1-\exp\{-i\alpha[\xi(\mathbf{p})-\xi(\mathbf{p}-\mathbf{q})+\omega]\}}{\xi(\mathbf{p})-\xi(\mathbf{p}-\mathbf{q})+\omega}. \quad (7)$$

Substituting (7) in (5) and averaging over the gauge fluctuations with the use of (2) we obtain the following formula [$n(p)$ is the Fermi distribution function]:

$$G(\epsilon,\mathbf{p})=i\int_0^{\infty}d\alpha\exp\{i\alpha[\epsilon-\xi(p)+i\delta_p]\}\times\exp\left[ig^2\int d^2\mathbf{q}\,d\omega\,D_{ij}(\omega,\mathbf{q})p_ip_j[1-n(\mathbf{p}-\mathbf{q})]\frac{1-\exp\{-i\alpha[\xi(\mathbf{p})-\xi(\mathbf{p}-\mathbf{q})+\omega]\}}{[\xi(\mathbf{p})-\xi(\mathbf{p}-\mathbf{q})+\omega]^2}\right]. \quad (8)$$

Expanding (8) in g^2 one generates all crossed graphs, including the most important "maximally crossed" ones [2], which give the dominant singular contributions.

The behavior of the Green function in the vicinity of the Fermi energy can be simply found at $p=p_F$:

$$G(\epsilon,p_F)=i\int_0^{\infty}d\alpha\exp[i\alpha(\epsilon+i\delta_p)-i^{1/3}\bar{g}^2\alpha^{1/3}]=-i\frac{\pi}{\epsilon}\frac{d^2}{dz^2}\mathrm{Hi}(z), \quad (9)$$

where $\epsilon>0$, $\bar{g}^2=g^2p_F/m\chi^{2/3}\gamma^{1/3}$ and $z=\exp(-2i\pi/3)\bar{g}^2/\epsilon^{1/3}$. The special function $\mathrm{Hi}(z)$ is defined in terms of Airy functions $\mathrm{Ai}(Z)$, $\mathrm{Bi}(z)$, as follows:

$$\mathrm{Hi}(z)=\tfrac{2}{3}\mathrm{Bi}(z)+\int_0^z[\mathrm{Ai}(t)\mathrm{Bi}(z)-\mathrm{Ai}(z)\mathrm{Bi}(t)]dt. \quad (10)$$

Now at $\epsilon\gg\bar{g}^6$ we reproduce the lowest order result of perturbation theory

$$G(\epsilon,p_F)=\frac{-i\pi}{\epsilon}c_1\left[1+c_2\frac{i^{-4/3}\bar{g}^2}{\epsilon^{1/3}}+\cdots\right], \quad (11)$$

where $c_1=2/[\sqrt{3}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})]$, and $c_2=\Gamma(\frac{1}{3})/3^{2/3}$. However, at $\epsilon\ll\bar{g}^6$ we find the asymptotic behavior

$$G(\epsilon,p_F)\sim-\pi^{-1/2}\bar{g}^{3/2}\epsilon^{-5/4}\exp\left[-\frac{2}{3}\frac{\bar{g}^3}{\epsilon^{1/2}}\right]. \quad (12)$$

Notice that the result (12) is essentially nonperturbative and nonexpandable in a power series in g^2. In fact, it implies a more drastic breakdown of the Fermi liquid theory (FLT) for the model (1) than Anderson's orthogonality catastrophe [1] or any kind of Luttinger liquid behavior. In particular, the wave function renormalization arising from (11) is $Z_{p_F}(\epsilon)\sim\exp(-\bar{g}^3/\epsilon^{1/2})$, shown in Fig. 2. Moreover, the one-particle distribution function $n(p)$ remains analytic near p_F, with a finite slope; $n(p)=\frac{1}{2}-\xi(p)/\bar{g}^6$, for $\xi(p)\ll\bar{g}^6$.

A better understanding of $G_{p_F}(\epsilon)$ is obtained by looking at its physical consequences. The perturbative high-energy limit has the dimensionless expansion parameter $\bar{g}^2\epsilon^{-1/3}$ (or, equivalently, $\epsilon/\bar{g}^6$). At order $\bar{g}^{2n}$, the leading contributions to this expansion are coming from max-

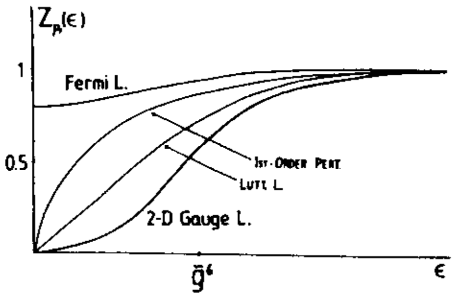

FIG. 2. A graph of $Z_{p_F}(\epsilon)$, the one-particle wave function renormalization. We also show, for comparison, the behavior of a Fermi liquid, a Luttinger liquid, and the extrapolation of the first order perturbative result (11) to zero energy.

imally crossed graphs [2] (with n screened gauge propagators). This expansion is around $z=0$ in (9); the Airy functions are analytic at $z=0$. We note that the energy scale $\bar{g}^6$ coming from the dimensionless parameter derives ultimately from the peculiar pole at $q\sim(i\omega)^{1/3}$ in $D_{ij}(q,\omega)$; this "anomalous skin effect" type pole is also responsible for the $\frac{1}{3}$ power of α in (9).

The opposite low-energy (or $|z|\gg 1$) limit is nonanalytic in $\bar{g}^2\epsilon^{-1/3}$ (recall that Airy functions $\sim\exp\{\frac{2}{3}z^{3/2}\}$, in the wedge of interest, when $|z|\to\infty$), and in fact (12) can be recovered directly from (9) via saddle-point integration. Thus $\epsilon=0$ is a singular point of $G_{p_F}(\epsilon)$, so that even though the eikonal expansion is a sum of crossed diagrams (each of order $\bar{g}^{2n}$), the low-energy limit is not analytic in $\bar{g}^2$; we have a genuinely divergent series.

Any physical quantity depending on the one-particle Green function will thus show quite a different behavior as $\epsilon\to 0$ from the perturbative result in (11). For example, the de Haas–van Alphen (dHvA) oscillatory magnetization can be calculated using the same eikonal technique. We have verified explicitly that a factor $\exp[-\pi/6(\bar{g}^6/\omega_c)r]$ appears in the rth harmonic of the dHvA amplitude (ω_c is the cyclotron frequency); this is analogous to the "Dingle factor" which would appear if impurity scattering were to smooth out the Fermi surface over an energy $\sim\bar{g}^6$. Here, however, the smoothing in $n(p)$ arises from the interaction itself, as described above.

One can also use the eikonal technique to calculate the entire partition function and various multipoint Green functions summing up the most infrared divergent contributions. It is particularly important to establish that the various response functions are properly related by Ward identities. Gauge invariance can be verified by the observation that within the leading eikonal approximation the irreducible three-point vertex function $\Lambda^\mu(p,q)$ obeys the Ward identity $[q_\mu=(\omega,\mathbf{q})]$:

$$q_\mu\Lambda^\mu(p,q)=G^{-1}(p)-G^{-1}(p+q)\,, \tag{13}$$

where $G(p)$ is taken from (8). This is also a check on the relation between the wave-function renormalization $Z_p(\epsilon)$ and the temporal component of $\Lambda^\mu(p,q)$, i.e., that $Z_p(\epsilon)=\Lambda^0(p,q)|_{q\to 0,\omega/q\to 0}$. This relation is an important check of self-consistency [2].

A closely related check on the application of the eikonal technique is the calculation of the two-particle Green function which describes correlations in particle-particle as well as particle-hole channels. A useful and efficient way to derive this is via Schwinger's "bilinear shift operator" [9]:

$$G_2(x_1,x_2;x_3,x_4;\mathbf{A}(x))=\exp\left\{\frac{i}{2}\int_{x_1}^{x_3}dx\int_{x_2}^{x_4}dx'\frac{\delta}{\delta A_i(x)}D_{ij}(x-x')\frac{\delta}{\delta A_j(x')}\right\}$$
$$\times[G(x_1,x_3;\mathbf{A}(x))G(x_2,x_4;\mathbf{A}(x))-(x_3\leftrightarrow x_4)]\,. \tag{14}$$

In the context of the eikonal expansion this equation plays the same role for problems involving singular interactions, as the Bethe-Salpeter equation. The averaging of (14) over the gauge fluctuations governed by (2) then generates insertions of the propagator $D_{ij}(x-x')$ joining two fermion lines in all possible ways, including the all-important vertex corrections [2], which again preserves the relevant Ward identities.

One may also calculate correlation functions starting from (14). As one might expect from the Ward identities the singularity in G does not necessarily appear in the response functions; in fact we have found that the static compressibility is finite and regular at low momenta. A full characterization of this system must therefore await a study of all the correlation functions.

The above results indicate a breakdown of FLT in the model (1). Moreover, the low-energy behavior found within the eikonal approximation appears to be quite different from either the "orthogonality catastrophe" [10] [which involves exponentiation of logarithmic divergences to give $Z(\epsilon)\sim\epsilon^\eta$] or the usual phase instabilities (such as pairing or charge density wave). The initial deviation from FLT is captured by the lowest IR-divergent diagram [Fig. 1(b)], but at $\epsilon\sim\bar{g}^6$ this power-law behavior turns into the exponential asymptotics of Eq. (12), as shown in Fig. 2. In view of this observation we conclude that those physical systems which can be adequately described by this gauge model should not be treated in the Fermi liquid framework. Physical systems which are be-

lieved to be described by the 2D gauge model include the $\nu=\frac{1}{2}$ fractional quantum Hall effect [11]; it has also been argued that the normal state of high-temperature superconductors may be describable in these terms.

Another interesting point which arises from this calculation is that the fermions in this gauge model appear to be beyond the reach of the bosonization methods recently developed by Haldane [12] for higher-dimensional Fermi liquids; this is because the spectrum is no longer linear in the vicinity of the Fermi surface. A similar conclusion was reached recently, by applying Haldane's methods directly [13]. It would be interesting to see whether some generalization of these methods could be applied to this kind of problem (or indeed to other kinds of singular interaction).

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Eikonal approximation in the theory of two-dimensional fermions with long-range current-current interactions

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We study the behavior of response functions of two-dimensional (2D) fermions interacting via a long-range transverse gauge field in the eikonal approximation. We observe that an exponentially vanishing wave-function renormalization prevents divergences in the density-density correlation function and the pairing susceptibility. The wave-function renormalization also shows up in a suppression of the de Haas-van Alphen effect. Elaborating on an observation recently made by Ioffe, Lidsky, and Altschuler, we also infer an effective bosonic description which makes it possible to reproduce our eikonal results in terms of free 2D bosons.

I. INTRODUCTION

Recently, much attention has been paid to the general problem of the existence of a metal-like non-Fermi-liquid ground state for interacting fermions in $D \geq 2$. Stimulated by Anderson's "Luttinger liquid" hypothesis about the ground state of the (2D) two-dimensional Hubbard model,[1] the previous studies mainly concentrated on short-ranged interactions. In this framework Anderson's hypothesis was disputed by a number of papers which claim the absence of anomalies in the conventional perturbation expansion and the validity of the Landau-Fermi-liquid picture for quasiparticle excitations.[2,3] However these conclusions cannot be simply extended onto the case of long-ranged interactions which are, in fact, of primary physical importance.

A 2D example provided by the Coulomb interaction $V(q) \sim \frac{1}{q}$ was recently shown to demonstrate some features of a breakdown of the regular perturbation expansion due to the pecularity of the Debye screening in 2D.[4]

Another relevant example is provided by the retarded current-current interaction of charged fermions mediated by a transverse gauge field. It was noticed in Ref. 5 and then elaborated in Refs. 6 and 7 that a relativistic transverse electromagnetic interaction in 3D metals leads to anomalies in perturbation theory.

In 2D a similar problem of the interaction via the transverse gauge field arises in the context of the modern gauge theory of strongly correlated electrons in doped Mott insulators[8] which is supposed to be an adequate description of the normal properties of high-T_c compounds. It has been also argued that the $\nu = 1/2$ fractional quantum Hall effect can be effectively described by the same kind of theory.[9]

The gauge model of nonrelativistic spinless fermions with chemical potential μ is given by the Hamiltonian written in the gauge $A_0 = 0$:

$$H = \Psi^\dagger \left(\frac{1}{2m}(-i\nabla - g\mathbf{A})^2 - \mu \right) \Psi + \frac{1}{2}(\nabla \times \mathbf{A})^2 + \frac{1}{2c^2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 . \quad (1.1)$$

Intending to concentrate on the effects of the transverse gauge field $A_\perp(k) = \frac{\mathbf{k}\times\mathbf{A}}{k}$, we shall discard the longitudinal component $A_\parallel(k) = \frac{\mathbf{k}\mathbf{A}}{k}$ responsible for the density-density Coulomb interaction, which was shown in Ref. 4 to lead to weaker singularities.

On the bare level the gauge field propagator

$$D^{(0)}_{ij}(\omega, \mathbf{q}) = \langle A_i(\omega, \mathbf{q}) A_j(-\omega, -\mathbf{q})\rangle = \frac{\delta_{ij} - \frac{q_i q_j}{q^2}}{-\omega^2 c^{-2} + q^2}$$

has a pole corresponding to a propagating mode $\omega = cq$. However in a metallic state with gapless charge excitations the gauge field spectrum becomes strongly renormalized due to particle-hole excitations. The gauge field dispersion corresponds to an overdamped mode $\omega \sim iq^3$:

$$D_{ij}(\omega, q) = \frac{\delta_{ij} - \frac{q_i q_j}{q^2}}{i\gamma\frac{\omega}{q} + \chi q^2} . \quad (1.2)$$

Formally this form of the propagator can be understood as a result of the fermion polarization (Fig. 1) which is governed by the Landau damping term, the coefficient γ being some function of the coupling constant. The random phase approximation (RPA) result is $\gamma_{\rm RPA} \sim g^2$ while $\chi_{\rm RPA} = 1 + O(g^2)$. Moreover at small ω and q the form of the gauge field propagator is quite insensitive to details of the charge excitation spectrum unless it develops a gap.

In the lowest order the fermion self-energy is given by the diagram of Fig. 2. In 2D this expression was calcu-

FIG. 1. RPA fermion polarization corrections to the gauge propagator.

FIG. 2. Fermion self-energy corrections.

lated in the form[6,9]

$$\Sigma(\epsilon) \approx -g^2 \frac{p_F}{m\chi^{2/3}\gamma^{1/3}}(i\epsilon)^{2/3} \quad (1.3)$$

if $\epsilon \gg \frac{\xi_p^3}{\mu^2}$, where $\xi_p = \frac{p^2}{2m} - \mu$ is the bare quasiparticle spectrum, otherwise the self-energy behaves as $\mathrm{Re}\Sigma(\epsilon, p) \sim -g^2 \frac{\epsilon}{\xi_p}$, $\mathrm{Im}\Sigma(\epsilon, p) \sim -g^2(\epsilon^2/\xi_p^4)\mathrm{sgn}\epsilon$.

Despite the obviously singular character of the correction (1.3) it was argued in Ref. 9 that the system still could be considered in the framework of some modified Landau-Fermi-liquid theory. Namely, it was conjectured that the fermion Green function still has a pole corresponding to the renormalized quasiparticle dispersion $\epsilon(p) \sim (p - p_F)^{3/2}$, although both real and imaginary parts of the spectrum were supposed to be of the same order.

Indeed, it can be easily shown that the functional form of $\Sigma(\epsilon)$ given by (1.3) does not change if one takes into account only completely uncrossed (ladder) diagrams.[10,11] However, one can neglect the vertex corrections to (1.3) only at $\epsilon \gg \frac{\xi^2}{\gamma}$, otherwise one cannot fulfill the Ward identity

$$\Lambda(\epsilon, p) = 1 - \frac{\partial\Sigma(\epsilon, p)}{\partial\epsilon} , \quad (1.4)$$

where $\Lambda(\epsilon, p)$ stands for the irreducible three-point vertex at zero transferred momentum $\Lambda(\epsilon, \mathbf{p}; \frac{\omega}{k} \to 0, \mathbf{k} \to 0)$. Equation (1.4) implies that close to the Fermi surface there exists a whole series of infrared divergent terms, the expression (1.3) being the first term of this series. Obviously, a complete summation of higher order self-energy and vertex corrections requires essentially nonperturbative methods. In our previous paper[12] we first applied the so-called eikonal approximation to study the behavoir of the one-particle Green function. In the present paper we undertake a consistent summation of higher order corrections to two-point response functions by means of the eikonal approximation.

The use of this approximation in the case of the conventional relativistic 3D electrodynamics provides an elegant and efficient way to find the well-known nonpole infrared asymptotics of the one-particle Green function as well as the double-logarithmic asymptotics of the three-point vertex function.[13]

In our case of nonrelativistic 2D fermions with singular interactions a general possibility in applying the eikonal approximation follows from the fact that for quasiparticles near the Fermi surface the small angle scattering becomes dominant. More exactly, the validity of the eikonal approximation is restricted by those amplitudes which receive their main contributions from excitations in the vicinity of the Fermi surface.

In addition, in the case of the transverse gauge interaction the overdamped dispersion (1.2) provides essentially different scales for energy and momentum transfer

$$\omega \ll v_F q_\parallel \sim \frac{1}{m}\left(\frac{\gamma\omega}{\chi}\right)^{2/3} \ll v_F q_\perp \sim v_F\left(\frac{\gamma\omega}{\chi}\right)^{1/3} , \quad (1.5)$$

where $q_{\parallel,\perp}$ are the longitudinal and transverse components of $\mathbf{q}$ with respect to the particle's momentum $\mathbf{p}$.

This circumstance opens a very interesting avenue of research initiated by recent discussions of a possibility in formulating a consistent bosonization procedure in $D > 1$.[14-17] It was argued in Ref. 18 that it might be possible to describe a long-wavelength behavior of 2D fermions interacting via the transverse gauge field in terms of free bosons. This conjecture was made in the framework of the 2D generalization of the bosonization procedure originally proposed by Haldane.[15]

In a recent paper by Ioffe, Lidsky, and Altshuler[19] it was explicitly shown how an effective bosonic quasi-1D description arises in the course of a straightforward diagrammatic calculation of the one-particle Green function. In the present paper we elaborate this observation and demonstrate that the results which can be found by means of the effective bosonic description are in agreement with those obtained in the eikonal approximation.

II. ONE-PARTICLE PROPERTIES

A formal eikonal expansion of the one-particle Green function starts from a derivation of the Green function $G(\mathbf{r}, \mathbf{r}'; t, t'; \mathbf{A})$ in a given external field $\mathbf{A}(\mathbf{r}, t)$

$$\left(i\frac{\partial}{\partial t} - \frac{1}{2m}(-i\nabla - g\mathbf{A})^2 + \mu \right) G(\mathbf{r}, \mathbf{r}'; t, t'; \mathbf{A}) = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t') . \quad (2.1)$$

For details of this procedure we refer to our previous paper.[12] To find the one-particle Green function one then has to average $G(\mathbf{r}, \mathbf{r}'; t, t'; \mathbf{A})$ over Gaussian gauge fluctuations which are governed by the propagator $D_{ij}(\omega, \mathbf{q}) = \langle A_i(\omega, \mathbf{q}) A_j(-\omega, -\mathbf{q})\rangle$ given in (1.2).

The resulting expression can be written in the integral form (n_p is the Fermi distribution function):

$$G(\epsilon, \mathbf{p}) = i\int_0^\infty d\alpha e^{i\alpha(\epsilon - \xi_p + i\delta_p)} \exp\left(ig^2 \int \frac{d\mathbf{q}d\omega}{(2\pi)^3} D_{ij}(\omega, \mathbf{q}) \frac{p_i p_j}{m^2} \frac{1 - e^{i\alpha(\xi_p - \xi_{p-q} + \omega)}}{(\xi_{\mathbf{p}} - \xi_{\mathbf{p}-\mathbf{q}} + \omega)^2} \right) . \quad (2.2)$$

Expanding (2.2) in g^2 one generates all crossed graphs including the so-called "maximally crossed" ones which are supposed to give the dominant contributions in the case of singular (long-range) interactions.[20]

When calculating the exponential "Debye-Waller factor" in (2.2) it is important to notice that the integrations over longitudinal and transversal components of $\mathbf{q}$ can be done separately because of the condition (1.5).

The behavior of the Green function in the vicinity of the Fermi energy can be simply found at $p = p_F$.[12] Estimating the integral over α one can see that at $\epsilon \gg \tilde{g}^6$ the lowest order results of perturbation theory are easily reproduced

$$G(\epsilon, p_F) = \frac{1}{\epsilon}\left(1 - \frac{\tilde{g}^2 \epsilon^{2/3}}{\epsilon^{1/3}} + \cdots\right). \quad (2.3)$$

On the contrary, at $\epsilon \ll \tilde{g}^6$ we find an essentially nonperturbative asymptotic behavior

$$G(\epsilon, p_F) \sim \frac{\tilde{g}^{3/2}}{\epsilon^{5/4}} \exp\left(-\frac{\tilde{g}^3}{\epsilon^{1/2}}\right). \quad (2.4)$$

Note that the asymptotics (2.4) is basically a result of the saddle point approximation in the integral over α in (2.2), hence it cannot be expanded in a power series in $\tilde{g}^2$.

It is worthwhile mentioning that the asymptotic behavior (2.4) implies a more radical breakdown of the Fermi-liquid theory for the model (1.1) than any kind of Luttinger liquid behavior proposed, for instance, in the Anderson's scenario of the "Tomographic Luttinger Liquid."[1]

In particular, the wave-function renormalization found from (2.4) vanishes exponentially on the Fermi surface: $Z_{p_F}(\epsilon) \sim \frac{\tilde{g}^{3/2}}{\epsilon^{1/4}} \exp(-\frac{\tilde{g}^3}{\epsilon^{1/2}})$. As a consequence, the distribution function $n(p)$ remains analytic in the vicinity of the Fermi surface and has a finite slope

$$n(p) = \frac{1}{2} - \frac{v_F}{\tilde{g}^6}(p - p_F). \quad (2.5)$$

The unusual low-energy behavior implied by (2.4) does affect physical observables which are sensitive to one-particle properties near the Fermi surface. In particular, it leads to a drastic suppression of oscillations of the orbital magnetization [de Haas-van Alphen effect (dHvA)] in a weak external magnetic field, when $\omega_c \ll \tilde{g}^6$ (and $\omega_c = \frac{H}{m}$ is the cyclotron frequency). The oscillatory part of the magnetic moment is conventionally calculated from the thermodynamic potential, but it can be also found in terms of the one-particle Green function

$$\tilde{M}(H) = \frac{\partial}{\partial H}\int_{-\infty}^{\mu} d\mu \,\mathrm{Tr} \int_{-\infty}^{\mu} d\epsilon \,\mathrm{Im} G(\epsilon; \mathbf{r}, \mathbf{r}'). \quad (2.6)$$

When $\omega_c \ll \tilde{g}^6$, the Green function can be taken in the semiclassical form (we use the Landau gauge for the background field):

$$G(\epsilon; \mathbf{r}, \mathbf{r}') \approx \tilde{G}(\epsilon; \mathbf{r} - \mathbf{r}') e^{i\frac{H}{2}(x'-x)(y'+y)}. \quad (2.7)$$

Then to calculate the trace in (2.6) one can use the basis of states provided by Landau levels and to apply the Poisson summation formula.[21] The result is given by the expression

$$\tilde{M}(H) \sim p_F^2 \sum_{r=1}^{\infty} \frac{I_r}{r} \sin\left[2\pi r\left(\frac{F}{H} - \phi\right)\right], \quad (2.8)$$

where the phase factor ϕ depends on the fermion spectrum far from the Fermi surface, and $F = p_f^2/2$ is the "dHvA frequency" of the orbit in $\mathbf{k}$ space. For the case of free fermions the amplitude I_r of the rth harmonic is equal to 1, giving the characteristic sawtooth structure (for 2D fermions) of $\tilde{M}(H)$. If g is finite, then for $\omega_c \ll \tilde{g}^6$, one has

$$I_r = \frac{2\pi i r}{H}\int_{-\infty}^{0} d\epsilon \exp\left(\frac{2\pi i r}{H}[\epsilon - \Sigma_{p_F}(\epsilon)]\right) \quad (2.9)$$

which is evaluated to give

$$\tilde{M}(H) \sim p_F^2 \frac{\tilde{g}^3}{H^{1/2}} \sum_{r=1}^{\infty} \frac{1}{r^{1/2}} \exp\left(-\frac{\pi}{6}\frac{\tilde{g}^6}{\omega_c} r\right) \times \sin\left[2\pi r\left(\frac{F}{H} - \phi\right)\right]. \quad (2.10)$$

Under the condition $\omega_c \ll \tilde{g}^6$, only the first ($r = 1$) harmonic is important because of the exponential suppression of amplitudes of higher harmonics. The exponential form of the amplitudes I_r in (2.10) looks similar to either the case of impurity scattering (with the identification of scattering time as $\tau \sim \tilde{g}^{-6}$) or the case of finite temperatures $T \sim \tilde{g}^6$. However the entire result (2.10) is not identical because of the magnetic field dependent prefactor, and the extra factor of $r^{1/2}$.

The result (2.10) is physically transparent, because according to (2.5) the jump of $n(p)$ on the Fermi surface is smoothed out over an interval of order $\tilde{g}^6$. It is quite important to notice that the formula (2.17) manifests an effect of the transverse gauge field on the gauge invariant quantitity $\tilde{M}(H)$. It takes place because the infrared corrections studied in this section affect the modulus of the one-particle Green function rather than its phase.

III. TWO-POINT CORRELATION FUNCTIONS

One can also use the eikonal technique to calculate the entire partition function and various multipoint Green functions summing up the most infrared divergent contributions. In particular, this method is capable of studying correlations in particle-particle as well as particle-hole channels. An important consistency check is a fulfillment of the Ward identity (1.4).[20]

A useful and efficient way to derive the appropriate expression for the four-point Green function in an external field $\mathbf{A}(x)$ is via Schwinger's "bilinear shift operator:"[22]

$$G_2[x_1, x_2; x_3, x_4; \mathbf{A}(x)] = \exp\left(\frac{i}{2}g^2 \int_{x_1}^{x_3} dx \int_{x_1}^{x_4} dx' \frac{\delta}{\delta A_i(x)} D_{ij}(x - x') \frac{\delta}{\delta A_j(x')}\right) \times \{G[x_1, x_3; \mathbf{A}(x)] G[x_2, x_4; \mathbf{A}(x)] - (x_3 \leftrightarrow x_4)\} \quad (3.1)$$

where $G[x, x'; \mathbf{A}(x)]$ is a solution of (2.1). In the context of the eikonal expansion this equation plays the same role as the Bethe-Salpeter equation in Fermi-liquid theory. The averaging of (3.1) over the gauge fluctuations governed by (1.2) then generates insertions of the propagator $D_{ij}(x - x')$ joining two fermion lines in all possible ways, including vertex corrections, which preserves the relevant Ward identity (1.4).

Fourier transform of the expression (3.1) gives a two-particle Green function

$$G_2(p, p'; p + k, p' - k) = \int d^3x e^{ikx} \int_0^{\infty} d\alpha_1 \int_0^{\infty} d\alpha_2 e^{i\alpha_1(\epsilon - \xi_p + i\delta) - i\alpha_2(\epsilon' - \xi_{p'} - i\delta)} \times \exp\left[i\frac{g^2}{m^2}\int \frac{d\mathbf{q}d\omega}{(2\pi)^3} D_{ij}(\omega, \mathbf{q})\left(p_i p_j \frac{1 - e^{2i\alpha_1(\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega)}}{(\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega)^2} + p_i' p_j' \frac{1 - e^{2i\alpha_2(\xi_{\mathbf{p}'} - \xi_{\mathbf{p}'-\mathbf{q}} - \omega)}}{(\xi_{\mathbf{p}'} - \xi_{\mathbf{p}'-\mathbf{q}} - \omega)^2} - 2e^{iqx} p_i p_j' \frac{1 - e^{i\alpha_1(\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega)}}{\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega} \frac{1 - e^{i\alpha_2(\xi_{\mathbf{p}'} - \xi_{\mathbf{p}'-\mathbf{q}} - \omega)}}{\xi_{\mathbf{p}'} - \xi_{\mathbf{p}'-\mathbf{q}} - \omega}\right)\right]. \quad (3.2)$$

Contracting two pairs of external lines in a way corresponding to the particle-hole channel we obtain the following eikonal formula for the density-density correlation function:

$$K(\Omega, \mathbf{Q}) = \int d^2p n_\mathbf{p} \int_0^{\infty} d\alpha e^{i\alpha(\Omega + \xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{Q}} + i\delta)} \times \exp\left[i\frac{g^2}{m^2}\int \frac{d^2\mathbf{q}d\omega}{(2\pi)^3} D_\perp(\omega, \mathbf{q})\left((p_\perp + Q_\perp)(1 - n_{\mathbf{p}+\mathbf{Q}+\mathbf{q}})\frac{1 - e^{i\alpha(\xi_{\mathbf{p}+\mathbf{Q}} - \xi_{\mathbf{p}+\mathbf{Q}+\mathbf{q}} + \omega)}}{\xi_{\mathbf{p}+\mathbf{Q}} - \xi_{\mathbf{p}+\mathbf{Q}+\mathbf{q}} + \omega} - p_\perp n_{\mathbf{p}+\mathbf{q}} \frac{1 - e^{i\alpha(\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega)}}{\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{q}} + \omega}\right)^2\right] + (\Omega \to -\Omega). \quad (3.3)$$

Calculating the integral in the exponent at small $\mathbf{Q}$ and Ω we find that, in agreement with the Ward identities, the self-energy and vertex corrections almost compensate each other and the result reads (hereafter we include irrelevant numerical factors in the definition of $\tilde{g}$)

$$K(\Omega, \mathbf{Q}) = \int d^2p n_\mathbf{p} \int_0^{\infty} d\alpha e^{i\alpha(\Omega + \xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{Q}} + i\delta)} \times \exp\left(-\tilde{g}^2 \frac{Q^2}{p}\alpha^{1/3}\right) + (\Omega \to -\Omega). \quad (3.4)$$

One can see that the exponential singularities of the one-particle Green functions (2.4) are integrated out and have no effect at small $\mathbf{Q}$. As a result, the compressibility $K(0, \mathbf{Q} \to 0)$ as well as a Fourier transform of the equal time correlator $\int d\Omega K(\Omega, \mathbf{Q} \to 0)$ remain finite. At small g these almost coincide with the results obtained for free fermions. In addition, we observe that the diffusion pole of the scattering amplitude found in Ref. 19 [such that $\Gamma(\epsilon, p; \omega, q) \sim \frac{(\omega+\epsilon)^{2/3} + \epsilon^{2/3}}{q_\parallel + \omega^{1/3} q_\perp}$] as a result of a summation of "fan-shaped" diagrams, does not affect the density-density correlator (3.3).

