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"College on Atmospheric Boundary Layer and Air Pollution Modelling"
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Lecture 4

"Modelling and Parameterization of the Stably Stratified Boundary Layer"

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Please note: These notes are intended for internal distribution only.

1.2 The Entrainment Equation for Convection

Geophysical examples of penetrative convection are the growth of a mixed layer in the atmosphere as a result of warming of a land surface on sunny summer days, development of cumuli and stratus-cumuli, deepening of the upper mixed layer in the ocean or in a lake under the action of cooling of the water surface or its salinization, for instance, due to evaporation. In all these cases, increase of the depth h of the convectively mixed boundary layer takes place on a background of stable stratification in undisturbed fluid: temperature inversion in the atmosphere; thermo- and halocline in water bodies. In the process, the growth of h is strongly influenced by the transmission of kinetic energy to the outer boundary of the convective boundary layer which is expended to overcome the forces of negative buoyancy in the entrained fluid, i.e., for penetration of turbulence into the stably stratified layer.

The most reliable sources of data on penetrative convection are laboratory experiments. Most of these are devoted to the study of convection in water whose initial stratification is linear or two-layer. For simplicity, we shall consider these regimes.

We shall use the linearized equation of state for water:

$$\varrho_w = \varrho_{w0}[1 + a_s(s - s_0) - a_T(T - T_0)] , \tag{1.49}$$

where T is temperature, s is salinity, ϱ_w is density, ϱ_{w0} is its value at $T = T_0$ and $s = s_0$, a_s and a_T are expansion coefficients related to salinity and temperature, respectively. Let us introduce the buoyancy

$$b = g(\varrho_{w0} - \varrho_w)/\varrho_{w0} = \beta(T - T_0) - \alpha(s - s_0) , \tag{1.50}$$

where g is the acceleration due to gravity, $\beta = ga_T$ and $\alpha = ga_s$ are buoyancy parameters. We will assume one of two initial stratifications: two-layer

$$b = \{b_{s0} \quad \text{at} \quad 0 < z < h_0 , \quad b_{s0} + \Delta b_0 \quad \text{at} \quad h_0 \le z < D\} , \tag{1.51a}$$

linear

$$b = b_{s0} + N^2 z \quad \text{at} \quad 0 < z < D . \tag{1.51b}$$

Here z is the height above the vessel bottom, D is the total depth of the water layer in the vessel, b_{s0} is buoyancy at the bottom, h_0 and Δb_0 the initial height of the interface and the initial buoyancy increment in the two-layer system, N is the initial buoyancy frequency in the linear system: $N^2 = -g\varrho_{w0}^{-1}(d\varrho_w/dz)|_{t=0}$.

Heating water from below generates convection. Due to the convective motions, a well-mixed layer forms and grows in the lower part of the vessel. This layer appears to be homogeneous, or at least nearly homogeneous, so that temperature, salinity and buoyancy can be considered as constant with respect to the height, $T = \bar{T}$, $s = \bar{s}$, $b = \bar{b}$, except in a thin layer near the bottom, of a thickness $\delta_a \sim 1\,\text{cm}$, where the temperature drops sharply with height from a near-bottom value T_s down to a value near $\bar{T}$. In the case of two-layer initial stratification, convection first spreads very quickly through the lower homogeneous layer $0 < z < h_0$. Then entrainment begins involving the less dense water lying above h_0, which is accompanied by a slow increase of the thickness h of the mixed layer and decrease of the buoyancy increment Δb at its upper boundary ($z = h$). In addition, the buoyancy increment is gradually smeared out along the vertical direction, thus, instead of a stepped vertical profile, a smoothed one appears. This can be devided into four main parts: near-bottom, well-mixed, intermediate and non-turbulized layers. In the case when the initial stratification is linear, continuous growth of the mixed layer due to entrainment is followed by the appearence of a buoyancy increment at its outer boundary. Thus, the vertical buoyancy profile is devided into the four parts with the only difference that the non-turbulized layer is now not homogeneous but linearly stratified.

In both systems, between the mixed layer and non-turbulized layer one should observe a turbulent entrainment layer (TEL) where the process of entrainment of the fluid from above occurs by turbulence. This process generates a buoyancy flux opposite to the near-bottom one.

