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Lecture 3

"How Can We Calculate the CBL Height"

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1.1 Vertical Structure and Energy Transfer Under Neutral and Stable Conditions

Similarity theory has been used in studies of PBL since the early 1960s (Kazanskii, Monin 1960). Examples of real results obtained in this way are the resistance and heat/mass transfer laws. For the steady-state regime with neutral stratification (Kazanskii, Monin 1961), steady-state stratified regime (Zilitinkevich, Laikhtman, Monin, 1967; Zilitinkevich 1967) nonequilibrium regimes (Zilitinkevich, Deardorff 1974; Zilitinkevich 1975a). Establishment of these laws created new problems. It became necessary to determine, first, the universal functions of the stratification parameter (the so-called A, B, C and D functions) included therein, and second, the PBL depth h.

Under unstable stratification in a fluid penetrative convection develops, so that a steady-state dynamic regime is practically impossible. In this case, the behavior of h as a function of time or of the horizontal coordinate is governed by an equation derived in the theory of penetrative convection (Sect. 1.2 and in more detail, Zilitinkevich 1987, 1991).

In contrast, in cases of neutral and stable stratification, a stationary, or at least quasi-stationary regime is not only possible theoretically, but is even usually observed. Indeed, in evening and night hours, with the increase of hydrostatic stability the atmospheric PBL collapses: its upper boundary falls, while turbulence in the overlying region degenerates. In the same way, the depth of the upper mixed layer in a water body is reduced during daytime warming. The process of turbulence degeneration at the outer boundary of a collapsing PBL occurs rather quickly. Its charateristic period does not exceed tens of minutes. In comparison, changes in the PBL (having a typical period of several hours) proceed very slowly. That is why it can be considered as quasi-stationary. This means that the PBL structure at each moment of time can be determined by parameters characterizing an instantaneous situation. The analysis of the momentum equations, together with similarity and dimensionality considerations, allowed defining the explicit form of the dependence of the depth of the stably stratified PBL on some governing parameters (Zilitinkevich 1972).

The papers presented below (Zilitinkevich 1989a, b) are devoted in particular to the steady-state regime under neutral und stable conditions. In this case, on the basis of the similarity theory asymptotic formulas have been derived for the A, B and C functions corresponding to the regime of strong stability (Zilitinkevich 1975a), and a series of empirical approximations of these functions have been constructed based on atmospheric data (Zilitinkevich, Chalikov 1968b; Clarke 1970, 1973; Clarke et al. 1971; Clarke, Hess 1974; Wippermann 1972a, b, c; Soloviev 1973, 1974; Deacon 1973; Melgarejo, Deardorff 1974, 1975; Arya 1975, 1977; Yamada 1976; Hess et al. 1981; Nieuwstadt 1981; Greenhut, Brook 1983). There are also several expressions for A, B and C functions obtained after *Zilitinkevich* (1975a) in the framework of certain theoretical PBL models. Nevertheless, the resistance problem under stable conditions cannot be considered as being solved, primarily because of the very wide spread in the existing empirical evaluations of the A, B and C functions. There are two main causes for this wide range of values: (a) the approximate character of the theory which does not take into account the non-stationarity of horizontal inhomogeneity of real meteorological fields; (b) errors in processing of experimental data due to uncertainty in determining the PBL depth. The first cause can be eliminated to a certain degree on the basis of the approach suggested by *Zilitinkevich* and *Deardorff* (1974) and *Zilitinkevich* (1975a). However, this does not lead to a significant reduction in the spread of empirical data on the universal functions and in the resistance and heat/mass transfer laws. Quite probably, the second cause is more important. If so, the current position can be improved by joint determination of the universal functions A, B and C and of the universal function characterizing the dimensionless depth of the PBL. The main objective of the discussed papers is to provide a theoretical approach to such a procedure. The second aim is to determine velocity and temperature profiles in a stratified PBL. Until now, the second problem has been solved almost exclusively on the basis of turbulent viscosity models or second-order closure models, i.e., on the basis of more or less arbitrary closure hypotheses, the solutions being cumbersome and sometimes detached from experiments.

The Ekman Boundary Layer. Under stationary and horizontally homogeneous conditions, the velocity components u and v along the horizontal axes x and y satisfy the Ekman equations

$$f(v - U_g \sin\alpha_*) + \frac{d\tau_x}{dz} = 0\,, \quad -f(u - U_g\cos\alpha_*) + \frac{d\tau_y}{dz} = 0\,, \qquad (1.1)$$

where z is height, f is the Coriolis parameter, $U_g \cos\alpha_*$ and $U_g \sin\alpha_*$ are components of the geostrophic wind vector (U_g is the modulus of this vector, α_* is the angle between it and the x-axis, τ_x and τ_y are components of the vertical turbulent flux of momentum per unit mass). Their near-surface values τ_{xs} and τ_{ys} (components of friction stress on the underlying surface) appear to be the main characteristics of the dynamic interaction between the atmosphere and the underlying surface.

Let us direct the x-axis along the surface friction vector. Then τ_{ys} reduces to zero, while α_* represents the angle through the wind direction turns in the PBL. Surface friction will be characterized by the angle α_* and the friction velocity $u_* = \sqrt{\tau_{xs}}$.

We shall take the PBL to be the layer $0 < z < h$, at whose upper boundary the turbulence dies out so that the vertical turbulent momentum fluxes reduce to zero:

$$\tau_x(h) = \tau_y(h) = 0 \qquad (1.2)$$

and the velocity components become the geostrophic ones:

$$u(h) = U_g \cos\alpha_*\,, \quad v(h) = U_g \sin\alpha_*\,. \qquad (1.3)$$

Integrating (1.1) over z from 0 to h according to the boundary conditions (1.2, 3), we obtain

$$\int_0^h u\,dz = hu(h)\,, \quad \int_0^h v\,dz = hv(h) + u_*^2/f\,. \qquad (1.4)$$

Another simple integral relation is obtained for the dissipation rate of the mean flow kinetic energy (i.e., for the rate of its transformation into turbulent energy) determined according to the formula

$$\varepsilon = \tau_x du/dz + \tau_y dv/dz\,. \qquad (1.5)$$

Integrating (1.5) over z from 0 to h and taking into account (1.1–4), we obtain

$$\int_0^h \varepsilon\,dz = U_g u_*^2 \cos\alpha_*\,. \qquad (1.6)$$

Equations (1.4, 6) follow only from (1.1) and the natural boundary conditions, i.e., they do not depend on any similarity hypotheses, nor on the presence or absence of buoyancy forces influencing the turbulent structure of the PBL.

