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**Workshop on Commutative Algebra
and its Relation to
Combinatorics and Computer Algebra**

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Toric ideals

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These are preliminary lecture notes, intended only for distribution to participants

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Trieste, May '94 B. Sturmfels
Lecture #1

Toric Ideals and Regular Triangulations

(a) Gröbner Basics:

If $I \subset k[x_1, \dots, x_n]$ is a homogeneous ideal and $\omega \in \mathbb{Q}^n$ then $in_\omega(I) = \langle in_\omega(f) : f \in I \rangle$ is the <u>initial ideal</u>. For almost all ω this is a monomial ideal, in which case we say that ω is a <u>term order</u> for I. The <u>initial complex</u> $\Delta_\omega(I)$ is the simplicial complex on $\{x_1, \dots, x_n\}$ defined by $Rad(in_\omega(I))$.

<u>Example</u>: Ideal of Twisted Cubic Curve: $I = I_2 \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$ $(n=4)$

$\omega = (0,2,0,1)$ is a term order, $in_\omega(I) = \langle x_2^2, x_2x_3, x_2x_4, x_1x_4^2 \rangle$

$Rad(in_\omega(I)) = \langle x_2, x_1x_4 \rangle = \langle x_2, x_1 \rangle \cap \langle x_2, x_4 \rangle$, $\Delta_\omega(I) =$ 1 — 3 — 4

Equivalence relation on $\mathbb{Q}^n$: $\omega \sim \omega' :\Leftrightarrow in_\omega(I) = in_{\omega'}(I)$

Classes are convex polyhedral cones, defining the <u>Gröbner fan</u> of I.

(b) Toric Ideals:

$:\Leftrightarrow$ presentation ideal I of a semigroup algebra $k[S] = k[\underline{x}]/I$

$S = \mathbb{N}\mathcal{A}$, $\mathcal{A} = \{a_1, \dots, a_n\}$, $a_i \in \mathbb{Z}^d$.

<u>Assumption</u>: S graded $\Leftrightarrow$ I homog. $\Leftrightarrow \exists\, \varphi \in (\mathbb{Q}^n)^*$: $\varphi(a_1) = \dots = \varphi(a_n) = 1$

Corresponding convex polytope $Q = conv(\mathcal{A})$ has $\dim \leq d-1$.

<u>Proposition</u>: If $X_\mathcal{A} = Proj(k[S]) \subset \mathbb{P}^{n-1}$ then $\dim(X_\mathcal{A}) = \dim(Q)$ and $degree(X_\mathcal{A}) = Vol(Q)$. "normalized volume"

<u>Example</u>: $\mathcal{A} = \{(3,0), (2,1), (1,2), (0,3)\}$ 1 — 2 — 3 — 4 Q

$\dim(X_\mathcal{A}) = 1$, $degree(X_\mathcal{A}) = 3$

$$\sum \lambda_i = \sum \mu_j$$

<u>Lemma</u>: $I_\mathcal{A} = \left\langle \prod_{i=1}^{n} x_i^{\lambda_i} - \prod_{j=1}^{n} x_j^{\mu_j} \; : \; \lambda_i, \mu_j \in \mathbb{N}, \; \sum \lambda_i a_i = \sum \mu_j a_j \right\rangle$

<u>Proof</u>: "$\supseteq$" is clear. In fact, RHS is the largest binomial ideal contained in $I_\mathcal{A}$. For "$\subseteq$" consider $J_\mathcal{A} = \langle x_i - \underline{t}^{a_i} : i=1,\dots,n \rangle \subset k[\underline{x}, \underline{t}^{\pm 1}]$, observe $I_\mathcal{A} = J_\mathcal{A} \cap k[\underline{x}]$, and use that the Buchberger algorithm preserves binomiality to conclude that $I_\mathcal{A}$ is binomial. ■

(c) Regular Triangulations

A <u>subdivision</u> of $(Q, \mathcal{A})$ is a collection Δ of subsets of $\mathcal{A}$ such that $\{ \operatorname{conv}(T) : T \in \Delta \}$ is a polyhedral complex whose union equals Q. It is called a <u>triangulation</u> if $\dim(\operatorname{conv}(T)) = \#(T) - 1$ for all $T \in \Delta$.