Parameterization of the Buoyancy Profile. Let us assume that with the development of convection, i.e., with the upward displacement of the mixed layer upper boundary $z = h$, the buoyancy profile changes, keeping a self-similarity of the following form:

$$b = \begin{cases} \bar{b} & \text{at} \quad z \le h - \delta/2 \\ \bar{b} + \delta b \phi\left(\dfrac{z - h + \delta/2}{\delta}\right) & \text{at} \quad h - \delta/2 \le z \le h + \delta/2 \\ b_{s0} + \Delta b_0 \text{ (a)} \quad \text{or} \quad b_{s0} + N^2 z \text{ (b)} & \text{at} \quad h + \delta/2 \le z \le D , \end{cases} \tag{1.52}$$

where δ is the depth of the thermo- or halocline, ϕ is a universal function (Kitaigorodskii, Miropolsky 1970; Linden 1975; Wyatt 1978) satisfying the conditions $\phi(0) = 0$ and $\phi(1) = 1$, δb is the buoyancy difference across the thermocline:

$$\delta b = \{b_{s0} + \Delta b - \bar{b} \text{ (a)} \quad \text{or} \quad b_{s0} + N^2(h + \delta/2) - \bar{b} \text{ (b)}\} .$$

The representation (1.52) is evidently inaccurate in a near-bottom layer of thickness $\delta_a \sim 1$ cm. However, if $\delta_a \ll h$, in deriving the total buoyancy budget equation for the whole layer $0 < z < h$ the contribution of the near-bottom part into integrals can be neglected.

Evolution of the profile (1.52) should satisfy the buoyancy transfer equation:

$$\partial b/\partial t = -\partial B/\partial z \, , \tag{1.53}$$

where t is time, $B = \beta Q - \alpha S$ is the vertical buoyancy flux, Q and S are the kinematic heat flux and the salinity flux.

We shall represent the buoyancy difference in the TEL by the formula $\delta b = \Delta b + N^2\delta/2$ where Δb is the difference between the value of the buoyancy which is obtained by linear extrapolation from the nondisturbed region down to the level $z = h$ and the mean value of buoyancy in the mixed layer (Fig. 1.7). Neglecting molecular heat and mass transfer (i.e., adopting $B = 0$ at $z > h + \delta/2$) and using the representation (1.52), termwise integration of (1.53) over z from 0 to D and from 0 to h, after proceeding in the obtained equations to the limit at $\delta \to 0$ gives the following total buoyancy budget equation:

$$\frac{d}{dt}\left(\frac{1}{2}N^2h^2 - h\Delta b\right) = B_s \, , \tag{1.54}$$

where $B_s = B|_{z=0}$ is the buoyancy flux through the bottom (the prescribed parameter), and the following expression for the vertical buoyancy flux through the outer boundary of the mixed layer:

$$B_h = B|_{z=h-0} = -\Delta b \, dh/dt \, . \tag{1.55}$$

Thus, a continuous approximation of the buoyancy profile is replaced by a discrete one:

$$b = \begin{cases} \bar{b} = \{b_{s0} + \Delta b_0 - \Delta b \text{ (a)} \quad \text{or} \quad b_{s0} + N^2h - \Delta b \text{ (b)}\} & \text{at} \quad z < h \, , \\ \{b_{s0} + \Delta b_0 \text{ (a)} \quad \text{or} \quad b_{s0} + N^2 z \text{ (b)}\} & \text{at} \quad z > h \, , \end{cases} \tag{1.56}$$

