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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

UNITED NATIONS INDUSTRIAL DEVELOPMENT ORGANIZATION

INTERNATIONAL CENTRE FOR SCIENCE AND HIGH TECHNOLOGY
c/o INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS 34100 TRIESTE (ITALY) VIA GRIGNANO, 9 (ADRIATICO PALACE) P.O. BOX 586 TELEPHONE 040-224572 TELEFAX 040-224575 TELEX 460449 APH I

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"Momentum and Energy Budgets for Atmospheric Boundary Layers"

A.P. VAN ULDEN
Royal Netherlands Meteorological Institute
de Bilt, The Netherlands

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MOMENTUM AND ENERGY BUDGETS FOR ATMOSPHERIC BOUNDARY LAYERS

A.P. van Ulden
KNMI

1. INTRODUCTION

In this note we consider horizontally homogeneous atmospheric boundary layers in steady state. From the vertically integrated equations for momentum and kinetic energy of the mean flow and for turbulence kinetic energy we deduce a number of important characteristics of boundary layers. Many of these characteristics are also relevant for oceanic boundary layers.

Horizontal homogeneity is assumed in the sense that horizontal gradients of mean dynamic variables vanish except for the pressure gradient terms. Moreover, it is assumed that horizontal pressure gradients do not vary with height and that the mean flow is hydrostatic.

2. MOMENTUM BUDGET

Since

$$\partial\bar{w}/\partial z = -\partial\bar{u}/\partial x - \partial\bar{v}/\partial y = 0 \qquad (1)$$

it follows that

$$\bar{w} = \bar{w}_{surface} = 0\,. \qquad (2)$$

Thus there are two momentum equations:

$$\frac{\partial\bar{u}}{\partial t} = 0 = -\frac{1}{\rho_o}\frac{\partial P_o}{\partial x} + f\bar{v} - \frac{\partial\overline{u'w'}}{\partial z}\,, \qquad (3a)$$

$$\frac{\partial\bar{v}}{\partial t} = 0 = -\frac{1}{\rho_o}\frac{\partial P_o}{\partial y} - f\bar{u} - \frac{\partial\overline{v'w'}}{\partial z}\,. \qquad (3b)$$

First we consider laminar flow for which $\overline{u'w'} = \overline{v'w'} = 0$.
For such flow (3a), (3b) yield:

$$f\bar{v} = \frac{1}{\rho_o}\frac{\partial P_o}{\partial x} \equiv f v_g \, , \tag{4a}$$

and

$$f\bar{u} = \frac{-1}{\rho_o}\frac{\partial P_o}{\partial y} \equiv f u_g \, . \tag{4.b}$$

These equations define the two components u_g, v_g of the geostrophic wind. Thus horizontally homogeneous steady flow without friction is geostrophic. There is a balance between the pressure force and the Coriolis force. The wind blows parallel with isobars.

Next we consider a boundary layer with depth h. For convenience we choose the x-axis parallel with the shear-stress at the surface. Then the surface boundary conditions for the shear stress are:

$$-\left(\overline{u'w'}\right)_o \equiv u_*^2 \, , \qquad -\left(\overline{v'w'}\right)_o = 0 \qquad \text{at } z = z_o \, , \tag{5a}$$

where u_* is the friction velocity and z_o the surface roughness length. At the top of the boundary layer both stress components vanish. Thus

$$-\left(\overline{u'w'}\right)_h = -\left(\overline{v'w'}\right)_h = 0 \qquad \text{at } z = h \, . \tag{5b}$$

With these boundary conditions we integrate the momentum equations over the boundary layer depth and obtain directly:

$$f\bar{\bar{v}} = \frac{1}{\rho_o}\frac{\partial P_o}{\partial x} + \frac{u_*^2}{h} \equiv f v_g + f\bar{\bar{v}}_a \tag{6a}$$

$$f\bar{\bar{u}} = -\frac{1}{\rho_o}\frac{\partial P_o}{\partial y} = f u_g \, . \tag{6b}$$