It was first pointed out in Ref. 23 that the lowest order vertex correction to $\Lambda(\Omega, Q)$ is logarithmically divergent at $Q \approx 2p_F$. At transferred momentum Q close to $2p_F$ the momenta of scattered particles $-\mathbf{p}$ and $\mathbf{p} + \mathbf{Q}$ have almost the same direction and then the current-current interaction provides an attraction which may lead to a physical singularity of (3.3) at $Q \to 2p_F$. Summing up the infrared divergent terms one obtains a 2D counterpart of the powerlike Kohn singularity at $\frac{\tilde{g}^6}{\mu} \ll \frac{|Q - 2p_F|}{p_F} \ll 1$ when the behavior of $K(0, \mathbf{Q})$ is mainly governed by the vertex corrections (Fig. 3)

$$K(0, \mathbf{Q}) = \int d^2p n_\mathbf{p} \int_0^{\infty} d\alpha e^{i\alpha(\xi_\mathbf{p} - \xi_{\mathbf{p}+\mathbf{Q}} + i\delta)} \times \exp\left(\frac{g^2}{\gamma}\ln(\mu\alpha) - \tilde{g}^2 p\alpha^{1/3}\right) \sim \text{const} + |Q - 2p_F|^{\frac{1-\gamma}{\gamma}}, \quad (3.5)$$

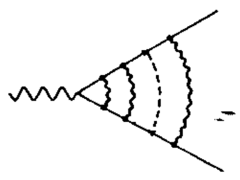

FIG. 3. Ladder-type vertex corrections.

where $\eta \sim \frac{g^2}{\gamma}$ and Q approaches $2p_F$ from above.

In the opposite regime $\frac{|Q-2p_F|}{p_F} < \frac{\tilde{g}^6}{\mu}$ the last term in the exponent associated with the wave-function renormalization dominates and renders the integral in (3.5) finite: $K(0, 2p_F) \sim \mathrm{const} + \tilde{g}^{3(1-\eta)}$.

Some remnant of the weak "$2p_F$ singularity" also appears at $\frac{|Q-2p_F|}{p_F} \gg \frac{\tilde{g}^6}{\mu}$ as a nonanalytic contribution to the Fourier transform of the equal time correlator

$$\int dx e^{i\mathbf{Q}\mathbf{x}} \langle \Psi^\dagger \Psi(\mathbf{x}) \Psi^\dagger \Psi(0) \rangle$$
$$= \int d\Omega K(\Omega, \mathbf{Q})$$
$$\sim \left(\max\left\{ \frac{|Q-2p_F|}{p_F}, \frac{\tilde{g}^6}{\mu} \right\} \right)^{2-\eta}. \tag{3.6}$$

The results (3.5) and (3.6) are consistent with the singular behavior of the renormalized vertex $\Lambda(\Omega, Q) \sim (\max\{\frac{|2p_F - Q|}{p_F}, (\frac{\Omega}{\mu})^{1/3}\})^{-\eta}$ found in Ref. 19. The density response function (3.6) manifests a kind of a crossover at critical value $\eta_c(g) = 2$. This value of the critical coupling constant is in agreement with the estimate obtained by the authors of Ref. 19 who argued that at some $\eta_c \sim 1$ a phase transition occurs. However it was also mentioned in Ref. 19 that the statement about the phase transition is based on diagrammatic calculations which assume $\eta \ll 1$ and, strictly speaking, do not account for all crossed diagrams becoming important at $\eta \sim 1$.

On the other hand, the eikonal approximation *is* supposed to provide a proper account of crossed diagrams. However, we only observe a crossover behavior of the density response function (3.6) at $Q \approx 2p_F$ and not a real singularity. The origin of this behavior is the exponential wave-function renormalization (2.4) which cuts off the powerlike divergency of $\Lambda(\Omega, Q \approx 2p_F)$.

In view of this we are inclined to the conclusion that a real phase transition at finite couplings does not seem likely. However, one cannot rule out the possibility of some kind of topological phase transition which does not manifest itself in correlations of any local field operators.[24]

It might be also interesting to study pairing correlations in the particle-particle channel. In contrast to the aforementioned Kohn singularity, which appears to be quite sensitive to the effective phase volume of scattered fermions, the singularity in the Cooper channel depends primarily on the sign of interaction. Indeed the Cooper instability persists even in the two-particle problem ($p_F \to 0$).

The corresponding (nongauge invariant) correlation function is given by the formula

$$\Delta(\Omega, \mathbf{Q}) = \int d^2p (1 - n_{\mathbf{p}} - n_{-\mathbf{p}+\mathbf{Q}}) \int_0^\infty d\alpha e^{i\alpha(\Omega - \xi_{\mathbf{p}} - \xi_{\mathbf{p}-\mathbf{Q}} + i\delta)}$$
$$\times \exp\left[i\frac{g^2}{m^2} \int \frac{d\mathbf{q} d\omega}{(2\pi)^3} D_\perp(\omega, \mathbf{q}) \left((-p_\perp + Q_\perp)(1 - n_{-\mathbf{p}+\mathbf{Q}+\mathbf{q}}) \right.\right.$$
$$\left.\left. \times \frac{1 - e^{i\alpha(\xi_{\mathbf{p}-\mathbf{Q}} - \xi_{-\mathbf{p}+\mathbf{Q}+\mathbf{q}} + \omega)}}{\xi_{\mathbf{p}-\mathbf{Q}} - \xi_{-\mathbf{p}+\mathbf{Q}+\mathbf{q}} + \omega} + p_\perp (1 - n_{\mathbf{p}-\mathbf{q}}) \frac{1 - e^{i\alpha(\xi_{\mathbf{p}} - \xi_{\mathbf{p}-\mathbf{q}} - \omega)}}{\xi_{\mathbf{p}} - \xi_{\mathbf{p}-\mathbf{q}} - \omega} \right)^2 \right]. \tag{3.7}$$

By calculating (3.7) at small Q we find no essential difference with the case of free fermions

$$\Delta(\Omega, \mathbf{Q}) = \int d^2p (1 - n_{\mathbf{p}} - n_{-\mathbf{p}+\mathbf{Q}}) \int_0^\infty d\alpha e^{i\alpha(\Omega - \xi_{\mathbf{p}} - \xi_{\mathbf{p}+\mathbf{Q}} + i\delta)} \exp\left(-\tilde{g}^2 \frac{Q^2}{p} \alpha^{1/3} \right) \sim \ln \frac{\mu}{\max\{\Omega, \frac{Q^2}{2m}\}}. \tag{3.8}$$

Although the expression (3.8) is logarithmically divergent it does not lead to any instability because the current-current interaction is repulsive at small Q. Indeed one could only expect to find some singularity of the effective vertex $\Gamma(p, p', k)$ if momenta of scattered particles are parallel and the current-current interaction becomes attractive, that is when $Q = |\mathbf{p} + \mathbf{p}'|$ approaches $2p_F$. One also has to have $\Omega \sim \mu$ otherwise as Q becomes comparable with p_F the Cooper loop ceases to be logarithmically divergent. In this regime (3.7) yields the result which is similar to (3.5): at $|\Omega - \frac{Q^2}{2m}| \gg \tilde{g}^6$ the pairing correlation function has a nonanalytic powerlike intermediate asymptotics while at $|\Omega - \frac{Q^2}{2m}| < \tilde{g}^6$ it remains finite even for large $\eta(\tilde{g})$:

$$\Delta(\Omega, \mathbf{Q}) = \int d^2p (1 - n_{\mathbf{p}} - n_{-\mathbf{p}+\mathbf{Q}}) \int_0^\infty d\alpha e^{i\alpha(\Omega - \xi_{\mathbf{p}} - \xi_{\mathbf{p}+\mathbf{q}} + i\delta)}$$
$$\times \exp\left(\frac{g^2}{\gamma} \ln(\mu\alpha) - \tilde{g}^2 p \alpha^{1/3} \right) \sim \frac{1}{\eta} \left(\max\left\{ \left| \Omega - \frac{Q^2}{2m} \right|, \tilde{g}^6 \right\} \right)^{-\eta}. \tag{3.9}$$

The anomalous exponent $\eta \sim \frac{g^2}{\gamma}$ coincides with the one in (3.5) although this equality holds only for spinless fermions.

Notice that the effect of the exponentially vanishing Z factor on (3.9) is exactly the same as in the case of the gauge invariant density-density correlation function. This should not be surprising because it derives from the modulus of the one-particle Green function rather than its phase.

A similar conclusion can be drawn about the equal time pairing susceptibility

$$\int dx e^{i\mathbf{Q}\mathbf{x}} \langle \Psi^\dagger \Psi^\dagger(\mathbf{x}) \Psi \Psi(0) \rangle$$
$$= \int d\Omega \Delta(\Omega, \mathbf{Q})$$
$$\sim \left(\max\left\{ \frac{|2p_F - Q|}{p_F}, \frac{\tilde{g}^6}{\mu} \right\} \right)^{2-\eta}. \tag{3.10}$$

Thus we do not find an instability in the Cooper channel either.

Nevertheless in the course of our eikonal calculations we discovered that the exponential factors appearing in (3.3) and (3.7) are related to each other and can be understood as correlation functions of some vertex operators:$\exp(i\Phi)$: built from a free boson field Φ. It is this property of the eikonal calculus which enables one to apply an effective bosonic description to study the problem (1.1).

IV. EFFECTIVE BOSONIC DESCRIPTION

In the Hamiltonian approach an attempt to construct a bosonic theory which captures mostly relevant small scattering angle processes and then reproduces long-wavelength properties of the model (1.1) was made in Ref. 17. Later on the authors of Ref. 19 explicitly showed how a quasi-1D (bosonizable) effective Lagrangian description occurs in the context of a straightforward diagrammatic calculation of the one-particle Green function. The Green function obtained in this manner is equivalent to our formula (2.4).

More precisely, in Ref. 19 the parameter γ was used to control the diagrammatic expansion instead of our coupling constant g which was, in turn, put equal to unity. This alternative choice of the expansion parameter corresponds to the case of the so-called "zero current" theory which follows from (1.1) in the strong coupling limit $g \to \infty$. At $\gamma \gg 1$ it turns out that within the weak coupling ($g \ll 1$) eikonal calculations the two expansions can be linked together by the relation $\gamma \sim g^{-6}$.

It was noticed in Ref. 19 that the asymptotic behavior (2.4) can be found by means of the effective quasi-1D bosonic theory at $\gamma < 1$.

The above consideration shows that at $\gamma < 1$ the asymptotics (2.4) simply extends over the entire energy range $\epsilon \sim \mu$. In the opposite case $\gamma \sim g^{-6} \gg 1$, the necessary condition $\epsilon \ll \tilde{g}^6$ for the nonperturbative asymptotics (2.4) to hold means a small scattering angle

$$\theta \sim q_\perp / p \sim \left(\frac{\gamma\epsilon}{\chi} \right)^{1/3} \Big/ p_F \ll 1. \tag{4.1}$$

It is the condition (4.1) which provides an effective quasi-1D dynamics along the direction of the particle's momentum $\mathbf{p}$. In other words, the eikonal condition (4.1) means that the space of momenta $\mathbf{p}$ can be effectively split into 1D "rays" in accordance with the heuristic picture of the "Luttinger Tomographic Projection."[1] However (and just as for Anderson's model[1,20]), there is still a "residual coupling" between these rays, as we shall see.

The 1D bosonic theory of Ref. 19 was invented primarily for the calculation of $G(\epsilon, \mathbf{p})$. However it is highly tempting to formulate a refined bosonic description which would be capable to reproduce the two-point response functions as well.

In the 3D case of free fermions this problem was first addressed by Luther[13] who showed that one can recover two-point correlation functions

$$\langle \Psi^\dagger(x) \cdots \Psi(x) \Psi^\dagger(0) \cdots \Psi(0) \rangle$$

substituting fermion operators by the bosonic ones

$$\Psi(\mathbf{x}) \to \int \frac{d\mathbf{n}}{2\pi} e^{ip_F \mathbf{n}\cdot\mathbf{x}} \exp i\Phi_{\mathbf{n}}(\mathbf{x}). \tag{4.2}$$

The bosonic field $\Phi_{\mathbf{n}}(\mathbf{x})$ should be understood as a chiral (right moving) field associated with the direction pointed by the unit vector $\mathbf{n}$ normal to the spherical Fermi surface which parametrizes a continuous manifold of Fermi points. This representation can be further generalized to the case of arbitrary shape of the Fermi surface and can also be used to treat interacting fermions.[15]

To get a complete correspondence between fermionic and bosonic operators as well as the corresponding Hilbert spaces one has to solve a subtle (and ambiguous) problem of operator ordering in $D > 1$.[14] Fortunately, to deal with two-point response functions one only needs to establish a relationship between currents.

Using a bosonic field $\Phi_{\mathbf{n}}(\mathbf{x})$ one can construct a corresponding (chiral) current operator $J_{\mathbf{n}}(\mathbf{x}) = (\mathbf{n}\nabla)\Phi_{\mathbf{n}}(\mathbf{x})$ obeying commutation relations[15]

$$[J_{\mathbf{n}}(\mathbf{x}), J_{\mathbf{n}'}(\mathbf{x}')] = \frac{1}{2\pi} \delta^2(\mathbf{n} - \mathbf{n}')(\mathbf{n}\nabla)\delta(\mathbf{x} - \mathbf{x}'). \tag{4.3}$$

To proceed with a conventional quantization $\Phi_{\mathbf{n}}(\mathbf{x})$ can be expanded in the form

$$\Phi_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{n}\mathbf{q}>0} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{J_{\mathbf{n}}(\mathbf{q})}{i\sqrt{\mathbf{n}\mathbf{q}\Lambda}}, \tag{4.4}$$

where $\Lambda \ll p_F$ stands for an ultraviolet cutoff for a transverse component of $\mathbf{q}$.

On the basis of the commutation relations (4.3) one can identify $J_{\mathbf{n}}(\mathbf{x})$ with a bilinear product of fermion operators: $J_{\mathbf{n}}(\mathbf{q}) = \sum_{|\mathbf{p}-p_F\mathbf{n}|<\Lambda} \Psi^\dagger_{\mathbf{p}+\mathbf{q}} \Psi_{\mathbf{p}}$. The transverse component of the space current coupled with $A_\perp(\mathbf{k})$ is given by the dual gradient: $\Psi^\dagger(\mathbf{x})(\frac{\mathbf{n}\nabla}{i})\Psi(\mathbf{x}) = \frac{\partial\Phi_{\mathbf{n}}(\mathbf{x})}{\partial t}$.

In contrast to the case of the one-particle Green function it turns out that to reproduce the response functions of the preceding section the intended bosonic theory has to incorporate kinetic couplings between bosons $\Phi_{\mathbf{n}}$ with different $\mathbf{n}$. This is necessary to account for vertex corrections on an equal footing with self-energy ones.

The effective bosonic action can be written in the form (v_F is put equal to unity)

$$S = \frac{1}{2}\int d\omega d^2q\left(\int \frac{d\mathbf{n}}{2\pi}\Phi_\mathbf{n}(-\omega,-\mathbf{q})(\mathbf{qn})(\omega - \mathbf{nq})\Phi_\mathbf{n}(\omega,\mathbf{q}) + \tilde{g}^2\omega^{5/3}\int\frac{d\mathbf{n}}{2\pi}\int\frac{d\mathbf{n}'}{2\pi}\Phi_\mathbf{n}(-\omega,-\mathbf{q})(\mathbf{nn}')\Phi_{\mathbf{n}'}(\omega,\mathbf{q})\right). \quad (4.5)$$

To avoid a double counting of loop corrections which were already accounted for in (1.2) one can introduce N bosonic replicas $\Phi^a_\mathbf{n}, a = 1, 2, ..., N$ and then put N equal to zero.[19] In other words, one has to exclude any renormalizations of a kinetic part of the bosonic Lagrangian. In the limit $N \to 0$ the boson propagator following from (4.5) becomes equivalent to the expression

$$\langle\Phi^a_\mathbf{n}(\omega,\mathbf{q})\Phi^b_{\mathbf{n}'}(-\omega,-\mathbf{q})\rangle = \frac{\delta_{ab}\delta_{\mathbf{n},\mathbf{n}'}}{\Lambda(\mathbf{nq})(\omega-\mathbf{nq})} + \frac{\tilde{g}^2}{\Lambda\omega^{1/3}}\frac{(\mathbf{nn}')}{[\omega - \mathbf{nq} - (\gamma\omega)^{2/3}][\omega - \mathbf{n}'\mathbf{q} - (\gamma\omega)^{2/3}]}. \quad (4.6)$$

More strictly, the formulas (4.5) and (4.6) enable one to reproduce those one- and two-particle eikonal amplitudes which are dominated by configurations of momenta obeying one of the conditions $|\mathbf{n}' \pm \mathbf{n}| \ll 1$. In this case all integrations over the transverse components of transferred momenta $q_\perp = \mathbf{q}\times\mathbf{n}$ simply cancel the factors Λ in the denominators.

On calculating the one-particle Green function with the use of the Lagrangian (4.5) one ends up with a quasi-1D theory with right and left moving bosons (two opposite Fermi points) associated with directions of $\mathbf{p}$ and $-\mathbf{p}$. The form of the Lagrangian (4.5) makes it possible to clarify the physical origin of the exponential asymptotics (2.4). The well-known powerlike behavior of 1D one-particle Green functions, as well as correlation functions, takes place for a quite special family of Lagrangians bilinear in current operators, the most popular example being the Sugawara construction which is a local bilinear form $L =: J_LJ_L - J_RJ_R :$. An example of a nonlocal coupling with the strength $V(x) \sim \frac{1}{x^2}$ is provided by the Calogero model. In all these cases the 1 + 1-dimensional theory possesses conformal invariance. As soon as this symmetry is lost the decay of 1D correlators ceases to be powerlike [see, for instance, the example of the Coulomb interaction $V(x) \sim \frac{1}{x}$ recently considered in Ref. 25].

Thus in the effective model (4.5) one simply encounters the more general situation where the Lagrangian bilinear form is nonlocal in time. It is obvious that to get a powerlike behavior one has to have not only an effectively one-dimensional dynamics but also a conformally invariant one.

As another example, we sketch the derivation of the asymptotics (3.6) using the bosonic theory (4.5):

$$\int d\Omega K(\Omega,\mathbf{Q}) = \int d^2\mathbf{x}e^{i\mathbf{Qx}}\langle\Psi^\dagger\Psi(\mathbf{x})\Psi^\dagger\Psi(0)\rangle = \int d\mathbf{x}\int\frac{d\mathbf{n}}{2\pi}\int\frac{d\mathbf{n}'}{2\pi}e^{i[\mathbf{Q}+p_F(\mathbf{n}'-\mathbf{n})]\cdot\mathbf{x}}$$
$$\times\exp\left(-\frac{1}{2}\int d\omega\int d^2q(\langle\Phi_\mathbf{n}\Phi_\mathbf{n}\rangle + \langle\Phi_{\mathbf{n}'}\Phi_{\mathbf{n}'}\rangle - 2\langle\Phi_\mathbf{n}\Phi_{\mathbf{n}'}\rangle)(1-\cos\mathbf{qx})\right). \quad (4.7)$$

The exponent in (4.7) can be found in the following approximate form:

$$\exp\left[-\int_0^\infty dq_\parallel\int_0^\infty d\omega[2 - \cos(q_\parallel\mathbf{nx}) - \cos(q_\parallel\mathbf{n}'\mathbf{x})]\left(\frac{1}{(\omega+q_\parallel)^2} + \frac{\tilde{g}^2}{\omega^{1/3}(\omega+q_\parallel)^2}\right.\right.$$
$$\left.\left.+\frac{\tilde{g}^2}{\omega^{1/3}[\omega - q_\parallel - (\gamma\omega)^{2/3}][\omega + q_\parallel - (\gamma\omega)^{2/3}]}\right)\right]$$
$$\sim\frac{(\max\{\mathbf{n}\cdot\mathbf{x},\mathbf{n}'\cdot\mathbf{x}\})^\eta}{(\mathbf{n}\cdot\mathbf{x}+i\delta)(\mathbf{n}'\cdot\mathbf{x}-i\delta)}\exp(-\tilde{g}^2x^{1/3}). \quad (4.8)$$

At $g = 0$ and $Q \approx 2p_F$ (4.7) and (4.8) yield

$$\int d\Omega K_0(\Omega,\mathbf{Q}) = \int d\mathbf{x}\int\frac{d\mathbf{n}}{(\mathbf{nx}+i\delta)}\int\frac{d\mathbf{n}'}{(\mathbf{n}'\mathbf{x}-i\delta)}$$
$$\times e^{i[\mathbf{Q}+p_F(\mathbf{n}'-\mathbf{n})]\cdot\mathbf{x}} \sim (Q-2p_F)^2. \quad (4.9)$$

The consistency of the applied approximations is achieved due to the fact that in the relevant region of integration $\mathbf{n}' \approx -\mathbf{n}$. Performing the calculation of (4.7) at finite values of g and using (4.9) we recover the result (3.6).

As was shown in Ref. 15, all the information about long-wavelength properties (in particular, the equilibrium thermodynamics) is encoded in eigenvalues of the generalized Landau equation for collective bosonic modes

$$(\omega - \mathbf{nq})\Phi_\mathbf{n}(\omega,\mathbf{q}) = \tilde{g}^2\omega^{2/3}\int\frac{d\mathbf{n}'}{2\pi}(\mathbf{n}\cdot\mathbf{n}')\Phi_{\mathbf{n}'}(\omega,\mathbf{q}). \quad (4.10)$$

Formally one can find two independent solutions of Eq. (4.10)

$$\Phi^{\parallel,\perp}_\mathbf{n}(\omega,\mathbf{q}) = \frac{q_{\parallel,\perp}}{\omega - \mathbf{nq} + i\delta}f(\omega). \quad (4.11)$$

The spectra of the corresponding collective modes are given by the equations

$$1 = \tilde{g}^2\omega^{2/3}\int\frac{1\pm\cos 2\phi'}{\omega - q\cos\phi' + i\delta}\frac{d\phi'}{2\pi}. \quad (4.12)$$

The dispersion of the mode associated with $\Phi^\parallel_\mathbf{n}$ is close to linear $[\omega = q - O(\tilde{g}^4q^{1/3})]$ at $\omega \gg \tilde{g}^6$. In the opposite limit $\omega < \tilde{g}^6$ it acquires a form $\omega \sim (\frac{q}{\tilde{g}})^{6/5}$. The solution corresponding to the second mode $\Phi^\perp_\mathbf{n}$ can only be found at $\omega < \tilde{g}^6$. Its spectrum demonstrates even a stronger nonlinearity: $\omega \sim (\frac{q}{\tilde{g}})^{3/2}$. Notice that in contrast to fermionic (one-particle) excitations which are completely incoherent the bosonic branches of the spectrum correspond to real quasiparticles.

Using the effective free boson description we can calculate the specific heat $C_V(T)$ as a function of temperature

$$C_V(T) = \frac{\partial}{\partial T}\sum_{\lambda=\parallel,\perp}\int\frac{d\mathbf{n}}{2\pi\Lambda}\int d^2\mathbf{q}\frac{\omega_\lambda(\mathbf{n}\cdot\mathbf{q})}{\exp(\frac{\omega_\lambda(\mathbf{n}\cdot\mathbf{q})}{T}) - 1}, \quad (4.13)$$

where $\omega_\lambda(q)$ denote different solutions of (4.12). Estimating the integral in (4.13) we obtain that at $T \gg \tilde{g}^6$ an ordinary Fermi-liquid result holds $[C_V(T) \sim T]$ which is solely due to the contribution of the mode $\Phi^\parallel_\mathbf{n}$. On the contrary, at low temperatures $T < \tilde{g}^6$ the specific heat becomes nonlinear and it is mainly determined by the $\Phi^\perp_\mathbf{n}$ contribution: $C_V(T) \sim \tilde{g}^3T^{2/3}$.

Remarkably this estimate coincides with the result of the RPA calculations[8,9] (we are reminded that the RPA result is nonanalytic in g since $\gamma_{RPA} \sim g^2$). In addition, we confirm the hypothesis made in Ref. 18 that the RPA result remains valid in an effective bosonic theory as well. The reason is that RPA contributions basically represent the effect of states with an arbitrary number of low-energy particle-hole pairs and the bosonization scheme takes an account of just this subspace of the entire Hilbert space.[16,17]

It is worthwhile mentioning that the bosonic representation can be also used to investigate the transport properties of the model (1.1) which are supposed to be quite unusual.

V. CONCLUSIONS

The above results provide further support for the statements about a breakdown of Fermi-liquid theory in the 2D model (1.1) made in Refs. 12 and 19. Moreover, the low-energy behavior found within the eikonal approximation appears to be quite different from the "orthogonality catastrophe"[1] [which involves exponentiation of logarithmic divergences to give $Z(\epsilon) \sim \epsilon^\eta$]. The initial deviation from the Fermi-liquid theory is demonstrated by the lowest infrared divergent diagram (Fig. 2), but at $\epsilon \sim \tilde{g}^6$ this power-law behavior turns into the exponential asymptotics of Eq. (2.4).

One might think that an exponential behavior of the one-particle Green function (2.4) in the vicinity of the Fermi surface is an artifact caused by a gauge noninvariance of the object. However we show that the trace of the exponentially decaying Z factor does appear in both gauge invariant and noninvariant response functions which typically receive their singular contributions from momenta close to the Fermi surface. In particular, due to this fact we do not find real divergencies of susceptibilities in both particle-particle and particle-hole channels which would demonstrate a tendency toward pairing or a formation of charge density wave. Moreover, the behavior of the one-particle Green function is reflected in experiments which are sensitive to the behavior of those fermions near the Fermi energy; thus it leads to a dramatic suppression of the oscillations of orbital magnetization in a weak external magnetic field, in the dHvA effect.

It may happen, of course, that an intrinsic instability of the model (1.1) cannot be detected as a divergent susceptibility of some local order parameter and then the corresponding phase transition is not of second order. As a plausible example one could consider the intriguing possibility of a spontaneous generation of uniform magnetic flux commensurate with the particle's density first discussed by Wiegmann.[24]

We have also shown that to recover the results of the eikonal approximation, capturing the most relevant features of the long-wavelength dynamics of the model (1.1), one has to use the effective bosonic Lagrangian which is not purely one dimensional.

The very existence of the approximate free boson representation means a possibility of a partial diagonalization of the problem involved in the scaling limit.

In other words, using this representation one is able to restore the (non-Fermi liquidlike) properties of the low-energy particle-hole subspace of the entire Hilbert space. This is supposed to be an intrinsic feature of any bosonization scheme.

As an example of the application of this technique we have found the spectrum of the bosonic collective mode governing particle-hole dynamics, and its contribution to specific heat.

We intend to further address these and other related issues (for instance, a generalization on the case of fermions with spin) elsewhere.

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Some aspects of singular interactions in condensed Fermi systems

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Abstract. — This article gives a fairly detailed survey of some of the problems raised when the interaction energy $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ between 2 fermionic quasiparticles (in 2 dimensions) is singular when $|\underline{k} - \underline{k}'| \to 0$. Before dealing with singular interactions, it is shown how a <u>non-singular</u> $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ leads to a 2-dimensional Fermi liquid theory, which is internally consistent, at least as far as its infrared properties are concerned. The quasiparticle properties are calculated in detail. The question of whether singular interactions arise for the dilute Fermi gas, with short-range repulsive interactions, is investigated perturbatively. One finds a weak singularity in $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ when the dimensionality $D = 2$, but it does not destabilize the Fermi liquid. A more sophisticated analysis is then given, to all orders in the interaction, using the Lippman-Schwinger equation as well as a phase shift analysis for a <u>finite</u> box. The conclusion is that any breakdown of Fermi liquid theory must come from non-perturbative effects. An examination is then made of some of the consequences arising if a singular interaction is introduced — the form proposed by Anderson is used as an example. A hierarchy of singular terms arise in all quantities — this is shown for the self-energy, and also the 3–point and 4–point scattering functions. These may be summed in a perfectly consistent manner. Most attention is given to the particle-hole channel, since it appears to lead to results different from those of Anderson. Nevertheless it appears that it is possible to derive a sensible theory starting from a singular effective Hamiltonian — although Fermi Liquid theory breaks down, all fermionic quantities may be calculated consistently. Finally, the effect of a magnetic field (which cuts off the infrared divergences) is investigated, and the de Haas-van Alphen amplitude calculated, for such a singular Fermionic system.

Prologue.

This article is written in honour of Rammal Rammal, for the symposium held in his memory on the 22 May, 1992, in Grenoble. I am very glad to have this opportunity to honour him, for not only is he now generally recognized as having been the finest condensed matter theorist of his generation in France; he was also a very fine man, whose loss was difficult to absorb.

My personal experience of Rammal began with my first visit to Grenoble in late 1985. Amongst the many of hundreds of anonymous physicists in the various labs, Rammal was immediately noticeable by his hospitable overtures to conversation, and I recall with pleasure several wide-ranging discussions of localisation theory at this time. Later, in Autumn 1987, it was his idea to begin a collaboration, looking at RVB states on triangular lattices (with an eye on ^{3}He films). Alas, this work was cut short by his terrible heart attack in early 1988. He was much more fragile thereafter, but in the summer and autumn of 1990, we were able to resume extended discussions, particularly on one of his favourite topics, the "W.A.H." problem. I was looking forward to renewing the debate on my return to Grenoble at the beginning of June last year, only to learn upon landing in France of his sudden and tragic death.

One is left with striking recollections. His love for the jab, feint, and thrust of argument and debate (scientific or otherwise), punctuated by blasts of irreverent and sometimes ribald humour. An incredible sense of humour, this, which ceded to nothing and nobody — I will never forget being entertained for nearly 6 hours one day in his office, after he had begun by announcing "how much work he had to do". And a passionate nature, which when channeled by his wide-ranging curiosity and love of enquiry, led to a constant flow of ideas. One wishes now that he had been less driven — although, given his character and background, would this have been possible?

Above all one is left in frank admiration of his extraordinary courage and tenacity, and his relentless refusal to let things as they were. His was a spirit too strong for his body; it is still an inspiration to those who where lucky enough to have known him.

The following contribution deals with a set of themes which fascinated Rammal, and I regret very much not being able to discuss it with him. He would have improved it considerably — having first, of course, tried to pull it to pieces!

1. Introduction.

The idea that we may describe the low-energy excitations of an interacting fermion system as fermionic quasiparticles, interacting via an interaction energy function $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, is one of the fundamental lynchpins of modern physics. Elaborated in the form of the Fermi liquid theory (FLT) of Landau [1], this idea is generally believed to describe ^{3}He and metals in 3 dimensions. Various modifications which incorporate BCS pairing [2], or other kinds of order parameter built "on top" of the basic FLT structure, allow the idea to be extended throughout 3 dimensions, and a similar idea appears implicitly or explicitly in nuclear physics and quantum field theory. One can of course also develop theories for bosonic systems involving interacting bosonic quasiparticles, in 2 and 3 dimensions.

