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Cohen-Macaulay Blowup Algebras and their Equations

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These are preliminary lecture notes, intended only for distribution to participants

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Lecture Notes on

Cohen–Macaulay Blowup Algebras and their Equations

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Abstract

Blowup algebras realize as rings of functions the process of blowing–up a variety along a subvariety. These lectures will focus on Rees rings of ideals, the most ubiquitous of those algebras. They will look at the numerical invariants, special divisors and attached algebras whose interplay assists in understanding the Cohen–Macaulay property. Emphasis will be placed on determining these invariants and properties from a description of the ring and ideals by generators and relations. Open problems and basic techniques will be stressed at the expense of individual results.

Contents

1 Introduction

The class of rings, called **blowup algebras**, appear in many constructions in Commutative Algebra and Algebraic Geometry. They represent fibrations of a variety with fibers which are often affine spaces; a polynomial ring $R[T_1, \ldots, T_n]$ is the notorious example of such algebras. Its uses include counterexamples to Hilbert's 14^{th} Problem, the determination of the minimal number of equations needed to define algebraic varieties, the computation of some invariants of Lie groups, and several others. An impetus for their systematic study has been the long list of beautiful Cohen–Macaulay algebras produced by the various processes. Finally, they provide a testing ground for several computational methods in Commutative Algebra.

Filtrations and Rees algebras

A *filtration* of a ring R is a family $\mathcal{F}$ of subgroups F_i of R indexed by some set S. The most useful kinds are indexed by an ordered monoid S, are *multiplicative*,

$$F_i \cdot F_j \subset F_{i+j}, \; i, j \in S,$$

and are either *increasing* or *decreasing*, that $F_i \subset F_j$ if $i \leq j$ or conversely.

The Rees algebra of $\mathcal{F}$ is the graded ring

$$R(\mathcal{F}) := \bigoplus_{i \in S} F_i,$$

with natural addition and multiplication. If the filtration is decreasing, there is another algebra attached to it, the associated graded ring

$$\text{gr}_{\mathcal{F}}(R) := \bigoplus_{i \in S} F_i / F_{<i},$$

with $F_{<i} = \bigcup_{j<i} F_j$. If the filtration is increasing, the associated graded ring is defined similarly by changing the sign of i.

I–adic filtrations

The algebras we will study arise from special filtrations of a commutative ring, multiplicative decreasing $\mathbb{N}$–filtrations $\mathcal{F}$ $\{R_n, \; n \in \mathbb{N}\}$ of R where each R_n is an ideal of R.

$$R_m \cdot R_n \subset R_{m+n}.$$

Its Rees algebra can be coded as a subring of the polynomial ring

$$R(\mathcal{F}) = \sum_{n \in \mathbb{N}} R_n t^n \subset R[t].$$

In addition to the associated graded ring as above, we also have the extended Rees algebra

$$R_e(\mathcal{F}) = R(\mathcal{F})[t^{-1}] = \sum_{n\in\mathbb{Z}} R_n t^n \subset R[t, t^{-1}].$$

These representations are useful when computing Krull dimensions. Very important is the isomorphism

$$R_e(\mathcal{F})/(t^{-1}) \simeq \mathrm{gr}_{\mathcal{F}}(R) = \bigoplus_{n=0}^{\infty} R_n/R_{n+1}.$$

It provides a mechanism to pass properties from $\mathrm{gr}_{\mathcal{F}}(R)$ to R itself (but we leave this statement purposely obscure for the reader to puzzle it out).

A major example is the I–adic filtration of an ideal I: $R_n = I^n$, $n \geq 0$. Its **Rees algebra**, which will be denoted by $R[It]$ or $\mathcal{R}(I)$, has its significance centered on the fact that it provides an algebraic realization for the classical notion of blowing–up a variety along a subvariety, and plays an important role in the birational study of algebraic varieties, particularly in the study of desingularization.

Symmetric algebras

The ancestors of these rings, **symmetric algebras**, have several other interesting descendants. Given a commutative ring R and an R–module E, the symmetric algebra of E is an R–algebra $S(E)$ which together with a R–module homomorphism

$$\pi : E \to S(E)$$

solves the following universal problem. For a commutative R-algebra B and any R–module homomorphism $\varphi : E \to B$, there exists a unique R–algebra homomorphism $\Phi : S(E) \to B$ that makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & B \\ \pi \downarrow & \nearrow \Phi & \\ S(E) & & \end{array}$$

commutative. Thus, if E is a free module, $S(E)$ is a polynomial ring $R[T_1, \ldots, T_n]$, one variable for each element in a given basis of E. More generally, when E is given by the presentation

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0, \ \varphi = (a_{ij}),$$

its symmetric algebra is the quotient of the polynomial ring $R[T_1, \cdots, T_n]$ by the ideal $J(E)$ generated by the 1-forms

$$f_j = a_{1j}T_1 + \cdots + a_{nj}T_n, \ j = 1, \ldots, m.$$

Conversely, any quotient ring of a polynomial ring $R[T_1, \ldots, T_n]/J$, with J generated by 1-forms in the T_i's, is the symmetric algebra of a module. Like the classical blowup, the morphism

$$\mathrm{Spec}(S(E)) \to \mathrm{Spec}(R)$$

is a fibration of Spec(R) by a family of hyperplanes. The case of a vector bundle, when E is a projective module, already warrants interest.