Here $\bar{\bar{u}}$ and $\bar{\bar{v}}$ are the vertically averaged components of the mean wind and $\bar{\bar{v}}_a$ the vertically averaged ageostrophic wind which is defined as:

$$\bar{\bar{v}}_a \equiv u_*^2 / fh \, . \tag{7}$$

It thus follows, that - in this coordinate system - the average $\bar{u}$ component is in geostrophic balance, while the average $\bar{v}$ component shows a balance between the vertically averaged pressure acceleration, Coriolis acceleration and the acceleration by the mean vertical stress gradient. Thus $\bar{\bar{v}}$ has both a geostropic and an ageostrophic component. In figure 1 we illustrate this in a vector diagram. Note that $\bar{\bar{v}}$ and v_g are negative in this coordinate system. In figure 2 we show typical profiles of $\bar{u}$ and $\bar{v}$.

3. KINETIC ENERGY OF MEAN FLOW

The kinetic energy equation for the mean flow is easily obtained by multiplying (3a) and (3b) with $\bar{u}$ and $\bar{v}$ respectively. This yields:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}^2 + \frac{1}{2}\bar{v}^2\right) = 0 = -\frac{\bar{u}}{\rho_o}\frac{\partial P_o}{\partial x} - \frac{\bar{v}}{\rho_o}\frac{\partial P_o}{\partial y} - \bar{u}\frac{\partial \overline{u'w'}}{\partial z} - \bar{v}\frac{\partial \overline{v'w'}}{\partial z} . \tag{8}$$

In this equation the first two terms on the right hand side give the production of mean kinetic energy and the last two terms energy losses due to the interaction between the mean flow and the vertical stress gradients. These losses are connected with the production of turbulence kinetic energy, as we will discuss later. Note that the mean kinetic energy equation does not contain terms related to the Coriolis acceleration. The reason for this is that the Coriolis force is always at a right angle with the wind vector and therefore cannot produce or destroy energy.

Next we average (8) over the boundary layer depth. This gives:

$$0 = -\frac{\bar{\bar{u}}}{\rho_o}\frac{\partial P_o}{\partial x} - \frac{\bar{\bar{v}}}{\rho_o}\frac{\partial P_o}{\partial y} - \frac{1}{h}\int_{z_o}^{h}\left(-\overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)dz . \tag{9}$$

Note that we have used that $-\bar{u}\,\partial\overline{u'w'}/\partial z = -\partial\bar{u}\,\overline{u'w'}/\partial z + \overline{u'w'}\,\partial\bar{u}/\partial z$ and that $\bar{u}\,\overline{u'w'}$ vanishes both at the surface and at the top of the boundary layer. The v component is treated in a similar manner. The last term in (8) is the vertically averaged production of turbulence by the interaction between the shear stress and vertical gradients of the mean wind (see also next section). The first two terms give the production of mean kinetic energy. We see that only wind components in the direction of the pressure gradient produce kinetic energy. Since by definition these wind components are the ageostrophic components of the wind, it follows that only the ageostrophic wind is involved in the production of kinetic energy. We can clarify this by choosing the coordinate system as in the previous section, thus with the x-axis in the direction of the surface stress. Using (6a) and (6b) we find that in this coordinate system:

$$-\frac{\bar{\bar{u}}}{\rho_o}\frac{\partial P_o}{\partial x} - \frac{\bar{\bar{v}}}{\rho_o}\frac{\partial P_o}{\partial y} = -\frac{\bar{\bar{v}}_a}{\rho_o}\frac{\partial P_o}{\partial y} . \tag{10}$$

As expected, the geostrophic components vanish. With help of (6b) and (7) we may also write this as:

$$-\frac{\bar{\bar{v}}_a}{\rho_o}\frac{\partial P_o}{\partial y} = \frac{u_*^2}{h}\bar{\bar{u}} . \tag{11}$$

The right hand side of this interesting result can be interpreted as follows. It gives the work done by the mean wind against the vertically averaged stress gradient, which is equal to the work done by the stress on a characteristic vertical velocity gradient. Thsu the right hand side of (11) gives the production of turbulence kinetic energy. Thus (11) simply states that the production of mean kinetic energy by the pressure gradient and the ageostrophic wind equals the production of turbulence kinetic energy by the surface stress and the mean wind in the direction of the surface stress.