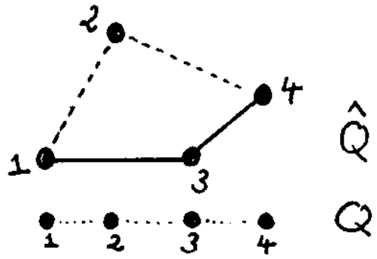
Every $\omega \in \mathbb{Q}^n$ defines a subdivision Δ^ω as follows:

$$T \in \Delta^\omega \Leftrightarrow \exists \, \ell \in (\mathbb{Q}^d)^*, c \in \mathbb{Q} : \begin{array}{ll} c + \ell(a_i) = \omega_i & \text{if } a_i \in T \\ c + \ell(a_j) < \omega_j & \text{if } a_j \notin T \end{array} \qquad (*)$$

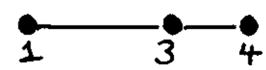
Geometrically: Δ^ω is the lower envelope of

$$\hat{Q} = \operatorname{conv}\{ (a_i, \omega_i) \in \mathbb{Q}^{d+1} : i = 1, \dots, n \}$$

<u>Example</u>



$$\Delta^\omega = \{ \{1,3\}, \{3,4\}, \{1\}, \{3\}, \{4\}, \emptyset \}$$



Subdivisions of the form Δ^ω are called <u>regular</u>.

<u>Question</u>: How many subdivisions (resp. triangulations) does our example have? Are all of them regular?

Equivalence relation on $\mathbb{Q}^n$: $\omega \sim \omega' \; :\Leftrightarrow \; \Delta^\omega = \Delta^{\omega'}$

Classes are convex polyhedral cones, defining the <u>secondary fan</u>.

(d) **THEOREM**: If $\omega \in \mathbb{Q}^n$ is a term order for $I_{\mathcal{A}}$ then $\Delta_\omega(I_{\mathcal{A}}) = \Delta^\omega$

Corollary: The Gröbner fan refines the secondary fan; equality holds if all initial ideals are square-free. $\rightsquigarrow$ Get a one-to-many correspondence between regular triangulations of $(Q, \mathcal{A})$ and initial ideals of $I_{\mathcal{A}}$.

Proof of Thm: By Linear Programming Duality, the negation of $(*)$ is equivalent to
$\exists \lambda \in \mathbb{Q}^n$: $\sum \lambda_i = 0$, $\sum \lambda_i a_i = 0$, $\lambda_j \geq 0$ for $j \notin T$, $\sum_{j \notin T} \lambda_j \omega_j \leq \sum_{i \in T} \lambda_i \omega_i$
Replacing $\mathbb{Q}^n$ by $\mathbb{Z}^n$, this is equivalent to
$\exists \lambda \in \mathbb{Z}^n$: $\prod_{j \notin T} x_j^{\lambda_j} - \prod_{i \in T} x_i^{\lambda_i} \in I_{\mathcal{A}}$ and $\prod_{i \in T} x_i^{\lambda_i} \in in_\omega(I_{\mathcal{A}})$

$$\Leftrightarrow \prod_{i \in T} x_i \in Rad(in_\omega(I_{\mathcal{A}})) \Leftrightarrow T \notin \Delta_\omega(I_{\mathcal{A}}). \qquad \blacksquare$$

By a refinement of this argument one can show:
for each maximal simplex $T \in \Delta^\omega$, the multiplicity of $\langle x_i : i \notin T \rangle$ in $in_\omega(I_{\mathcal{A}})$ equals $Vol(T)$ $\Rightarrow$ Proposition.