Leaving aside for the moment the issues surrounding 1−dimensional systems, it is really quite remarkable that until very recently nobody had thought to ask what would happen if the interaction function $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ were a <u>singular</u> function of its arguments (i.e., it diverged, or some derivative diverged, etc.). Although at first sight this question seems to make no sense (in, eg., the context of FLT), we shall see in this paper that in fact such singular interaction functions are quite meaningful physically, even in the context of FLT; and we shall see how to deal with them theoretically (see also Ref. [3]).

The idea that such singular interactions might exist amongst 2−dimensional fermions was suggested by Anderson [4], who discussed one particular singular form, for which $f_{\underline{k}\underline{k}'} \sim \underline{k}' \cdot (\underline{k} - \underline{k}')/|\underline{k} - \underline{k}'|^2$. We shall examine this form in this article. Meanwhile, the assertion of Anderson suggests another more difficult question, viz., under what physical circumstances can such singular interactions arise? This question is highly controversial at the moment, particularly in 2 dimensions − in this article I can give no definite answer, but only partial results (see Sect. 3). Thus most attention will be focussed here on describing how to deal with singular interactions, once they arise, and deducing some of their physical properties. This will

be done using a method first described in reference [3], which deals systematically with the hierarchy of infra-red singularities arising from such singular terms in $f_{\underline{k}\underline{k}'}$, as well as showing that the theory is self-consistent (by re-deriving $f_{\underline{k}\underline{k}'}$, after a summation of singular terms).

We shall find that the theory of "singular interacting Fermi liquids" is very different from the usual FLT. In fact, as conjectured by Anderson [4], there are certain resemblances to the properties of 1−dimensional interacting Fermi systems [5], in which the fermionic quasiparticle description breaks down completely — in fact, as $\omega \to 0$ the fermionic quasiparticles disappear completely. However the detailed connection between the two kinds of systems is not at present clear, and some of the results described here appear to disagree with those of Anderson.

Once one has admitted that singular interactions are possible, a rather rich new vein of physical ideas and possibilities is opened up, both to the theorist and the experimentalist. The most obvious applications are to known 2−dimensional fermionic systems, and attention has concentrated on the new superconductors. However there are probably better systems available in which to test these ideas - one example which springs to mind is the 2−dimensional ^{3}He film [6]. Particularly interesting is the role of applied magnetic fields. This is because the crucial effect of the singular $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is to create a hierarchy of IR divergences in the low-energy physical properties of the system (which eventually destroys the fermionic quasiparticles) and a magnetic field can be used to at least partially suppress these divergences. The net result is shown in figure 1, in which the quantity $Z(\omega)$ (the fermionic quasiparticle wave-function renormalization factor) is shown for finite H, and $H = 0$ (and temperature $T = 0$). Thus we see the very interesting experimental possibility of passing between the singular interacting system, and an ordinary spin-polarized Fermi liquid, just by switching on a magnetic field. In fact this sort of suppression of infrared (IR) effects by a field has already been suggested [7] for 3−dimensional Fermi liquids (with non-singular interactions), but these IR effects were rather weak (they certainly do not destroy the FLT ground state!). The modifications due to H are much more drastic for singular systems, and so some discussion is given (in Sect. 5) of what might be seen in the dHvA effect for a charged system. Unfortunately, apart from the dHvA effect, the effects are not quite so dramatic as one might hope, and for really strong field effects it seems that one ought to go to transport measurements rather than thermodynamic ones. These are more subtle to calculate, and at the time of writing I have no complete results.

Other interesting spin-offs suggested by the problem at issue concern the possible crossover between singular and non-singular behavior as a function of dimensionality (at $T = 0$ this should involve a phase transition). A rather controversial example at the moment is the "coupled chain problem", in which 1−dimensional Luttinger liquid chains are coupled by a perpendicular hopping $t_\perp$; recent discussions of this can be found in reference [8].

More abstract (but arguably more interesting and fundamental) spin-offs from the questions addressed here, concern our understanding of the different kinds of condensed matter system which may exist, and how we should classify and describe them. This problem of "generic states" can be approached in various ways. It is often useful to start from some sort of general "effective Hamiltonian" for the class of systems under discussion, and we are interested in something like

$$\mathcal{H}^{\rm int}_{\rm eff} = \frac{1}{2}\sum_{\underline{k}\underline{k}'}\sum_{\underline{q}} \left[f^S_{\underline{k}\underline{k}'}\delta\hat{\rho}_{\underline{k}}(\underline{q})\delta\hat{\rho}_{\underline{k}'}(-\underline{q}) + f^A_{\underline{k}\underline{k}'}\,\delta\hat{\underline{\sigma}}_{\underline{k}}(\underline{q})\cdot\delta\,\hat{\underline{\sigma}}_{\underline{k}'}(-\underline{q})\right] \tag{1}$$

for a system lacking spin-orbit coupling; here $\delta\hat{\rho}_{\underline{k}}(\underline{q})$ describes density fluctuations in the system, and $\delta\hat{\underline{\sigma}}_{\underline{k}}(\underline{q})$ describes spin fluctuations; $f^S_{\underline{k}\underline{k}'}$ and $f^A_{\underline{k}\underline{k}'}$ are the interaction energy functions, which may or may not be singular. In the context of FLT, such Hamiltonians have been discussed before [9, 10], and the formulation of FLT, for normal and superfluid systems, in terms

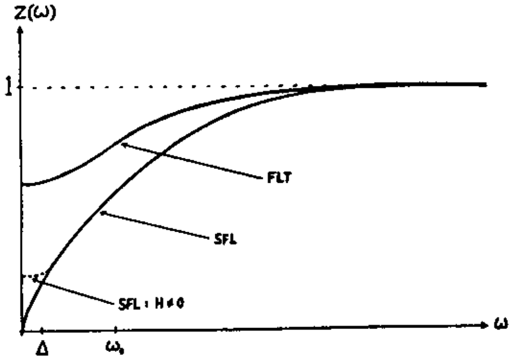

Fig. 1. — The fermionic quasiparticle wave-function renormalization $z(\omega)$, shown for a typical Fermi liquid (FLT), for the type of singular interaction dealt with in this paper (SFL), and for the same singular Fermi liquid now in an applied field (SFL: $H \neq 0$). All of these are at temperature $T = 0$; and Δ is the Zeeman splitting when $H \neq 0$.

of "dynamic molecular fields", [11] is closely related to this; very recently Haldane [12] has discussed fermion systems in 1, 2, and 3 dimensions in terms of a bosonized H_{eff} very similar to (1.1) (notice that $\delta\hat{\rho}_{\underline{k}}(\underline{q})$ and $\delta\hat{\underline{\sigma}}_{\underline{k}}(\underline{q})$ are bosonic operators).

What one would like to understand, both for singular and non- singular $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, is the general behavior open to $\mathcal{H}^{\rm int}_{\rm eff}$. Such a question is often formulated in the language of renormalization group (RNG) theory. It has been common in recent years to think of the FLT ground state as a fixed point, described in its vicinity by an effective Hamiltonian like $\mathcal{H}^{\rm int}_{\rm eff}$. Even for non-singular interactions there are certain difficulties in this point of view [10, 13]. For example, there exists in principle an infinite set of "FLT parameters" (for an isotropic Fermi liquid, these are the F_l^S and F_l^A, derived from $N(0)f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ on the Fermi surface S_F), but if we consider these as "irrelevant variables", describing the approach toward the fixed point, then it is hard to understand the infinite set of "Pomeranchuk instabilities" which arise at finite values of the $F_l^\lambda(\eta = S,\ A)$, in fact when $F_l^\lambda \to -(2l+1)$ from above in 3-d. Two possible ways [10] one might resolve this problem are to either (a) insist that the real irrelevant variables are actually the A_l^λ (where $A_l^\lambda = F_l^\lambda / \left[1 + F_l^\lambda/(2l+1)\right]$), so instability arises for infinite A_l^λ; or (b) suppose that the F_l^λ are marginally relevant [14]. In any case, it is clear that the extension of $\mathcal{H}^{\rm int}_{\rm eff}$ to include singular interactions must throw some light on this question, by enormously enlarging our space of possible effective Hamiltonians. In fact, as will be briefly discussed later (Sect. 6), it seems as though certain singular forms for $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ lead to "non-normalizable" but physically meaningful theories, lying quite outside the usual RNG machinery. Since the methods of RNG theory have so far been singularly unhelpful in understanding quantum liquids like ^{3}He which lack an order parameter, this may well turn out to be rather useful.

We shall not use RNG methods in the paper - in fact most of the discussion will rely ultimately on a technique which has hardly been exploited in such problems before, that of the "eikonal expansion". This semi-classical approximation has been previously exploited in high-energy field theory calculations [15], where it takes advantage of the Lorentz construction of wave-packets; here it turns out to be useful for many singular interactions in the low-energy limit, for quite different physical reasons (essentially because forward scattering dominates as $\omega \to 0$, because of phase space restrictions). Thus this method turns out to be the natural way to set up a description of singular interactions, even those having a non-normalisable character.

However in this paper we will not follow this method. This is partly because there is no space in what is already a very long paper, but also because the diagrammatic methods, which will be used instead, allow a much clearer understanding of the hierarchy of IR terms which result from singular interactions.

Finally, on a more speculative note, one may ask whether it is possible to set up some kind of "Statistical Quasiparticle theory" for singular interactions. This was done for non-singular Fermi liquids by Balian and de Dominicis [16], and it is fairly easy to generalize to 2-dimensional Fermi liquids (see Sect. 2) and to Fermi liquids in magnetic fields [7, 17]. The obvious obstacle for singular interactions is the large degeneracy involved as $\omega \to 0$ (almost as bad as that occurring in the Fractional Hall effect), and the associated statistics transmutation. Nevertheless the possibility is a tantalizing one, particularly in view of the possible connections to a semi-classical description in terms of the closed orbits of the equivalent classical Hamiltonian.

The plan of the paper is as follows. In section 2, I discuss the theory of the low-energy properties of a 2-dimensional system of fermions in which $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is non-singular. This leads to a Fermi liquid theory having a form somewhat different from the 3-dimensional theory. Nevertheless, at least as far as the low-energy properties are concerned, it is internally consistent (as usual, one can say little about the high-energy behaviour, which could conceivably cause a breakdown of Fermi liquid behaviour). In section 3, a rather thorough discussion is given of the perturbation theory results for a 2-d dilute Fermi gas with short-range repulsive interactions; this is based largely on work done with G. Beydaghyan and N.V. Prokofev. In sections 4 and 5, I give the results and calculational details of a partial investigation of the singular interaction advocated by Anderson, again in 2 dimensions. In section 4 the theory is developed for zero magnetic field — the results were already reported in reference [3], but with an error which is corrected here. In section 5, I explore the effect of the IR cut-off in a finite magnetic field, with particular attention paid to the dHvA effect. Section 6 concludes the paper. The general idea of this paper is to survey the important themes arising in the discussion of singular interactions, whilst giving enough details of the calculations so that interested readers may follow them. Some of these details are relegated to 3 Appendices, following after section 6. I emphasize here that this article is by no means complete — although the discussions of FLT and the dilute Fermi gas in $D = 2$ dimensions at least come to a conclusion, much more could be added to them; and as we shall see, the work described, in sections 4 and 5, on singular interactions is really only a beginning on what promises to be a very interesting topic of research.

2. Regular interactions: FLT in $D = 2$.

In this section we assume regular interactions between the fermionic quasiparticles, so that

$$f^{\sigma\sigma'}_{\underline{k}\underline{k}'} \to \tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'} \tag{2.1}$$

where $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is bounded and has no singular behavior as a function of its arguments, or any derivatives of these arguments. From now on in this paper the tilda "~" will denote quantities derived from or associated with such non-singular interactions.

We shall investigate the consistency of such a description in 2 dimensions (as Landau and many others did for 3 dimensions), and derive some of the main properties in terms of $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$.

Some of the results, in their general form at least, have already been given previously, but almost always in the context of perturbation theory. It is of course possible to show that if perturbation theory is valid, then a 2-dimensional FLT will follow (see next section); but here we wish to go to the next stage, and see what FLT then leads to in 2 dimensions. The results will also be important when we come to non-singular interactions. We concentrate on 2 dimensions for reasons which will become clear later in this paper.

A good place to start is by investigating consistency. There are a number of different ways of doing this — here we just use one and start with the Bethe-Salpeter equation for fermionic quasiparticle-quasiparticle scattering, which is of course central to the formulation of microscopic FLT (note that this is not quite the same as phenomenological FLT [13]). At low energy transfer ν, and momentum transfer $\underline{q}$, with the "3-vector" $Q = (\underline{q}, \nu)$, we have a quasiparticle T-matrix $\tilde{t}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ for non-singular interactions which takes the form (in zero applied field)

$$\tilde{t}^{\lambda}_{\underline{k}\,\underline{k}'}(Q) = \tilde{f}^{\lambda}_{\underline{k}\,\underline{k}'} + \sum_{\underline{k}''} \tilde{f}^{\lambda}_{\underline{k}\,\underline{k}''} \frac{\underline{q} \cdot \tilde{\underline{v}}_{\underline{k}''}}{\underline{q} \cdot \tilde{\underline{v}}_{\underline{k}''} - \nu + i\delta} \, \tilde{n}^{(0)}_{k''} \, \tilde{t}^{\lambda}_{\underline{k}''\,\underline{k}'}(Q) \tag{2.2}$$

in which $\tilde{n}^{(0)}_{k''} = \tilde{n}^{(0)}(\tilde{\epsilon}_{k''})$ is the density of states for quasiparticles as a function of the non-singular quasiparticle energy $\tilde{\epsilon}_k$. The function $\tilde{t}^{\lambda}_{\underline{k}\,\underline{k}'}(Q)$ is related to the usual 4-point vertex $\tilde{\Gamma}^{\lambda}(P, P; Q)$ by

$$\tilde{t}^{\lambda}_{\underline{k}\,\underline{k}'}(Q) = \tilde{N}(0)\tilde{z}_k\tilde{z}_{k'}\tilde{\Gamma}^{\lambda}\left(\underline{k},\ \tilde{\epsilon}_{\underline{k}};\underline{k}',\ \tilde{\epsilon}_{\underline{k}'};Q\right) \equiv \tilde{N}(0)\tilde{T}^{\lambda}_{kk'}(Q) \tag{2.3}$$

around the Fermi surface S_{F}; and the equivalent equation for the 4-point vertex is

$$\tilde{\Gamma}^{\lambda}(P,\ P';Q) = \tilde{I}(P,\ P') + \int \frac{d\epsilon}{2\pi} \int \frac{d^2k}{(2\pi)^2} \tilde{I}(P,\ K)\tilde{R}(K;Q)\tilde{\Gamma}(K,\ P';Q) \tag{2.4}$$

where the frequency integral $\int d\epsilon$ has yet to be done; integration over the factor $\tilde{R}(K;Q) = \tilde{G}(K+Q)\tilde{G}(K)$ gives the usual singular factor in (2.2), under the assumption that, at $T = 0$ and low Q (i.e., $q \ll k_{\mathrm{F}}$, $\nu \ll \epsilon_{\mathrm{F}}$) :

$$\tilde{R}(K,Q) \sim \frac{\tilde{z}^2_k}{\tilde{v}_k}\left(\frac{q \cdot \tilde{v}_k}{q \cdot \tilde{v}_k - \nu + i\delta}\right)\delta(\epsilon - \mu)\delta(k - k_f) + \phi(K) \tag{2.5}$$

where $\phi(K)$ is regular. All of this is standard, of course; the difference between 2 and 3 dimensions becomes immediately apparent if we integrate $\tilde{R}(K,Q)$ over frequency ϵ and momentum k to find a lowest-order susceptibility $\tilde{\chi}^0_D(Q)$; when the dimensionality $D = 3$,

$$\tilde{\chi}^{(0)}_3(Q) \sim \tilde{N}_3(0)\left[1 - \tilde{\eta}/2\,\ln\left|\frac{1-\tilde{\eta}}{1+\tilde{\eta}}\right| + i\frac{\pi}{2}\tilde{\eta}\theta\left(1 - |\tilde{\eta}|\right)\right] \tag{2.6}$$

for low Q (here $\tilde{\eta} = \nu/q\tilde{v}_{\mathrm{F}}$) : but when $D = 2$,

$$\tilde{\chi}^0_2(Q) \sim \tilde{N}_2(0)\left[1 - \tilde{\eta}\left(\frac{\theta\left(\tilde{\eta}^2 - 1\right)}{\left[\tilde{\eta}^2 - 1\right]^{\frac{1}{2}}} + i\frac{\theta\left(1 - \tilde{\eta}^2\right)}{\left[1 - \tilde{\eta}^2\right]^{\frac{1}{2}}}\right)\right] \tag{2.7}$$

and we see that $\mathrm{Im}\,\tilde{\chi}^0_2(Q)$ actually diverges as $\nu \to q\tilde{v}_{\mathrm{F}}$, a much stronger singularity than in the usual 3-dimensional Lindhard function in (2.6) [in these expressions, $\tilde{N}_3(0)$ and $\tilde{N}_2(0)$

are the usual renormalised densities of states; $\tilde{N}_3(0) = p^2_{\mathrm{F}}/\pi^2\tilde{v}_{\mathrm{F}}$ and $\tilde{N}_2(0) = \tilde{m}^*/2\pi$]. There is also a much weaker behavior around the point $Q \sim (2k_{\mathrm{F}},0)$, corresponding to low-energy excitation of quasiparticles right across the Fermi surface. In $D = 2$, for example, one finds that

$$\begin{aligned}\mathrm{Im}\,\tilde{\chi}^0_2(q \to 2k_{\mathrm{F}}, \nu = 0) \to \frac{\tilde{N}_2(0)}{q}\Big\{&\left[(2 - q/k_{\mathrm{F}}) + \omega/2\epsilon_{\mathrm{F}}\right]^{\frac{1}{2}}\theta\left[(2 - q/k_{\mathrm{F}}) + \omega/2\epsilon_{\mathrm{F}}\right] \\ &- \left[(2 - q/k_{\mathrm{F}}) - \omega/2\epsilon_{\mathrm{F}}\right]^{\frac{1}{2}}\theta\left[(2 - q/k_{\mathrm{F}}) - \omega/2\epsilon_{\mathrm{F}}\right]\Big\}\end{aligned} \tag{2.8}$$

This behavior can be visualised by plotting $\mathrm{Im}\,\tilde{X}^0_2(Q)$ in the $(|q|, \nu)$ plane (see Fig. 2); it will play no role in what follows.

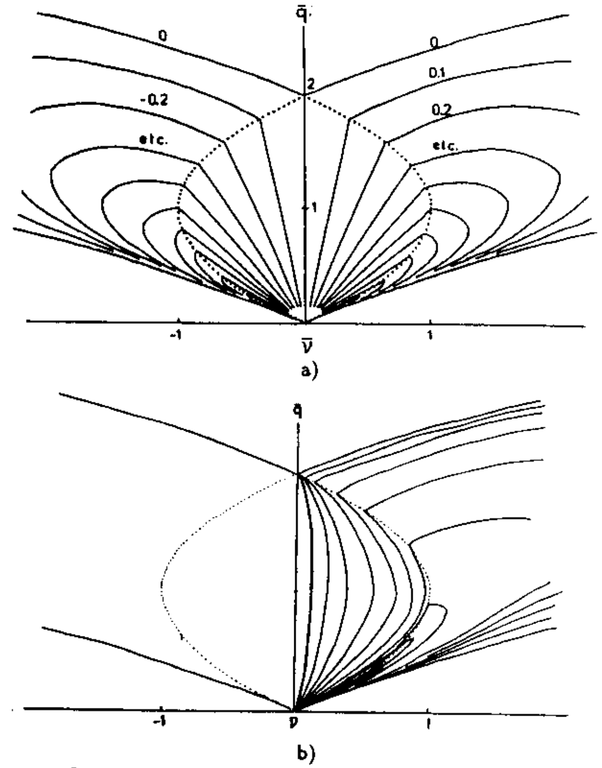

Fig. 2. — Plots of $\mathrm{Im}\tilde{\chi}^0_\sigma(q;\nu + i\delta)$ in the $(q,\ \nu)$ plane; these plots are simply contour maps. In (a) the 3-dimensional Lindhard function is shown, and in (b), the 2-dimensional version of this function (similar graphs can be found in Ref. [18]).

We may now reasonably ask whether the enhanced singular behavior in (2.7) is capable of leading to a breakdown of FLT in $D = 2$. A fairly convincing way of seeing that it will <u>not</u> lead to a breakdown, is by calculating the form of the fermionic self-energy at low energy ω. For a general $\tilde f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ to be a constant i.e., $\tilde f^{\sigma\sigma'}_{\underline{k}\underline{k}'} \to \tilde f^A_0$, for definiteness we will assume the coupling to be in the spin antisymmetric channel, and let $\tilde f^A_0 \equiv \tilde f_0$. A realistic calculation, incorporating all the different spin and angular momentum channels merely disguises the essential results (in a paper currently under preparation, this kind of calculation is done, and the general theory of Fermi Liquids in 2 dimensions is given in full detail).

Thus we wish to determine the form of $\tilde\Sigma_p(\omega + i\delta)$ as $p \to p_F$, $\omega \to 0$ (with ω measured from the Fermi energy). This will be deduced mainly from a calculation of Im $\tilde\Sigma_p(\epsilon_p + i\delta)$ (i.e., the "on-shell" imaginary part of $\tilde\Sigma_p(\omega)$).

We start by considering the lowest-order "reduced graph" contribution to Im $\tilde\Sigma_p(\epsilon_p + i\delta)$, shown in figure 3a, both in terms of the 4-point vertex $\tilde\Gamma$, and also in terms of the complete dynamic susceptibility $\tilde\chi^A_{00}(Q)$ [which we henceforth write as $\tilde\chi(Q)$, in this model calculation]. As shown in Appendix A, one can calculate Im $\tilde\Sigma_p(\epsilon_p)$ in these two different ways, but as we shall presently see, it is more convenient to calculate in terms of $\tilde\chi(Q)$. We then have in this simple isotropic model a lowest-order [in $\tilde\chi(Q)$] contribution to Im $\tilde\Sigma_p(\epsilon_p + i\delta)$ of the form [18]

$$\mathrm{Im}\left[\delta^{(1)}\tilde\Sigma_p(\omega+i\delta)\right] = \frac{\pi}{\tilde z_p}\sum_{\underline{p}'}\sum_{\underline{q}} |\tilde t_0(Q)|^2\, \eta_{\underline{p}'}\left(1-\eta_{\underline{p}-\underline{q}}\right)\left(1-\eta_{\underline{p}'+\underline{q}}\right)\delta\left(\omega+\epsilon_{\underline{p}'}-\epsilon_{\underline{p}-\underline{q}}-\epsilon_{\underline{p}'+\underline{q}}\right)$$

$$\equiv \frac{1}{\tilde z_p}\int_0^\infty \mathrm{d}\nu \sum_{\underline{q}}\left(1-\eta_{\underline{p}-\underline{q}}\right)\tilde f_0^2\, \mathrm{Im}\,\tilde\chi(q,\nu+i\delta)\,\delta\left(\nu-\left(\omega-\epsilon_{\underline{p}-\underline{q}}\right)\right) \qquad (2.9)$$

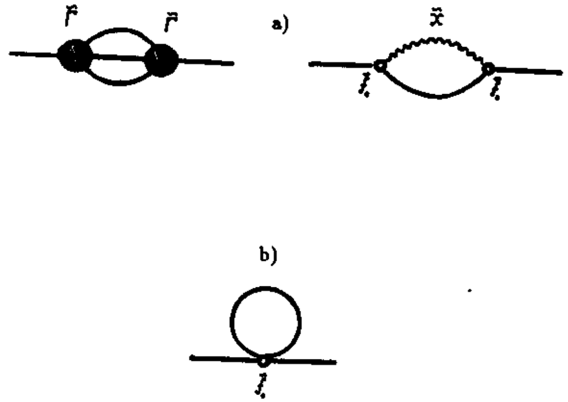

Fig. 3. — Low-order contributions to $\tilde\Sigma_p(\omega)$ for non-singular interactions. In (a) the lowest-order contribution to Im$\tilde\Sigma_p(\omega+i\delta)$ is shown in two different ways; in terms of the quasiparticle t-matrix, and in terms of the dynamic fluctuation propagator $\tilde\chi(Q)$, in the case where only one Legendre component $\tilde f_0$ of $\tilde f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is used (see Appendix A, and also Ref. [18]). In (b) the lowest order contribution to Re$\tilde\Sigma_p(\omega)$ is shown. Note that in (a), the calculation of Im$\tilde\Sigma_p(\omega+i\delta)$ proceeds by calculating the "reduced graph" obtained by cutting the graph in all possible ways (the only cut is shown here).

where $\eta_p \equiv \eta(\epsilon_p)$ is again the Fermi function; and (in this isotropic approximation)

$$\tilde t_0(Q) = \tilde f_0 / \left(1 + \tilde f_0 \tilde\chi^0_D(Q)\right) \qquad (2.10)$$

$$\tilde\chi(Q) = \tilde\chi^0_D / \left(1 + \tilde f_0 \tilde\chi^0_D(Q)\right) \qquad (2.11)$$

In (2.11), the "$\delta^{(1)}$" signifies the contribution first-order in $\tilde\chi(Q)$. We notice that the graph in figure 3b gives no contribution to Im $\tilde\Sigma_p(\omega + i\delta)$, although it obviously gives a contribution to Re $\tilde\Sigma_p(\omega)$; we return to this point below.

Equation (2.12) is easily evaluated and we find a standard result [19] in 3 dimensions (at $T = 0$):

$$\mathrm{Im}\left[\delta^{(1)}\tilde\Sigma_p(\omega+i\delta)\right] \sim \omega^2 \qquad (2.12)$$

However in $D = 2$ we find (Appendix A)

$$\mathrm{Im}\left[\delta^{(1)}\tilde\Sigma_p(\omega+i\delta)\right] \sim \omega^2 \ln\left|\frac{\omega}{\omega_0}\right| \qquad (2.13)$$

(the exact expression is given in the appendix – see equation (A.15)). There are two main differences between these two expressions: first, when $D = 2$ the leading term in Im $\left(\delta^{(1)}\tilde\Sigma\right)$ is $\sim \omega^2\ln\omega$ instead of the usual ω^2; and second, the leading IR correction to this, which in 3 dimensions is $\sim |\omega|^3$, is now also $\sim \omega^2\ln\left|\frac{\omega}{\omega_0}\right|$. This IR term, in $D = 3$, is just that responsible for the notorious $T^3\ln T$ terms [9, 20] in $C_v(T)$; its IR nature is shown by the way it is suppressed by a magnetic field [7]. The 2 dimensional term is stronger, and will show the same field suppression — it leads to a term in $C_v(T) \sim T^2$.

At this point one is tempted to conclude, solely from equation (2.13), that FLT is clearly internally consistent when $D = 2$, simply because the ratio Im $\left[\delta^{(1)}\tilde\Sigma(\omega)\right]$ /Re$\tilde\Sigma(\omega)$ tends to zero with ω. However this conclusion would be premature (although it is probably correct in the end!). It is rather instructive to see what lacunae exist in the argument, and how they can be dealt with, up to a point. There are essentially three points that must be taken care of, and they are:

(i) We must investigate the effect of higher fluctuation graphs.
(ii) We should look at 3–particle, 4–particle, etc., scattering processes.
(iii) We should check the behavior of Re$\Sigma_p(\omega)$ directly.

We shall deal with these in turn.

(i) Higher fluctuations: The nature of these contributions can be understood intuitively by referring back to the effective Hamiltonian in equation (1.1). This describes interactions between bosonic fluctuations, and it is natural to ask whether the higher-order interactions between the bosons will give significant corrections to the result in (2.12) and (2.13). From a formal point of view, in calculating the effect of these on the fermionic self-energy, it is convenient to work in terms of the "dynamic molecular fields" [11] which can be generated from (1.1), and which mediate the interactions between the fermions. These fields are again bosonic, and in fact the relevant graphs can be generated from the effective interaction in (1.1) just by breaking up

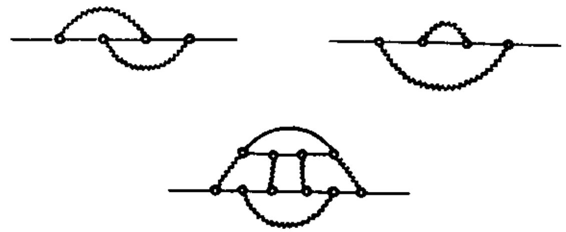

Fig. 4. — Higher-order fluctuation contributions to $\tilde{\Sigma}_p(\omega + i\delta)$: these are discussed in the text and also in appendix A.

one of the bosonic lines into its fermionic constituents (plus all permutations over the original bosonic graph). This then generates the series shown in figure 4, for the fermionic self-energy.

Now the point here is that if FLT is to be a consistent theory, then these higher contributions must not seriously affect the results already derived in (2.12) and (2.13). That this is true is easily seen by noticing that for regular interactions $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, higher-order graphs in the fluctuation fields $\tilde{\chi}(Q)$ lead to terms higher order in ω, in the low-energy expansion of Im $\tilde{\Sigma}(\omega)$, at least when $D = 2$ or 3. This point is shown in more detail in Appendix A. Thus this consistency check is satisfied — we shall see that for singular interactions it most definitely is not.