4. TURBULENCE KINETIC ENERGY

4.1 Introduction

The turbulence kinetic energy equation reads (e.g. Stull, 5.1a)

$$\frac{\partial \bar{e}}{\partial t} = 0 = -\overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z} + \frac{g}{\theta_v}\overline{w'\theta_v'} - \frac{\partial \overline{w'e'}}{\partial z} - \frac{1}{\rho_o}\frac{\partial \overline{w'p'}}{\partial z} - \varepsilon . \tag{12}$$

Here

$$\bar{e} = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) ,$$

is the mean turbulent kinetic energy, while e' denotes the fluctuations from this mean value. ε represents the viscous dissipation and θ_v the virtual potential temperature. The vertical average of (12) is:

$$0 = \frac{1}{h}\int_{z_o}^{h}\left(-\overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z}\right)dz + \frac{1}{h}\int_{z_o}^{h}\frac{g}{\theta_v}\overline{w'\theta_v'}\,dz - \frac{1}{h}\int_{z_o}^{h}\varepsilon\,dz . \tag{13}$$

Here the first term at the right hand side gives the shear production (SP), the second term the buoyant production (BP), which can be negative, and the last term the average dissipation (D) which is always a loss term.

The average shear production has been evaluated in the previous section. It is given by

$$SP = \frac{u_*^2 \bar{\bar{u}}}{h} . \tag{14}$$

The other two terms will be modelled in the following for neutral, stable and unstable boundary layers. It should be noted that (14) does not depend on the stability of the boundary layer.

4.2 The neutral boundary layer

First we consider the neutral boundary layer in which by definition the buoyant production is absent. We thus have:

$$SP_n = D_n \tag{15}$$

where the subscript stands for neutral. The dissipation is modelled by using the concept of the energy cascade. Thus the local dissipation at the height z is taken equal to the energy cascade rate at this height:

$$\varepsilon(z) \cong \frac{\sigma_w^3}{l_m}, \qquad (16)$$

where

$$\sigma_w^2(z) = \left(\overline{w'w'}\right)_z$$

is the variance at the height z and l_m the length scale of the largest eddies at this height. Since $l_m \propto z$ close to the ground and $l_m \propto h\text{-}z$ close to the boundary layer height we model l_m as:

$$\frac{1}{l_m} = \frac{1}{z} + \frac{1}{h-z}. \qquad (17)$$

This is equivalent to:

$$l_m = z\left(1 - \frac{z}{h}\right). \qquad (18)$$

Near the surface, we may use surface layer similarity theory to evaluate (16). For a coordinate system with the x-axis parallel with the surface stress, the surface layer energy budget reads:

$$-\overline{u'w'}\,\frac{\partial \bar{u}}{\partial z} = \varepsilon .$$

Using that close to the ground $-\overline{u'w'} \cong u_*^2$ and $\partial\bar{u}/\partial z \cong u_*/kz$, we find that in a neutral surface-layer

$$\varepsilon \sim \frac{u_*^3}{kz} \qquad \text{for } h >> z . \qquad (19)$$

In the boundary layer σ_w decreases with height, until it vanishes at z = h. Observations show that approximately

$$\sigma_w^3 \sim \frac{u_*^3}{k}\left(1 - \frac{z}{h}\right)^p, \qquad (20)$$

where $p \cong 2$ is an empirical coefficient. Using this and (18) we find that

$$D_n \cong \frac{1}{h}\int_{z_o}^{h} \frac{u_*^3}{kz}\left(1-\frac{z}{h}\right)^{p-1} dz \cong \frac{u_*^3}{kh}\left(\ln\frac{h}{z_o} - A\right), \tag{21}$$

where $A \cong 1$ is an empirical constant. From (14), (21) and (6b) we now directly obtain:

$$\bar{\bar{u}} = \frac{u_*}{k}\left(\ln\frac{h}{z_o} - A\right) = u_g . \tag{22}$$

This is one form of the well known geostrophic drag law (Stull, 9.8 a).