Corollary: $in_\omega(I_{\mathcal{A}})$ is $\left\{\begin{matrix}\text{radical} \\ \text{square-free}\end{matrix}\right.$ $\Leftrightarrow$ each max. simplex of Δ^ω has unit volum

Corollary: If $(\mathcal{A}, Q)$ admits a regular triangulation into unit simplices, then $k[\underline{x}]/I_{\mathcal{A}}$ is a Cohen-Macaulay ring

Proof: Δ^ω is a shellable (hence Cohen-Macaulay) simplicial complex. $\blacksquare$

"Homework": Describe the triangulations induced by the lexicographic resp. reverse lexicographic term orders!

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Lecture #2

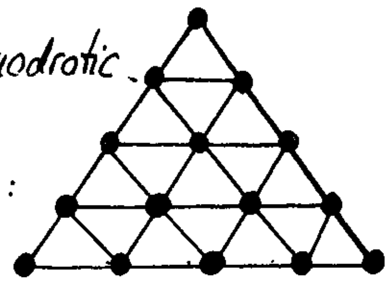
My favorite Gröbner basis for the Veronese ideal

The Veronese ideal is $I_{\mathcal{A}}$ where $\mathcal{A} = \{(a_0,\dots,a_d) \in \mathbb{N}^{d+1} : a_0+\dots+a_d = r\}$

Question: What is a "nice" Gröbner basis for $I_{\mathcal{A}}$?

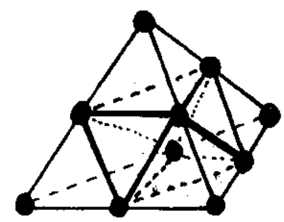
Answer: $(d=2,\ r=4)$

Take the square-free quadratic Gröbner basis defined by the "standard" triangulation:



For a different answer see [Eisenbud-Reeves-Totaro]

: How can this be generalized to $d \geq 3$?



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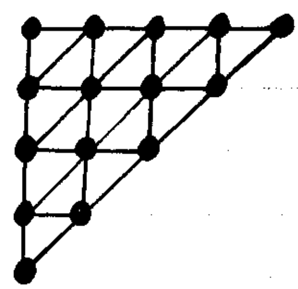
: Knudsen-Mumford-triangulation [KKMS]

"Break the symmetry; think Weil chamber"



Algebraic construction:

For each $\mathbf{a} = (a_0,\dots,a_d) \in \mathcal{A}$ we abbreviate $\hat{a}_i := a_0 + a_1 + \dots + a_i$ and $\hat{\mathbf{a}} := (\hat{a}_0, \hat{a}_1, \dots, \hat{a}_{d-1}) \in \mathbb{N}^d$, and we introduce a variable $x[\hat{\mathbf{a}}]$.

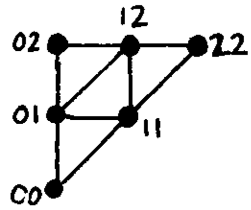
Considering the ideal $I_{\mathcal{A}}$ in these variables, we get the relations

(1) $$x[\hat{a}] \cdot x[\hat{b}] - x\left[\left\lfloor \tfrac{\hat{a}+\hat{b}}{2} \right\rfloor\right] \cdot x\left[\left\lceil \tfrac{\hat{a}+\hat{b}}{2} \right\rceil\right] \quad \text{for all } a,b \in \mathcal{A}$$

This relation is nonzero iff

(2) $$\exists\, i : |\hat{a}_i - \hat{b}_i| \geq 2 \quad \underline{\text{or}} \quad \exists\, i,j : (\hat{a}_i - \hat{b}_i)(\hat{a}_j - \hat{b}_j) < 0$$

THEOREM: The relations (1) for $a, b \in \mathcal{A}$ satisfying (2) are the reduced Gröbner basis for the Veronese ideal w.r.t. to a suitable term order.