If the source of buoyancy forces is at the underlying surface (say it is caused by cooling or heating of the surface), the effect of these forces in the PBL is characterized by the near-surface value of the vertical buoyancy flux B_s[1] or

[1] In the atmosphere $B_s = \beta_s H_s/c_p\rho_s + 0.608\, gE_s/\rho_s$, where H_s and E_s are vertical turbulent fluxes of sensible heat and water vapor, ρ_s is air density, c_p is the specific heat capacity of air at constant pressure, g is the acceleration due to gravity, $\beta_s = g/T_s$ is the temperature buoyancy parameter, T_s is the absolute temperature (index s indicates the near-surface value).

by some parameter based on it, e.g., the Monin-Obukhov length scale L or dimensionless stratification parameter μ:

$$L = \frac{-u_*^3}{kB_s}, \quad \mu = \frac{-k^2 B_s}{|f| u_*^2}, \tag{1.7}$$

where k is the von Karman constant whose conventional value is $k = 0.4$ (Högstrom 1985).

Strictly speaking, at $B_s \neq 0$ a steady-state regime is impossible in the PBL because of the accumulation or loss of buoyancy (warming or cooling), which is described first of all by the non-stationary heat transfer equation. However, due to the action of the Coriolis force, the wind field can follow this transient process in a quasi-stationary fashion, as if forgetting the previous history of its evolution at each stage. That is why the assumption of steady-state flow is justified in the PBL even with non-stationary changes of the buoyancy field.

With neutral stratification ($B_s = 0$), buoyancy forces are absent; with stable stratification ($B_s < 0$), they suppress turbulence, so the only mechanism in both cases for producing turbulence is the wind shear. In the neutral case, the depth of the steady state PBL is expressed by the *Rossby-Montgomery* (1935) formula:

$$h = \Lambda_0 u_* / |f|, \tag{1.8}$$

where Λ is a dimensionless universal constant which is usually assumed to be equal to 0.3. An idea of the range of its empirical evaluations is given in Table 1.1. It is worth noting that (1.8) is virtually the initial point of the *Kazanskii* and *Monin* (1960) similarity theory according to which the only length scale characterizing the steady-state barotropic neutrally stratified PBL is the Ekman scale $u_*/|f|$ and the stratification effect in PBL is characterized by B_s, L or μ.

With stable stratification the list of parameters determining the structure of the steady-state PBL is supplemented by the near-surface value of the buoyancy flux B_s. This means that the PBL depth h can be represented by a formula similar to (1.8), as before, but the dimensionless coefficient on the right-hand side will no longer be a constant Λ_0, but a universal function of the dimensionless stratification parameter $\Lambda(\mu)$. As was already mentioned, in the case of sufficiently strong stability ($\mu \gg 1$), the asymptotic behavior of this function can be determined by analysis of the momentum equations together with similarity considerations. This gives the following expression for h (Zilitinkevich 1972):

$$h = C_h u_*^2 |f B_s|^{-1/2}, \tag{1.9}$$

where C_h is a dimensionless universal constant of the order of unity. A summary of empirical evaluations of this constant is presented in Table 1.2.

All of them are based on meteorological measurements with the exception of the first theoretical evaluation (Zilitinkevich 1972) marked with an asterisk and the last limnological evaluation (Zilitinkevich, Rumyantsev 1990).

The simplest interpolation formula combining (1.8) and (1.9) has the form

$$\frac{|f|h}{u_*} = \Lambda(\mu) = \left(\frac{1}{\Lambda_0} + \frac{\mu^{1/2}}{kC_h}\right)^{-1}. \tag{1.10}$$

Of course, it is desirable to have an empirically based expression for the function $\Lambda(\mu)$ in the transition region $0 < \mu < 10$. However, (1.10) can serve as quite a good meteorological approximation because the stable stratification, which takes place in the atmospheric PBL during a great part of the night, is characterized by values of μ of the order of tens or even hundreds.

As to the oceanic and lacustrine PBL, values of μ of the order of unity or several units are typical with stable stratification. It is quite possible that (1.10) will be found to be too crude in these cases (for example, see the empirical data in Fig. 4 of Felzenbaum, 1980). It should be mentioned that following *Zilitinkevich* (1972), a number of papers appeared where expressions for the function $\Lambda(\mu)$ satisfying the asymptotic formula (1.9) are obtained in the framework of theoretical PBL models. However, such expressions, until they are verified by experimental data, can hardly be considered to be more reliable than interpolation formulas as simple as (1.10).

Table 1.1. Dimensionless constant Λ_0 in the Rossby and Montgomery formula (1.8) for the depth of a neutrally stratifield PBL

Reference	Λ_0
Measurements in the ocean	
Rossby and Montgomery (1935)	0.2
Kitaigorodskii (1973)	0.1–0.3
Measurements in the atmosphere	
Gill (1967)	0.1
Clarke (1970)	0.2
Deardorff (1972)	0.35
Tennekes (1973b)	0.3
Yamada (1976)	0.3
Arya (1978)	0.3
Laboratory Experiments	
Caldwell et al. (1972)	0.3

Table 1.2. Dimensionless constant C_h in the Zilitinkevich expression (1.9) for the depth of the stably stratified PBL

Reference	C_h
Zilitinkevich (1972)*	O(1)
Businger and Arya (1974)	1.14
Arya (1977)	1.58
Caughey et al. (1979)	1.11
Nieuwstadt (1981)	0..63
Garrat (1982)	0.55
Caughey (1982)	1.10
Nieuwstadt (1984)	0.55
Zilitinkevich, Rumyantsev (1990)	1

Neutral Stratification. In this case, the velocity profile satisfies the near-surface logarithmic law at $z \ll h$:

$$u(z) = \frac{u_*}{k} \ln \frac{z}{z_{0u}} , \quad v(z) = 0 , \tag{1.11}$$

where z_{0u} is the roughness parameter relative to wind; and the velocity defect law at $z \gg z_{0u}$ is

$$U_g \cos \alpha_* - u(z) = \frac{u_*}{k} \phi_u(\zeta) , \quad U_g \sin \alpha_* - v(z) = \frac{-u_*}{k} \phi_v(\zeta) \operatorname{sign} f , \tag{1.12}$$

where ζ is a dimensionless height:

$$\zeta = z/h , \tag{1.13}$$

ϕ_u and ϕ_v are universal functions satisfying, in accordance with (1.3), the conditions $\phi_u(1) = \phi_v(1) = 0$.