(ii) Multi-particle scattering: The next set graphs that must be checked for consistency go completely outside the framework of the effective Hamiltonian (1.1), since they involve 3-particle or higher multi-particle scattering. Some typical examples are shown in figure 5. Now the usual Landau expansion of the energy of a Fermionic liquid

$$E = E_0 + \sum_{\underline{k}\sigma} \epsilon_{\underline{k}\sigma}\delta\eta_{\underline{k}\sigma} + \frac{1}{2}\sum_{\underline{k}\sigma}\sum_{\underline{k}'\sigma'} \tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}\delta\eta_{\underline{k}\sigma}\delta\eta_{\underline{k}'\sigma'} + \ldots \tag{2.14}$$

can of course be supplemented by higher terms:

$$\delta E^{(3)} = \frac{1}{6}\sum_{\underline{k}_1\sigma_1}\sum_{\underline{k}_2\sigma_2}\sum_{\underline{k}_3\sigma_3} \tilde{g}^{\sigma_1\sigma_2\sigma_3}_{k_1k_2k_3}\delta\eta_{\underline{k}_1\sigma_1}\delta\eta_{\underline{k}_2\sigma_2}\delta\eta_{\underline{k}_3\sigma_3} \tag{2.15}$$

$$\delta E^{(4)} = \frac{1}{24}\sum_{\underline{k}_1\sigma_1}\sum_{\underline{k}_2\sigma_2}\sum_{\underline{k}_3\sigma_3}\sum_{\underline{k}_4\sigma_4} \tilde{h}^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\underline{k}_1\underline{k}_2\underline{k}_3\underline{k}_4}\,\delta\eta_{\underline{k}_1\sigma_1}\,\delta\eta_{\underline{k}_2\sigma_2}\,\delta\eta_{\underline{k}_3\sigma_3}\,\delta\eta_{\underline{k}_4\sigma_4} \tag{2.16}$$

corresponding to the 6-point and 8-point vertices in figure 5a; we assume a system of unit volume. The $\tilde{g}-$ and $\tilde{h}-$ functions can be related to the 6-point and 8-point vertices in a way similar to the relation between $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ and $\tilde{\Gamma}(K,\ K';Q)$ in the Landau theory [1]; thus, for example; the 3-particle T-matrix can be defined in · way analogous to (2.3), in terms of the 6-point vertex $\Gamma_6(P_1P_2P_3;Q_1Q_2)$, as shown in figur· 5a:

$$\tilde{T}_{\underline{p}_1\underline{p}_2\underline{p}_3}(Q_1Q_2) = \tilde{z}_{p_1}\tilde{z}_{p_2}\tilde{z}_{p_3}\tilde{\Gamma}\left(\underline{p}_1\ \epsilon_{p_1}\ ;\ \underline{p}_2\ \epsilon_{p_2}\ ;\underline{p}_3,\ \epsilon_{\underline{p}_3}\ ;Q_1,\ Q_2\right) \tag{2.17}$$

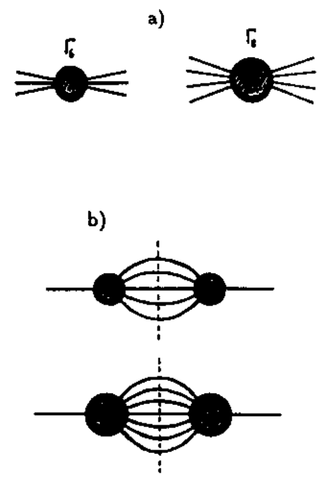

Fig. 5. — Some multi-particle processes that can contribute to $\tilde{\Sigma}_p(\omega)$. In (a) the complete 6-point and 8-point vertices are shown — these go into higher terms in the Landau expansion. In (b), the lowest-order contributions to Im$\tilde{\Sigma}_p(\omega)$, coming from the 6-point and 8-point vertices, are shown in reduced graph form.

and then for low momentum transfers $Q_1,\ Q_2$, we have

$$\tilde{g}_{p_1p_2p_3} = \left(\tilde{N}(0)\right)^2 \lim_{Q_1\to 0}\lim_{Q_2\to 0}\lim_{q_1/\nu_1\to 0}\lim_{q_2/\nu_2\to 0}\tilde{T}_{\underline{p}_1\underline{p}_2\underline{p}_3}(q_1,\ \nu_1\ ;\ q_2,\ \nu_2) \tag{2.18}$$

(the spin indices in (2.17) and (2.18) are suppressed for simplicity). If for the moment we ignore Γ_8, and higher vertices, equations (2.14) – (2.16) imply a quasiparticle energy functional

$$\tilde{E}_{k\sigma}\{\eta_{k\sigma}\} = \epsilon_{\underline{k}\sigma} + \sum_{\underline{k}'\underline{\sigma}'}\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}\delta\eta_{\underline{k}'\sigma'} + \frac{1}{2}\sum_{\underline{k}'\underline{\sigma}'}\sum_{\underline{k}''\underline{\sigma}''} g^{\sigma\sigma'\sigma''}_{\underline{k}\underline{k}'\underline{k}''}\delta\eta_{\underline{k}'\sigma'}\delta\eta_{\underline{k}''\sigma''} \tag{2.19}$$

and an effective interaction between quasiparticles of the form

$$\tilde{V}^{\sigma\sigma'}_{\underline{k}\underline{k}'}\{\eta_{\underline{k}\sigma}\} = \tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'} + \sum_{k''\sigma''}\tilde{g}^{\sigma\sigma'\sigma''}_{kk'k''}\delta\eta_{k''\sigma''} \tag{2.20}$$

Now the standard and well-known reason why we may ignore these higher contributions, in the context of FLT, is that at least at low ω and low T, the expansions in (2.14), (2.19), and (2.20) are expansions in the "small" deviation $\delta\eta_{\underline{k}\sigma}$ from the $T = 0$ fermionic quasiparticle distribution. Of course at higher T and ω, the single expansion in (2.14) will break down, and this will happen roughly when $\tilde{g}\delta\eta \sim \tilde{f}$. Notice, however, that this argumentation is only valid if the $\tilde{g}$- and $\tilde{h}$-functions are Non-Singular. If not, then not only does the whole FLT framework break down, but so does the more general effective Hamiltonian (1.1) (in which $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is not necessarily singular).

A number of remarks are in order at this point. First, we notice form (2.19) and (2.20) that, just as the quasiparticle energy in the usual FLT depends non-trivially on $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, in the same way it acquires less than non-trivial dependence on the $\bar{g}$-function if we go to higher orders in, say, energy ω; and likewise the original f-function now also becomes a functional of $\delta\eta_{\underline{k}\sigma}$. Thus when calculating terms beyond (2.12) and (2.13) in $\mathrm{Im}\tilde{\Sigma}(\omega)$, we must not only calculate the higher fluctuation terms, but also terms arising from the g-, h-, etc. functions.

A second remark, which is well known to anyone working with the Fadeev equations for 3-particle scattering, is that the form of the f-function also feeds back into the higher vertices; all of those contributions to higher vertices which are 2-particle irreducible will bring the f-function in. Thus these contributions to the $\bar{g}$- and h-functions can be calculated. What we cannot calculate in the original FLT framework are the interactions which are 3-particle irreducible and higher. In the same way as for FLT, it then follows that these irreducible functions will have to be introduced phenomenologically. However, provided they are regular, it is immediately obvious that they will not change the results in either (2.12) or (2.13), because the extra integrations over energy in figure 5b increase the power of ω, in the usual way.

However, this brings us to our third remark, which is that there is no obvious reason to suppose that, even if the 2-quasiparticle interaction energy $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is regular, that a higher function like $g^{\sigma\sigma'\sigma''}_{\underline{k}\underline{k}'\underline{k}''}$ must also be regular. If for some reason the contribution to the g-function coming from the irreducible 3-point vertex happens to be singular, then there is not a lot we can do without going outside the framework of our original effective Hamiltonian (1.1). This in fact takes us beyond the scope of this article.

Since there is not a great deal we can say here about these higher functions, we will assume from now on that they are benign (except where singularities in $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ imply otherwise). This proviso applies with even more force to Re $\tilde{\Sigma}_p(\omega)$.

(iii) Real part of $\tilde{\Sigma}(\omega)$: Up to now we have been discussing the imaginary part of $\tilde{\Sigma}(\omega)$ in order to show that

$$\lim_{\omega\to 0} \mathrm{Im}\left(\tilde{\Sigma}_p(\omega+i\delta)\right)/\mathrm{Re}\tilde{\Sigma}_p(\omega+i\delta) = 0 \tag{2.21}$$

If we assume that $\mathrm{Re}\tilde{\Sigma}(\omega) \sim \omega$, then this equation is certainly correct, as we have seen above; provided that $\tilde{f}$, $\bar{g}$, $\bar{h}$, and all higher interaction fractions are non-singular, then $\mathrm{Im}\tilde{\Sigma}_p(\omega+i\delta) \sim \omega^2 \ln\omega$, and (2.21) is correct.

However one must be careful here, because our conclusions about the low-energy behavior of $\tilde{\Sigma}_p(\omega)$ do not necessarily tell us much about the high-energy behavior, and so we cannot deduce anything about $\mathrm{Re}\tilde{\Sigma}_p(\omega)$, for low ω, from our results for $\mathrm{Im}\tilde{\Sigma}_p(\omega)$ (except for IR singular terms, such as the $|\omega|^3$ term in 3 dimensions, which leads to an $\omega^3\ln\omega$ term in $\mathrm{Re}\tilde{\Sigma}_p(\omega)$; in 2 dimensions, the $\omega^2\ln\omega$ term in $\mathrm{Im}\tilde{\Sigma}_p(\omega)$ leads to a term $\sim \omega$ in $\mathrm{Re}\tilde{\Sigma}_p(\omega)$). Thus we must be wary that even though our results for $\mathrm{Im}\tilde{\Sigma}_p(\omega)$ may be consistent with FLT, the form of $\mathrm{Re}\tilde{\Sigma}_p(\omega)$ may not be.

It is not hard to find examples of such a situation. The most obvious is provided by the Pomeranchuk instabilities which arise in both 3 dimensions and 2 dimensions when

$$\tilde{F}^{\lambda}_l \to -(2l+1) \tag{2.22}$$

from above. A typical such instability, involving $\tilde{F}^A_0$, is the ferromagnetic instability, but in fact its appearance will be visible in $\mathrm{Im}\tilde{\Sigma}_p(\omega)$, since the coefficients of the various terms in the low-ω expansion of $\mathrm{Im}\tilde{\Sigma}_p(\omega)$ depend on $\tilde{A}^A_0$, which diverges in the limit (2.22). Much more subtle is the instability which can occur if $\tilde{m}^* \to 0$, when $\tilde{F}^s_l \to -3$; there is no soft mode accompanying this Pomeranchuk instability, and so the approach towards this instability is not obvious in $\mathrm{Im}\tilde{\Sigma}_p(\omega)$.

An example which is nearer to the theme of this article was provided recently by Khodel and Shaginyan [21], who pointed out that any interaction $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ which is reasonably strongly peaked in the forward direction will give an instability in the very lowest-order contribution to $\mathrm{Re}\tilde{\Sigma}_p(\omega)$, shown in figure 3b. The interaction does not have to be a singular function of $|\underline{k}-\underline{k}'|$ for this instability, although it is clearly easier to achieve instability if it is.

It is actually much more difficult to check the behaviour of $\mathrm{Re}\tilde{\Sigma}_p(\omega)$ at low energy then it is $\mathrm{Im}\tilde{\Sigma}_p(\omega)$, since the higher multi-fluctuation contributions to $\tilde{\Sigma}_p(\omega)$ of figure 3a can all contribute to the leading-order (in ω) behavior of $\mathrm{Re}\tilde{\Sigma}_p(\omega)$ (although, as we have seen, they contribute to steadily higher-order terms in $\mathrm{Im}\tilde{\Sigma}_p(\omega+i\delta)$). Thus it is rather obvious that, for example, the calculation of Khodel and Shaginyan by no means establishes a real physical instability in their model (see also Nozières [21]); to establish this, one would have to sum all the higher-order terms. However in fact the forward scattering peak in their model actually leaves it open to treatment by exactly the same approximation which we are going to use here for genuinely singular interactions, and so this should allow at least a partial solution. Naturally, in the framework of perturbation theory this will correspond to a sum to infinite order of the multi-fluctuation contributions — we shall see in section 4 how this is done.

This concludes our analysis of the self-consistency of FLT in 2 dimensions. We see that there is nothing in the behaviour of $\mathrm{Im}\tilde{\Sigma}_p(\omega)$ which leads one to suspect a breakdown of FLT, and provided we can be sure that $\mathrm{Re}\tilde{\Sigma}_p(\omega)$ is well-behaved, then FLT is a perfectly good theory. As such it has been developed by a number of authors, particularly Miyake & Mullin [22], who noticed in particular that the properties of a spin-polarized Fermi liquid would be rather interesting in 2 dimensions. The discussion in this section (and in Appendix A) confirms that calculations in a $D = 2$ FLT are perfectly consistent, and that criticisms of this approach are quite unjustified [23], provided we assume $\tilde{f}^{\sigma\sigma'}_{kk'}$ is not singular. As such, FLT can be developed in all of its usual aspects, including transport theory [22], the theory of non-equilibrium and high-frequency phenomena, and even superfluid pairing. One may also develop a "Statistical Quasiparticle" (SQP) theory to describe its equilibrium properties. All of this is interesting, particularly for spin-polarized systems, but is too far off the main theme of this article, so we leave it for now.

At this point two natural questions arise, viz., (a) is there any way we can justify a non-singular $\tilde{f}^{\sigma\sigma'}_{kk'}$ from more microscopic considerations (at least for some class of systems); and (b) how do we treat singular $\tilde{f}^{\sigma\sigma'}_{kk'}$ forms? In the next section we look at question (a), where we find that perturbation theory indicates that $\tilde{f}^{\sigma\sigma'}_{kk'}$ must be regular in $D = 2$, at least for isotropic and translationally invariant systems (provided, of course, we ignore the superconducting instability). However, this does not exclude a non-perturbative contribution to $\tilde{f}^{\sigma\sigma'}_{kk'}$ which is singular – moreover, we shall see that the general effect of unscreened interactions is to give rise to singular forms for $\tilde{f}^{\sigma\sigma'}_{kk'}$, and so it then becomes necessary to examine singular functions anyway; this is done in section 4.

3. Microscopic origin of singular interactions.

In the last section a few consequences of the assumption that $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ was regular were studied; in particular, it was found that for $D = 2$, such an assumption was perfectly consistent with an FLT. However, we would really like to known a little bit more about $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ — apart from anything else, we would like to know if it is regular for some class of models and, if possible, what form it actually takes for a few particular models.

From the outset it is very important to realize that this is an extremely difficult question; I emphasize this because the various subtleties of the question are often ignored in the literature (usually in order to avoid compromising some approximation technique, such as, eg., the RPA). This is true in $D = 3$ as well as $D = 2$, although things generally get worse as D decreases. Amongst the difficulties and pitfalls one may include the following:

(i) Although one may succeed in showing that FLT results from perturbation theory for some model interaction, for some temperature range, even to all orders in perturbation, this is by no means a proof that FLT survives a perturbation theory expansion in 2 dimensions for a set of fermions interacting via realistic short-range repulsive forces, even in the absence of a lattice, right down to $T = 0$. Some features of this will be discussed below — we note that similar arguments were given in 3 dimensions some 30 years ago, in some very well-known papers [24] (see also Ref. [16]). Moreover this does not rule out the breakdown of FLT from non-perturbative terms; in 2 dimensions we already have the example of the Fractional Hall Effect to remind us of this (although the conditions prevailing there are rather different from those assumed here). In general we have less reason to believe in the results of perturbation theory in 2 dimensions than in 3 dimensions.

(ii) The discussion of internal consistency given in the last section has certain loopholes, as we already saw in some detail. These loopholes apply with even greater force to approximate calculations in 2 dimensions, so that their internal consistency in no way guarantees even the approximate validity of their results. Many such approximations are in any case internally inconsistent [13, 18]. This remark applies most importantly in the extrapolation of techniques, which may be fairly accurate in weak coupling, to the strong coupling regime (i.e., in going from the dilute to the dense case).

(iii) The apparent agreement of simple experimental features of a system's behaviour with FLT behaviour should never be used to justify FLT for that system (although it very often has been, both for heavy fermions in 3 dimensions, and for high-T_c superconductors and ^{3}He films in 2 dimensions). Thus the existence of a linear $C_v(T)$ at very low T, and a constant $\chi(T)$ in the same regime, is usually taken to be a proof that FLT applies to normal heavy fermions. This is sheer nonsense — not only do Luttinger liquids give the same low-T behaviour, but it is also possible to invent quite innocent and banal interactions between 3-dimensional fermions which will give the same behaviour, and yet for which FLT is invalid (see, eg., Ref. [21]). For heavy fermions experimental proof of FLT behaviour is still very far away (compare, eg., Ref. [17]), and one can give theoretical arguments that they should not be Fermi liquids at all. In fact the only system in nature for which reasonably good evidence exists for FLT behaviour is ^{3}He liquid in 3 dimensions, above the superfluid transition [1]. We shall return to this point again, particularly in section 6; meanwhile we note that a similar argument applies to any apparent agreement of experiment with other theories (such as, eg., ^{3}He properties with Gutzwiller theory).

Given all of these pitfalls, the value of a perturbation - theoretic investigation may seem questionable. Nevertheless it is important to see what results it gives, and so in the first part of this section we elucidate this question, and show that perturbation theory does indicate FLT behaviour is obeyed for short-ranged repulsive interactions in 2 dimensions, provided we ignore superconducting instabilities of the Kohn-Luttinger variety [25]. This then demonstrates, as stated in Ref. [3], that singular forms for $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ like that postulated by Anderson [4], must be non-perturbative in origin. After making some rather speculative remarks on how such a term might arise for this system, I then go on to point out that for some other systems in 2 dimensions, one can show the existence of singular terms. Thus, even if Anderson's arguments turn out to be incorrect for, say, ^{3}He films, it is clearly still of interest to investigate the consequences if a singular $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ exists.

3.1 PERTURBATION THEORY FOR DILUTE INTERACTING GAS.

— Several recent papers have been devoted to the investigation of the dilute interacting Fermi gas (DIFG) in 2 dimensions [26-28], and there are also some older papers of interest [22, 29]. Most of these calculations have been carried out using some kind of approximation (Ref. [28] is however exact in the dilute limit), and they all arrive at the conclusion that FLT is correct, at least if perturbation theory applies.

Here we have two things to add to these analyses. First, a calculation, analogous to that of Abrikosov and Khalatnikov [30], will be done for the f-function $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ to 2nd-order in the appropriate dimensionless coupling constant. As we shall see it is important that this be done for arbitrary values of $\underline{k}$ and $\underline{k}'$, not just for $\underline{k}$ and $\underline{k}'$ on the Fermi surface. The results of this are quite illuminating; we then go on to give a more general analysis of the problem in terms of the Lippman-Schwinger equation. The first analysis is based on work done by Beydaghyan and myself [31], and the second on work done with Prokofev [32].

We start from the usual short-range Hamiltonian

$$\mathcal{H} = \sum_{\underline{k},\sigma} \frac{\hbar^2 k^2}{2m} c^+_{\underline{k}\sigma} c_{\underline{k}\sigma} + \frac{1}{\Omega} \sum_{\underline{k}\,\underline{k}'\underline{q}} U(\underline{q}) c^+_{\underline{k}\uparrow} c^+_{\underline{k}'\downarrow} c_{\underline{k}'+\underline{q}\downarrow} c_{\underline{k}-\underline{q}\uparrow} \tag{3.1}$$

in which the interaction $U(r)$ is assumed to be of very short range — we shall work in the s-wave scattering channel only, and the Fourier transfer of $U(r)$. Define now the dimensionless interaction constant α by

$$\alpha = (8\pi q)^{1/2} a_0(q) \tag{3.2}$$

where $a_0(\underline{q})$ is the s-wave "scattering length" (actually in $D = 2$, $a_0(\underline{q})$ has the dimensions $L^{1/2}$). Now to first order in α (or in $U(\underline{q})$), we have a ground state energy shift

$$\Delta E^{(1)} = \frac{\hbar^2}{\Omega m} \alpha \sum_{\underline{k}\,\underline{k}'} n^0_{\underline{k}\uparrow} n^0_{\underline{k}'\downarrow} \tag{3.3}$$

$$\alpha = \frac{m}{\hbar^2} \overline{U}_0 \equiv \frac{m}{\hbar^2} \int d^2 r U(r) \tag{3.4}$$

To second order in α, we have an energy shift

$$\Delta E^{(2)} = \frac{-2m}{\Omega^2} \left(\frac{\hbar^2 \alpha}{m}\right)^2 \sum_{\underline{k}_1 \underline{k}_2 \underline{q}} \frac{n^0_{\underline{k}_1\uparrow} n^0_{\underline{k}_2\downarrow} \left[n^0_{\underline{k}_2+\underline{q}\uparrow} + n^0_{\underline{k}_1-\underline{q}\downarrow}\right]}{k_1^2 + k_2^2 - (\underline{k}_2 + \underline{q})^2 - (\underline{k}_1 - \underline{q})^2} \tag{3.5}$$

with the parameter α now defined in terms of U_0 as

$$\alpha = \frac{m}{\hbar^2} \left[\overline{U}_0 + \frac{1}{2} \frac{\overline{U}_0^2}{\Omega} \sum_{q} \frac{1}{k_1^2 + k_2^2 - (\underline{k}_2 + \underline{q})^2 - (\underline{k}_1 - \underline{q})^2} \right] \tag{3.6}$$

The parameter α now depends on $\underline{k}_1$ and $\underline{k}_2$, but for low densities we will ignore this. So far the development is very similar to that in 3 dimensions [30], with α playing the role of a renormalized (physical) interaction parameter; the only real difference is that $a_0(\underline{q})$ now depends on momentum, and in fact diverges like $q^{-1/2}$ as $q \to 0$.

We now wish, in the framework of this second order renormalized perturbation theory, to calculate the interaction energy function $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ between quasiparticle states in the presence of the background Fermi sea. This is easily shown to be

$$\Delta^{(2)} f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2} = \delta^2 \Delta E^{(2)} / \delta n^0_{p_1\sigma_1} \delta n^0_{p_2\sigma_2} \tag{3.7}$$

$$= -\left(\frac{\hbar^2\alpha}{m}\right)^2 \left[I_1^{\sigma_1\sigma_2}\left(\underline{p}_1\underline{p}_2\right) + \left(I_2\left(\underline{p}_1\underline{p}_2\right) + I_2\left(\underline{p}_2\underline{p}_1\right)\right)\right] \tag{3.8}$$

where the integrals I_1 and I_2 are

$$I_1^{\sigma_1\sigma_2}\left(\underline{p}_1\underline{p}_2\right) = \frac{2m}{\hbar^2\Omega^2} \sum_{\underline{k}} \frac{4\hat{Q}^{\sigma_1\sigma_2}}{p_1^2 - k^2 - p_2^2 - \left(\underline{p}_1 + \underline{p}_2 + \underline{k}\right)^2} \eta_i^0 \tag{3.9}$$

$$I_2\left(\underline{p}_1\underline{p}_2\right) = \frac{2m}{\hbar^2\Omega^2} \sum_{\underline{k}} \frac{1}{p_1^2 + k^2 - p_2^2 - \left(\underline{p}_1 - \underline{p}_2 + \underline{k}\right)^2} \eta_i^0 \tag{3.10}$$

and $\hat{Q}^{\sigma_1\sigma_2}$ is the Dirac spin exchange operator. In figure 6 the diagrammatic equivalents of these terms are shown; note that I_1 is singular in the Cooper channel $\left(\underline{p}_1 + \underline{p}_2 \to 0\right)$, whilst I_2 is singular in the crossed channel $\left(\left|\underline{p}_1 - \underline{p}_2\right| \to 0\right)$. There is of course no contribution in the zero sound channel, since $f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2}$ is irreducible in this channel. In (3.8) we have ignored the term of first order in α, since it is independent of $\underline{p}_1$ and $\underline{p}_2$; it is just given by $(\hbar^2\alpha/m\Omega)\,\hat{Q}^{\sigma_1\sigma_2}$.

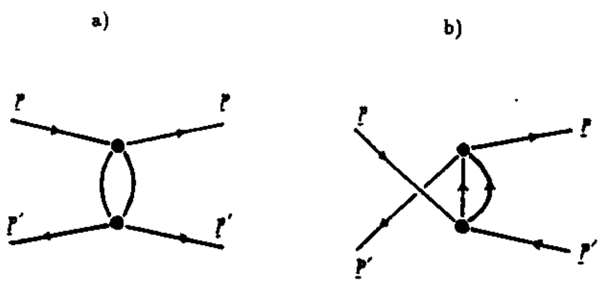

Fig. 6. — Diagrammatic representation of the terms contributing to $f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2}$, to second order in the dimensionless coupling α. In (a) we see the Cooper term I_1 of equation (3.9), and in (b) the crossed channel term I_2 of equation (3.10).

Now at this point we could make the using simplifying manoeuvre of putting $\underline{p}_1$ and $\underline{p}_2$ on the Fermi circle S_F to calculate the leading ($T = 0$) FLT properties. This, however, would be throwing out the baby with the bathwater, since it is precisely the singular terms off the Fermi surface that we are interested in (recall that, for example, Anderson's form [4] actually disappears if $\left|\underline{p}_1\right| = \left|\underline{p}_2\right|$!). Thus we shall evaluate (3.8) for arbitrary wave-vectors. Some details of this derivation are given in Appendix B; the more complete analysis, including the effects of spin polarization, appears in reference [31].

The Cooper contribution, coming from I_1, gives a rather unwieldy expression, which I give here for completeness:

$$\Delta^{(2)}_{\text{Cooper}} f^{\sigma_1\sigma_1}_{\underline{p}_1\underline{p}_2} = \frac{\hbar^2}{\pi m\Omega}\alpha^2\hat{Q}^{\sigma_1\sigma_1}\left\{ \ln\left|\frac{\underline{p}_1+\underline{p}_2}{\underline{p}_1-\underline{p}_2}\right| + \right.$$
$$\ln\left|\frac{\left[\left(p_F^2-u_+\right)\left(p_F^2-u_-\right)\right]^{\frac{1}{2}} + p_F^2 - \frac{1}{2}\left(p_1^2+p_2^2\right)}{\frac{1}{2}\left|p_1+p_2\right|\left|p_1-p_2\right|}2\right| \theta\left(u_- - p_F^2\right)\operatorname{sign}\left(p_1\cdot p_2\right) +$$
$$\left. +\theta\left(p_F^2-u_+\right)\cdot\right\} \tag{3.11}$$

where u_+ and u_- are given by

$$u_\pm = \frac{1}{4}\left(\left|\underline{p}_1+\underline{p}_2\right| \pm \left|\underline{p}_1-\underline{p}_2\right|\right)^2 \tag{3.12}$$

This function is not very important for our purposes; it has the usual singularity for $\left|\underline{p}_1+\underline{p}_2\right| \to 0$, as is most easily seen by letting $\underline{p}_1$ and $\underline{p}_2$ go to the Fermi surface; the resulting function then has the form

$$\Delta^{(2)}_{\text{Cooper}} f^{\sigma_1\sigma_2}\left(\theta_{p_1p_2}\right) = \frac{\hbar^2}{\pi m\Omega}\hat{Q}^{\sigma_1\sigma_2}\frac{1}{2}\ln\left|\frac{1+\cos\theta p_1p_2}{1-\cos\theta p_1p_2}\right| \tag{3.13}$$

The singularity for $\theta p_1 p_2 = \pi$ is genuine as can be shown in the usual way by summing the logs to infinite order, either directly [30] or by using one-loop renormalization group methods [5]. Its form is basically the same as in 3 dimensions [30].

The crossed channel contribution is more interesting; it is given by

$$\Delta^{(2)}_{\text{Crossed}} f_{\underline{p}_1\underline{p}_2} = \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{p_F}{\left|\underline{p}_1-\underline{p}_2\right|}\left\{\left[|s_1| - \left(s_1^2-1\right)^{\frac{1}{2}}\theta\left(|s_1|-1\right)\right]\operatorname{sign}\left(s_1\right) + \right.$$
$$\left. + \left[|s_2| - \left(s_2^2-1\right)^{\frac{1}{2}}\theta\left(|s_2|-1\right)\right]\operatorname{sign}\left(s_2\right)\right\} \tag{3.14}$$

where

$$s_1 = \frac{1}{p_F}\frac{\underline{p}_1\cdot\left(\underline{p}_2-\underline{p}_1\right)}{\left|p_1-p_2\right|} \qquad s_2 = \frac{1}{p_F}\frac{p_2\cdot\left(\underline{p}_1-\underline{p}_2\right)}{\left|\underline{p}_1-\underline{p}_2\right|} \tag{3.15}$$

We notice immediately the similarity to Anderson's term, already mentioned in the introduction. However the resemblance turns out to be insufficient to produce any dramatic results, as we shall see. Let us begin by analyzing the behavior of (3.14) for different values of $\underline{p}_1$ and $\underline{p}_2$.

First, we notice that if both $\underline{p}_1$ and $\underline{p}_2$ are below the Fermi circle, then the function is quite anodyne, since then s_1, $s_2 < 1$, and

$$\frac{p_F}{\left|\underline{p}_1 - \underline{p}_2\right|}\left[\frac{\underline{p}_1 \cdot \left(\underline{p}_2 - \underline{p}_1\right)}{\left|\underline{p}_1 - \underline{p}_2\right|} + \frac{\underline{p}_2 \cdot \left(\underline{p}_1 - \underline{p}_2\right)}{|p_1 - p_2|}\right] = -1 \tag{3.16}$$

Thus this term can only cause a trivial change in the ground state energy.

Things become more interesting once we have excited quasiparticles. Suppose, for example, that $\left|\underline{p}_1\right| > p_F$, whereas $|p_2| < p_F$. Then inside the sliver of momentum space shown in figure 7a, we have $|s_1| > 1$, $|s_2| < 1$, and $\mathrm{sign}(s_1) = -$, $\mathrm{sign}(s_2) = +$. This then implies that

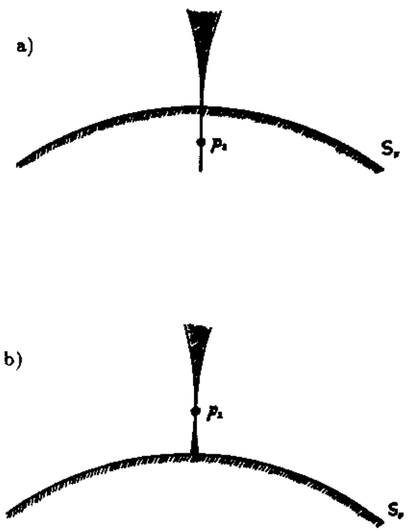

Fig. 7. — Regions in momentum space where interesting singular behaviour exists in $\Delta^{(2)} f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2}$, coming from the crossed channel. In (a), we see the situation for $\left|\underline{p}_2\right| < p_F$, $|p_1| > p_F$; singular behaviour exists for $\underline{p}_1$ inside the shaded region (for any p_2 on the vertical axis). In (b) we have $|p_1|$ and $|p_2| > p_F$, and the singular region is again shown for p_1, with fixed $\underline{p}_2$. These singularities are much weaker than those discussed by Anderson (see text).

$$\Delta^{(2)} f_{p_1p_2} \to \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{p_F}{\left|\underline{p}_1 - \underline{p}_2\right|}\left[|s_2| - \left(|s_1| - (s_1^2 - 1)^{\frac{1}{2}}\right)\right]$$
$$\sim \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{\underline{p}_2 \cdot (p_1 - p_2)}{|p_1 - p_2|^2} \tag{3.17}$$

in this small region of momentum space. A similar answer with $\underline{p}_1 \leftrightarrow \underline{p}_2$, obviously follows for $\left|\underline{p}_2\right| > p_F$, $\left|\underline{p}_1\right| < p_F$. Finally, if we let both $\left|\underline{p}_1\right|, \left|\underline{p}_2\right| > p_F$, then we have, for $\left|\underline{p}_1\right|$ and $\left|\underline{p}_2\right|$ close to p_F, the structure depicted in figure 7b, and inside the sliver of momentum space shown, we have

$$\Delta^{(2)} f_{p_1p_2} \to \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{p_F}{\left|\underline{p}_1 - \underline{p}_2\right|}\left\{(s_1 + s_2) + \left[(s_1^2 - 1)^{\frac{1}{2}} - (s_2^2 - 1)^{\frac{1}{2}}\right]\mathrm{sign}\,(|p_1| - |p_2|)\right\}$$
$$\sim \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{p_F}{\left|\underline{p}_1 - \underline{p}_2\right|}\left[(s_1^2 - 1)^{\frac{1}{2}} - (s_2^2 - 1)^{\frac{1}{2}}\right]\mathrm{sign}\,(|p_1| - |p_2|) \tag{3.18}$$

In both these cases, if $\underline{p}_1$ is outside the slivers of momentum space depicted in figure 7, $\Delta^{(2)} f_{p_1p_2}$ is completely regular. Inside these slivers it is singular, but the singularity is actually rather weak; in fact, from (3.17) and (3.18) we can summarize the singular part of $\Delta^{(2)} f_{p_1p_2}$, as $|p_1 - p_2| \to 0$, by

$$\Delta^{(2)} f_{p_1p_2} \sim \frac{\hbar^2\alpha^2}{2\pi m\Omega}\frac{p_F}{\left|\underline{p}_1 - \underline{p}_2\right|}\left\{\left[\left(\frac{|p_1| - p_F}{p_F}\right)^{\frac{1}{2}}\theta(p_1 - p_F)\,\theta(p_F - p_2) + \left(\underline{p}_1 \leftrightarrow \underline{p}_2\right)\right]\right.$$
$$\left. + \left[\left(\frac{\left|\underline{p}_1 - p_F\right|}{p_F}\right)^{\frac{1}{2}} - \left(\frac{\left|\underline{p}_2 - p_F\right|}{p_F}\right)^{\frac{1}{2}}\right]\mathrm{sign}\left(\left|\underline{p}_1\right| - \left|\underline{p}_2\right|\right)\theta(p_a - p_F)\,\theta(p_2 - p_F)\right\} \tag{3.19}$$

in which we see that the $1/(p_1 - p_2)$ divergence is suppressed by the factor $[(|p| - p_F)/p_F]^{\frac{1}{2}}$; this is in addition to the restriction of this weak singularity to the slivers of momentum space in Figure 7.