Example $(d=2, r=2)$

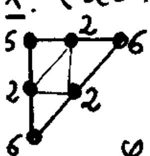
$x[00]x[02]-x[01]^2$, $x[00]x[12]-x[01]x[11]$
$x[00]x[22]-x[11]^2$, $x[01]x[22]-x[11]x[12]$
$x[02]x[22]-x[12]^2$, $x[02]x[11]-x[01]x[12]$

Proof: We define the weight of the variable $x[\hat{a}]$ to be

$$\omega(a) = \sum_{\substack{0\le i_1\le i_2\le d\\ 1\le k\le r-1}} |a_{i_1}+a_{i_1+1}+\dots+a_{i_2}-k|$$

For all relations (1) this breaks the tie and selects the first monomial as the initial monomial. (Check!)

Ex. $(d=r=2)$ $\omega(a_0,a_1,a_2) = |a_0-1|+|a_0+a_1-1|+|a_0+a_1+a_2-1|+|a_1-1|$



$+|a_1+a_2-1|+|a_2-1| = 2\cdot|a_0-1|+2|a_1-1|+2|a_2-1|$

Let $\mathcal{M}$ be the ideal generated by all monomials $x[\hat{a}]x[\hat{b}]$, where $(\hat{a},\hat{b})\in(2)$. Clearly, $\mathcal{M}\subseteq in_\omega(I_\mathcal{A})$, and we need to show equality.

A monomial m does not lie in $\mathcal{M}$ iff there exists a variable $x[\hat{a}]$ and a permutation σ of $\{0,1,\dots,d-1\}$ such that

$$m = x[\hat{a}]^{i_0}x[\hat{a}+e_{\sigma(0)}]^{i_1}\times\dots\times x[\hat{a}+e_{\sigma(0)}+\dots+e_{\sigma(d-1)}]^{i_d}$$

This shows that the minimal primes of $\mathcal{M}$ all have affine dimension $d+1$, and that they are indexed by pairs $(a,\sigma)\in\mathcal{A}\times S_d$ such that $\hat{a}_{\sigma(i)}\in\{0,1,\dots,d-1\}$ and $\hat{a}_{\sigma(i)}<\hat{a}_{\sigma(i+1)}$ whenever $\sigma(i)<\sigma(i+1)$.

Ex. $(d=r=2)$ Pairs: $((00),21)$, $((01),21)$, $((01),12)$, $((11),21)$

The number of such pairs (a,σ) equals r^d = the degree of the Veronese ($\rightsquigarrow$ bijection in [KKMS, p.120, Lemma 2.5]). Hence $\mathcal{M}$ is an equidimensional radical ideal contained in a homogeneous ideal $in_\omega(I_\mathcal{A})$ of the same degree and dimension. $\Rightarrow$ They are equal. ∎

In the previous situation: $\mathcal{A}$ = weights in the $GL_{d+1}(\mathbb{C})$-module $S^r\mathbb{C}^{d+1}$

Next: the analogous construction for the weights in $\Lambda^r\mathbb{C}^{d+1}$:

$$\mathcal{A} = \{ e_{i_1} + \cdots + e_{i_r} \; : \; 0 \le i_1 < \cdots < i_r \le d \}.$$

The polytope $Q = \mathrm{conv}(\mathcal{A})$ is the r-th hypersimplex.

Represent the variables by brackets $[i_1 \cdots i_r]$ and the monomials by tableaux $\to$ $\begin{bmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \\ \vdots & & \vdots \\ k_1 & \cdots & k_r \end{bmatrix}$