Overlapping of (1.11) and (1.12) in the region $z_{0u} \ll z \ll h$ (where both are valid) with due regard for (1.8) leads to the resistance law:

$$\ln(C_g \text{Ro}) - B_0 = \sqrt{(k/C_g)^2 - A_0^2} , \quad \sin \alpha_* = \frac{-A_0}{k} C_g \operatorname{sign} f , \tag{1.14}$$

where Ro is the near-surface Rossby number, C_g is the geostrophic drag coefficient,

$$\text{Ro} = U_g/|f| z_{0u} , \quad C_g = u_*/U_g , \tag{1.15}$$

and A_0 and B_0 are dimensionless universal constants. The first formula in (1.14) was obtained by *Kasanskii* and *Monin* (1961), the second one, almost simultaneously by *Zilitinkevich, Laikhtman,* and *Monin* (1967), *Csanady* (1967), *Gill* (1967, 1968), *Blackadar* and *Tennekes* (1968). Then a number of experimental data were processed to evaluate A_0 and B_0. A summary is given in Table 1.3, where with the exception of the first theoretical evaluation marked with an asterisk (Kasanskii, Monin 1961) and the laboratory evaluation (Caldwell et al. 1972), all the others are based on meteorological observations. They give an average $A_0 = 4.5$, $B_0 = 1.7$.

According to (1.11–15) and (1.8), the velocity profile in the whole region $z \le h$ allows the representation

$$u(z) = \frac{u_*}{k} \left[\ln \frac{z}{z_{0u}} + f_u(\zeta) \right] , \quad v(z) = \frac{-u_*}{k} f_v(\zeta) \operatorname{sign} f , \tag{1.16}$$

where f_u and f_v are universal functions of ζ expressed by the functions ϕ_u and ϕ_v: $f_u = -B_0 - \ln(\Lambda_0 \zeta) - \phi_u$, $f_v = A_0 - \phi_v$ and obeying the conditions

$$f_u(0) = f_v(0) = 0 \tag{1.17}$$

$$f_u(1) = -B_0 - \ln \Lambda_0 , \quad f_v(1) = A_0 . \tag{1.18}$$

Table 1.3. Dimensionless constants A_0 and B_0 in the Rossby-number similarity theory resistance law (1.14) for a neutrally stratified PBL

Reference	A_0	B_0
Kazanskii and Monin (1961)*	1.8	1.7
Zilitinkevich, Laikhtman, Monin (1967)	5.3	1.5
Gill (1967)	4.7	1.7
Csanady (1967)	4.8	1.7
Zilitinkevich and Chalikov (1968b)	5	2
Clarke (1970)	4.5	0.9
Wippermann (1970)	4.5	0.9
Clarke et al. (1971)	5	1
Plate (1971)	4.3	1.7
Caldwell et al. (1972)	2.5	2.2
Deacon (1973)	4.7 ± 0.15	1.9 ± 0.35
Clarke (1973)	4.9, 6.1	4.7
Clarke and Hess (1974)	4.3 ± 0.7	1.1 ± 0.5
Yamada (1976)	3.020	1.855
Nieuwstadt (1981)	1.9	2.3
Hess et al. (1981)	4.3 ± 0.7	1.1 ± 0.5

Taking into account conditions (1.17), we shall approximate the functions f_u and f_v by the following quadratic polynomials:[2]

$$f_u(\zeta) = b_0 \zeta + b_0^* \zeta^2 , \quad f_v(\zeta) = a_0 \zeta + a_0^* \zeta^2 . \tag{1.19}$$

For the determination of the coefficients b_0, b_0^*, a_0 and a_0^*, we shall use first of all the conditions (1.4). Then substituting u and v from (1.16, 19) and taking into account (1.8) we obtain

$$b_0^* = -\frac{3}{2} - \frac{3}{4} b_0 , \quad a_0^* = \frac{3k}{2\Lambda_0} - \frac{3}{4} a_0 . \tag{1.20}$$

Further, after substituting (1.19, 20) into (1.18) we obtain

$$b_0 = 6 - 4B_0 - 4 \ln \Lambda_0 , \quad a_0 = 4A_0 - 6k/\Lambda_0 . \tag{1.21}$$

In this way, in the adopted log-polynomial approximation, the velocity profiles for a neutrally stratified PBL are obtained. They are determined by four dimensionless constants: k, Λ_0, A_0 and B_0. If we adopt the values

$$k = 0.4 , \quad \Lambda_0 = 0.3 , \quad A_0 = 4.5 , \quad B_0 = 1.7 \quad \text{(At)}$$

recommended above for the atmospheric PBL, then from (1.20, 21) we obtain $a_0 = 10$, $a_0^* = -5.5$, $b_0 = 4$, $b_0^* = -4.5$.

From the laboratory values

$$k = 0.4 \quad \Lambda_0 = 0.3 , \quad A_0 = 2.5 , \quad B_0 = 2.2 \quad \text{(Lab)}$$

[2] A representation of this type was suggested by *Long* (1974) where, in contrast to (1.19), f_u is assumed to be an even function while f_v is an odd function.

we obtain $a_0 = 2$, $a_0^* = 0.5$, $b_0 = 2$, $b_0^* = -3$.