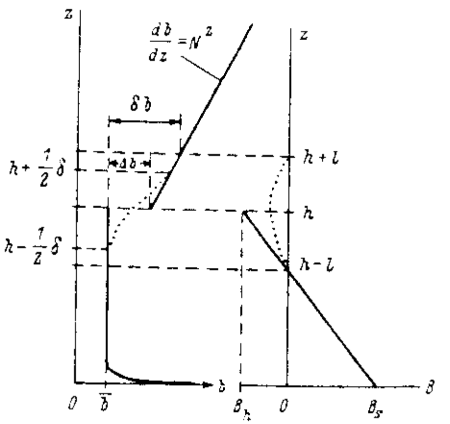

Fig. 1.7. Vertical profiles of buoyancy b and vertical buoyancy flux B with non-stationary penetrative convection. Real profiles are shown by *dotted curves*; *solid lines* correspond to parameterization via (1.56, 57)

and the buoyancy flux profile, in accordance with (1.53), takes the form

$$B = \begin{cases} (1 - z/h)B_s + (z/h)B_h & \text{at} \quad z < h \, , \\ 0 & \text{at} \quad z > h \, . \end{cases} \tag{1.57}$$

Since $B_s > 0$ and $B_h < 0$, it follows from (1.57) that the buoyancy flux changes sign at the level $z = h - l$, where l is expressed by an apparent "geometric" formula (Stull 1976b):

$$l/(h - l) = -B_h/B_s \, , \tag{1.58}$$

It is understood that the real profiles $b(z)$ and $B(z)$ are continuous. According to laboratory experiments (e.g. Deardorff et al. 1980), $B(z)$ is a linear function almost in the entire region where $B > 0$ and in the region where $B < 0$, $B(z)$ is practically symmetric relative to the extremum point $z = h$. Therefore, the region $h - l < z < h$ should be interpreted as a halved turbulent entrainment layer (TEL), the value of l, as a half-thickness of TEL, the value of B_h, as a conditional parameter characterizing our schematized $B(z)$ profile, viz., as the value of B obtained by linear extrapolation from the mixed layer up to the level $z = h$.

Thus, the approach based on the approximation of the buoyancy profile by a discontinuous function, i.e., on reducing the thermocline or halocline to the surface $z = h$, does not at all mean that the turbulent entrainment layer is also reduced to a surface. The TEL schematization consists only in that the real continuous $B(z)$ profile, negative in the region $h - l < z < h + l$ and having an extremum at $z = h$ is replaced by a discontinuous profile as in (1.57).

Simple Theoretical Models. The main parameter characterizing penetrative convection is the entrainment coefficient $A_E = -B_h/B_s$. According to (1.55) the following equation holds:

$$dh/dt = A_E B_s/\Delta b \, , \tag{1.59}$$

which at known A_E forms, together with (1.54), a closed system for determining h and Δb using the prescribed $B_s(t)$ and N. A number of theoretical models of turbulent penetrative convection can be formulated in terms of the parameter A_E. One of the acceptable hypotheses was suggested by *Ball* (1960), namely that $B_h = -B_s$, i.e. $A_E = 1$. Such a regime corresponds to the maximum possible degree of entrainment.

The other extreme case, corresponding to the minimum degree of entrainment, will be referred to as a *marginal convection* when the buoyancy change at the outer boundary of the mixed layer is not formed, so that $\Delta b = 0$ and $A_E = 0$, and hence, the growth of the layer depth is described by a reduced form of (1.54): $dh/dt = B_s/(N^2h)$. This regime was first discussed by *Zubov* (1945).

The intermediate hypothesis $A_E = C_1 < 1$ was suggested simultaneously by *Betts* (1973), *Carson* (1973) and *Tennekes* (1973a). The corresponding equation for the dimensionless entrainment rate $E = w_*^{-1} dh/dt$ can be written as

$$E = C_1 \mathrm{Ri}_1^{-1} , \tag{1.60}$$

where $\mathrm{Ri}_1 = h\Delta b/w_*^2$ is the Richardson number based on the buoyancy increment Δb and the convective velocity scale $w_* = (B_s h)^{1/3}$. Many papers, based on laboratory experiments and atmospheric or hydrospheric measurements, are devoted to determining the dimensionless constant C_1, i.e., to determining the parameter A_E. Various empirical estimates of A_E lying between 0 and 1 have been obtained, but most of them are around 0.2 (Stull 1976a).