From this we see that 2nd-order perturbation theory in α gives no support for the kind of singular behavior advocated by Anderson. As already mentioned, a number of other perturbation-theoretic analyses have also appeared recently [26-29], and these come to the same conclusion. All of these papers deal, as done here, with the low-density gas, and all, with the exception of Fabrizio *et al.* [28], use some low-density approximation scheme (usually the T-matrix approximation of Galitski [33]).

However at this point one might try and argue that there may be more subtle singular terms in $f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2}$, lurking at higher orders in α, or beyond the reach of a 1-loop or T-matrix approximation. Thus if we really want to answer this question (still within the context of perturbation theory!) it seems to be necessary to go a little farther.

Now in fact one can do this, starting from the Lippmann-Schwinger equation [32]; this approach also has the advantage of clarifying the essentially non-perturbative nature of any singular term more dramatic than (3.18), and also shows why we get (3.19). We shall thereby see that 2nd-order perturbation theory in α is enough to describe the essential structure of $f^{\sigma_1\sigma_2}_{\underline{p}_1\underline{p}_2}$, at least as far as the crossed channel is concerned; thus there is no need to go beyond this in trying to understand the DIFG for positive α.

Let us start again from equation (3.1), and consider the scattering between states $|p_1\rangle$ and $|p_2\rangle$, in the presence of the background Fermi sea (we ignore spin here). Assuming $U(q) \to U_0$, we have for the outgoing 2-particle wave-function

$$\Psi(\underline{r}) \sim U_0 \sum_{\underline{Q}} \frac{\left(1 - n_{\underline{k}_0+\underline{Q}}\right)\left(1 - n_{\underline{k}_0-\underline{Q}}\right)}{\epsilon - \left(E_{\underline{k}_0+\underline{Q}} + E_{\underline{k}_0-\underline{Q}}\right)}\varphi_{\underline{Q}}(r) \tag{3.20}$$

where $n_{\underline{k}_0+\underline{Q}}$ are Fermi functions, and $2k_0 = \underline{p}_1+\underline{p}_2$; the final states $|\underline{p}'_1\rangle$ and $|\underline{p}'_2\rangle$ are summed over here, with $2\underline{Q} = \underline{p}'_1 - \underline{p}'_2$. Notice that $\Psi(\underline{r})$ in (3.20) is nothing but the solution to the 2-particle Schrödinger equation

$$L^2 = U_0 \sum_{\underline{Q}} \frac{\left(1 - n_{\underline{k}_0+\underline{Q}}\right)\left(1 - n_{\underline{k}_0-\underline{Q}}\right)}{\epsilon - \left(E_{\underline{k}_0+\underline{Q}} + E_{\underline{k}_0-\underline{Q}}\right)} \tag{3.21}$$

in a square box of side L (I shall come to the question of the box shape below).

The first question to answer is how we treat the $\sum_Q$ sums in (3.20) and (3.21). Let us start by assuming we can deal naively with these sums, not worrying too much about the discrete nature of phase space imposed by our finite box — we shall drop this assumption presently. In this case the right-hand side of (3.21) is just the sum of a P.P (principal part) integral, dealing with states far from the mass shell, and a mass shell part which must be directly evaluated as a discrete sum. Assuming that $\underline{p}_1 - \underline{p}_2 = 2\underline{Q}_0$, we have

$$L^2/U_0 = P\int_{-\infty}^{\infty} \mathrm{d}x \frac{\tilde{\rho}(x)}{\epsilon - x} + \pi \frac{\tilde{\rho}(Q_0)}{\tan\left[\pi \Delta E(Q_0)\,\tilde{\rho}(Q_0)\right]} \tag{3.22}$$

where the density of states $\tilde{\rho}(x)$ is the "Pauli restricted density of states" in which the factors $\left(1 - n_{\underline{k}_0+\underline{Q}}\right)\left(1 - n_{\underline{k}_0-\underline{Q}}\right)$ operate:

$$\mathrm{P}\int \mathrm{d}x \frac{\tilde{\rho}(x)}{\epsilon - x} \equiv \mathrm{P}\int \mathrm{d}\theta \int Q\mathrm{d}Q \frac{\left(1 - n_{\underline{k}_0+\underline{Q}}\right)\left(1 - n_{\underline{k}_0-\underline{Q}}\right)}{\epsilon - \left(E_{\underline{k}_0+\underline{Q}} + E_{\underline{k}_0-\underline{Q}}\right)} \tag{3.23}$$

The mass shell term sum in (3.22) results from a sum over the levels nearby the energy $x(Q_0)$ we are interested in; we assume a free-particle spectrum, so $x(Q) = Q^2/m$ (recall this is a two-particle energy). The energy $E(Q_0)$ is then shifted by $E(Q_0) = x(Q_0) + \Delta E(Q_0)$, by the interaction U_0; the form of the mass shell term results from the simple formula

$$\sum_{m=-\infty}^{\infty} \frac{1}{\alpha - m} = \pi/\tan \pi\alpha \tag{3.24}$$

Now a naive determination of the P.P. integral involves a restricted density of states having the form (recall $Q^2 = mx$) :

$$\begin{aligned}\tilde{\rho}(Q) = N_2(0)\Big\{&\theta\left((\underline{k}_0 - \underline{k}_F)^2 - Q^2\right) + \theta\left(Q^2 - (k_0 + k_F)^2\right)\\ &+ \frac{2}{\pi}\theta\left(Q^2 - (\underline{k}_0 - \underline{k}_F)^2\right)\theta\left((\underline{k}_0 + \underline{k}_F)^2 - Q^2\right)\sin^{-1}\left[\frac{(k_0 - k_F)k_F + \frac{1}{2}Q^2}{k_F Q}\right]\Big\}\end{aligned} \tag{3.25}$$

which is shown in figure 8a. In figure 8b we see how this function arises — the diagram shows how the Pauli restrictions operate to reduce $\tilde{\rho}(x)$ from the free one-particle density of states $N_2(0)$ ($= mL^2/4\pi$ for a square box).

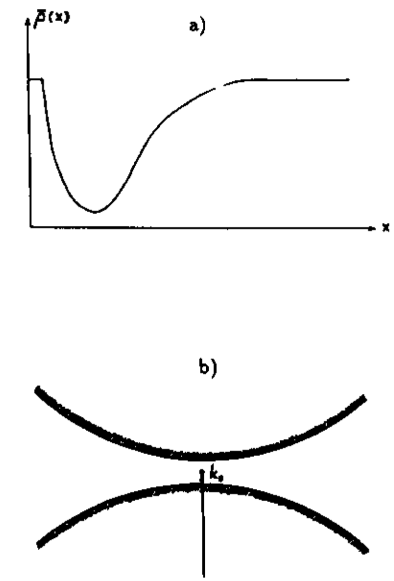

Fig. 8. — The "Pauli-restricted density of states" $\tilde{\rho}(x)$, for energy $x = Q^2/m$, of a pair of fermions with initial momentum $\underline{p}_1 + \underline{p}_2 = 2\underline{k}_0$; we assume $|\underline{k}_0| > k_F$ here (for $|k_0| < k_F$, see discussion in text). The momentum difference $\underline{p}'_1 + \underline{p}'_2$ between the final states is $2Q$. In (a) the function $\tilde{\rho}(x)$ is plotted from equation (3.25); and in (b) the phase space restrictions which produce this curve are shown — the fermion pair can only scatter outside the 2 Fermi circles.

It is useful to see how these formula work in the low-density gas. Then, as is well known [29], the P.P. integral increases logarithmically as the density goes to zero. In this case we can use the approximation.

$$\pi\tilde{\rho}(Q_0)/\tan\left[\pi\tilde{\rho}(Q_0)\,\Delta E(Q_0)\right] \sim 1/\Delta E(Q_0) \tag{3.26}$$

which leads immediately to an energy shift

$$\Delta E(Q_0) \sim \frac{U_0/L^2}{1 + (U_0/L^2)\,N_2(0)F(Q_0)} \tag{3.27}$$

where $F(Q_0)$ is

$$\begin{aligned} F(Q_0) &\sim \left\{2\ln\left|\frac{2(\underline{k}_0 - \underline{k}_F)}{Q_0}\right| + \ln\left|\frac{\omega_0}{\epsilon_F}\right|\right\} \quad (Q_0 \ll k_0 - k_F)\\ &\sim \ln\left|\frac{\omega_0}{\epsilon_F}\right| \quad (Q_0 \gg k_0 - k_F)\end{aligned} \tag{3.28}$$

This is of course just a standard renormalization; ω_0 is a crossover related to U_0.

Having seen how we treat the sum over Q (at least naively), let us now return to the scattering equation (3.20) for the 2-particle wave-function. Certainly if perturbation theory is valid, we ought to treat the $\sum_Q$ in the same way here as we did for (3.21). It then remains to decide on boundary condition for (3.20). Let us first assume retarded outgoing boundary conditions, and examine the solution for $\Psi(\underline{r})$ when $|\underline{r}|$ is large, but still well away from the boundary (i.e. $k_0 r \gg 1$, but $r \ll L$). Then the retarded solution $\Psi^R(\underline{r})$ has the form, at least for the low-density case (assuming (3.26))

$$\Psi^R(\underline{r}) \sim \left\{ \frac{e^{i\underline{Q}_0\cdot\underline{r}}}{\Delta E(Q_0)} + \mathrm{P}\int Q\mathrm{d}Q\, e^{i\underline{Q}\cdot\underline{r}} \frac{2\tilde{\rho}(Q)}{Q_0^2 - Q^2} \right\} \tag{3.29}$$

which for large r just gives a phase shift

$$\delta(Q_0) = \pi\tilde{\rho}(Q_0)\,\Delta E(Q_0) \tag{3.30}$$

with $\tilde{\rho}(Q_0)$ already given by (3.25).

Before going on, it is work making 3 remarks here, viz.,

(i) There is nothing surprising about (3.30), viewed as a relationship between a phase shift, an energy shift, and a density of states; this form is generally valid, even when perturbation theory breaks down. What is important here is that it involves the restricted density of states $\tilde{\rho}(x)$, and not the unrestricted density of states $N_2(0)$.

(ii) One might argue that to determine relations like (3.30), involving stationary properties like the density of states, one ought not to use outgoing boundary conditions, but rather the standing wave conditions appropriate to the actual eigenstates of the system in a box. This distinction is essentially that between the dynamical quasiparticles and statistical quasiparticles of the system, already mentioned in section 2. We shall see how to deal with it below.

(iii) If we start from a finite box, then in reality there is a discretization of momentum space involved. It then becomes important to decide whether or not to calculate phase shifts, etc., before or after taking the limit $L \to \infty$. If we calculate δ_0 before taking the limit, then we ought to take explicit account of this discretization.

These 3 remarks become all the more important when we see what are the consequences of (3.30), which are rather subtle. Notice that $|\underline{k}_0|$ may be greater or less than k_F. Consider first the case where $|\underline{k}_0| > k_F$. Then we see that the value of the phase shift $\delta(Q_0)$ as $Q_0 \to 0$ is actually ambiguous — it depends on the order in which we take the limits $Q_0 \to 0$ and $(k_0 - k_F) \to 0$. If we first let $Q_0 \to 0$, and then $(k_0 - k_F)$, then we get a density of states at zero energy which is a constant, $N_2(0)$. This limit corresponds to letting the 2 particles approach each other first, and then letting them both go down to the Fermi circle S_F; obviously this always leaves a small circular region of momentum space, around $\underline{k}_0$, free of Pauli restrictions (see Fig. 8b again). In the case of the dilute Fermi gas, we can then apply (3.27) and (3.28) for the energy shift to get (recall $N_2(0) = mL^2/4\pi$) :

$$\lim_{Q_0 \to 0} \delta_0(Q_0)\,\Big|_{\frac{Q_0}{k_0 - k_F} < 1} \sim \frac{mU_0/4}{1 + (mU_0/4\pi F(Q_0))} \tag{3.31}$$

where $F(Q_0)$ depends on the ratio between $(k_0 - k_F)$ and Q_0, as we saw in (3.28). If we hold this ratio to a finite constant $\lambda = (Q_0/(k_0 - k_F)) < 1$, then $\delta_0(Q_0) \to$ constant, as $Q_0 \to 0$.

Note, rather amusingly, that if $\lambda \to 0$, then so does $\delta_0(Q_0)$, as $Q \to 0$; but this is not because $\tilde{\rho}(Q_0)$ is going to zero, but rather because ΔE_{Q_0} is.

Let us now consider the case where $\lambda \gg 1$, ie., we let $(k_0 - k_F) \to 0$ more quickly than Q_0. This corresponds to letting the two quasiparticles stay apart even as we let $\underline{k}_0$ approach the Fermi circle. Now in the extreme limit where $\lambda \to \infty$, ie., we let $(k_0 - k_F) = 0$ with a finite Q_0, it is clear that the density of states now behaves as

$$\lim_{Q_0 \to 0} \tilde{\rho}(Q_0)\,\Big|_{(k_0 - k_F) = 0} = \frac{N_2(0)}{\pi k_F} Q_0 = \frac{mL^2}{4\pi^2 k_F} Q_0 \tag{3.32}$$

and this square root behavior in energy is reflected in the behavior of the phase shift; we will find

$$\lim_{Q_0 \to 0} \delta_0(Q_0)\,\Big|_{(k_0 - k_F)} \sim Q_0/k_F \tag{3.33}$$

Finally, consider the situation where Q_0 is finite, but $(|k_0| - k_F) < 0$; the two states $\left|\underline{p}_1\right\rangle$ and $\left|\underline{p}_2\right\rangle$ are widely separated, but $|p_1|$ and $|p_2|$ are close to k_F. In this case the 2 Fermi circles in figure 8b now overlap, and $\underline{Q}_0$ must always exceed a minimum value $Q_{\min}$, where

$$Q_{\min}^2 = k_F^2 - k_0^2 \tag{3.34}$$

In this case it is easy to see that we get the following limiting behavior, assuming $Q_{\min} \ll k_F$:

$$\delta_0(Q_0) \sim Q_0/k_F \qquad (\text{for } k_F \gg Q_0 \gg Q_{\min}) \tag{3.35}$$

$$\delta_0(Q_0) \sim \frac{Q_0 - Q_{\min}}{k_F} \qquad (\text{for } k_F \gg Q_{\min} \gg Q_0 - Q_{\min}) \tag{3.36}$$

Summarizing, we see that if we keep $\underline{p}_1$ and $\underline{p}_2$ separated whilst letting the pair energy go to zero, we get zero phase shift; but if we let $\underline{p}_1$ and $\underline{p}_2$ approach each other at the same time as we let their combined energy go to zero, we get in general a finite phase shift.

This peculiar behaviour is very puzzling, and requires that we take a closer look at the various limiting processes. It thus also requires us to be more careful about our treatment of the density of states in a finite box (since the continuum limit is so peculiar). In dealing with a finite box we must pay particular attention to the effect of boundary conditions on our wave functions (and thence the density of states). Thus in effect, we are thrown back onto the remarks raised just after equation (3.30).

I shall start by assuming that it does not matter whether we choose a square or a circular box, and freely go from one to another, as convenience requires; in both cases perfectly reflecting boundaries are assumed. This assumption needs to be justified, and this will be done once all the arguments have been made.

Consider then figure 9a, where the vector $\mathbf{k}_0$ approaches the Fermi circle for a system inside a square box of side L; we assume, to start with, that $\mathbf{k}_0$ is parallel to the box orientation. The question we wish to answer is — which limit, (3.31) or (3.33), is appropriate to this case?

If we assume that figure 9 gives a correct representation of the situation, then it is easy to see that this would imply a finite phase shift as $\mathbf{k}_0 \to k_F$, for the allowed states in figure 9a are like that in a 1-dimensional problem. If for $\mathbf{k}_0$ we choose the lowest state outside the Fermi circle, for example, then it will lie at an energy E_0 such that $(E_0 - 2\epsilon_F) \sim \epsilon_F/N$, where N is the number of particles per unit volume ($N = k_F^2 L^2/4\pi$, since the points in k-space are separated

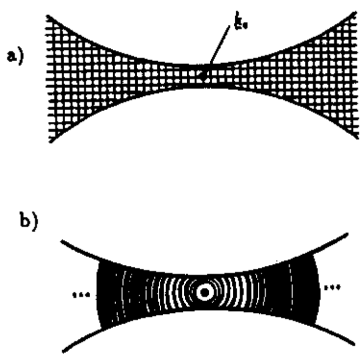

Fig. 9. — Available scattering states in "discretized momentum space", as $|\underline{k}_0| \to k_F$. In (a) we show the available states (at the intersections of the grid lines), with the particular choice that $\underline{k}_0$ is parallel to the momentum space and grid. In (b) we angular average — the weight to be attached to each final state is proportional to the length of curved line associated with it (the angular average is accomplished by rotating the grid).

by $2\pi/L$). Then (see Fig. 9a, for the pair states $\left|\underline{p}_1'\underline{p}_2'\right\rangle$ with small Q (i.e. $Q < k_F N^{-\frac{1}{4}}$), equation (3.21) becomes

$$\frac{N}{U_0} \simeq \sum_{n=1}^{M} \frac{1}{\Delta E_0 - \frac{4\pi^2}{mL^2} n^2} \sim \frac{1}{\Delta E_0} \tag{3.37}$$

leading to an energy shift $\Delta E_0 \sim U_0/N$ (in (3.37), the sum is cut off at $Q_1 \sim k_F N^{-1/4}$, corresponding to a maximum n given by $M \simeq (k_F L)^{1/2}$; higher Q-vectors in the sum can obviously be absorbed into the principal part). The same kind of analysis gives for the Lippmann-Schwinger wave-function

$$\Psi(\underline{r}) \to \frac{\text{const}}{\Delta E_0} \left\{ 1 - \frac{e^{2\pi i r/L}}{4\pi^2/mL^2} + \ldots \right\} e^{iQ_0 . r} \tag{3.38}$$

which implies a finite phase-shift $\delta_0 \sim \pi m U_0/k_F^2$.

This argument seems to indicate that as $k_0 \to k_F$, the limit (3.31), with a finite phase-shift, is the appropriate one to take. However in fact the argument is quite misleading, and equations (3.37) and (3.38) are wrong! The reason is slightly subtle, and lies in the assumption, in figure 9a, that the orientation of the lattice of Q-vectors in this figure is parallel to k_0. Now in fact we can take k_0 in any direction, and a proper expectation value ought to average over these (it is certainly difficult to imagine any physical measurement capable of going beyond this angular average!). In figure 9b we see what happens if we take this angular average (by rotating the coordinate frame!); the resulting density of states is shown in figure 10a. It is obvious that the angular average renders the privileged configuration of figure 9a unimportant, and we end up with $\delta_0 (k_0 \to k_F) = 0$, i.e., it is really the limit (3.33) that is appropriate.

In the same way, for a finite box, the density of states one finds, after this angular average, for a k_0 some distance above k_F is shown in figure 10b. The reader immediately notices that

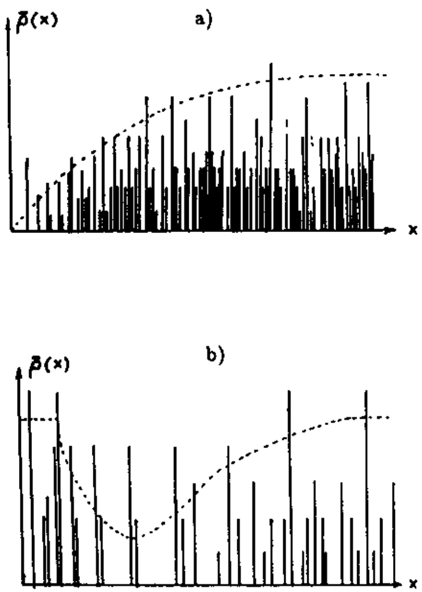

Fig. 10. — The density of states $\tilde{\rho}(x)$ for a finite box. In (a) the density of states for $|\underline{k}_0| \to k_F$ is shown, derived from the procedure illustrated in figure 9. In (b) we see $\tilde{\rho}(x)$ for a finite $(|\underline{k}_0| - k_F)$; this is just a "fractal version" of figure 8a, with fluctuations $\sim N^{-1/2}$.

figures 10a and 10b are merely "fractal" versions of the smoothed functions $\bar{\rho}(x)$ that one finds in the limit $L \to \infty$ (compare Fig. 10b with Fig. 8a), and the fluctuations in $\tilde{\rho}(x)$ for a finite box are just the usual deviations, of order $N^{-1/2}$, that one expects for a finite system.

There are 2 lacunae left to fill in these arguments — we must also show (i) that computing the phase shift using standing wave boundary conditions does not alter the result, and (ii) show that changing the box shape makes no difference either.

In fact both of these points may be dealt with using rather old ideas, due to Newton and Fukuda, and de Witt [34]; the famous "orthogonality catastophe" analysis of Anderson [35], for the problem of an impurity scatterer in a Fermi gas, is based on the same ideas. Since these ideas are well known, I will only outline here how the argument works for our problem. It is convenient to start now from a circular box, and look at the standing-wave solutions for the two-particle wave-function (not the 1-particle wave-function!); for $1 \ll Q_0 r \ll L$, these have the form

$$\Psi_l(r) \sim \frac{1}{\sqrt{r}} \sin\left[Q_0 r - \frac{\pi l}{2} + \delta_l(Q_0)\right] \tag{3.39}$$

But then, in the usual way [34] we may derive the relation between the phase shift $\delta_l(Q_0)$, the energy shift $\Delta E(Q_0)$, and the density of states, as

$$\delta_l(Q_0) = \pi \rho(Q_0) \Delta E_l(Q_0) \tag{3.40}$$

The immediate question is then — what is $\rho(Q_0)$ in this equation? Perturbation theory says it must be the Pauli-restricted density of states, as in (3.30). However, as we shall briefly discuss below, Anderson [14] has denied that this is correct. What we have demonstrated here is that, within the context of perturbation theory, changing to standing waves does not change the arguments, nor the conclusion that the Fermi surface phase shift is zero.

In the same way, changing the box shape will rearrange the "fractal" structure of the energy levels, but will not change the angular-averaged results — the arguments of Fukuda and Newton [34] are easily adapted to our 2-dimensional, 2-particle problem, to show this.

Let us now summarize the arguments in this sub-section. Within the context of perturbation theory, and the validity of the Lippmann-Schwinger equation, the arguments above purport to show that for the case of short-range repulsive interactions, we cannot get a singular form for the effective interaction like that postulated by Anderson — there is a weak singularity, but it is sufficiently weak that it gives a phase shift which goes to zero as $k_0 \to k_F$.

To conclude this discussion it is perhaps worthwhile to indicate at what point in the above arguments one might expect that non-perturbative terms might change things. It is hard to imagine the general relation (3.30) breaking down; the weak point is rather in the density of states which we use in this expression. The use of $\tilde{\rho}(x)$ depends essentially on the Pauli restrictions operating on the intermediate states in the scattering expressions - and we have already seen how delicate is the limit $(|k_0| - k_F) \to 0$. Clearly if we could arrange a finite density of states in this limit, the results would be quite different — we would get a finite phase shift, δ_0, and this would imply a change in the energy of a quasiparticle $|k\rangle$, coming from a quasiparticle $|\underline{k}'\rangle$, given by

$$\delta\epsilon_k = \frac{\delta_0}{\pi} \frac{\underline{k}' \cdot (\underline{k} - \underline{k}')}{\left|\underline{k} - \underline{k}'\right|^2} \tag{3.41}$$

Now in fact it has been argued by Anderson that in considering the limit $k_0 \to k_F$, $Q \to 0$, we should not use the Pauli-restricted $\tilde{\rho}(Q)$, but rather an unrestricted density of states $N_2(0)$. The reasoning is that this process involves intermediate states with identical momenta, and that the standard rules of perturbation theory do not apply here. This argument is strongly reminiscent of the points made over 30 years ago by Bloch, Balian, and de Dominicis [16], concerning the $T = 0$ limit of perturbation theory, but it is not the same — the simplest way to see this is by noting that their arguments, and those of Kohn and Luttinger [36], when applied to an isotropic system in either $D = 2$ or 3 dimensions, give zero energy shift. Thus Anderson's argument is inherently non-perturbative (as we have already seen), whilst the results of Refs. 16 and 36 are perturbative.

In the opinion of this author, the verdict on non-perturbative contribution to $f_{\underline{k}\,\underline{k}'}$ must still be regarded as open for the DIFG, and the subtlety of even the perturbative contributions makes it unlikely that it will be resolved quickly. However it is interesting to note that for some models the existence of a singular interaction is much easier to establish. One obvious example is the case where we have unscreened gauge interactions, in 2 dimensions and even higher, and of course the various anyon models are a somewhat extreme example of this. In such examples one again has singular behaviour in the forward scattering direction (in the anyon case, via the Aharonov-Bohm effect). A related example, provided to this author by Kveschenko (private communication) arises in the effective interaction of 2-dimensional fermions moving in a random magnetic field — this apparently gives a form for $f_{\underline{k}\,\underline{k}'}$ identical to Anderson's. Thus, in any case, it is clearly necessary to study the consequences of a finite phase shift, and the concomitant energy shift in (3.41).

4. Singular interactions.

We now come to the second part of this paper, in which we look at the effects of the singular interaction described by Anderson [4]. It has the form

$$f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'} \sim \frac{\delta_0}{\pi N(0)} \frac{\underline{k}' \cdot (\underline{k} - \underline{k}')}{\left|\underline{k} - \underline{k}'\right|^2} \delta_{\sigma,\sigma'} \tag{4.1}$$

and describes the interaction energy between fermionic quasiparticle states $|\underline{k}\sigma\rangle$ and $|\underline{k}'\sigma'\rangle$. Written in this form, $f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ is not symmetric in $\underline{k}$ and $\underline{k}'$, and this creates problems of interpretation; moreover, the symmetric sum is not even singular!

One way of resolving these problems is as follows. We have already seen, in section 3, that a second-order perturbation theoretic calculation of $f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ gives a form that has no singular behaviour at all (at least as a function of $|\underline{k} - \underline{k}'|$) if both $|\underline{k} < k_F|$, and $|\underline{k}'| < k_F$. It is only when one or both of the quasiparticles are above the Fermi surface that singular behaviour emerges. Now in calculating quantities like $\Sigma_p(\omega)$ at $T = 0$, or $\Gamma(P,\ P';\ Q)$ at $T = 0$, we shall in fact only be interested in situations where a single quasiparticle is excited above S_F, and interacting either with the ground state Fermi sea, or else the Fermi sea minus one hole. Thus we will enforce the constraint in $f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$, that $|\underline{k}\rangle$ is above k_F, but $|\underline{k}'\rangle$ is below k_F, at least as far as the singular part of the interaction is concerned; if both $|\underline{k}| < k_F$ and $|\underline{k}'| < k_F$, we assume $f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ to be non-singular. Assuming only singular behaviour if $\sigma' = -\sigma$ (compare reference 4) we have, instead of 4.1), the form

$$f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'} = \tilde{f}^{\sigma\sigma'}_{\underline{k}\,\underline{k}'} + \lambda^{\sigma\sigma'}_{\underline{k}\,\underline{k}'} \tag{4.2}$$

$$\lambda^{\sigma\sigma'}_{\underline{k}\,\underline{k}'} = \frac{\delta_0}{\pi \tilde{N}(0)} \delta_{\sigma,-\sigma'} \frac{\underline{k}' \cdot (\underline{k} - \underline{k}')}{\left|\underline{k} - \underline{k}'\right|} \theta\left(|\underline{k}| - k_F\right) \theta\left(k_F - |k'|\right) \tag{4.3}$$

in which, as usual, the superscrip "tilde" indicates quantities renormalised by non-singular (FLT) interactions only, these being described by $\tilde{N}(0)$; all singular contributions to $f^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ are contained in $\lambda^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ (note the change of notation from Ref. [3], where $\lambda^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$ was denoted instead by $g^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$).

Most of the calculations described in this paper will use a diagrammatic method which consistently sums the singular terms arising from $\lambda^{\sigma\sigma'}_{\underline{k}\,\underline{k}'}$. This apparently leaves the results open to two criticisms, viz.:

(a) That the method is only approximate, and that some important terms may have been left out - or that there is some inconsistency or double-counting hidden in the calculation;

(b) That the methods is not very general, and that while it may work for Anderson's interaction form, there are other singular interaction forms of interest for which it may be useless.

In fact both criticisms are unwarranted, but this cannot be convincingly demonstrated using diagrammatic methods only. The way to show the generality and accuracy of the diagrammatic method depends on the physical observation that the singularities we are interested in occur when $\underline{k} \to \underline{k}'$, i.e. for comoving quasiparticles. This suggests that we systematically isolate those terms occurring in this limit, and in fact there is a method for doing so, i.e., the semi-classical eikonal expansion [15]. Quite remarkably, the leading term of this expansion gives a series identical in structure to the diagrammatic sum of singular graphs. The higher terms in this expansion are non-singular, and so we can completely justify the diagrammatic results.