Now it is not difficult to derive expressions for profiles of vertical momentum fluxes. For this it is sufficient to substitute $U_g \cos\alpha_*$, $U_g \sin\alpha_*$ after (1.3), and u, v after (1.16, 19) into (1.1) and then integrate over z. The components of the dimensionless velocity defect and dimensionless vertical momentum flux calculated in this way versus dimensionless height are shown in Figs. 1.1, 2 together with *Caldwell* et al. (1972) collection of atmospheric measurements and laboratory data as well as data obtained by *Deardorff* (1970) in three-dimensional numerical simulations.

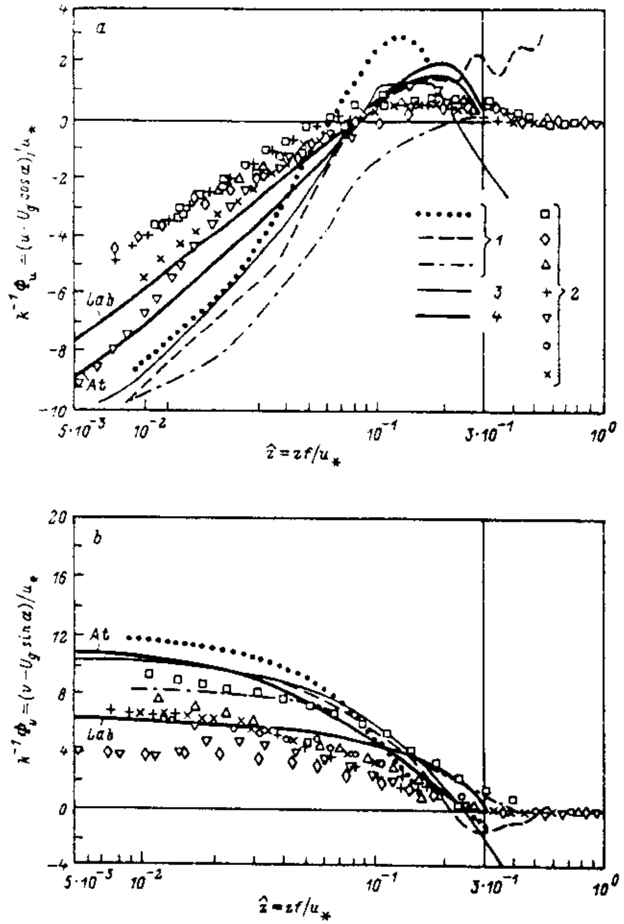

Fig. 1.1. Dimensionless profiles of velocity defects in a neutrally stratified PBL. (a) Component along the friction stress direction at the surface; (b) component perpendicular to this direction. Empirical data (1 – atmospheric measurements, 2 – laboratory experiments) and results of the *Deardorff* (1970) numerical experiments (3) are taken from *Caldwell* et al. (1972). Heavy curves 4 ("At" and "Lab") are theoretical

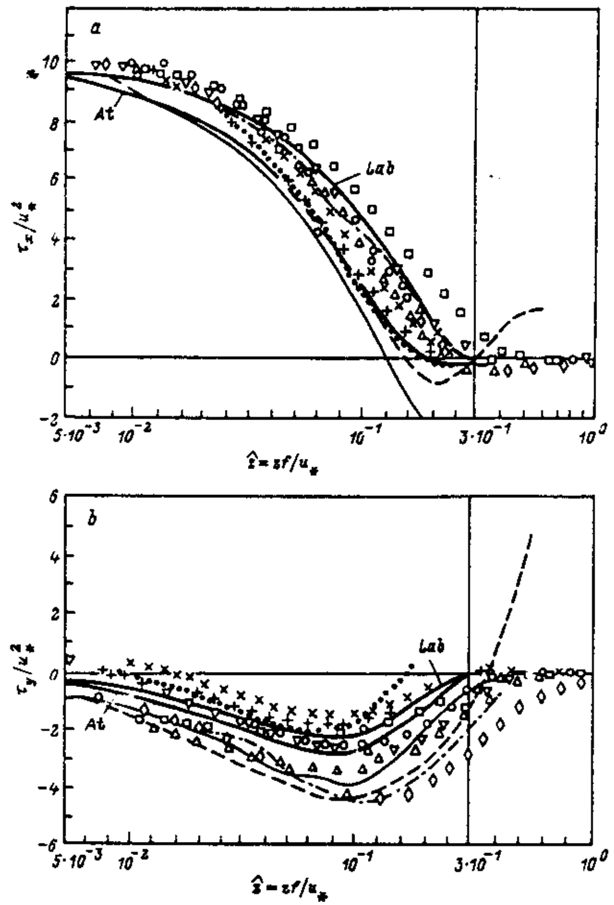

Fig. 1.2. Dimensionless profiles of vertical momentum fluxes in a neutrally stratified PBL. (a) Component along the direction of friction stress at the surface; (b) component perpendicular to this direction. Curves and symbols are defined as in Fig. 1.1

Resistance with Stable Stratification. In this case, the number of parameters characterizing the PBL is supplemented by the buoyancy flux B_s and hence by the length scale L or the stratification parameter μ determined by (1.7). Near the surface, as before, the logarithmic velocity profile (1.11) is used; however, the region where it is valid is limited not only to $z \ll h$, but also to $z \ll L$. A velocity defect law similar to (1.12) similar holds true, but now ϕ_u and ϕ_v are functions of not one argument ζ but of two: ζ and μ. In addition, as was already mentioned, the PBL depth h depends not only on u_* and f, but also on B_s, i.e., it is expressed in the form $|f|h/u_* = \Lambda(\mu)$.

With due regard to these properties, combining the velocity defect law with the near-surface logarithmic velocity profile leads to the resistance law:

$$\ln(C_g \mathrm{Ro}) - B(\mu) = \sqrt{(k/C_g)^2 - A^2(\mu)}\,, \quad \sin\alpha_* = \frac{-A(\mu)}{k} C_g \,\mathrm{sign} f\,, \quad (1.22)$$

which differs from (1.14) by replacement of the universal constants A_0 and B_0 by the universal functions $A(\mu)$ and $B(\mu)$. This law was derived by *Zilitinkevich, Laikhtman, Monin* (1967), while the functions A and B were first determined from experimental data by *Zilitinkevich* and *Chalikov* (1968b). Following these works, several empirical evaluations of A and B functions were carried out.