Theorem The quadratic relations

$$\begin{bmatrix} \cdots i \cdots j \cdots \\ \cdots k \cdots l \cdots \end{bmatrix} - \begin{bmatrix} \cdots i \cdots k \cdots \\ \cdots j \cdots l \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cdots i \cdots l \cdots \\ \cdots j \cdots k \cdots \end{bmatrix} - \begin{bmatrix} \cdots i \cdots k \cdots \\ \cdots j \cdots l \cdots \end{bmatrix}$$

where $0 \le i < j < k < l \le d$ are a reduced Gröbner basis of the ideal

$$I_{\mathcal{A}} = \ker\Big(k\big[[i_1 \cdots i_r] : 0 \le i_1 < \cdots < i_r \le d \big] \to k[t_0, \ldots, t_d] \Big)$$
$$[i_1 \cdots i_r] \mapsto t_{i_1} \cdots t_{i_r}$$

Example $(r=2, d=3)$ G.B. $[01][23] \to [02][13]$, $[03][12] \to [02][13]$

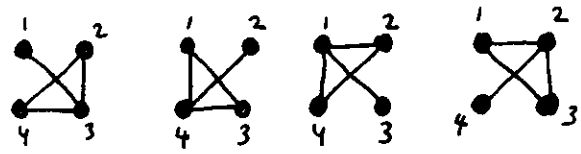
Q: What is the corresponding regular triangulation of the octahedron ??

Corollary: $k\big[t_{i_1} \cdots t_{i_r} : 0 \le i_1 < \cdots < i_r \le d \big]$ is a Cohen-Macaulay ring.

In this situation a tableau is "standard" if $i_1 < j_1 < \cdots < k_1 < j_2 < \cdots < i_r < \cdots < k_r$

- The "standard" tableaux form a k-linear basis mod $I_{\mathcal{A}}$.
- Their supports are the simplices in the corresponding triangulation.

Combinatorial interpretation in terms of "thrackles" for $r=2$: [DeLoera-St.-Thomas]



"Homework": Extend this construction to Segre varieties!

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ecture #3

HOW TO DEFORM A GRASSMANNIAN INTO A POLYTOPE

The passage from a projective scheme to its initial scheme with respect to a set of weights is the main operation of Gröbner bases theory. When the weights are sufficiently generic then the initial ideal is generated by monomials. This note deals with a situation where the initial ideal is not monomial but *toric*, that is, prime and generated by binomials.

1. From sagbi bases to Gröbner bases. Let R be a finitely-generated graded subalgebra of the polynomial ring $k[x_1, \ldots, x_n]$, and let $\prec$ be a monomial order. The ***initial algebra*** $in_\prec(R)$ is the k-vector space spanned by $\{in_\prec(f) : f \in R\}$. If we assume that $in_\prec(R)$ is finitely generated as a k-algebra, then there exists a subset $\mathcal{S} = \{f_1, f_2, \ldots, f_m\}$ of R such that

$$in_\prec(R) \quad = \quad k\big[in_\prec(f_1), in_\prec(f_2), \ldots, in_\prec(f_m)\big]. \tag{1}$$

This condition implies $R = k[f_1, f_2, \ldots, f_m]$. Following Robbiano-Sweedler [RS], we call $\mathcal{S}$ is a ***sagbi-basis*** of the subalgebra R. One of the most important open problems of the subject is to find conditions which guarantee that $in_\prec(R)$ is finitely generated. See [RS, Examples 1.2 and 4.11] for simple examples where $in_\prec(R)$ is not finitely generated.

We now define two ideals in the polynomial ring $S = k[y_1, y_2, \ldots, y_m]$. Let ϕ denote the surjective k-algebra homomorphism from S onto R defined by $\phi(y_i) = f_i$, and let I denote the kernel of ϕ. Similarly, we let ϕ' denote the map from S onto $in_\prec(R)$ defined by $\phi(y_i) = in_\prec(f_i)$, and we write I' for the kernel of ϕ. The ideal I' is the toric ideal associated with the non-negative integer $m \times n$-matrix π whose i-th column is the exponent vector of the monomial $in_\prec(f_i)$. We remark that both ideals I and I' are homogeneous if we define the degree of y_i to be the total degree of f_i.