Obviously, the eikonal expansion applies by its very nature to all singular interactions of the kind specified, and so we have arrived at a technique which can be used to attack many of the general problems described in the introduction.

There is not space here to describe both the diagrammatic methods and the eikonal expansion in all their detail, and in this paper we shall concentrate on the diagrammatic methods. This is done partly because it was the first method of the two to be found [3], but also because it is easier to understand both the hierarchy of IR terms involved in the theory, and the relationship to non-singular FLT, in terms of this approach. The understanding of the IR terms also has extra benefits when we come to apply a magnetic field, since the principal effect of the field is to suppress the IR divergencies. In another paper, currently under preparation, I will explain the link to the eikonal approximation more fully.

Another lacuna in the present paper will become clearer in the detailed calculations presented below (and in the Appendices). The existence of a singular interaction $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ will affect physical quantities through processes in all 3 scattering channels (i.e., the direct or "zero sound" channel, the crossed channel, and the Cooper channel). This in turn means that we must in general consider processes involving both particle-particle and particle-hole fluctuation propagators, both for low momentum and also for momenta near $2k_F$, when looking at the infra-red physics. However, in this paper I will only discuss processes involving the particle-hole fluctuation propagator. Since the infra-red divergences in the various channels will be coupled this makes the present work incomplete (the calculations were not finished by the deadline for this article).

However the calculations which are described in this section (and which were originally reported in Ref. [3]) are still of considerable interest, for several reasons, viz.,

(a) We find singular behaviour in the low-order (in λ) graphs for physical quantities such as $\Sigma_p(\omega)$, or $A_p(\omega)$; it is not immediately obvious how to deal with higher-order graphs, and the method for summing all singular terms is given here — this method seems to apply to all channels in the same way.

(b) It is found that the resulting completely summed expression does not have the same form as that given by Anderson [4]; in fact it is more singular (the statement to the contrary, in Ref. [3], is incorrect, as we shall see below).

(c) The conclusions which we draw, in section 5, concerning the effect of spin polarisation, are mostly unaffected by our neglect of processes involving particle-particle fluctuation propagators.

Thus the calculations described in section 4 can be regarded as a partial solution to the problem of a systematic discussion of the consequences of a singular $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$. The reason they are being reported in this form is essentially because of (b) just above, i.e., they show that the consequences of the singular $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ hypothesized by Anderson seem more subtle than was at first believed.

4.1 LOW-ORDER SINGULAR GRAPHS. — The basic idea of the diagrammatic method is to start from the set of non-singular graphs calculated from $\tilde{f}^{\sigma\sigma'}_{kk'}$ (see Sect. 2 and Appendix A), and to then systematically insert singular vertices $\lambda^{\sigma\sigma'}_{kk'}$ into these graphs in all possible ways. As we shall see later in the section, a set of dominant graphs emerges in this process; but in this sub-section we shall simply look at the lowest-order contributions (in $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$). The results turn out to be useful for 2 reasons - they give us a clue as to what may happen at higher orders, and, as we shall see in the next sub-section, all the dominant graphs (at any order) can be shown to be simple functions of the lowest-order graphs. The physical reason for this last result only becomes clear in the eikonal expansion, and it is in fact much easier to prove using

this expansion.

Let us start by considering graphs for $\Sigma_p(\omega)$ in which a single singular vertex λ, of the form given by (4.3), is inserted into the set of otherwise non-singular graphs, for Im Σ_p, of Appendix A. The vertex λ can of course be inserted anywhere in any of the graphs of figures 3, 4 and 5, but is fairly easy to see that the most singular low-energy contribution to Im $\Sigma_p(\omega+i\delta)$ of all of these comes from the graph shown in figure 11a. Inserting λ anywhere else in this graph, or into higher-order fluctuation graphs, or graphs involving 3-particle or higher vertices, would simply involve extra integrations over the arguments of $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, and reduce the effect of its singularity.

In Appendix B the graph of figure 11b is evaluated, to find

$$\mathrm{Im}\left[\Delta^{(1)}\Sigma_p(\omega+i\delta)\right] \sim \frac{p_F}{4\tilde{v}_F}\frac{\tilde{A}_0}{1+\tilde{F}_0}\left(\frac{\delta_0}{\pi}\right)\omega\theta(\omega) \tag{4.4}$$

implying a behaviour for the real part $\sim \omega\ln\omega$. This, amusingly, shows that $\Delta^{(1)}\Sigma_p(\omega)$ behaves in the same way as that postulated by the "Marginal Fermi Liquid" (MFL) phenomenology of Varma *et al.* [37]. However it would be quite clearly inconsistent to stop here (in fact it is not even clear that the MFL phenomenology is internally consistent itself [38]), and this leads us to look for the most divergent contribution to Im Σ at 2nd order in λ. This is given by figure 11b, and one finds (Appendix C) that

$$\mathrm{Im}\left[\Delta^{(2)}\Sigma_p(\omega+i\delta)\right] \sim \frac{1}{\pi}\left(\frac{1}{1+\tilde{F}_0}\right)^2 \frac{p_F^2}{8\hbar^2\tilde{N}(0)}\left(\frac{\delta_0}{\pi}\right)^2 \theta(\omega)\ln\left|\frac{\omega}{\tilde{\omega}_0}\right| \tag{4.5}$$

in which $\tilde{\omega}_0$ is the same "crossover" energy scale that we introduced in section 2 (and Appendix A); we recall that $\tilde{\omega}_0$ cannot be calculated within our theory, since it is a high-energy quantity.

Fig. 11. — Low-order singular contributions to Im $\Sigma_p(\omega+i\delta)$, coming from the singular vertex $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$. In (a) the leading 1st-order (in λ) contribution is shown, and in (b) the leading 2nd-order contribution. The hatched line indicates a cut of the graph to produce a reduced graph. The vertex λ is indicated by the heavy square.

Equation (4.5) shows a much more serious breakdown of FLT at 2nd order in λ than at first order, since Im $[\Delta^{(2)}\Sigma]$ actually diverges as $\omega \to 0$; in contrast, $\Delta^{(1)}\Sigma_p(\omega)$ is at the dividing line between complete breakdown of FLT and its obeisance, since the criterion of equation (2.21) is (only just) satisfied (again, this assumes that (4.4) can be taken seriously as it stands!). Thus at first glance our theory appears to be pathological, with ever-worsening kinds of divergence at higher orders in λ; and even if this prognosis proves to be too pessimistic, it is still not clear how we are to proceed from here.

When described in this way, the problem appears to have a certain formal analogy to the X-ray edge [39] or Kondo problems [40]; it is necessary to be able to handle the low-order IR divergences, in a consistent way, to all orders. However, although we see that there are interesting formal correspondences between the results here, to higher orders, and the famous "orthogonality catastrophe" of Anderson [35], the analogy is not complete, and moreover the physics is quite different. In the case of the impurity problem discussed by Anderson in 1967, there is no recoil, and the problem involves independent fermions. Here we have indistinguishable fermions scattering off, and interchanging with, each other; and they can recoil (subject to Pauli restrictions). The crucial physical point is again that made above, that the main singular scattering occurs between almost comoving particles. This feature can persist even in the absence of the logarithmic IR divergences of the Kondo variety, and so our problem is really quite different.

Having said this, it should be noted that there is a clear difference between the particle-particle and particle-hole channels in this respect. We have already seen the phase space restrictions operating in the particle-particle scattering channel – they are the same as those discussed in the context of the Lippmann-Schwinger equation, in section 3 (compare Fig. 8(b)). As $|\underline{p}|$ approaches p_F from above, the phase space restrictions become very strong, forcing the intermediate states in the particle-particle channel "off to the side", so that for an intermediate state $|\underline{k}_1, \underline{k}_2>$, $\underline{k}_1 - \underline{k}_2$ is roughly perpendicular to $\underline{k}_1$ and $\underline{k}_2$. In this sense Pauli restrictions prevent significant recoil of the 2 quasiparticles, as already discussed by Anderson [4]. However the phase space restrictions operating in the particle-hole channel are much weaker, as we saw in the calculation of $\Delta^{(2)}\Sigma_p(\omega)$ (compare Fig. C1). This is the reason why we get a $\ln|\omega|$ singularity in $\mathrm{Im}\left[\Delta^{(2)}\Sigma_p(\omega)\right]$, which will give us a stronger singularity in $\Sigma_p(\omega)$ and $z(\omega)$ than that derived by Anderson.

Before dealing with high-order graphs, let us note that the same log singularity appears, at 2nd order in λ, in the other quasiparticle vertex functions. In particular, if we calculate the 3-point vertex $\Lambda(P, K)$, describing the interaction between a fermionic quasiparticle $|p, \omega>$ with an external density perturbation carrying 3-momentum $K = (\underline{k}, \Omega)$, we find that the 2nd-order term has the same singularity as $\Delta^{(2)}\Sigma_p(\omega)$; denoting this term by $\Delta^{(2)}\Lambda_p^0(\omega)$ in the important limit where $K \to 0$, $\Omega/k \to 0$, one finds

$$\mathrm{Im}\left[\Delta^{(2)}\Lambda_p(\omega)\right] = \frac{p_F v_F}{4} \frac{\theta(\omega)}{\left(1+\tilde{F}_0\right)^2} \left(\frac{\delta_0}{\pi}\right)^2 \ln\left|\frac{\omega}{\omega_0}\right| \tag{4.6}$$

(see equation (C.20) of Appendix C).

This concludes the discussion of the lowest-order graphs in the singular interaction $\lambda_{\underline{k}\,\underline{k}'}^{\sigma\sigma'}$; we now see how to extend this work to all orders.

4.2 Consistently summing the divergences.

— In order to get a handle on the behaviour of the higher order (in λ) singular graphs, it is useful to briefly consider what form the 3rd- and 4th-order contributions will take. It is not difficult to convince oneself that the most important of all of these will be the 4th-order contributions shown in figure 12a; all others are less IR divergent. These graphs may be rather laboriously evaluated, but in fact this is not necessary, as we now see. Consider the expressions for the two 4th-order terms, which may be written as

$$\Delta^{(4)}\Sigma_p(\omega) \sim \frac{1}{\bar{z}_p} \mathrm{Tr}\left\{\left(\frac{\delta_0}{\pi \tilde{N}(0)}\right)^4 \chi_{00}(\tilde{\eta}_1)\, \chi_{00}(\tilde{\eta}_2) \left[\Psi_1 + \Psi_2\right] F_4\{n_j\}\right\} \tag{4.7}$$

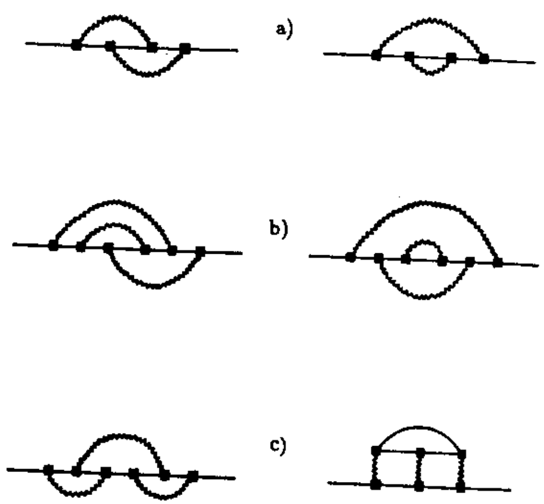

Fig. 12. — Higher-order contributions to $\mathrm{Im}\,\Sigma_p(\omega + i\delta)$, coming from $\lambda_{\underline{k}\,\underline{k}'}^{\sigma\sigma'}$. In (a) the two maximally-crossed" 4th-order (in λ) graphs are shown. In (b) a couple of maximally-crossed 6th-order graphs are shown (there are others), and in (c) a couple of 6th-order graphs, that are not maximally-crossed, are shown for comparison.

where the trace is just

$$\mathrm{Tr} \equiv \int \frac{\mathrm{d}\nu_1}{\pi} \int \frac{\mathrm{d}\nu_2}{\pi} \sum_{\underline{q}_1} \sum_{\underline{q}_2} \tag{4.8}$$

and the statistical factor $F_4\{n_j\}$ is

$$F_4\{n_j\} = \left(1 - n_{\underline{p}-\underline{q}_1}\right)\left(1 - n_{\underline{p}-\underline{q}_2}\right)\left(1 - n_{p-(\underline{q}_1+\underline{q}_2)}\right) \delta\left(\nu_1 - q_1 \tilde{v}_F \mu_1\right) \delta\left(\nu_2 - q_2\, \tilde{v}_F\, \mu_2\right) \tag{4.9}$$

The integrands Ψ_1 and Ψ_2 refer to the 2 different diagrams of figure 12a, and take the form

$$\Psi_1 = \left(\frac{p_F \mu_1/q_1}{q_1 \tilde{v}_F \mu_1 - \nu_1 + i\delta}\right)^2 \frac{(p_F \mu_2/q_2)^2}{\tilde{v}_F (q_1\mu_1 + q_2\mu_2) - (\nu_1 + \nu_2) + i\delta} \tag{4.10}$$

$$\Psi_2 = \frac{p_F \mu_1/q_1}{q_1 \tilde{v}_F \mu_1 - \nu_1 + i\delta} \frac{p_F \mu_2/q_2}{q_2 \tilde{v}_F \mu_2 - \nu_2 + i\delta} \frac{(p_F \mu_1/q_1)(p_F \mu_2/q_2)}{\tilde{v}_F (q_1\mu_1 + q_2\mu_2) - (\nu_1 + \nu_2) + i\delta} \tag{4.11}$$

But now notice that we can write $\Psi_1 + \Psi_2$ in a symmetrized form, as

$$\Psi_1 + \Psi_2 = \frac{1}{2}\left(\frac{\Psi_1 + \Psi_2}{\Psi_1^2 \Psi_2^2}\right) \tag{4.12}$$

$$\Psi_j = \tilde{v}_F q_j \mu_j - \nu_j + i\delta \tag{4.13}$$

However this kind of factorization means that we may now write the leading 4th-order term in the form

$$\Delta^{(4)}\Sigma_p(\omega) \sim -\frac{\Delta^{(2)}\Sigma_p(\omega)}{2}\left\{\int d\nu \sum_{q} \frac{\lambda_{p,\, p-q}\lambda_{p,\, p+q}}{(\omega - \epsilon_{p-q})^2}(1 - n_{p-q})\, \mathrm{Im}\, \tilde{\chi}_{00}(Q)\right\} \tag{4.14}$$

and we notice that the term in the curly brackets is just $\partial/\partial\omega\,(\mathrm{Re}\left[\Delta^{(2)}\Sigma_p(\omega)\right])$.

Now this kind of analysis is easily extended to higher-order graphs. One finds that the graphs with add powers of λ are always less divergent than those of even power, and moreover the graphs of even power sum to give

$$\Delta^{(2n)}\Sigma_p(\omega) = \frac{\Delta^{(2)}\Sigma_p(\omega)}{n!}\left\{-\frac{\partial}{\partial\omega}\mathrm{Re}\left[\Delta^{(2)}\Sigma_p(\omega)\right]\right\}^{n-1} \tag{4.15}$$

The way in which this is done is just a simple extension of what we have already done for the 4th-order diagrams, and so I do not give the details. There are in fact other ways of getting to the same answer. For example, one can isolate the logarithmic divergence in the reduced graphs for $\mathrm{Im}\Sigma_p(\omega + i\delta)$, as was done in reference [3]; this leads rapidly to the same answer, using the techniques of reduced graphs [41]. Alternatively, one may do the symmetrization by permuting the internal indices of a graph of arbitrary order, and using the well-known formula

$$\sum_{\mathrm{perm}} \frac{1}{(\Psi_1 + \Psi_2 + \Psi_n)\ldots(\Psi_1 + \Psi_2)\Psi_1} = \prod_{j=1}^{n} \frac{1}{\Psi_j} \tag{4.16}$$

All of these techniques consist in fact in the summation of what we may call the "maximally crossed" diagrams for $\Sigma_p(\omega)$ (this usage of the term comes from QED; notice that it is not the same as that employed in, eg., the theory of weak localisation in 2 dimensions). The maximally crossed diagrams are defined such that a diagram containing $2n$ fluctuations is maximally crossed if n fluctuations are "emitted" from the fermionic line before any are absorbed. The definition is illustrated for $\Sigma_p(\omega)$ in figure 12.

Another way of deriving equations like (4.14) is that using the eikonal approximation, which readily yields the exponential whose series expansion has terms of this form – as noted previously, I will not give a detailed discussion of this method here.

One may analyze other field-theoretical quantities in a similar way. At this point it is important to note that one must make sure, in calculating any singular quantity in this theory, to incorporate all of the leading divergent terms on the same level. The mathematical problem is much the same as one meets in dealing with any divergent series, and so any operations like integration or differentiation should be avoided.

Thus, for example, one might try and extract the quasiparticle renormalisation factor $z(\omega)$ directly from the series for $z(\omega)$, using

$$\frac{\partial}{\partial\omega}[\mathrm{Re}\ \Sigma(\omega)] = 1 - z^{-1}(\omega) \tag{4.17}$$

However this would be quite incorrect, since the differentiation of the singular terms makes them of different order in divergence. In fact one should proceed by noting that $\partial\Sigma/\partial\omega$ can be related to the set of maximally-crossed 3 point vertices, using the Ward identity

$$(1 - \partial\Sigma_p(\omega)/\partial\omega) = \Lambda^0(p,\ \omega) \tag{4.18}$$

Here $\Lambda^0(p,\ \omega)$ is a limit of the usual 3-point vertex $\Lambda^K(\underline{p},\ \omega)$, describing the interaction of a quasiparticle of momentum $\underline{p}$, frequency ω, with an external scalar field carrying 3-momentum $K = (\underline{k},\ \Omega)$; the symbol 0 indicates the limit $K \to 0$, and $\Omega/k \to 0$.

Thus a calculation of $z(\omega)$ must start from a summation of the maximally-crossed graphs for $\Lambda^Q(p,\ \omega)$, these latter taking the form shown in figure 13. The procedure is the same as for $\Sigma_p(\omega)$, and one finds a series analogous to (4.15):

$$\Lambda_p^0(\omega) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[-\Delta^{(2)}\ \Lambda_p^0(\omega)\right]^n \tag{4.19}$$

Thus we get the all-important exponentiation of $\Delta^{(2)}\Lambda_p(\omega)$. This quantity is calculated in Appendix C (see equations (C.20) and (C.22)), and so we arrive finally at the result which was already depicted in figure 1 (see the introduction), that is

$$z_p(\omega) \sim \exp\left\{-\frac{p_F\tilde{v}_F}{8}\frac{1}{\left(1 + \tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2 \ln^2\left|\frac{\tilde{\omega}_0}{\omega}\right|\right\} \tag{4.20}$$

This expression is different from that given in reference [3]; in that letter the imaginary part of $\Delta^{(2)}\Lambda_p^0(\omega)$ was erroneously exponentiated, when in fact one should exponentiate the real part (compare (4.17)). The difference is very important, for it means that our result (4.20) is now quite different from Anderson's result for $z(\omega)$, which was

$$\begin{aligned} z_{\mathrm{And}}(\omega) &= \exp\left\{\left(\frac{\delta_0}{\pi}\right)^2 \ln\left|\frac{\omega}{\tilde{\omega}_0}\right|\right\} \\ &= \left|\frac{\omega}{\tilde{\omega}_0}\right|^{(\delta_0/\pi)^2} \end{aligned} \tag{4.21}$$

Hence the statements in reference [3], concerning the relationship between the calculation here and Anderson's result in reference [4], can now be seen to be incorrect – in fact we have found a stronger singularity in (4.20) than Anderson's power law form.

One may also analyse the 4-point vertex in a similar way (Fig. 13(b)). Now, however, a rather interesting point emerges, for when calculating either the irreductible 4-point vertex $I(P,\ P')$, or the full vertex $\Gamma(P,\ P';\ Q)$ (produced by iteration in the zero sound channel, as in equation (2.4)), we must take account of "vertex corrections", i.e., maximally crossed graphs in which a set of internal fluctuations spans the vertex (or vertices) which connect the two interacting particles. But this simply renormalizes the total graph by a factor $z_p^{-1}(\epsilon_p)\, z_{p'}^{-1}(\epsilon_{p'})$, one factor for each vertex, since these vertex corrections are nothing but those appearing in the 3-point vertex.

If we now recalculate the interaction function $f_{\underline{k}\underline{k}'}^{\sigma\sigma'}$ between the fermions, using the well-known formula of Landau that

$$f_{\underline{k}\underline{k}'}^{\sigma\sigma'} \sim \lim_{\substack{Q \to 0 \\ q/\nu \to 0}} z_k\, z_{k'}\, \Gamma_{\underline{k}\underline{k}'}^{\sigma\sigma'}(Q) \to \lambda_{\underline{k}\underline{k}'}^{\sigma\sigma'} \tag{4.22}$$

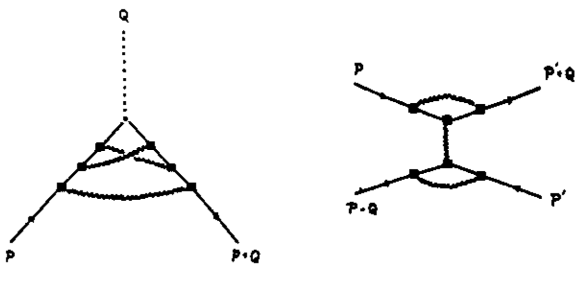

Fig. 13. — A maximally-crossed graph for the 3-point vertex $\Lambda(P, Q)$, 6th-order in λ; and (b) the same for the 4-point vertex $\Gamma(P, P'; Q)$, showing vertex corrections.

then of course we immediately find that we just get back $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$. But this is very important, because it demonstrates the internal consistency of our theory; if we had not gotten $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ back in (4.22), then the initial form chosen for the interaction function would have been incorrect, and/or our diagrammatic sums would have been inconsistent with each other. As an aside, it would be noted that this consistency requirement is not at all satisfied in the MFL phenomenology [37]. This is clear from the fact that they only consider the first term in what clearly must be an infinite series of non-analytic terms for $\Sigma_p(\omega)$; it is also clear from the simple observation that the MFL model is not a conserving approximation (which can be seen most quickly by feeding it into the recursive set of equations which describe such approximations – compare Ref. [18], section 3).

Let us now summarize what has been done so far in this section. The structure of the hierarchy of singular terms created by $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ is now clear. As stated in the beginning of the section, we have only dealt with the renormalization of $z(\omega)$ due to particle-hole fluctuations – however, as we have seen, the effect of these is stronger than the particle-particle fluctuations which appear to have been assumed by Anderson in his discussion of this problem. Thus, while it is clear that a complete solution of this problem will have to deal with the coupling between channels, we appear to have found the dominant singular contributions to $\Sigma_p(\omega)$, $z_p(\omega)$, etc., in the calculation just described.

4.3 FURTHER REMARKS. — The nature of the technique just described suggests that there may well be important corrections which we have missed. Indeed it it quite typical, when summing the set of dominant terms in this kind of divergent series, to find a set of sub-dominant terms which may also be singular, although less strongly so. Such a feature is well known in, eg., the Kondo problem, and so one would like to know whether a similar thing could happen here (in the Kondo problem, the sub-dominant terms are rather important, since they change the energy scale of the problem).

A related point is that, having shown that the fermions disappear from the picture as $\omega \to 0$, we now would like to know what replaces them! The obvious hypothesis, first made by Anderson, is that what is happening here is essentially the same as what occurs in the one-dimensional liquid; in this case we should think of the ω, $T \to 0$ state of the system as a "tomographic Luttinger liquid" in which the low-energy sector of the spectrum is entirely

bosonic in character. However it is not clear that the results of this section bear this out – we seem to have found an even more singular behaviour than Anderson for $z_p(\omega)$. However there is one obvious feature that we have in common with Anderson, which is that the fermions to disappear as $\omega \to 0$, and it seems likely that the low-energy limit of the theory must be describable by some effective bosonic Hamiltonian.

When it comes to any experimental test of this work, we must of course realize that we have calculated everything here in terms of the original fermionic quasiparticles. Any experiment which allows us to focus exclusively on the fermions will naturally be describable by calculations like those in the last section. One obvious example of such an experiment is the dHvA effect, to be discussed in the next section; and there are others. But the vast majority of physical phenomena involve the bosonic sector as well, and the theory given above would give quite misleading answers for even simple thermodynamic quantities like $C_v(T)$, $\chi(T)$, or compressibility. The situation is of course even worse with transport properties. The problem is easily seen when one recalls that in a 1-dimensional Luttinger liquid, $C_v(T)$ and $\chi(T)$ have FLT-like behaviour, despite the disappearance of the fermions as $T \to 0$.

Nevertheless the situation is not completely hopeless. It turns out, as mentioned in the introduction, that if we either calculate the scattering amplitude for fermion-fermion interactions (i.e., $\Gamma(P, P'; Q)$, or alternatively the fermion propagator $G_p(\omega)$, in a path integral formalism, using $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ as the irreductible 4-point interaction function, then the leading term, incorporating only the classical external paths, is simply an exponentiation of the 2nd-order (in λ) terms we have already calculated, provided we only incorporate processes involving particle-hole fluctuations. This exponentiation of course appears quite naturally in the path integral formation, and the important point is that if we now calculate the leading correction of this "eikonal approximation" for either Γ or G, we find that it is non-singular. The eikonal expansion is essentially an expansion in small scattering angles, and in our case each higher power is of order $(|p - p_F|/p_F)^{1/2}$ relative to the previous term. The full details of this expansion are very lengthy, and will be described elsewhere; we simply note here that this result shows that we need not worry about the sub-dominant terms, and that the answer (4.19) for $z|\omega|$ is essentially exact as $\omega \to 0$. One also realizes, from this perspective, that the full sum of diagrams we have calculated is inevitable for problems with strong forward scattering; from this point of view it is rather obvious that in problems such as that discussed by Khodel and Shaginyan [21], or in interacting anyon theories, calculations done only to finite order in the relevant renormalised interaction are bound to give misleading results (in both of these cases, this implies we must go beyond Hartree/Fock or RPA approximation, which are of zeroth- and first- order, respectively, in the relevant fluctuation propagator).

5. Finite field effects, and some experiments.

It will be clear to the reader from section 3 that our understanding of the origin of singular terms in $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ leaves much to be desired; and this makes a comparison between theory and experiment somewhat difficult. Nevertheless, since it has been asserted by Anderson [4] that the singular interaction discussed in the last section ought to occur quite generally in 2 dimensions, it is worthwhile exploring what important experimental consequences this should have. The main point to be made in this section is that the infrared cut-off imposed by an applied magnetic field should make it a useful tool for the testing of Anderson's assertion. The basic physical idea is very simple – since the IR cut-off suppresses the hierarchy of divergences we encountered in the last section, the effect of a field will be to restore Fermi-liquid behaviour (as shown in Fig. 1 of the introduction).

Now the effect on the fermionic degrees of freedom is relatively easily calculated, as we shall see: Thus any experiment which focusses on these degrees of freedom can be understood fairly quickly – as an example, I will sketch the implications for a dHvA (de Haas-van Alphen) experiment.

The dHvA results are rather interesting. However, the dHvA effect is one of the few which we can deal with here, since, as already noted in the last section, most experiments on Luttinger-type liquids necessarily involve both the fermionic and bosonic degrees of freedom, in such a way that equilibrium properties often differ little from that of a Fermi liquid. Thus the effect of applying a magnetic field on $C_v(T)$ or $\chi(T)$ may be small. Much more dramatic changes should occur in the transport properties; and so although we cannot yet say how these will behave when $H = 0$, we can at least say how they should behave for large fields, where things again depend only on the fermionic sector.

Thus in what follows we compute the effect, on the fermionic propagator, of a magnetic field, and thence on the dHvA effect.

5.1 FERMIONIC PROPAGATOR. — The calculation of $\Sigma_p(\omega)$ in a finite field will be done here in the same way as in zero field, i.e., we first calculate the behaviour for a non-singular interaction $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, and then add in the singular interactions.

(a) Non-singular SPFLT result: We recognize from the outset that for a non-singular $\tilde{f}^{\sigma\sigma'}_{\underline{k}\underline{k}'}$, we are simply dealing with a problem in "spin-polarized Fermi liquid theory" (SPFLT), in 2 dimensions. As such, the calculation is very similar to the 3-dimensional calculation [7], and so I only outline here how it works in 2 dimensions [42].

The lowest-order contribution to Im $\tilde{\Sigma}_p(\omega,\ H)$ splits into 2 parts, coming from longitudinal and transverse fluctuations respectively. We will again assume, as in Appendix A, that there is only one interaction parameter of importance, which we will take to be the spin-antisymmetric parameter $\tilde{f}^A_0$; this is because the interesting effects can all be understood in terms of $\tilde{f}^A_0$, at least in their general structure.

Now unless we apply very large fields indeed, the effects on any longitudinal processes will be small; moreover, the changes in the Landau parameters (including $\tilde{f}^A_0$) will also be small (a careful discussion of these points would parallel precisely that given in Ref. [7] and 42 for the $D = 3$ case). Under these conditions equation (A.9) of appendix A will be replaced by

$$\delta^{(1)}\tilde{\Sigma}^\sigma(\omega,\ H) \sim \frac{\left(\tilde{f}^A_0\right)^2}{\tilde{z}_p}\sum_{\underline{q}}\int \mathrm{d}\nu\left(1-\eta^{-\sigma}_{\underline{p}-\underline{q}}\right)\ {}^{\perp}\chi^{\sigma,\,-\sigma}_{00}(Q)\delta\left[\nu-\left(\omega-\epsilon_{\underline{p}-\underline{q}}\right)-\sigma\Delta\right] \qquad (5.1)$$

where Δ is the Zeeman splitting between up and down-spins (enhanced by the molecular field coming from $\tilde{f}^A_0$), and ${}^{\perp}\chi^{\sigma,\,-\sigma}_{00}(Q,\ H)$ is the transverse spin-spin response function. This response function is rather more complicated than its zero-field limit (Eq. (A.10)). At $q = 0$ its spectral weight is entirely exhausted by the field-induced spin-wave, so that

$${}^{\perp}\chi^{\sigma,\,-\sigma}_{00}(q=0,\ \nu+i\delta) = \frac{\sigma m}{\nu+\sigma\omega_L+i\delta} \qquad (5.2)$$

The spin wave lies at the Larmor frequency ω_L (in the absence of spin-orbit coupling), and has spectral weight m, where $m = n^{\uparrow} - n^{\downarrow}$ is the magnetization density. However at finite frequency ${}^{\perp}\chi^{\sigma,\,-\sigma}_{00}(\underline{q},\ \nu)$ is a sum of a spin wave term and an incoherent transverse fluctuation,

whose form can be given exactly for the region of low q and low ν that we are interested in:

$$\mathrm{Im}\ {}^{\perp}\chi^{\sigma,\,-\sigma}_{00}(\underline{q},\ \nu+i\delta) \sim \Big\{m\theta\left(p^{\sigma}_{\mathrm{F}}-p^{-\sigma}_{\mathrm{F}}-q\right)\delta\left(\nu+\sigma\omega_{\mathrm{L}}(q)\right)+ \frac{\pi\ \tilde{N}_2(0)}{\left(1+\tilde{F}^A_0\right)^2}\frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\theta\left(q\bar{v}_{\mathrm{F}}-|\nu+\sigma\Delta|\right)\frac{1-\bar{\eta}^2}{1+\left(\bar{A}^2_0-1\right)\bar{\eta}^2}\Big\} \qquad (5.3)$$

where $\omega_L(q)$ is the spin wave frequency, which joins the incoherent continuum at $(q,\ \nu) = \left(p^{\uparrow}_{\mathrm{F}}-p^{\downarrow}_{\mathrm{F}},\ 0\right)$, and $\bar{\eta} = \nu/q\bar{v}_{\mathrm{F}}$ as usual. A contour map of (5.3), for a typical (negative) value of $\tilde{F}_0$ is shown in figure 14; we see that in the integral (5.1) the spin wave is excluded by kinetic restrictions from the low-energy contribution to the self-energy $\tilde{\Sigma}$ (note that for these fairly low polarisations, $\Delta \sim \bar{v}_{\mathrm{F}}\left(p^{\uparrow}_{\mathrm{F}}-p^{\downarrow}_{\mathrm{F}}\right)$).