It is worth mentioning that a cause for confusion is that a designation of the universal functions in the resistance law different from those adopted in (1.22) is often used: A instead of B and B instead of A.

If in deriving the resistance law, we use a more definite form of the velocity defect law with due regard to the specific features of the PBL at strongly-stable stratification established in *Zilitinkevich* (1972), overlapping occurs not with the logarithmic near-surface law but with the linear law following from the similarity theory of *Monin* and *Obukhov* (1954) for the height region $L \ll z \ll h$. Then we obtain the following asymptotic expressions for the functions A and B at $\mu \gg 1$ (Zilitinkevich 1975a):

$$A(\mu) \to N_1\mu^{1/2}\,, \quad B(\mu) \to -N_2\mu^{1/2}\,, \tag{1.23}$$

where N_1 and N_2 are dimensionless universal constants.

Further, the function A and B have been determined by many authors within the framework of the turbulent viscosity or second-order closure models of the PBL. It is understood that in cases when these models agree with the similarity theories of *Monin* and *Obukhov* (1954), *Kazanskii* and *Monin* (1960) and *Zilitinkevich* and *Deardorff* (1974), the resulting expressions for $A(\mu)$ and $B(\mu)$ at $\mu \gg 1$ satisfy the asymptotic relations (1.23): $A \propto -B \propto \mu^{1/2}$.

More important is the fact that these relations have been verified by experimental data (Table 1.4).

Determination of Velocity Profiles and Universal Functions $A(\mu)$ and $B(\mu)$. As with neutral stratification, by using the resistance law and velocity defect law, the u and v profiles for the entire PBL region $z \le h$ can be represented in the form

$$u(z) = \frac{u_*}{k}\left[\ln\frac{z}{z_{0u}} + f_u(\zeta,\mu)\right]\,, \quad v(z) = \frac{-u_*}{k}f_v(\zeta,\mu)\,\mathrm{sign} f\,, \tag{1.24}$$

where $f_u = -B(\mu) - \ln[A(\mu)\zeta] - \phi_u$ and $f_v = A(\mu) - \phi_v$ are functions of the two arguments satisfying the conditions

$$f_u(0,\mu) = f_v(0,\mu) = 0 \tag{1.25}$$

$$f_u(1,\mu) = -B(\mu) - \ln A(\mu)\,, \quad f_v(1,\mu) = A(\mu)\,. \tag{1.26}$$

Let us determine these functions. Taking into account conditions (1.25), we approximate them by the following quadratic polynomials:

$$f_u(\zeta,\mu) = b(\mu)\zeta + b^*(\mu)\zeta^2\,, \quad f_v(\zeta,\mu) = a(\mu) + a^*(\mu)\zeta^2\,, \tag{1.27}$$

Table 1.4. Empirical approximations of the universal functions $A(\mu)$ and $B(\mu)$ in the resistance law (1.22) for a stably stratified PBL at $\mu \gg 1$ based on the main term in the Zilitinkevich asymptotic relations (1.23)

Reference	$A(\mu)$	$B(\mu)$
Zilitinkevich (1975a)	$4\mu^{1/2}$	$\ln\mu - 4\mu^{1/2}$
Yamada (1976)	$2.47(\mu - 16.6)^{1/2}$	$1.2\text{–}2.55(\mu - 26.6)^{1/2}$
Arya (1977)	$1.1 + 1.82\mu^{1/2}$	$2.96 + \ln\mu^{1/2} - 1.52\mu^{1/2}$
Long and Guffey (1977)	$2.55\mu^{1/2}$	$23.2 + \ln\mu - 5.28\mu^{1/2}$
Brost and Wyngaard (1978)	$2.21\mu^{1/2}$	$2.46 + \ln\mu^{1/2} - 1.42\mu^{1/2}$

whose coefficients are no longer constants as in (1.19), but functions of μ satisfying, according to conditions (1.4), the following relations:

$$b^*(\mu) = -\frac{3}{2} - \frac{3}{4}b(\mu)\,, a^*(\mu) = \frac{3k}{2A(\mu)} - \frac{3}{4}a(\mu)\,. \tag{1.28}$$

If the function $A(\mu)$ is known, determined, for example, by (1.10), two other functions remain to be determined: $b(\mu)$ and $a(\mu)$. In order to do this, let us use the following information: in the near-surface region at $z \ll h$ and $L \ll h$, the velocity profile should satisfy the logarithmic + linear law (Monin, Obukhov 1954):

$$u(z) = \frac{u_*}{k}\left(\ln\frac{z}{z_{0u}} + \beta_u\frac{z}{L}\right)\,, \quad v(z) = 0\,, \tag{1.29}$$

where β_u is a dimensionless constant of the order of 10. Numerous papers are devoted to its evaluation, summaries being given in *Zilitinkevich* (1970), *Monin* and *Yaglom* (1971), *Tennekes* (1982) and *Panin* (1985). Table 1.5 contains values of β_u from these summaries (recalculated with the use of the standard value of the von Karman constant $k = 0.4$) as well as the value of β_u obtained by interpretation of the data of laboratory experiments by *Chuang* and *Cermak* (1967) with account of the empirical dependence of the turbulent Prandtl number on stratification according to Fig. 1.22 of *Zilitinkevich* (1970). The causes of variability in the β_u empirical evaluation are discussed in *Byzova* and *Vyaltseva* (1988).

Table 1.5. Dimensionless constant β_u in the Monin and Obukhov logarithmic + linear law (1.29) for the wind profile in the near-surface layer

Reference	β_u
Summaries of meteorological measurements	
Zilitinkevich (1970)	11
Monin and Yaglom (1971)	10
Tennekes (1982)	4.7
Panin (1985)	6
Laboratory experiments	
Chuang and Cermak (1967)	14

Combining the approximation (1.24, 27) and the near-surface expression (1.29) leads to the conclusion that at $\mu \to \infty$, the function $b(\mu)$ behaves as follows:

$$b \sim \beta_u h/L \sim C_h \beta_u \mu^{1/2} , \tag{1.30}$$

and the function $a(\mu)$ should remain finite. The simplest expressions ensuring such behavior at large μ and satisfying the required conditions at $\mu = 0$ have the form

$$b(\mu) = b_0 + C_h \beta_u \mu^{1/2} , \quad a(\mu) = a_0 , \tag{1.31}$$

where a_0 and b_0 are dimensionless constants defined by (1.21).