The variation of A_E at small Ri_1 is explained by *Zilitinkevich* (1975b) where a nonstationary turbulence energy balance in the TEL was considered and the following corrected equation was derived:

$$E = C_1 (C_2 + \mathrm{Ri}_1)^{-1} , \tag{1.61}$$

At very small Ri_1 this expression becomes $E = C_1/C_2$ = const corresponding to penetration of the convective zone into a homogeneous fluid, and at big Ri_1 it is reduced to (1.60). In this model we have the following expression: $A_E = C_1 \mathrm{Ri}_1 (C_2 + \mathrm{Ri}_1)^{-1}$. Such behavior of A_E agrees with experimental data at small Ri_1, but not always at large Ri_1. According to some laboratory experiments (for instance Deardorff et al. 1969; Kantha 1979), A_E does not come to a plateau with increasing Ri_1, but after reaching a maximum, starts to decrease. In addition, in the experiments by *Turner* (1968) at very big Ri_1, the entrainment was found to follow $E \propto \mathrm{Ri}_1^{-3/2}$, contradicting both (1.60) and (1.61). Below a new theoretical model explaining these contradictions is suggested (Zilitinkevich 1987, 1991).

Entrainment Rate Equation. We shall use the energy balance equation of the turbulence kinetic energy e:

$$\frac{\partial e}{\partial t} = B - \frac{\partial}{\partial z}(F + P) - \varepsilon , \tag{1.62}$$

where F and P are vertical turbulent fluxes of kinetic energy and pressure fluctuations, respectively, ε is the viscous dissipation rate of kinetic energy. Then we integrate (1.62) over z from 0 to $h - 0$, i.e., over the entire depth of the mixed layer including the TEL. Considering that $F + P = 0$ at $z = 0$ (due to the apparent impossibility of turbulence transfer through the solid wall) and expressing the vertical buoyancy flux B by means of (1.55, 57), we obtain (Kitaigorodskii, Kozhelupova 1978):

$$\frac{d}{dt}\int_0^h e\,dz = \frac{h}{2}\left(B_s - \Delta b \frac{dh}{dt}\right) - (F + P)_h - \int_0^h \varepsilon\,dz . \tag{1.63}$$

According to similarity theory for turbulent boundary layers of time-dependent height (Zilitinkevich, Deardorff 1974) in the convection regime considered, dimensionless turbulence parameters, normalized by the scales of length h, velocity $w_* = (B_s h)^{1/3}$ and buoyancy $b_* = B_s/w_*$ (or temperature $T_* = Q_s/w_*$), appear to be universal functions of the dimensionless height z/h. This means that the vertical profiles of kinetic energy e and its dissipation rate ε are given by

$$e = w_* \phi_e(z/h) , \quad \varepsilon = B_s \phi_\varepsilon(z/h) , \tag{1.64}$$

where ϕ_e and ϕ_ε are universal functions. The correctness of these representations over most of the height of the convectively mixed layer is confirmed by laboratory experiments (Willis, Deardorff 1974; Deardorff, Willis 1985) and atmospheric measurements (Caughey 1982).

Escape of energy outside the limits of a convectively mixed layer is evidently due to a single mechanism, the radiation of gravitational internal waves into the adjoining stratified fluid layer. The maximum value of the vertical energy flux which can be caused by propagation of internal waves with amplitude l_i and length λ_i is expressed by (Thorpe 1973):

$$F_i = (3\pi\sqrt{3})^{-1} N^3 l_i^2 \lambda_i . \tag{1.65}$$

In this case internal waves are generated by disturbances in the TEL. That is why the typical wave amplitude l_i can be considered as being proportional to the typical amplitude of these disturbances which is equal to a half-depth of the TEL l, $l_i = C_l l$, where C_l = const.