Proposition. *The toric ideal I' is an initial ideal of I. More precisely, if $\omega \in \mathbf{R}^n$ is any weight vector representing the monomial order $\prec$ on $k[x_1, \ldots, x_n]$, then $in_{\pi(\omega)}(I) = I'$.*

Proof. If p lies in the ideal I then $p(f_1, \ldots, f_n)$ is the zero polynomial in $k[x_1, \ldots, x_n]$. In particular, terms of highest order cancel. This implies

$$in_{\pi(\omega)}(p)\big(in_\omega(f_1), \ldots, in_\omega(f_m)\big) \quad = \quad 0,$$

and therefore $in_{\pi(\omega)}(p) \in I'$. Thus the inclusion $in_{\pi(\omega)}(I) \subseteq I'$ holds even without the sagbi property. For the converse, let q be any binomial in I'. We consider the polynomial $q(f_1, \ldots, f_m)$ in $k[x_1, \ldots, x_n]$. By the sagbi property of $f_1, \ldots, f_m$, there exists $r \in S$ such $q(f_1, \ldots, f_m) = r(f_1, \ldots, f_m)$ and $q = in_{\pi(\omega)}(q - r)$. This proves $q \in in_{\pi(\omega)}(I)$ as desired.

Corollaries.
(1) Every reduced Gröbner basis of the I' lifts to a reduced Gröbner basis of I.
(2) Every regular triangulation of the cone $\pi(\mathbf{R}^n_+)$ is an initial complex of the ideal I'.
(3) The state polytope of I' is a face of the state polytope of I

(4) *The secondary polytope of* $\pi(\mathbf{R}_+^n)$ *is a face of the Chow polytope of* I.

Since the matrix π has non-negative entries, the cone $\pi(\mathbf{R}_+^n)$ is pointed. It can be identified with a polytope P of one dimension less. Statements (2) and (4) remain valid for the polytope P. The projective toric variety defined by I' is identified with the polytope P itself; this is done in a precise geometric way using the moment map (see [Ful, §4.2]). The flat deformation from I to I' is therefore a deformation from the projective scheme $Proj(R)$ into the polytope P. We shall specialize to the case where $Proj(R)$ is the Grassmannian $G_d(k^n)$ and P is the order polytope of a product of two chains $[n-d] \times [d]$.

2. The order polytope and its toric variety. The constructions and results summarized in the next two paragraphs are well-known in algebraic combinatorics. Most of them are due to R. Stanley and T. Hibi.

Let $\mathcal{P}$ be a finite poset and $\mathcal{L} = J(\mathcal{P})$ the distributive lattice of order ideals in $\mathcal{P}$. The *order polytope* P of $\mathcal{P}$ is the subset of $\mathbf{R}^{\mathcal{P}}$ consisting of all order-preserving maps from $\mathcal{P}$ to the interval $[0,1]$. The vertices of P are the 0-1-incidence vectors of the order ideals in P; so there is a natural bijection between the vertices of P and elements of $\mathcal{L}$.

We introduce variables $y_1, \ldots, y_m$ for the elements of $\mathcal{L}$. Let I' be the binomial ideal in $S = k[y_1, \ldots, y_m]$ generated by all relations $y_i \cdot y_j - (y_i \wedge y_j) \cdot (y_i \vee y_j)$. (Here "$\vee$" and "$\wedge$" denote the join and meet of the distributive lattice $\mathcal{L}$.) The ideal I' is a homogeneous prime ideal. It is the toric ideal associated with the order polytope. By a well-known result about algebras with straightening laws, the non-zero relations $y_i \cdot y_j - (y_i \wedge y_j) \cdot (y_i \vee y_j)$ for y_i and y_j are the reduced Gröbner basis with respect to the reverse lexicographic order induced by any linear extension of $\mathcal{L}$. The regular triangulation of P associated with this Gröbner basis equals the simplicial complex $\Delta(\mathcal{L})$ of chains in $\mathcal{L}$. The complex $\Delta(\mathcal{L})$ is shellable and every shelling of $\Delta(\mathcal{L})$ gives a Cohen-Macaulay basis for S/I'. The generating function for the number of such chains is the Hilbert function of I', or equivalently, the Hilbert polynomial of I' is the Erhart polynomial of the order polytope P.