Expressions (1.10, 24, 27, 28, 31) determine the velocity profiles in the log-polynomial approximation. Knowing these, exactly as in the neutral stratification, it is not difficult to obtain the vertical profiles of the momentum flux components. Finally, substituting the resulting expressions of f_u and f_v into (1.26) and taking into account (1.10, 21) we obtain

$$\begin{aligned} \Lambda(\mu) &= \Lambda_0 + \frac{3}{2C_h}\mu^{1/2} , \\ B(\mu) &= B_0 + \ln\left(1 + \frac{\Lambda_0 \mu^{1/2}}{kC_h}\right) - \frac{1}{4} C_h \beta_u \mu^{1/2}. \end{aligned} \tag{1.32}$$

The main terms in these expressions for the universal functions A and B agree with the theoretical asymptotes (1.23) and together with equations (1.10, 22) completely specify the resistance law. They contain six dimensionless universal constants: k, Λ_0, A_0, B_0, C_h, and β_u. The first four can be roughly estimated from measurements under conditions of neutral stratification. Let us use values based on atmospheric evaluations (At) and determine C_h and β_u by comparing the results of (1.32) with experimental data on the functions $A(\mu)$ and $B(\mu)$. The results of such a comparison are shown in Fig. 1.3. Expression (1.32) for A does not contain β_u. Therefore, Fig. 1.3a allows one to obtain an independent evaluation of C_h. Then β_u is found by means of Fig. 1.3b. Adopting the empirical approximation of *Yamada* (1976) as the most representative and reliable we obtain the values

$$C_h = 0.85 , \quad \beta_u = 12 , \tag{At'}$$

which are within the range of the direct evaluations presented in Tables 1.2 and 1.5.

According to (1.10, 32) and (At), (At′), the universal functions Λ, A and B have the form

$$\begin{aligned} \Lambda(\mu) &= \left(\frac{1}{0.3} + \frac{\sqrt{\mu}}{0.34}\right)^{-1} , \quad A(\mu) = 4.5 + \frac{3\sqrt{\mu}}{1.7} , \\ B(\mu) &= 1.7 + \ln\left(1 + \frac{3\sqrt{\mu}}{3.4}\right) - 2.55\sqrt{\mu} . \end{aligned} \tag{1.33}$$

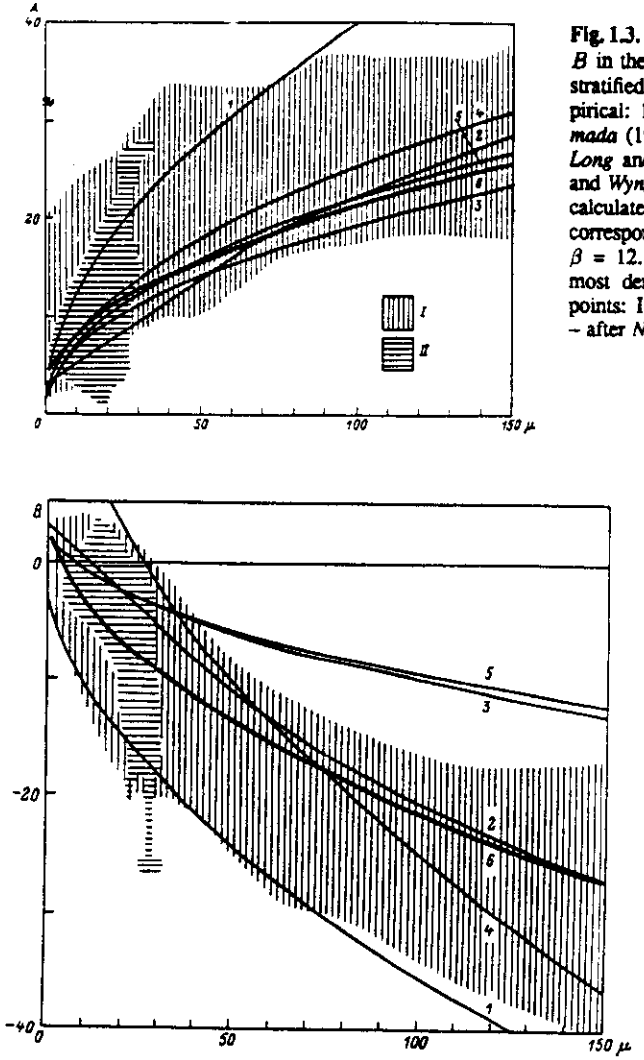

Fig. 1.3. Universal functions A and B in the resistance law for a stably stratified PBL. *Curves 1–5* are empirical: 1 – *Garrat* (1982), 2 – *Yamada* (1976), 3 – *Arya* (1977), 4 – *Long* and *Guffey* (1977), 5 – *Brost* and *Wyngaard* (1978). *Curves 6* are calculated according to (1.33), i.e. corresponding to C_h = 0.85 and β = 12. *Hatched regions* are those most densely filled with empirical points: I – after *Yamada* (1975), II – after *Nieuwstadt* (1981)

Calculation of Surface Friction and Energy Dissipation. When calculating the friction velocity u_* and the angle of turning of the wind α_* in the PBL with the help of the resistance law equations (1.22, 33), it should be borne in mind that the parameter μ defined by the second formula in (1.7) depends on u_*. Therefore it is convenient to present it in the form

$$\mu = \frac{M}{C_g^2} , \quad M = \frac{-k^2 B_t}{|f| U_g^2} , \tag{1.34}$$

where M is an external parameter of stratification which does not include values that need to be determined.

The dependences of the geostrophic drag coefficient C_g and angle α_* on M and another external parameter Ro calculated from (1.22, 33) are shown in Fig. 1.4.