Similarly it is natural to relate the characteristic length of radiated waves λ_i to the horizontal length of disturbances in the TEL $\lambda(h)$. At first glance, it is tempting to use for determining the latter an expression of the type of (1.64), $\lambda(z)/h = \phi_\lambda(z/h)$, from which follows $\lambda(h) \propto h$. However, this would be a mistake. The point is that such expressions are true in the major part of the convective mixed layer (and therefore are quite suitable for calculation of integrals), but not near $z = h$ where the *Zilitinkevich* and *Deardorff* (1974) similarity theory is not applicable. This breach of similarity in TEL is manifested in the increase of the range of the values of empirical estimates of the function $\phi_\lambda(z/h)$ as one approaches the $z = h$ level (Fig. 4.7 in Caughey 1982). The relation $\lambda(h) \propto h$ has also not been experimentally confirmed. Confirmation would have been achieved by the shape of an interface between turbulized and non-turbulized fluid on photographs of a convectively mixed layer (e.g. Deardorff et al. 1969). Judging by such visualization experiments, the relation of the horizontal dimensions of disturbances in TEL to the vertical ones is not very variable. This means that a half-depth l of TEL can serve as a scale of not only vertical but also of horizontal disturbances and hence, as a typical length of radiating internal waves: $\lambda_i = C_\lambda l$ where C_λ = const.

Substituting the derived expressions for l_i and λ_i into (1.65) for F_i and then substituting $(F + P)_h \propto F_i$ and (1.64) into (1.63), after identical transformations we have

$$(C_2 + \mathrm{Ri}_1)E + C_3 \mathrm{Ri}_2^{3/2}\left(\frac{\mathrm{Ri}_1 E}{1 + \mathrm{Ri}_1 E}\right)^3 = C_1 - \frac{2}{5} C_2 \mathrm{De} , \tag{1.66}$$

where $\mathrm{Ri}_2 = (Nh/w_*)^2/2$ is the Richardson number based on the buoyancy frequency, $\mathrm{De} = h(B_s w_*)^{-1} dB_s/dt$ is the nonstationarity parameter introduced in *Deardorff* et al. (1980) (we shall call it the Deardorff number), and C_1, C_2 and C_3 are dimensionless constants,

$$C_1 = 1 - 2\int_0^1 \phi_\epsilon(\zeta)d\zeta\,, \quad C_2 = \frac{10}{3}\int_0^1 \phi_\epsilon(\zeta)d\zeta\,, \quad C_3 \le \frac{2^{5/2}C_\epsilon C_\lambda}{3\pi\sqrt{3}}\,. \quad (1.67)$$

The sign $\le$ in the expression for C_3 reflects the fact that F_i is the maximum energy flux which can be provided by the internal waves propagation, so that $(F+P)_h \le F_i$.

Evaluation of Universal Constants and Asymptotic Forms of the Entrainment Equation. Using the data of the functions ϕ_ϵ and ϕ_e from the laboratory experiments of *Deardorff* and *Willis* (1985), we calculate the integrals for C_1 and C_2 which yields $C_1 = 0.2$ and $C_2 = 1.7$. The same estimates are obtained from the atmospheric data of *Caughey* (1982).

These direct estimates are in quite satisfactory agreement with indirect ones. For C_1 the agreement is ideal: $C_1 = 0.2$ is the commonly accepted value of the constant in the entrainment equations (1.60, 61) which are particular cases of (1.66); the same value was obtained by *Zilitinkevich* (1987) by analysis of the experimental results of *Deardorff* et al. (1980) from measurements of convective entrainment in linearly stratified and two-layer fluid systems (Figs. 1.8, 9).

For C_2, the direct estimate $C_2 = 1.7$ lies within the range of indirect ones. It is twice the experimental value of $C_2 = 0.83$ obtained by *Deardorff* et al. (1980) for convective entrainment in a homogeneous fluid as well as the *Zilitinkevich* (1987) value of $C_2 = 0.8$, based on an analysis of Deardorff data on a linearly stratified fluid (Fig. 1.8). However, it is very close to the *Driedonks* and *Tennekes* (1984) estimate of $C_2 = 1.5$, and 1.5 times *Tennekes*' (1975) earlier estimate $C_2 = 2.6$.

Assuming in the third formula of (1.67) that $C_\lambda \sim C_1 \sim 1$, we obtain the inequality $C_3 \lesssim 0.35$, which is in agreement with the estimate $C_3 = 0.1$ obtained from the experimental data of *Deardorff* and *Willis* (1980), Fig. 1.8.