We now consider the special case $\mathcal{P} = [n-d] \times [d]$, the product of an $(n-d)$-chain and a d-chain. Let $P(n,d)$ denote the order polytope of $[n-d] \times [d]$. The distributive lattice $J([n-d] \times [d])$ is isomorphic to the natural poset $\Delta(n,d)$ of d-element subsets of $\{1, 2, \ldots, n\}$. (This is sometimes called *Young's lattice.*) The isomorphism taking d-sets $\{i_1 < \ldots < i_d\} \in \Delta(n,d)$ to order ideals in the poset $[n-d] \times [d]$ is defined as follows. We introduce a linear map ψ from $\mathbf{R}^{n \times d}$ to $\mathbf{R}^{[n-d] \times [d]}$. If $A = (a_{ij})$ is any $n \times d$-matrix, then $\psi(A)$ is the $(n-d) \times d$-matrix whose (r,s)-entry equals $\sum_{i=j+1}^{j+r} a_{i,s}$. Each d-set $\{i_1 < \cdots < i_d\}$ in $\Delta(n,d)$ is coded by the matrix $E_{i_1, i_2, \ldots, i_d}$ with entries 1 in coordinates $(i_1, 1), (i_2, 2), \ldots, (i_d, d)$ and zeros elsewhere. Then $\psi(E_{i_1, i_2, \ldots, i_d})$ is the 0-1-incidence vector of an order ideal in $[n-d] \times [d]$, hence it encodes an element of the distributive lattice $J([n-d] \times [d]$. This construction proves in particular the following statement.

Proposition. *The order polytope* $P(n,d)$ *of the poset* $[n-d] \times [d]$ *is affinely isomorphic to the convex hull of the* $\binom{n}{d}$ *matrices* $E_{i_1, i_2, \ldots, i_d}$ *in* $\mathbf{R}^{n \times d}$.

We shall identify $P(n,d)$ with its affine image in $\mathbf{R}^{n\times d}$. We write Δ_{n-1} for the regular $(n-1)$-simplex that is the convex hull of the coordinate points in $\mathbf{R}^n$. The following lemma is proved easily by identifying matrix space $\mathbf{R}^{n\times d}$ with the d-fold product of $\mathbf{R}^n$ with itself. By a *subpolytope* of a polytope we mean the convex hull of a subset of the its vertices.

Lemma.

(1) The order polytope $P(n,d)$ is a subpolytope of the product of simplices $(\Delta_{n-1})^d$.

(2) If $(\lambda_1,\ldots,\lambda_d)$ is a partition of n, then the product of simplices $\Delta_{\lambda_1-1}\times\cdots\times\Delta_{\lambda_d-1}$ is a subpolytope of the order polytope $P(n,d)$.

Corollary. *The order polytope $P(n,2)$ is totally unimodular for all n. The order polytope $P(6,3)$ contains the regular 3-cube as subpolytope and hence is not totally unimodular.*

3. The Grassmannian. Let R denote the k-algebra generated by the $d\times d$-minors of an $n\times d$-matrix (x_{ij}) of indeterminates. Thus $Proj(R)$ is the Grassmannian of d-dimensional linear subspaces in k^n. We introduce a variable $[i_1,\ldots,i_d]$ for each maximal minor. The polynomial ring in these $\binom{n}{d}$ variables is denoted S and called the *bracket ring* (cf. [AIT,§3.1]). The kernel I of the natural map from S onto R is the *Grassmann-Plücker ideal.* By Theorem 3.2.9 of [AIT], the $d\times d$-minors are a sagbi basis for the subalgebra R with respect to the "diagonal term order" on $k[x_{ij}]$.