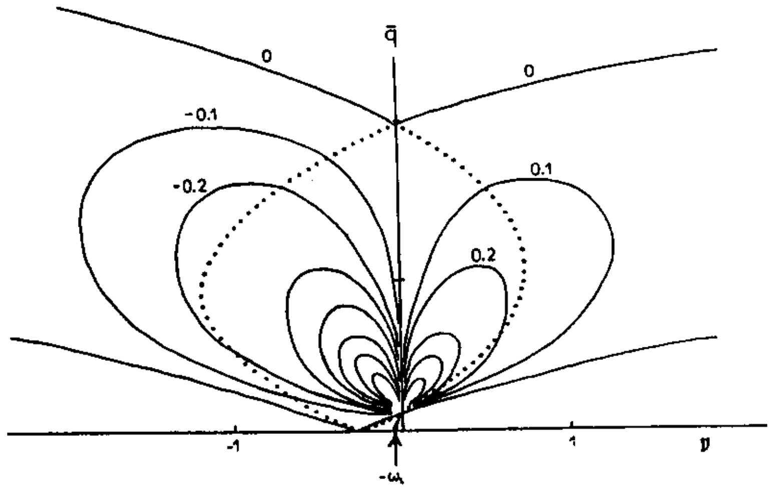

Fig. 14. — A contour map of Im${}^{\perp}\chi^{\sigma,\,-\sigma}(q,\ \nu+i\delta)$ for a spin polarised two-dimensional Fermi liquid, which has an enhanced spin susceptibility $\left(\tilde{F}^A_0 < 0\right)$. The spin wave has frequency $\nu = \omega_L(q)$; $\omega_L(q = 0)$ is the Larmor frequency.

From this point on the manipulations proceed in an analogous way to those in 3 dimensions, with the following results:

(a) The $\omega^2 \ln|\omega/\bar{\omega}_0|$ term is suppressed in Im $\tilde{\Sigma}_p(\omega+i\delta)$, up to energies $\omega \sim \Delta$; our calculation is done under the assumption that $\Delta \ll \bar{\omega}_0$.
(b) The coefficient of the ω^3 term in Im $\tilde{\Sigma}$ is changed by a factor $\sim \Delta/\bar{\omega}_0$.
(c) The effective mass is changed by the same order of magnitude, i.e., $\delta\tilde{m}^*/\tilde{m}^* \sim \Delta/\bar{\omega}_0$. A corresponding change occurs in the wave-function renormalization $\tilde{z}_p$.

(b) Effect of H on singular terms: The above results are for non-singular interactions. In a recalculation of the singular self-energy of section 4, we see that more care is needed, because the IR cut-off will of course have a profound effect on the singular $\lambda_{\underline{k}\underline{k}'}^{\sigma\sigma'}$. Thus a proper self-consistent treatent of $\Sigma_p(\omega,\ H)$ also must deal with the effects of the IR cut-off on the 4-point transverse vertex ${}^{\perp}\Gamma(P,\ P';\ Q)$, and this must be extended also to the 3-point vertices if we are to determine the effect of H on the wave-function renormalisation $z_p(H)$.

Now such a self-consistent calculation can be done, but I will not go through it here. This is because, as we shall see, the result is more or less obvious once the 2nd-order (in λ) graph for $\Sigma_{p(\omega)}$ is computed.

The crucial point is that we are assuming that $\lambda_{\underline{k}\underline{k}'}^{\sigma,\sigma'}$ only acts between opposite spins (cf. Eq. (4.1)).

Thus the Zeeman splitting will simply remove any singular part of $\lambda_{\underline{k}\underline{k}'}^{\sigma,\sigma'}$ from the integral (we refer, of course, to those singularities arising when $|\underline{k}-\underline{k}'|\to 0$). The mechanism is basically the same as that operating in the non-singular case, as we see if we generalize the singular equation (C.8) of appendix C to the spin-polarised regime:

$$\Delta^{(2)}\Sigma_p^{\uparrow}(\omega;\ H)\sim\frac{1}{\tilde z_p^{\uparrow}}\int \mathrm{d}\nu\sum_{\underline{q}}\left(\frac{\delta_0}{\pi N(0)}\frac{p\mu}{q}\right)^2\,{}^{\perp}\tilde\chi_{00}^{+-}(Q)\left(1-\eta^{\downarrow}_{\underline{p}-\underline{q}}\right)\delta\left[\nu-(\omega-\epsilon_{p-q})-\Delta\right]\quad(5.4)$$

an expansion which is easily transformed, using standard techniques, to

$$\Delta^{(2)}\Sigma_p^{\uparrow}(\omega;\ H)\sim\frac{p_{\mathrm F}^2}{2\pi\tilde v_{\mathrm F}\tilde z_{\mathrm F}}\left(\frac{\delta_0}{\pi\tilde N(0)}\right)^2\int_{\Delta/\tilde v_{\mathrm F}}^{q_c}\mathrm{d}q/q^2\int_0^{\infty}\mathrm{d}\nu\,(\mu^{2\perp}F(\omega;\ Q))\qquad(5.5)$$

where

$${}^{\perp}F(\omega;\ Q)=\left[\mathrm{Im}\,{}^{\perp}\tilde\chi_{00}^{+-}(q,\nu)\ln|\omega+\nu|-\mathrm{Im}\,{}^{\perp}\tilde\chi_{00}^{-+}(q,\ \nu)\ln|\omega-\nu|\right]\qquad(5.6)$$

Here $(+,\ -)\equiv(\uparrow,\ \downarrow)$, and the lower limit in the q-integral is just $\Delta/\tilde v_{\mathrm F}=\left(p_{\mathrm F}^{\uparrow}-p_{\mathrm F}^{\downarrow}\right)$. The region of integration around ($q=0$, $\nu=\pm\Delta$) is rendered benign because $\mathrm{Im}\,{}^{\perp}\chi_{00}$ disappears and so we immediately see that the effect of the lower cut-off in $\int\mathrm{d}q/q^2$ is to destroy the $\ln|\omega/\omega_0|$ divergence in $\mathrm{Im}\left[\Delta^{(2)}\Sigma_p(\omega)\right]$.

It will now be intuitively clear (and is easily verified in a more detailed investigation) that the effect of the IR cut-off is to cause a rapid crossover between singular behaviour for energies $\omega>\Delta$, and SPFLT behaviour for $\omega<\Delta$. Thus we arrive at the picture shown in the introduction (Fig. 1), in which the quasiparticle renormalisation $z(\omega)$ turns over at $\omega\sim\Delta$, and has a zero-energy asymptotic value

$$z(\omega=0;\ H)\sim\exp\left[\frac{p_{\mathrm F}\tilde v_{\mathrm F}}{8}\left(\frac{1}{1+\tilde F_0^{\mathrm A}}\right)^2\left(\frac{\delta_0}{\pi}\right)^2\ln^2\left|\frac{\Delta}{\omega_0}\right|\right]\qquad(5.7)$$

It is important to notice how this result must be obtained. It is not sufficient to simply calculate $\Delta^{(2)}\Sigma_p(\omega;\ H)$, and then derive $z_p(\omega;\ H)$ directly from this. Even though none of the higher contributions $\Delta^{(2n)}\Sigma_p(\omega;\ H)$ is singular, they must all be summed in order to get the correct answer.

From (5.7) we immediately see that for $\omega\ll\Delta$, the self-energy has the simple form of a SPFL, albeit with a strongly-renormalised mass (which is highly field-dependent!). In fact

$$\Sigma_p(\omega;\ H)\sim\left[\left(z_0^{-1}(H)-1\right)\omega-z_0^{-2}(H)\left(\frac{\omega}{\tilde\omega_0}\right)^2\right]\qquad(5.8)$$

for either spin; $z_0(H)$ is just $z(\omega=0,\ H)$ from (5.7).

With these results we may make a few experimental statements; but again, the reader is reminded that we have so far recovered only the fermion sector of the low-energy spectrum.

5.2 SOME IMPLICATIONS; THE dHvA EFFECT. — In what follows some of the consequences for the dHvA effect will be sketched – I do not give a comprehensive discussion here, since it is rather lengthy.

The standard expression for the dHvA effect, for a well-behaved Fermi liquid, is the usual Lifshitz-Kosevich (LK) expression [43] for the oscillatory magnetization $\underline{\tilde M}(H)$:

$$\underline{\tilde M}\sim\alpha H^{\frac12}\sum_{r=1}^{\infty}\frac{(-1)^r}{r^{\frac32}}D^r\,\psi_r\cos\left[2\pi\,r\frac{F}{H}\pm\frac{\pi}{4}\right]\cos\left(\pi r\frac{\tilde g^*\tilde m^*}{m_0}\right)\qquad(5.9)$$

where D is the Dingle factor, accounting for impurity scattering, and $\psi_0(\beta H)$ is a temperature factor, given by $\psi_r=\lambda_r/\sinh\lambda_r$, where

$$\lambda_r=2\pi^2r\tilde m^*/e\hbar\beta H\qquad(5.10)$$

The factors $\tilde g^*$ and $\tilde m^*$ are the spin "g-factor" and effective mass respectively, at the Fermi energy, and F is the "dHvA frequency" of the extremal orbit concerned (in general one will have several such orbits, of course); this extremal orbit has an area in k-space $\tilde A$, such that $F=(\hbar/2\pi e)\tilde A$. The constant α includes the curvature of the Fermi surface around this orbit:

$$\alpha=\frac{e\hbar}{4\pi^4}\left(\frac{2\pi e}{\hbar}\right)^{1/2}\frac{\tilde A}{\tilde m^*}\left|\frac{\partial^2\tilde A}{\partial k_z^2}\right|_{\mu}^{-1/2}\qquad(5.11)$$

As is by now well known, an expression like (5.9) is not necessarily expected to be accurate in the presence of strong many-body effects – although quite surprisingly, departures from the LK form are almost unknown, even in heavy fermion systems [44]. Part of the reason for this can be explained if we go to a statistical quasiparticle formulation of the dHvA effect [17] (although this does not necessarily explain the absence of such departures in heavy fermions, since we have no particular reason at present to believe that these are Fermi liquids!). In this formulation one sees that deviations will be small (of order $m_0/\tilde m^*$) for a typical Fermi liquid, since the Landau level splitting is reduced by this factor in strongly-interacting Fermi liquids. The statistical quasiparticle treatment starts from the rather important observation that the dHvA effect is an equilibrium effect; for a Fermi liquid (spin polarised or not) this means that the thermodynamic potential has an oscillatory part given by

$$\tilde\Omega=\frac{-eH}{4\pi^2\hbar\beta}\left\{\sum_{\nu,\sigma}\int\mathrm{d}k_z\,\ln\left[1+\exp\left[\beta\left(\mu-\varepsilon_{\nu}^{\sigma}(k_z)\right)\right]\right]\right\}_{\mathrm{osc}}\qquad(5.12)$$

where the $\varepsilon_{\nu}^{\sigma}(k_z)$ are the statistical quasiparticle energies; in this equation ν labels the Landau levels. In both the SQP formulation, and the LK formulation, the reduction to 2 dimensions can be done, usually without too much difficulty [45].

There is also a formulation of the dHvA effect in terms of the one-particle Green function [46, 47]. This formulation starts from the general expression for the thermodynamic potential, viz.

$$\Omega=\Phi\{G\}+\mathrm{Tr}[\ln(-1/G)+\Sigma G]\qquad(5.13)$$

and then throws away all contributions to $\tilde{\Omega}$ coming from Φ (the skeleton graphs) and ΣG. As Luttinger showed [46], these cancel to leading order in H. However, this cancellation is not valid to higher order in H (see Ref. [17]).

We now ask – which is the best way of understanding the dHvA effect when we have singular interactions of Anderson's form? It is clear that if the Landau level splitting $\hbar\tilde{\omega}_c^* = \hbar eH/\tilde{m}^*$ is considerably less than Δ, then the deviations from LK behaviour ought to be small (electrons are being transferred between levels spaced much closer in energy than the energy scale Δ over which the electronic parameters vary). However this is with a very important proviso – the mass $\tilde{m}^*$ in (5.9) is field-dependent, according to equations (5.7) and (5.8). This will lead to a dHvA mass plot which looks rather pathological, as shown in figure 15. Notice also that the LK formula, with the field-dependent $\tilde{m}^*(H)$ coming from (5.8), will only be valid in the low-temperature limit i.e., $kT \ll \Delta$; however if $\hbar\tilde{\omega}_c^* \ll \Delta$, then this is just the regime where large oscillations occur.

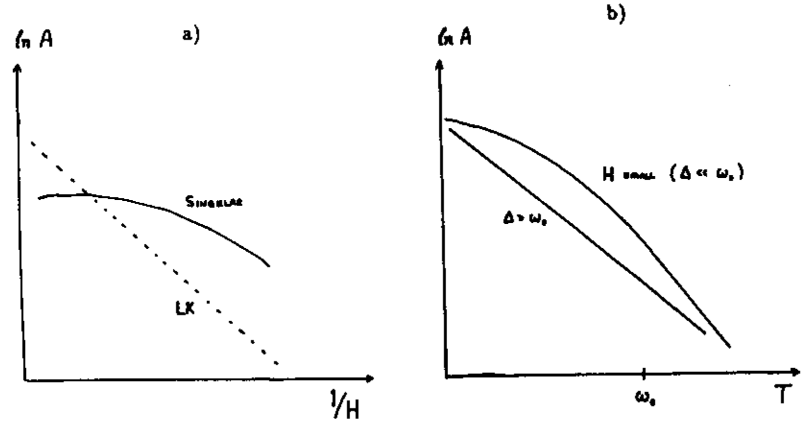

Fig. 15. — The sort of signal one might expect in a dHvA effect on a singular Fermi liquid. In (a) the "Dingle plot" is shown, for the lowest harmonic of the dHvA amplitude; and in (b) the "mass plot", for the same amplitude. These graphs are schematic only, given the approximation (ω, $T < \Delta$) inherent in equation (5.8).

The hypothesis that $\hbar\tilde{\omega}_c^* < \Delta$ is actually a fairly reasonable one, especially if $\tilde{F}_0^A < 0$, although a rigourous check is difficult until we can calculate $\tilde{g}^*(H)$; since $\chi(H,\ T)$ includes bosonic contributions, this is beyond the present theory.

Thus we can see that the dHvA effect ought to provide fairly unambiguous evidence of a breakdown of FLT, at least in the case where we are dealing with singular interactions of the Anderson type. Notice again that to derive a picture like that in figure 15 it is necessary to have made the self-consistent sum leading to (5.7); otherwise the behaviour would be quite different. Thus it would be rather dangerous to use the Fowler-Prange/Engelsberg-Simpson method here, in view of the remarks made above (and, of course, quite incorrect to simply substitute $\Delta^{(2)}\Sigma_p(\omega,\ H)$ into (5.13), from equation (5.5), under any circumstances [48]).

This result makes it clear that further investigation of high-T_c superconductors using the dHvA effect ought to be very useful. However what of neutral systems like 2-dimensional ^{3}He, where we do not have the charge to help us isolate the fermionic degrees of freedom? In this case, as already noted above, a test is much more difficult, since we do not expect a field to strongly affect thermodynamic properties like $\chi(H,\ T)$ and $C_V(H,\ T)$. The best hope appears to be in an examination of transport properties, and this will have to wait until we have a more complete theory.

6. Discussion and conclusions.

After the rather long development of this paper, it is perhaps useful to summarize the general perspective involved, and ask where all of this may be leading.

The general spirit of this paper has been to take singular interactions between quasi-particles seriously, and to explore their consequences. That the microscopic foundations of such an idea have yet to be established should be obvious from section 3 of this paper – but this does not prevent us from pursuing its consequences. These turn out to be rather interesting. If we take as an example the singular form proposed by Anderson, we see that although the resulting physics is rather strange, there is no obvious reason to believe that the theory is internally inconsistent (although clearly the result of section 4, that we obtain an exponentiation of $\ln^2|\omega/\omega_0|$ terms, means that this question needs to be looked at in more detail). Work on these lines (looking at the particle-particle channel, and the coupling between this and the particle-hole channel) is being completed at this time.

Of course one would like to find a cleaner way to deal with such singular interactions, and one which is moreover easily generalised to deal with the bosonic as well as fermionic excitations of the system (the latter disappearing as $\omega \to 0$). One way of doing this may well be the recent bosonization methods developed by Haldane [12]; another way may lie in the further development of the eikonal expansion which underlies the calculations of section 4. The latter method seems to be very well suited to any problem involving strong forward scattering, and this includes a multitude of interesting physical interactions, quite apart from the Anderson form discussed here. In addition to the form proposed by Khodel and Shaginyan, there are also those problems involving interactions unscreened at long range – these include the Fractional Hall effect, anyon systems, and 2-dimensional models of lattice fermions interacting via statistical gauge fields, as well as the more banal problem of the unscreened current-current part of the electromagnetic interaction, between electrons in a 2-dimensional conductor [49]. There are almost certainly other such problems.

In any case, no matter which methods are developed, it is clear that one important result of this kind of study is already obvious – for it has led to new insights and methods in the study of strongly-interacting fermions (notice that more conventional methods, like perturbation theory, $1/N$ expansions, or RNG methods, have been rather useless so far). This is just as well, for they will be needed in order to get to grips with the more detailed aspects of the new kinds of state produced by singular interactions – we still know very little about these, and we seen to be faced with a kind of infrared physics which is in some aspects quite new. It may well be, as speculated in the introduction, that further investigation will lead us into the realm of "non-renormalisable field theory". The few investigations so far of this kind of theory (see, eg., Parisi [50]), using methods such as $1/N$ expansions, indicate that fascinating things are in store – it would be very amusing if such ideas were to find their first realisation in the messy world of the cuprate superconductors.

Looking at more specific problems, it is clearly rather urgent that we understand whether and how non-perturbative terms may contribute to $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ for, eg., the dilute Fermi gas. One promising line of attack may be to consider the problem of weakly-coupled Luttinger liquid chains, already mentioned in the introduction (and Ref. [8]). If it could be established that Anderson's results for this problem were correct, this could be very strong evidence that singular terms prevail also for the $D = 2$ dilute gas.

Another question concerns the onset of superconductivity – to the best of my knowledge, there is no real understanding of the general effect of singular interactions on the superconducting transition. This seems a particularly difficult question in 2 dimensions, although Anderson has given a discussion of superconductivity in the 2-dimensional Hubbard model [51], starting from the singular $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ which was analyzed in section 4.

Finally, it is clear that none of this work will really be of much importance until it can be unambiguously linked with experiment. The field effects described in section 5 constitute an example of this, with at least one fairly striking preduction – but much more needs to be done.

Acknowledgements.

I would like to thank I. Affleck, P.W. Anderson, G. Beydaghyan, M. Dobroliubov, and I. Kogan for useful comments and discussion; and my understanding of the perturbative aspects of this problem benefitted enormously from discussion with N.V. Prokofev (see section 3). I also thank J. Maxwell for typing the very large manuscript.

Appendix A: calculation of Im $\tilde{\Sigma}_p(\omega)$.

In this Appendix the calculation of the non-singular self-energy $\tilde{\Sigma}_p(\omega + i\delta)$, for low ω, is described in more detail. It is useful, from time to time, to compare the results derived here for $D = 2$, with results from $D = 3$; these are of course much better known. In what follows we first calculate what turns out to be the leading contribution at low ω, to Im $\tilde{\Sigma}_p(\omega)$, coming from the single fluctuation graph of Figure 3a (which is re-displayed here as Fig. A1, in terms of the quasiparticle $\tilde{t}$-matrix). We then estimate the higher-order corrections coming from multi-fluctuation graph. Throughout this appendix we assume that interaction functions like $\tilde{f}^{\sigma\sigma'}_{kk'}$ are <u>non-singular</u>.

A.1 LOWEST-ORDER CONTRIBUTIONS. — As discussed in section 2 of the text, we classify the contribution to Im $\tilde{\Sigma}_p(\omega)$ according to how many "dynamic molecular field" fluctuations, coupling the fermionic quasiparticles, appear in the graph for Im $\tilde{\Sigma}_p(\omega)$ (recall Figs. 3 and 4). For a general quasiparticle $\tilde{t}$-matrix (i.e., a general $\tilde{f}^{\sigma\sigma'}_{kk'}$), we can write the lowest order contribution to Im $\tilde{\Sigma}_p(\omega)$ (which we call Im $\left[\delta^{(1)}\tilde{\Sigma}_p(\epsilon_p + i\delta)\right]$ to denote the contribution first order in the dynamic susceptibility) in the form

$$\mathrm{Im}\left[\delta^{(1)}\tilde{\Sigma}_p(\epsilon_p + i\delta)\right] = \frac{\pi}{\tilde{Z}_p}\sum_{\underline{p}'}\sum_{\underline{q}}\left|\tilde{t}_{\underline{p}\underline{p}'}(Q)\right|^2 n_{\underline{p}'}\left(1 - n_{\underline{p}-\underline{q}}\right)\left(1 - n_{\,+\underline{q}}\right)\delta\left(\omega + \epsilon_{p'} - \epsilon_{\underline{p}-\underline{q}} - \epsilon_{\underline{p}-\underline{q}}\right) \quad (A.1)$$

where the $\tilde{t}$-matrix is defined in terms of the complete 4-point vertex by equation (2.3); it is related to the interaction function $\tilde{f}^{\sigma\sigma'}_{kk'}$ by equation (2.2), the Bethe-Salpeter equation. For

the moment we ignore spin – we shall put it back in at the end of the calculation, since it makes no difference to the structure of the results derived here. Equation (A.1) is illustrated in figure A1.

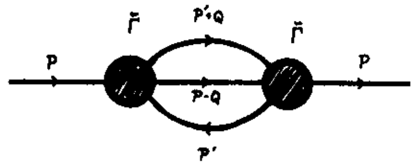

Fig. A1. — A diagramatic representation of equation (A.1), in reduced graph form.

Now instead of dealing with a general form for $\tilde{t}_{\underline{p}\underline{p}'}(q, \nu)$, involving an expansion in Legendre polynomials around the Fermi circle, it is very instructive to make the approximation used throughout the text, viz., that

$$\tilde{f}_{\underline{k}\underline{k}'} \to \tilde{f}_0 \quad (A.2a)$$

$$\tilde{t}_{\underline{k}\underline{k}'}(Q) \to \tilde{t}_0(Q) = \tilde{f}_0 / \left[1 + \tilde{f}_0\tilde{\chi}^0_2(Q)\right] \quad (A.2b)$$

where (A.2b) follows from (A.2a) using (2.2); the 2-dimensional Lindhard function is, for low Q, given by

$$\begin{aligned}\tilde{\chi}^0_2(\tilde{\eta}) &= \tilde{N}_2(0)\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{\cos\theta}{\cos\theta - \tilde{\eta} - i\delta} \\ &= \tilde{N}_2(0)\left[1 - \tilde{\eta}\left(\frac{\theta\,(\tilde{\eta}^2 - 1)}{[\tilde{\eta}^2 - 1]^{\frac{1}{2}}} + i\frac{\theta\,(1 - \tilde{\eta}^2)}{[1 - \tilde{\eta}^2]^{\frac{1}{2}}}\right)\right]\end{aligned} \quad (A.3)$$

where $\tilde{\eta} = \nu / q\tilde{v}_F$, and $\tilde{N}_2(0) = \tilde{m}^*/2\pi = p_F/2\pi\tilde{v}_F$ (and for the calculation we let $\hbar = 1$; where $\hbar$ is needed, it will be put into the final answers).

Now writing

$$\begin{aligned}\delta\left(\omega + \epsilon_{p'} - \epsilon_{p-q} - \epsilon_{p'q}\right) &= \int_{-\infty}^{\infty} d\nu\,\delta\left(\nu - \left(\omega - \epsilon_{\underline{p}-\underline{q}}\right)\right)\delta\left(\nu - \left(\epsilon_{\underline{p}'-\underline{q}} - \epsilon_{p'}\right)\right) \\ &= \int_{-\infty}^{\infty} d\nu\,\delta\left(\nu - q\tilde{v}_F\mu\right)\delta\left(\nu - q\tilde{v}_F\mu'\right)\end{aligned} \quad (A.4)$$

where $\mu = \hat{\underline{p}}\cdot\hat{\underline{q}}$, $\mu' = \hat{\underline{p}}'\cdot\hat{\underline{q}}$; we then have

$$\begin{aligned}\sum_{\underline{p}'}\left(1 - \eta_{\underline{p}'-\underline{q}}\right)\eta_{p'}\int d\nu\,\delta\left(\nu - q\tilde{v}_F\mu'\right) &= \frac{p_F}{2\pi}\int d\nu\int_{-\pi/2}^{\pi/2}\frac{d\theta'}{\pi}q\mu'\delta\left(\nu - q\tilde{v}_F\mu'\right) \\ &= \frac{p_F}{2\pi^2\tilde{v}_F}\int d\nu\frac{\nu}{\left(q^2\tilde{v}_F^2 - \nu^2\right)^{\frac{1}{2}}}\end{aligned} \quad (A.5)$$

Inserting this into (A.1), with $\tilde{t}$ given by (A.2b), we then get

$$\text{Im}\left[\delta^{(1)}\tilde{\Sigma}_p(\omega)\right] = \frac{2\pi}{\tilde{z}_p}\frac{\tilde{v}_F}{p_F}\sum_{\underline{q}}\int d\nu \frac{\nu}{(q^2 v_F^2 - \nu^2)^{\frac{1}{2}}}\left|\frac{\tilde{F}_0}{1+\tilde{F}_0\Omega_0(\tilde{\eta})}\right|^2 \times (1-\eta_{p-q})\,\delta(\nu - q\tilde{v}_F\mu)\,\theta(q\tilde{v}_F - \nu) \tag{A.6}$$

where $\tilde{F}_0 = \tilde{N}_2(0)\tilde{f}_0$, and also $\tilde{\chi}_2^0 = \tilde{N}_2(0)\Omega_0(\tilde{\eta})$ [so that $\Omega_0(\tilde{\eta}=0) = 1$]. It is useful to notice that, using

$$\text{Im}\left(\frac{\Omega_0(\tilde{\eta})}{1+\tilde{F}_0\Omega_0(\tilde{\eta})}\right) \sim \left|\frac{1}{1+\tilde{F}_0\Omega_0(\tilde{\eta})}\right|^2 \text{Im}\,\Omega_0(\tilde{\eta}) \tag{A.7}$$

(exact for $\tilde{\eta} \ll 1$), we can write (A.6) in the form

$$\text{Im}\left[\delta^{(1)}\tilde{\Sigma}_p(\omega)\right] \sim \frac{1}{\tilde{z}_p\tilde{N}_2(0)}\int d\nu \sum_{\underline{q}}\tilde{F}_0^2\,\text{Im}\left(\frac{\Omega_0(\tilde{\eta})}{1+\tilde{F}_0\Omega_0(\tilde{\eta})}\right)\theta(\omega - q\tilde{v}_F\mu)\,\delta(\nu - q\tilde{v}_F\mu) \tag{A.8}$$

However this is exactly the form we derive from figure A2a; here we see figure A.1 drawn explicitly as a 1-fluctuation graph, and given by

$$\delta^{(1)}\tilde{\Sigma}_p(\omega) = \frac{\tilde{f}_0^2}{\tilde{z}_p}\int d\nu \sum_{\underline{q}}\left(1-\eta_{\underline{p}-\underline{q}}\right)\tilde{\chi}_{00}(q,\ v)\,\delta(\nu - (\omega - \epsilon_{p-q})) \tag{A.9}$$

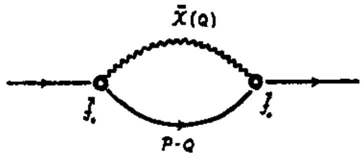

Fig. A2. — A redrawing of figure A1 as a 1-fluctuation graph, in the case where only one Landau parameter is important (see Eq. (A.8)).

where the dynamic fluctuation propagator $\tilde{\chi}_{00}(Q)$ is just

$$\tilde{\chi}_{00}(Q) = \tilde{\chi}_2^0(Q)/\left[1+\tilde{f}_0\chi_2^0(Q)\right] \tag{A.10}$$

We shall see below that the leading behavior (in ϵ_p) of Im $\left[\delta^{(1)}\tilde{\Sigma}\right]$ comes from the integration over small $\tilde{\eta}$.