4.3 The stable boundary layer

In the stable boundary layer the virtial potential temperature increases with height and vertical motions are affected by buoyant decellerations. This reduces the size of the largest eddies in comparison with the neutral case. This effect is usually modelled with the following empirical relation:

$$\frac{1}{l_m} \cong \frac{1}{z} + \frac{1}{l_b} . \tag{23}$$

Here l_b is the buoyancy length scale, which is the vertical distance over which air parcels can be displaced by eddies with scale l_b and energy $\sigma_w{}^2$.

The buoyancy length scale can be computed as follows:
An air parcel which is displaced upward over a distance l_b has a difference in virtual potential temperature

$$\Delta\theta_v \sim -l_b \frac{\partial\bar{\theta}_v}{\partial z}, \tag{24}$$

with respect to its environment. This corresponds with a potential energy difference

$$\Delta\, PE \sim -\frac{1}{2}\frac{g}{\theta_v}\Delta\theta_v\, l_b . \tag{25}$$

This potential energy is created at the cost of the turbulent kinetic energy of the transporting eddy:

$$\Delta\, TKE \sim \frac{1}{2}\sigma_w^2 . \tag{26}$$

Taking Δ PE = Δ TKE gives:

$$\sigma_w^2 = -\frac{g}{\theta_v}\Delta\theta_v\, l_b , \tag{27}$$

and

$$\frac{\sigma_w^3}{l_b} = -\frac{g}{\theta_v}\Delta\theta_v\,\sigma_w \sim -c_b\frac{g}{\theta_v}\overline{\theta_v'w'}\ , \tag{28}$$

where c_b is an empirical constant. Using (16), (20), (23) and (28) we obtain our model for the dissipation in stable conditions:

$$\varepsilon(z) = \frac{u_*^3}{kz}\left(1-\frac{z}{h}\right)^2 - c_b\frac{g}{\theta_v}\overline{w'\theta_v'}\ . \tag{29}$$

We see that in comparison with the neutral case (21) the dissipation is enhanced since $\overline{w'\theta_v'}$ is negative. To estimate the empirical constant c_b we use surface layer similarity. For the stable case surface layer similarity tells us that:

$$0 = \frac{u_*^3}{kz}\left(1+5\frac{z}{L}\right) + \frac{g}{\theta_v}\overline{w'\theta_v'} - \varepsilon\ , \tag{30}$$

where L is the Obukhov length. (30) applies when $z \ll h$. Since

$$L = -u_*^3 / \frac{g}{\theta_v}\overline{w'\theta_v'}\ . \tag{31}$$

We may rewrite (30) as

$$\varepsilon = \frac{u_*^3}{kz} - 4\frac{g}{\theta_v}\overline{w'\theta_v'}\ . \tag{32}$$

Equating (29) and (32) for $z \ll h$ we see that

$$c_b \cong 4\ . \tag{33}$$

We are now ready for our boundary layer budget. Experience learns that generally $\overline{w'\theta_v'}$ varies smoothly with height:

$$\overline{w'\theta_v'} \cong \left(\overline{w'\theta_v'}\right)_o\left(1-\frac{z}{h}\right)^q\ , \tag{34}$$

where $q \cong 1$ is an empirical coefficient. Using (14), (29) and (34) we obtain now for our energy budget:

$$SP_S = u_*^2 \bar{\bar{u}}$$

$$BP_S = B \frac{g}{\bar{\bar{\theta}}_v} \left(\overline{w'\theta_v'} \right)_o \qquad \text{with } B \sim \frac{1}{2}$$

$$D_S = \frac{u_*^3}{kh} \left(\ln \frac{h}{z_o} - A \right) - B\, c_b \frac{g}{\bar{\bar{\theta}}_v} \left(\overline{w'\theta_v'} \right)_o .$$

Using

$$0 = SP_S + BP_S - D_S$$

we find

$$\bar{\bar{u}} = \frac{u_*}{k} \left(\ln \frac{h}{z_o} - A \right) + B \left(c_b + 1 \right) \frac{g}{\bar{\bar{\theta}}_v} \left(\overline{w'\theta_v'} \right)_o . \tag{35}$$

This is the geostrophic drag law for stable boundary layers derived e.g. by Nieuwstadt (1984).