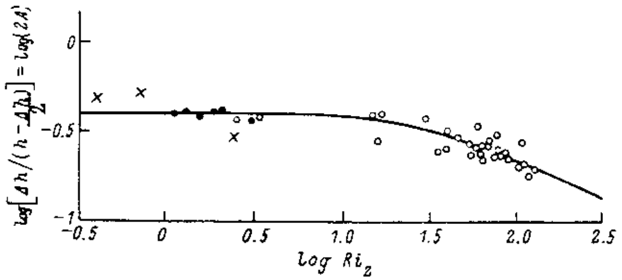

Fig. 1.8. Dependence of the TEL dimensionless depth $\Delta h/(h - \Delta h/2) = 2A_E$ on Richardson number $\mathrm{Ri}_2 = (Nh/w_*)^2/2$ based on the buoyancy frequency N. Data points are taken from the experiments of *Deardorff* et al. (1980). *Open circles* – linearly stratified fluid, *filled circles* – two-layer fluid, *crosses* – those of two-layer fluid when Ri_2 is not well defined. The *curve* is plotted by using (1.68) at $C_1 = 0.2$ and $C_3 = 0.1$

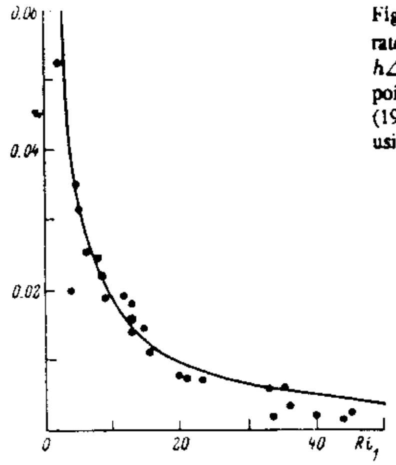

Fig. 1.9. Dependence of the dimensionless entrainment rate $E = w_*^{-1}dh/dt$ on the Richardson number $\mathrm{Ri}_1 = h\Delta b/w_*^2$ based on the buoyancy increment Δb. Data points are taken from the experiments of *Deardorff et al.* (1980) for a two-layer fluid system. The curve is calculated using (1.61) at $C_1 = 0.2$ and $C_2 = 0.8$

In the case of a two-layer fluid (when $N = 0$) at constant heating from below ($dB_s/dt = 0$, i.e. $\mathrm{De} = 0$), our entrainment equation (1.66) is reduced to the form of (1.61). The main objections to the latter are that at large Ri_1 (1.61) gives asymptotes $E \propto \mathrm{Ri}_1^{-1}$ and $A_E = -B_h/B_s = \mathrm{Ri}_1 E =$ const which are not always satisfied. Thus, in the experiments by *Turner* (1968) together with the foregoing expression $E \propto \mathrm{Ri}_1^{-1}$ the alternative $E \propto \mathrm{Ri}_1^{-3/2}$ was obtained; in many other experiments decrease of A_E with increase of Ri_1 was observed. At the same time, at Ri_1 values of the order of tens, typical for atmospheric and oceanic convectively mixed layers, calculations of h by means of (1.61) or even (1.60) as well as the relation $A_E =$ const are in fairly good agreement with the experimental data.

All these facts can be explained within the proposed model. For atmospheric and oceanic convectively mixed layers the values of $\mathrm{De} \sim 10^{-2}$ and $\mathrm{Ri}_1 \sim 1$ are rather typical, so (1.66) is reduced to (1.61), and if $\mathrm{Ri}_1 \gg C_2$, then it is reduced to (1.60). In the case of laboratory experiments with very well defined initial stratification, when Ri_1 and Ri_2 are big, under the condition $\mathrm{De} \ll 1$, (1.66) written in terms of $A_E = \mathrm{Ri}_1 E$ becomes

$$A_E + C_3\mathrm{Ri}_2^{3/2}\left(\frac{A_E}{1+A_E}\right)^3 = C_1\,, \quad (1.68)$$

whence at $\mathrm{Ri}_2 \gg 1$ there follow asymptotic formulas:

$$A_E = (C_1/C_2)^{1/3}\mathrm{Ri}_2^{-1/2}\,, \quad E = (C_1/C_2)^{1/3}\mathrm{Ri}_1^{-1}\mathrm{Ri}_2^{-1/2}\,. \quad (1.69)$$

The expression for A_E, due to the self-evident correlation between Ri_1 and Ri_2, explains the observed decrease of A_E with increase of Ri_1 in penetrative convection experiments. The expression for E gives exactly the Turner equation $E \propto \mathrm{Ri}_1^{-3/2}$. Indeed in Turner's experiments mixing was induced by oscillations of a grid immersed into water instead of by convection so that the following conditions were observed: $B_s = 0$ and, according to (1.54), $\Delta b = N^2 h/2$, i.e., $\mathrm{Ri}_1 = \mathrm{Ri}_2$.