Weights defining the diagonal term order are given, for instance, by the Vandermonde matrix $(\omega_{ij})=(i^j)$. For the bracket variable $[i_1,i_2,\ldots,i_d]$ this induces the weight $i_1+{i_2}^2+\cdots+{i_d}^{\,l}$. We call these the *Vandermonde weights* on the bracket ring. (They are denoted $\pi(\omega)$ in our general discussion in §1). From our discussion in §1 and §2 we derive the following result.

Theorem. *The initial ideal of the Grassmann-Plücker ideal I with respect to the Vandermonde weights equals the toric ideal of the order polytope $P(n,d)$.*

Corollaries.

(0) There exists a flat deformation taking the Grassmannian $G_d(k^n)$ into (the projective toric variety associated with) the order polytope $P(n,d)$.

(1) Every reduced Gröbner basis of the toric ideal of $P(n,d)$ lifts to a reduced Gröbner basis of the Grassmann-Plücker ideal I

(2) Every regular triangulation of $P(n,d)$ is an initial complex of the Grassmannian.

(3) The state polytope of $P(n,d)$ is a face of the state polytope of the Grassmannian.

(4) The secondary polytope of $P(n,d)$ is a face of the Chow polytope of the Grassmannian.

The classical straightening algorithms for the Grassmannians are all special cases of the Gröbner bases in (1). However, not all Gröbner bases arise in this fashion. To see this, let as look more closely at the case of rank $d=2$. The polytope $P(n,2)$ is totally unimodular; it is a subpolytope of $\Delta_{n-1} \times \Delta_{n-1}$. This implies the following two properties of the toric ideal I' associated with $P(n,2)$:

(a) The following set of circuits is a universal Gröbner basis:

$$[i_1 j_1][i_2 j_2] \cdots [i_r j_r] - [i_2 j_1][i_3 j_2] \cdots [i_1 j_r], \qquad (i_1, i_2 < j_1, i_2, i_3 < j_2, \ldots i_r, i_1 < j_r).$$

(b) All initial ideals of I' are square-free.

For $d = 2$ and $n = 6$ the Grassmannian has initial ideals which are not square-free. Example: the weight vector $\omega = (9, 56, 82, 40, 86, 95, 55, 85, 88, 88, 39, 46, 10, 26, 62)$ for the $\binom{6}{2} = 15$ brackets $[ij]$ in the usual lexicographic order, we get an initial ideal whose minimal generators include $[15][23]^2[46]$. By statement (b), this flat deformation of $G_2(k^6)$ does not factor through the deformation to the order polytope $P(6,2)$ given in the Theorem.

We also remark that statement (b) does not hold for $d \geq 3$; for instance, $P(6,3)$ contains a regular 3-cube as subpolytope, hence it has a regular triangulation one of whose simplices does not have unit volume, hence it has an initial ideal that is not square-free.

We close with an open problem: What is the maximum degree $F(n,d)$ appearing in any reduced Gröbner basis for the Grassmannian $G_d(k^n)$? Part (2) of the Lemma in §2 implies exponential lower bounds for $F(n,d)$ as d and n tend to infinity. For $d = 2$ fixed we get more precise information from statement (a) above. The maximum degree of any circuit is $n-2$, and therefore we recover the following inequality due to Brian Taylor:

$$F(n,2) \quad \geq \quad n-2.$$

For an explicit proof of this inequality we may consider the degree $n-2$ circuit

$$[13][24][35] \cdots [n-2, n] \; - \; [23][34][45] \cdots [1, n]. \qquad (*)$$

To construct an initial ideal of the Grassmannian which has one of the monomials in $(*)$ as minimal generator, simply start with the Vandermonde weights and break ties using an elimination order for the variables appearing in $(*)$. In light of the general setting developed here, it would be very interesting to know whether $F(n,2) = n-2$.

Some References (but not all)

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