4.4. The unstable boundary layer

We consider an unstable boundary layer capped by a temperature inversion near $z = h$. We consider the special case with no mean wind. Thus $SP = 0$. All turbulence is produced by buoyancy: $BP > 0$. Again we approximate $\overline{w'\theta_v'}$ by a smooth profile:

$$\overline{w'\theta_v'} = \left(\overline{w'\theta_v'} \right)_o \left(1 - \frac{z}{h} \right)^q , \tag{36}$$

with $q \cong 1$. This is a good assumption except for a limited region near the top of the boundary layer, where $\overline{w'\theta_v'}$ usually is slightly negative. Thus we get

$$BP \cong B \frac{g}{\bar{\bar{\theta}}_v} \left(\overline{w'\theta_v'} \right)_o \tag{37}$$

with $B \sim 1/2$.

Turbulence is produced in the bulk of the boundary layer by large eddies, with the greatest production near the ground. At the same time the produced turbulent energy is transported away from the ground such that the resulting σ_w profile is given by:

$$\sigma_w^3 \cong w_*^3 \frac{z}{h}\left(1-\frac{z}{h}\right)^r . \tag{38}$$

where w_* is a velocity scale for convective turbulence and $r \cong 2$ an empirical coefficient.

The length scale of turbulence is limited both by the distance from the ground and by the distance to the capping inversion; thus like in the neutral case we use:

$$\frac{1}{l_m} \sim \frac{1}{z} + \frac{1}{h-z} \Rightarrow l_m \sim z\left(1-\frac{z}{h}\right) .$$

For the dissipation we thus obtain:

$$D_c \sim \frac{1}{h}\int_o^h \frac{\sigma_w^3}{l_m} dz \sim \frac{w_*^3}{h}\int_o^1 \left(1-\frac{z}{h}\right)^{r-1} d\left(\frac{z}{h}\right) . \tag{39}$$

Thus

$$D_c \sim \frac{1}{2}\frac{w_*^3}{h} , \tag{40}$$

where the subscript c stands for convective. Using

$$BP_c = D_c , \tag{41}$$

we find that

$$w_* \sim \left(\frac{g}{\theta_v}\left(\overline{w'\theta_v'}\right)_o h\right)^{1/3} . \tag{42}$$

which is the well-known velocity scale for convective turbulence (Stull, 4.2a).