Experimental Verification. We shall use the results of the laboratory experiments of *Deardorff* et al. (1980). Unfortunately, the data from this paper allow one to calculate Ri_1 only to low accuracy. Therefore it is not feasible to determine the dependence of E on Ri_1, Ri_2 and De by using these data, i.e., to verify (1.66) directly. However, because in most of the experiments under discussion $\mathrm{Ri}_1 \gg 1$ and $\mathrm{De} \ll 1$, according to the suggested theory, the entrainment equation should have the form of (1.68), i.e., the entrainment coefficient A_E should be expressed by a universal function of Ri_2 only. The determination of such a function using empirical data is quite feasible. Moreover, since in the experiments under consideration the TEL depth Δh is very precisely known, we can use its value to calculate A_E by using an expression equivalent to (1.58), i.e., $\Delta h/(h - \Delta h/2) = 2A_E$. This allows us to avoid the use of $A_E = \mathrm{Ri}_1 E$ which is undesirable due to the above-mentioned large errors in the determination of Ri_1.

The results of this analysis, which verifies the entrainment equation (1.68) are shown in Fig. 1.8. The data of two experiments in a linearly stratified fluid (E1, E3, E4, E6–E11 and E14) and in a two-layer fluid (runs 2, 3 and 4 in experiments E2 and E5, run 4 in experiment E15, runs 2 and 3 in experiment E16) have been used.

The data points in Fig. 1.8 have a much lesser spread than on the original plot of *Deardorff* et al. (1980) where the dependence of $\Delta h/(h - \Delta h/2)$ on the Richardson number $\mathrm{Ri}_* = h\delta b/w_*^2$ is presented. In addition, the data points in Fig. 1.8 are approximated well by the theoretical curve corresponding to (1.68) at $C_1 = 0.2$ and $C_3 = 0.1$. The remaining scatter in the points of Fig. 1.8 is explained by measurement errors and the dependence of A_E on Ri_1: quite a few points correspond to the range of $\mathrm{Ri}_1 < 10$, when this dependence is considerable.

As was already mentioned, for the case of a two-layer fluid Ri_2 goes to zero, so at $\mathrm{De} \ll 1$ the entrainment equation is reduced to the form of (1.61). The verification of this equation using the data of *Deardorff* et al. (1980) is shown in Fig. 1.9. In this analysis the data of experiments E2, E5, E12, E13 (with the exception of the first run from which it was impossible to determine Ri_1), E15, E16, and experiment E11 for which Ri_2 is very small are used. The theoretical curve determined using (1.61) at $C_1 = 0.2$ and $C_2 = 0.8$ in the region $0 < \mathrm{Ri}_1 < 30$ agrees quite well with the data points. We would like to emphasize that the value of $C_1 = 0.2$ coincides with the value of this constant derived from the experiments in a linearly stratified fluid shown in Fig. 1.8, and the value of $C_2 = 0.8$ almost coincides with the estimate of $C_2 = C_1/0.24 = 0.83$ derived from the comparison of our asymptote $E = C_1/C_2$ with the empirical result $E = 0.24$ obtained in penetrative convection experiments of *Deardorff* et al. (1980) in a homogeneous fluid.

Thus the expression for the entrainment in a linearly stratified fluid (1.68) agrees with the experimental data in Fig. 1.8, and the equation for a two-layer fluid (1.61) agrees with the data in Fig. 1.9. The free entrainment relation $E = C_1/C_2$ following from (1.61) agrees with the empirical value of $E = 0.24$, the final, most appropriate numbers being $C_1 = 0.2$, $C_2 = 0.8$, $C_3 = 0.1$.