Turning now to the integration $\sum_q \equiv \int d^2q/4\pi^2$, we first carry out the angular integration, using

$$\int_0^1 d\mu \frac{1}{(1-\mu^2)^{\frac{1}{2}}}\theta(\omega - q\tilde{v}_F\mu)\,\theta(q\tilde{v}_F - \nu)\,\delta(\nu - q\tilde{v}_F\mu) = \frac{1}{(q^2\tilde{v}_F^2 - \nu^2)^{\frac{1}{2}}}\theta(\omega-\nu)\theta(q\tilde{v}_F - \nu) \tag{A.11}$$

and making the low $-\tilde{\eta}$ approximation $(1-\Omega_0(\tilde{\eta})) \sim i\tilde{\eta}/\left(1-\tilde{\eta}^2\right)^{\frac{1}{2}}$, so that

$$\left|\frac{\tilde{F}_0}{1+\tilde{F}_0\Omega_0(\tilde{\eta})}\right|^2 = \left|\frac{\tilde{A}_0}{1-\tilde{A}_0(1-\Omega_0(\tilde{\eta}))}\right|^2 \sim \tilde{A}_0^2\left|\frac{(1-\tilde{\eta}^2)}{(1-\tilde{\eta}^2)+\tilde{A}_0^2\tilde{\eta}^2}\right| \tag{A.12}$$

where the scattering parameter $\tilde{A}_0 = \tilde{F}_0/\left(1+\tilde{F}_0\right)$, we then find by substituting into (A.6) that

$$\text{Im}\left[\delta^{(1)}\tilde{\Sigma}_p(\omega)\right] = \frac{\tilde{A}_0^2}{2\tilde{z}_p p_F\tilde{v}_F}\int_0^\omega \nu d\nu \int_0^{q_c}\frac{dq}{q}\left(\frac{1}{1+\left(\tilde{A}_0^2-1\right)\tilde{\eta}^2}\right)\theta(\omega-\nu)\theta(q\tilde{v}_F - \nu) \tag{A.13}$$

which is conveniently rewritten as

$$\text{Im}\left[\delta^{(1)}\tilde{\Sigma}(\omega)\right] = \frac{\tilde{A}_0^2}{2\tilde{z}_p p_F\tilde{v}_F}\int_0^\omega \nu dv \int_{\nu/q_c\tilde{v}_F}^1 \frac{d\tilde{\eta}}{\tilde{\eta}}\left(\frac{1}{1+\left(\tilde{A}_0^2-1\right)\tilde{\eta}^2}\right) \tag{A.14}$$

In equations (A.13) and (A.14), an ultraviolet momentum cut off is imposed in the momentum integrations. It is of course impossible to determine q_c within the theory; its existence is equivalent to the existence of a typical frequency scale $\tilde{\omega}_0 = q_c\tilde{v}_F$, which is roughly the energy at which we cross over into the low-energy FLT regime, associated with the $l = 0$ channel we are presently discussing [eg. equation (A.2)]. In 3-dimensional ^{3}He liquid, $\tilde{\omega}_0$ is generally associated with the spin fluctuation energy (i.e. the channel associated with $\tilde{f}_0^A$), this being the lowest and most important energy scale [9, 18]; it must of course be determined from experiment, since there is no way of calculating $\tilde{f}_0^A$ for a dense Fermi liquid [14].

Now suppose we took the integrand in (A.14) to be unity; this would be tantamount to assuming that $\tilde{t}_0(Q)$ was independent of Q (or ignoring the Q-dependence of $\tilde{\chi}_{00}(Q)$ in (A.10), and would give

$$\begin{aligned}\text{Im}\left[\delta^{(1)}\tilde{\Sigma}(\omega)\right] &\sim \frac{\tilde{A}_0^2}{2\tilde{z}_p p_F v_F}\int_0^\omega \nu d\nu \int_{\nu/q_c\tilde{v}_F}^1 \frac{d\tilde{\eta}}{\tilde{\eta}} \\ &= \frac{\tilde{A}_0^2}{16\pi\tilde{v}_F^2\tilde{N}_2(0)}\left(\omega^2 + 2\omega^2 \ln\left|\frac{\tilde{\omega}_0}{\omega}\right|\right)\end{aligned} \tag{A.15}$$

with $\tilde{\omega}_0 = q_c\tilde{v}_F$ as above. Equation (A.15) already shows that the important IR non-analyticity is coming from the low-$\tilde{\eta}$ part of the integral, thereby justifying the approximation in (A.12). However (A.15) is note quite right, as we see if we now use the more accurate low-$\tilde{\eta}$ expansion in (A.14); this properly includes the $\tilde{\eta}$-dependence of $\tilde{t}_0(\tilde{\eta})$. However, rather amusingly, the only change produced if we now carry out the full integration in (A.14) is to renormalise $\tilde{\omega}_0$ to $\tilde{\omega}_0/\tilde{A}_0$. Since $\tilde{\omega}_0$ is unknown anyway, we ignore this renormalisation, and stick with (A.15).

From (A.15) we see that the character of the leading IR correction to the usual ω^2 term is actually larger than ω^2! To be sure, this does not imply a breakdown of FLT, since both the ω^2 and $\omega^2 \ln\omega$ terms in Im $\tilde{\Sigma}$ imply a term $\sim\omega$ in Re $\tilde{\Sigma}$, so that the criterion (2.21) for the validity of FLT is still satisfied.

However this result clearly shows the desirability of checking the expansion of Im $\tilde{\Sigma}(\omega)$, for low ω, by looking at the higher-order multi-fluctuation graphs. We note in passing that adding in the higher angular momentum channels in (A.1), and the spin sums, makes no difference to the form of (A.15). The spin sums lead simply to sums of the form

$$\left(\left|t^s_{\underline{p}\underline{p}'}(Q)\right|^2 + 3\left|t^A_{\underline{p}\underline{p}'}(Q)\right|^2\right) \tag{A.16}$$

in (A.1); and the angular momentum sums lead to highly unwieldy formulae, in which the ω^2 and $\omega^2 \ln\omega$ terms are multiplied by terms $\sim \tilde{A}_l^2$ summed over l; the explicit calculation of these here would take up too much space.

A.2 HIGHER CONTRIBUTIONS. — Explicit calculations of all of the higher terms would take up too much space; and we only look here at higher fluctuation terms, since if $f_{\underline{k}\underline{k}'}^{\sigma\sigma'}$ is regular, it is obvious that multi-particle contributions will be of higher order in ω than (A.15). The basic point to be made is rather obvious from an examination of either of the 2nd-order fluctuation graphs in figure 4; these are both 4th-order in $\tilde{A}_0$. Taking for example the "crossed graph" (as opposed to the "rainbow graph", where one fluctuation encloses the other), we have

$$\mathrm{Im}\left[\delta^{(2)}\bar{\Sigma}_p(\omega+i\delta)\right] \sim \frac{\tilde{A}_0^4}{\bar{z}_p \tilde{N}_{(0)}^2} \mathrm{Tr}\left\{F_2(\bar{\eta}_1,\ \bar{\eta}_2)\,\delta(\nu_1 - q_1\tilde{v}_F\mu_1)\,\delta(\nu_2 - q_2\tilde{v}_F\mu_2)\right\} \qquad (A.17)$$

where $\mathrm{Tr} \equiv \int \mathrm{d}\nu_1 \int \mathrm{d}\nu_2 \sum_{q_1}\sum_{q_2}$, and

$$F_2(\bar{\eta}_1,\ \bar{\eta}_2) = \mathrm{Im}\,\tilde{\chi}_{00}(Q_1)\,\mathrm{Im}\,\tilde{\chi}_{00}(Q_2) \times\left[\theta(\omega - q_1\tilde{v}_F\mu_1)\,\theta(\omega - q_2\tilde{v}_F\mu_2)\,\theta(\omega - \tilde{v}_F(q_1\mu_1 + q_2\mu_2))\right] \qquad (A.18)$$

This is easily transformed to

$$\frac{A_0^2}{2\bar{z}_p p_F^2}\int_0^\omega \nu_1\,\mathrm{d}\nu_1 \int_0^\omega \nu_2\mathrm{d}\nu_2 \int_{\nu_1/q_c\tilde{v}_F}^1 \mathrm{d}\bar{\eta}_1/\bar{\eta}_1 \int_{\nu_2/q_c\tilde{v}_F}^1 \mathrm{d}\bar{\eta}_2/\bar{\eta}_{12}\theta\left[\omega - \tilde{v}_F(q_1\mu_1 + q_2\mu_2)\right] \qquad (A.19)$$

from which we see that

$$\mathrm{Im}\left[\delta^{(2)}\bar{\Sigma}_p(\omega+i\delta)\right] \sim \left[\omega^4 \ln\omega + 0(\omega^4)\right] \qquad (A.20)$$

From this it is obvious that higher fluctuation graphs will not affect our basic conclusion, derived from (A.15), that the low-energy properties of this theory are consistent with a $D = 2$ FLT. We note, as already mentioned in the text, that nothing we have said here excludes the possibility that some high-energy process might have more serious effects.

Appendix B: perturbation theory for the DIFG.

In this appendix I give some details of the derivation of the main formulae in section 3.1, for the dilute interacting Fermi gas (DIFG). The model Hamiltonian is that of equation (3.1), and we wish to evaluate the function

$$\Delta^{(2)} f_{\underline{p}_1\underline{p}_2}^{\sigma_1\sigma_2} = \left(\Delta_{\mathrm{Cooper}}^{(2)} f_{\underline{p}_1\underline{p}_2}^{\sigma_1\sigma_2} + \Delta_{\mathrm{Crossed}}^{(2)} f_{\underline{p}_1\underline{p}_2}\right) \qquad (B.1)$$

where the Cooper channel and Crossed channel contribution are given respectively by

$$\Delta_{\mathrm{Cooper}}^{(2)} f_{\underline{p}_1\underline{p}_2}^{\sigma_1\sigma_2} = \left(\frac{\hbar^2\alpha}{m}\right)^2\left[\frac{2m}{\Omega^2}\sum_{\underline{k}} n_{\underline{k}}^0 \frac{4\hat{Q}^{\sigma_1\sigma_2}}{p_1^2 - k^2 - p_2^2 - \left(\underline{p}_1 + \underline{p}_2 + \underline{k}\right)^2}\right] \qquad (B.2)$$

with $\hat{Q}^{\sigma_1\sigma_2} = \frac{1}{4}(1 - \sigma_1\cdot\sigma_2)$, and

$$\Delta_{\mathrm{crossed}}^{(2)} f_{\underline{p}_1\underline{p}_2} = -\left(\frac{\hbar^2\alpha}{m}\right)^2\left[\frac{2m}{\hbar^2\Omega^2}\sum_k n_k^0 \frac{1}{p_1^2 + k^2 - p_2^2 - (p_1 - p_2 + k)^2} + \left(\underline{p}_1 \leftrightarrow \underline{p}_2\right)\right] \qquad (B.3)$$

The terms in square brackets, in (B.2) and (B.3), are just the integrals I_1 and I_2 of equations (3.9) and (3.10). The behavior of the 2 channels is quite different, and so we analyze them separately.

B.1 COOPER CHANNEL. — We start with the evaluation of (B.2), which was shown diagramatically in figure 6a. The integral I_1, is conveniently rewritten as

$$I_1 = \frac{4m}{\Omega}\sum_{\underline{k}} n_{\underline{k}}^0 \frac{4Q^{\sigma_1\sigma_2}}{k\left|\underline{p}_1 + \underline{p}_2\right|\cos\theta - \left(\underline{p}_1\cdot\underline{p}_2 + k^2\right)} \qquad (B.4)$$

where $\theta = \theta_{p_1p_2}$ is the angle between $\hat{p}_1$ and $\hat{p}_2$. Using now the result that

$$P\int_0^{2\pi}\frac{\mathrm{d}\theta}{\alpha - \beta\cos\theta} = \frac{2\pi}{(\alpha^2 - \beta^2)^{\frac{1}{2}}}\theta(\alpha^2 - \beta^2)\,\mathrm{sign}(\alpha) \qquad (B.5)$$

for the principal value integral (since we are interested in the energy shift, due to the interaction, of the state $|\underline{p}_1\sigma_1>$ due to $|\underline{p}_2\sigma_2>$, (or vice versa), we are uninterested in the imaginary part), we then find, at $T = 0$, that

$$I_1 = \frac{m}{\pi\hbar^2}\int k\mathrm{d}k\frac{\theta\left[(k^2 - u_+)(k^2 - u_-)\right]}{\left[(k^2 - u_+)(k^2 - u_-)\right]^{\frac{1}{2}}}\mathrm{sign}\left(\underline{p}_1\cdot\underline{p}_2 + k^2\right) \qquad (B.6)$$

where the roots $u_\pm$ are

$$u_\pm = \frac{1}{4}\left(\left|\underline{p}_1 + \underline{p}_2\right| \pm \left|\underline{p}_1 - \underline{p}_2\right|\right)^2 \qquad (B.7)$$

The integral (B.6) can be evaluated easily to find the answer given in equation (3.11) of the text for $\Delta_{\mathrm{Cooper}}^{(2)} f_{p_1p_2}^{\sigma_1\sigma_2}$.

B.2 CROSSED CHANNEL. — The integral I_2 coming from inside the square brackets in (B.3) is rewritten as

$$I_2\left(\underline{p}_1,\ \underline{p}_2\right) = \frac{2m}{\Omega}\sum_k n_k^0 \frac{1}{\left(\underline{p}_1\cdot\underline{p}_2 - p_2^2\right) - k\cdot\left(\underline{p}_1 - \underline{p}_2\right)} \qquad (B.8)$$

whence we immediately see the singular character as $\left|\underline{p}_1 - \underline{p}_2\right| \to 0$. The θ-integral is easily evaluated, using (B.5), to give

$$I_2\left(\underline{p}_1,\ \underline{p}_2\right) = \frac{2mp_F}{\pi\hbar^2}\frac{1}{|p_1 - p_2|}\int_0^1 \bar{k}\,\mathrm{d}\bar{k}\frac{\mathrm{sign}(s_2)}{(s_2^2 - \bar{k}^2)^{\frac{1}{2}}} \qquad (B.9)$$

where $\bar{k} = k/k_F$, and

$$s_2 = \frac{1}{p_F}\frac{p_2\cdot\left(\underline{p}_1 - \underline{p}_2\right)}{\left|\underline{p}_1 - \underline{p}_2\right|} \qquad (B.10)$$

and then we easily find

$$I_2(p_1p_2) = \frac{2m}{\pi\hbar^3}\frac{p_F}{|p_1-p_2|}\left\{|s_2| - (s_2^2-1)^{\frac{1}{2}}\,\theta(|s_2|-1)\right\} \quad (B.11)$$

If we now sum $I_2(\underline{p}_1\underline{p}_2)$ and $I_2(\underline{p}_2\underline{p}_1)$, we get the answer quoted in equation (3.14) for $\Delta^{(2)}_{\text{Crossed}} f_{\underline{p}_1\underline{p}_2}$.

With these functions we may calculate the quasiparticle energy, as well as thermodynamic and response functions, in the usual way; the calculations are also easily generalized to include spin polarization effects [31].

Appendix C: evaluation of low-order singular graphs.

In this appendix we give details of the evaluation of the lowest order (in λ) singular graphs; the results of these calculations are discussed in section 4. The idea is to systematically insert singular vertices λ into the "regular graphs" which have already been calculated using non-singular f-functions. As in most of the paper, we will assume a simple isotropic regular interaction $\tilde{f}_0$. The singular vertex $\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'}$ has the form specified in equation (4.3), viz.

$$\lambda^{\sigma\sigma'}_{\underline{k}\underline{k}'} = \frac{\delta_0}{\pi\tilde{N}(0)}\frac{\underline{k}'\cdot(\underline{k}-\underline{k}')}{|\underline{k}-\underline{k}'|^2}\,\delta_{\sigma,-\sigma'}\,\theta(|\underline{k}|-k_F)\,\theta(k_F-|\underline{k}'|) \quad (C.1)$$

so that for a quasiparticle $|\underline{p}\sigma\rangle$, with $|\underline{p}| > p_F$, interacting with another quasiparticle $|\underline{p}-\underline{q},\sigma\rangle$ with $|\underline{p}-\underline{q}| < p_F$, we have

$$\lambda^{\sigma,-\sigma}_{\underline{p},\underline{p}-\underline{q}} = \frac{\delta_0}{\pi\tilde{N}(0)}\left(\frac{\underline{p}\cdot\underline{q}}{q^2}\right) \equiv \frac{\delta_0}{\pi\tilde{N}(0)}\frac{p\mu}{q} \quad (C.2)$$

where $\mu = (\hat{\underline{p}}\cdot\hat{\underline{q}})$; the approximation (C.2) is valid for $q \ll p$, and in any case the remaining term is non-singular and therefore already included in $\tilde{f}^{\sigma,-\sigma}_{\underline{p},\underline{p}-\underline{q}}$. As discussed in the text, we only consider interactions with particle-hole fluctuations, and not the particle-particle channel.

C.1 Self-energy graphs.

(a) First order in λ :

We start off with the lowest-order graph, shown in figure 11a in its fluctuation propagator form. As discussed in the text, this graph gives the most singular contribution to Im $\Sigma_p(\omega)$ at first order in λ. It has the form (for $\omega > 0$) :

$$\Delta^{(1)}\Sigma_p(\omega) = \frac{1}{\tilde{z}_p}\int \mathrm{d}\nu \sum_{\underline{q}}\left(\tilde{f}_0\frac{\delta_0}{\pi\tilde{N}(0)}\frac{p\mu}{q}\right)\frac{\tilde{\chi}^0_2(\tilde{\eta})}{1+\tilde{f}_0\tilde{\chi}^0_2(\tilde{\eta})}(1-\eta_{p-q})\,\delta\left[\nu-(\omega-\epsilon_{p-q})\right] \quad (C.3)$$

By the same sorts of manœuvres used in appendix A, this gives

$$\operatorname{Im}\left[\Delta^{(1)}\Sigma_p(\omega+i\delta)\right] = \frac{\tilde{A}_0^2}{\tilde{z}_p\tilde{v}_F\tilde{F}_0}\frac{\delta_0 p_F}{4\pi^2 N(0)}\int \nu\mathrm{d}\nu\int\frac{\mathrm{d}q}{q}\int_0^1\mathrm{d}\mu\frac{\mu}{(1-\mu^2)^{1/2}}$$
$$\times\left(\frac{1}{1+\left(\tilde{A}_0^2-1\right)\tilde{\eta}^2}\right)(1-\tilde{\eta}^2)^{1/2}\;\theta(\omega-q\tilde{v}_F\mu)\,\theta(q\tilde{v}_F-\nu)\,\delta(\nu-q\tilde{v}_F\mu) \quad (C.4)$$

where we get $|p| = p_F$, and as before, $\tilde{\eta} = \nu/q\tilde{V}_F$. Using $\tilde{N}(0) = p_F/2\pi\tilde{v}_F$, and changing variables, this becomes

$$\operatorname{Im}\left[\Delta^{(1)}\Sigma_p(\omega+i\delta)\right] = \frac{\tilde{A}_0^2}{\tilde{z}_p\tilde{F}_0}\left(\frac{\delta_0}{2\pi}\right)\int_0^\omega\mathrm{d}\nu\int_{\nu/q_c\tilde{v}_F}^1\tilde{\eta}\mathrm{d}\tilde{\eta}\left(\frac{1}{1+\left(\tilde{A}_0^2-1\right)\tilde{\eta}^2}\right)\theta(\omega-\nu)\,\theta(q\tilde{v}_F-\nu) \quad (C.5)$$

where the approximation in (C.5) picks out the singular term; thus we get

$$\operatorname{Im}\left[\Delta^{(1)}\Sigma_p(\omega+i\delta)\right] = \frac{1}{4\tilde{z}_p}\frac{\tilde{A}_0}{1+\tilde{F}_0}\left(\frac{\delta_0}{\pi}\right)\left[|\omega| + 0(|\omega|)^3\right]\theta(\omega) \quad (C.6)$$

which is the same as equation (4.4). From (C.6) we may deduce a term in Re $\left[\Delta^{(1)}\Sigma_p(\omega)\right]$ of the form

$$\operatorname{Re}\left[\Delta^{(1)}\Sigma_p(\omega)\right] \sim \omega\ln\omega \quad (C.7)$$

which could of course have been derived directly from (C.3); this is the form used in the "Marginal Fermi Liquid" phenomenology.

(b) Second order in λ :

We turn now to the graph of figure 11b, which has the expression ($\omega > 0$) :

$$\Delta^{(2)}\Sigma_p(\omega) = \frac{1}{\tilde{z}_p}\int\mathrm{d}\nu\sum_q\left(\frac{\delta_0}{\pi\tilde{N}(0)}\frac{p\mu}{q}\right)^2\frac{\tilde{\chi}^0_2(\tilde{\eta})}{1+\tilde{f}_0\tilde{\chi}^0_2(\tilde{\eta})}(1-\eta_{p-q})\,\delta\left[\nu-(\omega-\epsilon_{p-q})\right] \quad (C.8)$$

so that the imaginary part is

$$\operatorname{Im}\left[\Delta^{(2)}\Sigma_p(\omega+i\delta)\right] = \frac{1}{\tilde{z}_p\tilde{N}(0)}\left(\frac{1}{1+\tilde{F}_0}\right)^2\frac{p_F^2}{4\pi^2\tilde{v}_F}\left(\frac{\delta_0}{\pi}\right)^2\int\nu\mathrm{d}\nu\int\frac{\mathrm{d}q}{q^2}\int_0^1\mathrm{d}\mu\,\frac{\mu^2}{(1-\mu^2)^{1/2}}$$
$$\times\left(\frac{1}{1+\left(\tilde{A}_0^2-1\right)\tilde{\eta}^2}\right)(1-\tilde{\eta}^2)^{1/2}\left(1-\eta_{\underline{p}-\underline{q}}\right)\delta(\nu-q\tilde{v}_F u)\,\theta(q\tilde{v}_F-\nu) \quad (C.9)$$

However we must be careful in the ensuing integrations, which do not commute because of the logarithmic singularity in the final answer. The correct way to proceed is to first do the ν-integration, which yields

$$\operatorname{Im}\left[\Delta^{(2)}\Sigma_p(\omega+i\delta)\right] \sim \frac{p_F^2}{4\pi\,\tilde{z}_p\tilde{N}(0)}\left(\frac{1}{1+\tilde{F}_0}\right)^2\left(\frac{\delta_0}{\pi}\right)^2\int\frac{\mathrm{d}q}{q}\int_{-1}^1\mu^3\mathrm{d}\mu\;\theta(\omega-q\tilde{v}_{F\mu}) \quad (C.10)$$

where, as usual, we drop the $\tilde{\eta}^2$ in the denominator of the fluctuation propagator term in (C.9). The region of integration in q-space is shown in figure C1, and then we see that

$$\int\frac{\mathrm{d}q}{q}\int\mu^3\mathrm{d}\mu\;\theta(\omega-q\tilde{v}_F\mu) \to \int_0^{\omega/\tilde{v}_F}\frac{\mathrm{d}q}{q}\int_{-1}^1\mu^3\,\mathrm{d}\mu + \int_{\omega/\tilde{v}_F}^{q_c}\frac{\mathrm{d}q}{q}\int_{-1}^{\omega/q\tilde{v}_F}\mu^3\mathrm{d}\mu$$
$$= \frac{1}{4}\ln\left|\frac{\omega}{q_c\tilde{v}_F}\right| + 0\left(\omega^4\right) \quad (C.11)$$

Fig. C1. — Region of integration, in $\underline{q}$-space, involved in equations (C.10)

and we see that the log term originates from the angular dependence of $\lambda^{\sigma\sigma'}_{pp'}$. Writing, as before, $q_c \tilde{v}_F = \tilde{\omega}_0$, we get

$$\begin{aligned}\mathrm{Im}\left[\Delta^{(2)}\Sigma_p(\omega + i\delta)\right] &= \frac{p_F^2}{16\pi\ \tilde{z}_p \tilde{N}(0)} \frac{\theta(\omega)}{\left(1+\tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2 \ln\left|\frac{\omega}{\tilde{\omega}_0}\right| \\ &= \frac{p_F \tilde{v}_F}{8\ \tilde{z}_p}\frac{\theta(\omega)}{\left(1+\tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2 \ln\left|\frac{\omega}{\tilde{\omega}_0}\right|\end{aligned} \quad (C.12)$$

which we use in equation (4.5).

C.2 GRAPHS FOR 3-POINT VERTEX. — We are concerned in general with graphs of the type shown in figure C2a which satisfy a Bethe-Salpeter equation relating $\Lambda(P,\ K)$ to the 4-point vertex $\Gamma(P,\ P';\ K)$:

$$\Lambda(P,\ K) = 1 + \int \frac{\mathrm{d}\epsilon'}{\pi}\sum_{\underline{p}'} G(P')G(P'+K)\Gamma(P,\ P';\ K) \quad (C.13)$$

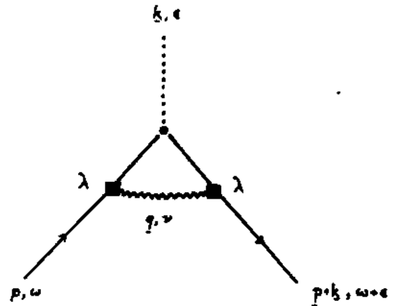

Fig. C2. — The contribution $\Delta^{(2)}\Lambda(P,\ K)$ which is calculated in this appendix (see Eq. (C.17)).

In the low-energy regime we are interested in, where the quasiparticle $|\underline{p}\rangle$ is just above the Fermi surface, this question can be written as (note: $K \equiv (\underline{k},\ \Omega)$) :

$$\Lambda^K_{\underline{p}}(\omega) = 1 + \sum_{\underline{q}} \frac{\underline{k}\cdot\underline{v}_{p-q}}{\underline{k}\cdot\underline{v}_{p-q} - \Omega}\left(1 - n_{\underline{p}-\underline{q}}\right) T_{\underline{p},\ \underline{p}-\underline{q}}(K) \quad (C.14)$$

where we have changed variables ($P' = P - Q$), and $T_{\underline{p}\underline{p}'}(K)$ is related to $\Gamma(P,\ P';\ K)$ by equation (2.3).

The vertex Λ is useful mainly because of the Ward identity

$$z^{-1}_{\underline{p}}(\omega) = \mathrm{Re}\ \Lambda^0_{\underline{p}}(\omega) \quad (C.15)$$

where

$$\Lambda^s_{\underline{p}}(\omega) = \lim_{K\to 0} \Lambda^K_{\underline{p}}(\omega)\ \Big|_{\Omega/k\cdot v_p = s} \quad (C.16)$$

but it is also useful to note that is has a very clear physical interpretation: $\Lambda^K_p(\omega) \equiv \Lambda(\underline{p},\ \omega;\ \underline{k},\ \Omega)$ is just the space-time Fourier Transform of the function $\Lambda(\underline{p},\ \omega;\ \underline{r},\ t)$ which describes the evolution in spacetime of the quasiparticle distribution function $\delta n_{\underline{p}(\underline{r},\ t)}$ due to an initial quasi-particle wave-packet centred around $|\underline{p}\rangle$ (see Ref. [52], and also Ref. [10]).

For the singular interaction of section 4 we are interested in the graph, 2nd order in λ, shown in figure C2b; in the same way as for $\Sigma_{\underline{p}}(\omega)$, we must sum the maximally-crossed diagrams for $\Lambda^K_p(\omega)$, and this graph, which we call $\Delta^{(2)}\Lambda^K_p(\omega)$, will then be exponentiated (see Eq. (4.19), and Fig. 13). One has (for $\omega > 0$) :

$$\mathrm{Im}\left[\Delta^{(2)}\Lambda^s_{\underline{p}}(\omega)\right] \sim \int\frac{\mathrm{d}\nu}{\pi}\sum_{\underline{q}}\lambda^2_{\underline{p},\ \underline{p}-\underline{q}}\frac{\underline{k}\cdot\underline{v}_{\underline{p}}}{\underline{k}\cdot\underline{v}_{\underline{p}} - \Omega}\ \mathrm{Im}\tilde{\chi}_{00}(Q)\theta\left(\omega - qv_F\mu\right)\delta\left(\nu - qv_F\mu\right) \quad (C.17)$$

where we write $\underline{v}_{\underline{p}-\underline{q}} \sim v_{\underline{p}}$, since we are only interested in low $\underline{q}$; this in fact allows us to take the Landau factor $J(s) = \underline{k}\cdot\underline{v}_{\underline{p}}/\left(\underline{k}\cdot\underline{v}_{\underline{p}} - \Omega\right)$ out of the integral altogether. $\Lambda(P,\ K)$ has complex analytic properties in the ω- and Ω-frequency planes, but we will ignore the behaviour in the Ω-plane. Assuming $\omega \to \omega + i\delta$, we can then split $\Lambda^K_p(\omega)$ into a real and imaginary part, and from (C.17) we have

$$\begin{aligned}&\mathrm{Im}\left[\Delta^{(2)}\Lambda^s_p(\omega + i\delta)\right] \sim J(s)\left(\frac{p_F\delta_0}{\pi\tilde{N}(0)}\right)^2 \\ &\int \mathrm{d}\nu\sum_{\underline{q}}\left(\frac{\mu}{q}\right)^2\left[\frac{\eta\tilde{N}(0)}{(1-\eta^2)^{1/2}}\left|\frac{1}{1+\tilde{F}_0\chi^0_2(\eta)}\right|^2\theta\left(q\tilde{v}_F - \nu\right)\right]\theta\left(\omega - q\tilde{v}_F\mu\right)\delta\left(\nu - q\tilde{v}_F\mu\right)\end{aligned} \quad (C.18)$$

with $\eta = \nu/q\tilde{v}_F$ as usual; we notice that this integral is basically the same as that in (C.9), and we evaluate it in the same way to get

$$\begin{aligned}\mathrm{Im}\left[\Delta^{(2)}\Lambda^s_p(\omega)\right] &\sim J(s)\frac{p_F^2}{8\pi\ \tilde{N}(0)}\frac{\theta(\omega)}{\left(1+\tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2\ln\left|\frac{\omega}{\tilde{\omega}_0}\right| \\ &\equiv J(s)\frac{p_F\tilde{v}_F}{4}\frac{\theta(\omega)}{\left(1+\tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2\ln\left|\frac{\omega}{\tilde{\omega}_0}\right|\end{aligned} \quad (C.19)$$

Since we are interested in the limit $s \to 0$, this gives

$$\mathrm{Im}\left[\Delta^{(2)}\Lambda^0_p(\omega)\right] = \frac{p_F\tilde{v}_F}{4}\frac{\theta(\omega)}{(1+F_0)^2}\left(\frac{\delta_0}{\pi}\right)^2 \ln\left|\frac{\omega}{\omega_0}\right| \tag{C.20}$$

We are more interested in $\mathrm{Re}\left[\Delta^{(2)}\Lambda^s_p(\omega)\right]$, which may be derived from (C.19) using the analytic properties of $\Lambda^s_p(\omega)$, or directly from (C.17). One finds, using the result that

$$P\int_{-\omega_0}^{\omega_0}\frac{dx}{\omega - x}\theta(x)\ln\left|\frac{x}{\omega_0}\right| \sim \frac{1}{2}\ln^2\left|\frac{\omega}{\omega_0}\right| \tag{C.21}$$

that $\mathrm{Re}\left[\Delta^{(2)}\ \Lambda^s_p(\omega)\right]$ has a $\ln^2\omega$ singularity; from (C.20) we have that

$$\mathrm{Re}\left[\Delta^{(2)}\Lambda^0_p(\omega)\right] = \frac{p_F\tilde{v}_F}{8}\frac{1}{\left(1+\tilde{F}_0\right)^2}\left(\frac{\delta_0}{\pi}\right)^2 \ln^2\left|\frac{\omega}{\tilde{\omega}_0}\right| \tag{C.22}$$

This result is crucial to the summation of singular terms for the quasiparticle renormalisation (see Eq. (4.2)).

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