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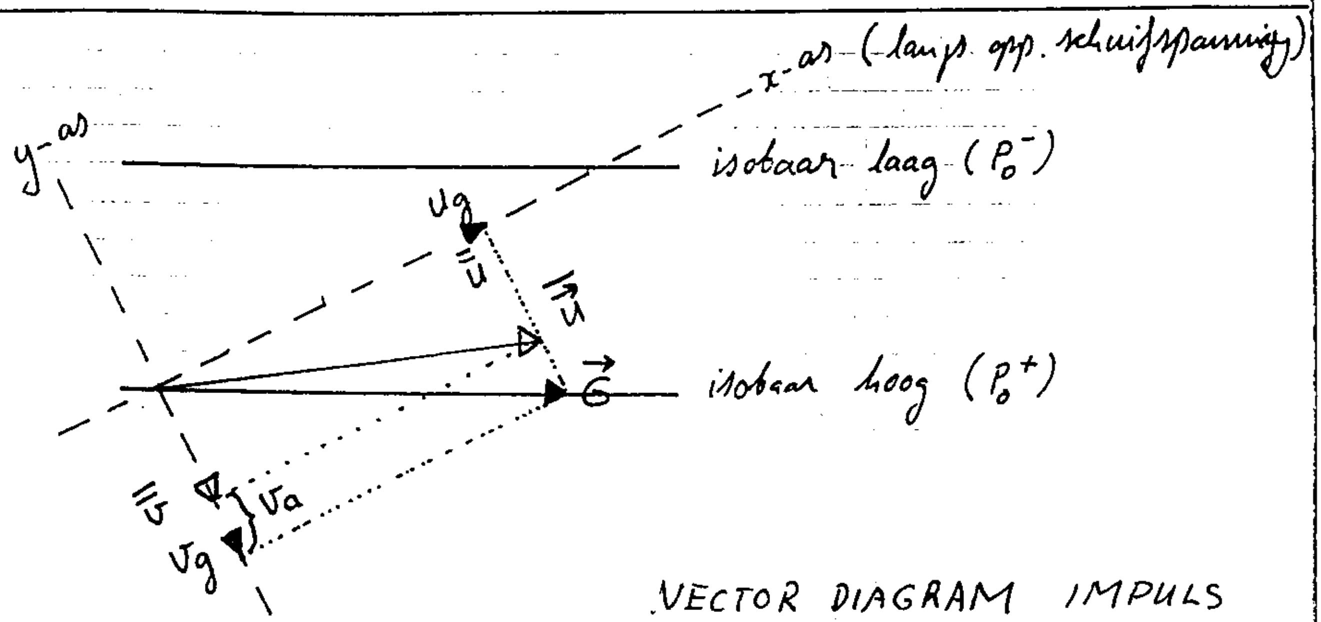


VECTOR DIAGRAM IMPULS ~~MOMENTUM~~ BUDGET (Noord. Halfrond)

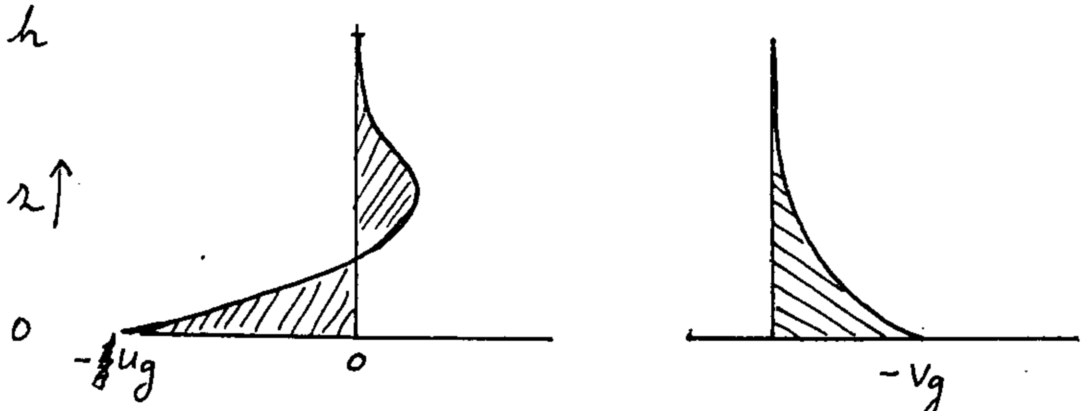
fig 1 : $P_o^+ > P_o^-$

$\vec{G}$: geowindvector

$\vec{\bar{u}}$: gemiddelde windvector.

$\bar{\bar{u}} = u_g$

$|\bar{\bar{v}}| = |v_g| - |v_a|$ N.B. $\bar{\bar{v}}$ negatief, V_g negatief. V_a positief.



$f(\bar{u} - u_g) \rightarrow$

gearceerde ~~vlak~~ opp. zijn gelijk

$f(\bar{v} - V_g) \rightarrow$

gearceerd opp. gelijk aan $\frac{u_*^2}{f}$

fig 2 SCHEMATISCHE WINDPROFIELEN IN GRENSLAAG