The sharp decrease of u_* with increase of M is quite visible. At the same location and with the same geostrophic wind, say, of $f = 10^{-4}\,\mathrm{s}^{-1}$, $z_{0u} = 0.1$ m, $U_g = 10\,\mathrm{m\,s}^{-1}$, i.e., Ro $= 10^6$, variations of M from 0 (evening neutral stratification) to 0.012 (distinctly stable stratification at night) are accompanied by a four-fold decrease of u_* from 0.4 to 0.1 m s^{-1}.

From the point of view of energetics, the main characteristic of the PBL is the total dissipation rate of the kinetic energy of the mean flow. Using (1.6) and the resistance law (1.22), we obtain

$$U_g^{-3} \int_0^h \varepsilon dz = C_g^2 \sqrt{1 - [A(\mu) C_g / k]^2}\ , \tag{1.35}$$

where the right-hand part, according to (1.22, 33, 34), can be represented as a function of the external parameters M and Ro. This function is shown in Fig. 1.5, from which it is seen that the dissipation rate decreases with increasing stability at an even greater rate than in the case of u_*. In the example discussed above, where M increases from 0 to 0.012, the value of $\int_0^h \varepsilon dz$ decreases twenty-fold from 1.4 to 0.07 (m/s)^3.

Thus, with stable stratification the dynamic interaction of the atmosphere with the underlying medium is very much weakened. It therefore seems to follow that in this case, there is no need to strive for accuracy in determining u_* and α_*. That assumption is true from the purely meteorological point of view which reduces the pole of the atmospheric PBL in the formation of the surface drag. And it is quite wrong from the viewpoint of the interaction of the atmosphere with a water reservoir. Actually, in the case of warm air advection, the friction

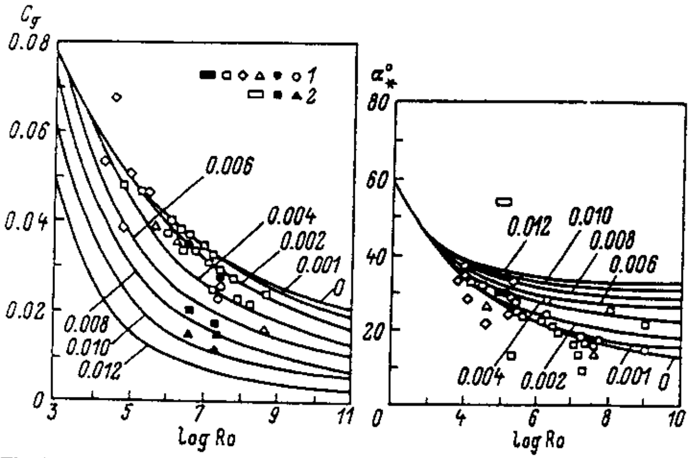

Fig. 1.4. Geostrophic drag coefficient $C_g = u_*/U_g$ and angle α_* of the wind turning in the PBL as functions of surface Rossby number Ro $= U_g/|f| z_{0u}$ for different values of the external stratification parameter $M = -k^2 B_s/|f| U^3$ (indicated at each curve), calculated from (1.22, 33, 34). The data points are from *Zilitinkevich* (1970): 1 – at neutral and 2 – at stable stratification

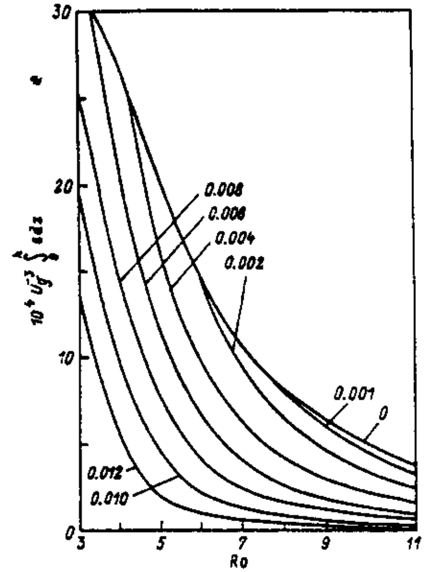

Fig. 1.5. Dimensionless dissipation rate of the kinetic energy of the mean flow in PBL $U_g^{-3} \int_0^h \varepsilon dz$ as a function of Ro at different M values calculated from (1.22, 33–35)

velocity u_* determines the depth of the upper mixed water layer, and hence the intensity of its warming, which increases sharply only at very small u_*. That is why the requirements for accuracy in determining u_* are not less, but on the contrary, more stringent.

Temperature Profile and Heat Transfer. According to the similarity theory (Zilitinkevich 1975a), in the case of a quasi-stationary PBL, the expression for the temperature defect

$$\theta_h - \theta(z) = \theta_* \phi_\theta(\zeta, \mu) \tag{1.36}$$

as well as for the heat transfer

$$(\theta_h - \theta_s)/\theta_* = \ln(u_*/|f| z_{0T}) - C(\mu) \tag{1.37}$$

are valid, where z is height, $\zeta = z/h$ is a dimensionless height, z_{0T} is the roughness parameter of the underlying surface with respect to temperature, θ is potential temperature, θ_h and θ_s are its values at the PBL upper boundary and at the underlying surface, respectively, $\theta_* = -Q_s/ku_*$ is the temperature scale, k is the von Karman constant[3], Q_s is the near-surface value of the vertical turbulent kinematic heat flux, ($Q_s = H_s/c_p\varrho_s$, where H_s is sensible heat flux), C is universal function of μ, and ϕ_θ is the universal function of ζ and μ.

From (1.10, 36, 37) an expression follows for the temperature profile:

$$\theta(z) = \theta_s + \theta_* \left[\ln(z/z_{0T}) + f_\theta(\zeta, \mu)\right]\ , \tag{1.38}$$

[3] According to some data, "the temperature von Karman constant" in the expression for θ_* is somewhat bigger than that for wind, 0.48.

where $f_\theta = -C - \ln(\zeta \Lambda) - \phi_\theta$ is a universal function of ζ and μ satisfying the conditions

$$f_\theta(0,\mu) = 0 , \quad f_\theta(1,\mu) = -C(\mu) - \ln \Lambda(\mu) . \tag{1.39}$$

Taking the first of these conditions, we approximate f_θ by a square polynomial of the following form:

$$f_\theta(\zeta,\mu) = c(\mu)\zeta + c^*(\mu)\zeta^2 . \tag{1.40}$$

Since density stratification is generated by heat exchange (or heat and moisture exchange) between the air flow and the underlying surface, it is natural to assume that when approaching the PBL upper boundary, the vertical gradient of potential temperature tends to zero:

$$d\theta/dz \to 0 \quad \text{at} \quad z \to h . \tag{1.41}$$

Observance of this condition is confirmed, e.g., by the data of the "Wangara" experiment: see Fig. 3 in *Clarke* (1974) or Fig. 1 in *Yamada* (1976). Substituting the expression for θ from (1.38) and (1.40) into (1.41), we have

$$c^*(\mu) = -\frac{1}{2}c(\mu) - \frac{1}{2} . \tag{1.42}$$

Now recall that under strongly stable stratification, the vertical temperature profile in the height interval $L \ll z \ll h$ should be expressed by the *Monin* and *Obukhov* (1954) logarithmic + linear law:

$$\theta(z) = \theta_s + \theta_*[\ln(z/z_{0T}) + \beta_\theta z/L] , \tag{1.43}$$

where β_θ is a dimensionless constant.

For (1.43) to be devided as an approximate result from (1.38) and (1.40) at $L \ll z \ll h$, the following relation should be satisfied: $c(\mu)z/h \sim \beta_\theta z/L$, i.e., at $\mu \gg 1$, the function $c(\mu)$ should have the following asymptotic behavior:

$$c(\mu) \sim C_h\beta_\theta\mu^{1/2} . \tag{1.44}$$

The simplest interpolation formula providing for a particular non-zero value of the function $c(\mu)$ with neutral stratification and the asymptote (1.44) for this function with strong stability has the form

$$c(\mu) = c_0 + C_h\beta_\theta\mu^{1/2} . \tag{1.45}$$

Formulas (1.38, 40, 42, 45) completely determine the vertical temperature profile and hence the function $C(\mu)$ in the heat transfer expression (1.37). According to (1.39, 40, 42, 45), the latter is expressed as

$$C(\mu) = \frac{1}{2} - \frac{1}{2}c(\mu) - \ln \Lambda(\mu) = C_0 + \ln\left(1 + \frac{\Lambda_0\mu^{1/2}}{kC_h}\right) - \frac{C_h\beta_\theta}{2}\mu^{1/2} , \tag{1.46}$$

where $C_0 = C(0)$ is a dimensionless constant related to c_0 by $C_0 = 1/2 - c_0/2 - \ln \Lambda_0$. We adopt the traditional values of the constants, $k = 0.4$, $\Lambda_0 = 0.3$, and the estimate $C_h = 0.85$ obtained above. There remains to determine C_0 (or c_0) and β_θ.

Equation (1.46) agrees with the asymptote $C(\mu) \sim -N_3\mu^{1/2}$ at $\mu \gg 1$ derived in *Zilitinkevich* (1975a). Resulting from the preliminary empirical estimate $N_3 = 6$, therefore, we would have $\beta_\theta \sim 2N_3/C_h = 14$. However, we shall use the more representative data using the function $C(\mu)$ found in *Yamada* (1976). These data embrace the interval of positive values of μ from zero to several hundreds and are approximated by Yamada by a particular empirical function. As seen in Fig. 1.6, within the rather great spread of values observed, equation (1.46) permits approximation of the same data with the following values of the constants in which we are interested:

$$C_0 = 3.7 , \quad \beta_\theta = 9 . \tag{1.47}$$

The recommended value of $C_0 = 3.7$ fully complies with the other available data on the function $C(\mu)$ at small μ [for instance, with the empirical graph from *Zilitinkevich* (1975a)], and corresponds to the value of $c_0 = -4$. The recommended value of $\beta_\theta = 9$ is within the range of estimates of this constant based on the verification of (1.43) on the basis of measurements in the near-surface air layer: $\beta_\theta = 11$ in *Zilitinkevich* (1970), $\beta_\theta = 9$ in *Panin* (1985) and $\beta_\theta = 4.7$ in *Tennekes* (1982).

According to (1.46, 47), the universal function $C(\mu)$ is

$$C(\mu) = 3.7 + \ln\left(1 + \frac{3\sqrt{\mu}}{3.4}\right) - 3.825\sqrt{\mu} . \tag{1.48}$$

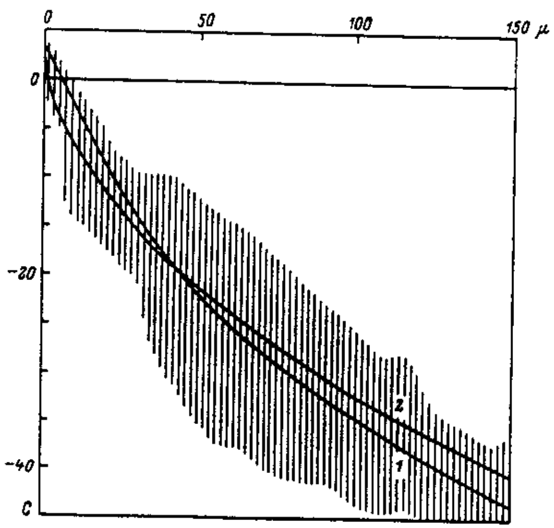

Fig. 1.6. Universal function in the heat transfer law for a stably stratified PBL. *The hatched region* is that most densely filled with empirical data points from *Yamada* (1976): *Curve 1* empirical (Yamada). *Curve 2* calculated from (1.48)

The heat transfer expression (1.37) and (1.48) combined with the equation for the resistance derived above, namely, with (1.22, 33), provides a convenient way for calculating the parameters of dynamic and thermal interaction between the atmosphere and the underlying medium in terms of external parameters such as the geostrophic wind velocity and temperature difference across the PBL. Approximation of the temperature profile by using (1.38, 40, 42, 45) can be applied in the construction of a parameterized model of the PBL diurnal cycle.