The other algebras are derived from $S(E)$ by effecting modifications on its components, some rather mild but others brutal. To show how this comes about, consider the case of ideals. For an ideal $I \subset R$, there is a canonical surjection

$$\alpha : S(I) \to \mathcal{R}(I).$$

If, further, R is an integral domain, the kernel of α is just the R-torsion submodule of $S(I)$. This suggests the definition of the Rees algebra $\mathcal{R}(E)$ of an R-module as $S(E)/T$, with T the (prime) ideal of the R-torsion elements of $S(E)$.

Another filtration is that associated to the symbolic powers of the ideal I. If I is a prime ideal, its nth symbolic power is the I-primary component of I^n. (There is a more general definition if I is not prime.) Its Rees algebra

$$\mathcal{R}_s(I) := \sum_{n \geq 0} I^{(n)} t^n,$$

the **symbolic Rees algebra** of I, which also represents a blowup, inherits more readily the divisorial properties of R, but has its usefulness limited because it is not always Noetherian. The presence of Noetherianess in $\mathcal{R}_s(I)$ is loosely linked to the number of equations necessary to define set-theoretically the subvariety $V(I)$ ([12]). In turn, the lack of Noetherianess of certain cases has been used to construct counterexamples to Hilbert's 14th Problem.

A common thread of the algebras derived from $S(E)$ is that each is obtained by the same process of taking the ring of global sections of Spec($S(E)$) on an appropriate affine open set (see [83] for details).

Aims

Within the realm of these algebras, let us chart up the territory to be covered here. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, let $I = (f_1, \ldots, f_n)$ be an ideal of positive height and denote by $\mathcal{R} = R[It]$ and $\mathcal{G} = \mathrm{gr}_I(R)$ its Rees algebra and associated graded ring. The main theme is the search for Cohen–Macaulay algebras. This provides a certain unity of purpose and a framework for techniques that may be used in other problem areas.

There are at least three main issues regarding the Cohen–Macaulay properties of the algebras $\mathcal{R}$ and $\mathcal{G}$. First, for which classes of ideals are the conditions achieved? It seems to occur in more varied ways than one seems able to catalog. Another area of interest is the study of the relation between $\mathcal{R}$ and $\mathcal{G}$. Here one has been able to pursue a more focused approach. Finally, there are rich connections to other problems, such as the theory of Hilbert–Samuel characteristic functions.

One of our purposes here is to examine the rich tableau on which the relationships between the Cohen–Macaulayness of $\mathcal{R}$ and of $\mathcal{G}$ are played out. More precisely, we will look at the hierarchy of conditions,

$$\boxed{\begin{array}{c} \mathcal{R} \text{ is Cohen–Macaulay} \\ \Downarrow \\ \mathcal{G} \text{ is Cohen–Macaulay} \\ \Downarrow \\ \text{Proj } \mathcal{G} \text{ is Cohen–Macaulay} \\ \Updownarrow \\ \text{Proj } \mathcal{R} \text{ is Cohen–Macaulay} \end{array}} \qquad (1)$$

looking for instances of equivalence.

From a computational perspective, the path to decide that a Rees algebra $\mathcal{R}$ is Cohen–Macaulay hardly ever goes by first checking that property in $\mathcal{G}$. Mixing the generators of an ideal and their relations may ask for gridlock in a computation. Nevertheless the interplay between $\mathcal{R}$ and $\mathcal{G}$ provides a rich set of guideposts in highlighting the significant numerical invariants of the algebras. These, in turn, may be addressed by more direct means.

The other main goal (time permitting) is to discuss methods for the construction of Cohen–Macaulay Rees algebras. This is really an open–ended area which we are just beginning to enter.

We sketch the content of these notes. We begin by introducing in Section 2 a large set of measures of size or behaviour of ideals and whose dependencies express arithmetical properties of Rees algebras. Section 3 describes some auxiliary structures useful to look and build ideals of low reduction numbers. The fruitful relationship between the Cohen–Macaulayness of $\mathcal{G}$ and $\mathcal{R}$ is then examined, first cohomologically by comparing the local cohomology of these two algebras mediated by that of R, and tandem by studying some divisors attached to $\mathcal{R}$. While the method of divisors is basically a mirror image of the former, some manipulations are more explicit in the language of divisors. Some applications to old and new results are then given. Next we introduce the *approximation complexes* and d–sequences: they provide for many new ways to build Cohen–Macaulay Rees algebras. Finally, we discuss how reductions inherit good properties from ideals with rich Koszul homology and in turn are used to obtain properties of the Rees algebras of the ideals themselves.

A word about the references

The extensive bibliography listed at end, even when not mentioned in the body of the lectures, is a source for related results and gives an indication of the activities in the area. An additional listing will be collected from the participants. An effort will be made to collect open problems and distribute.

These notes were prepared for the Workshop on Commutative Algebra, in May 1994 at the ICTP. Its preliminary, open character, reflects the need to await the interaction with other researchers at the Workshop before a more definitive set is written. Corrections in matters of attribution are especially welcomed.

2 Numerical invariants of a Rees algebra

We begin by recalling various integers associated to the Rees algebra of an ideal. We will seek to understand what information about the depth of $\mathcal{R}$ these numbers contain. At first glance, the invariants tend to give measures of how far is an ideal from being generated by analytically independent elements and reflect nuanced versions of this notion. These are best set in a local ring, so that throughout we assume $(R, \mathfrak{m})$ is a local ring of dimension d, and at places, that the residue field of R is infinite.

There are two general approaches to describing properties of the Rees algebra $\mathcal{R} = R[It]$ of an ideal I.

- Through a presentation of the algebra as a quotient of a polynomial

$$0 \rightarrow J \longrightarrow R[T_1, \ldots, T_n] \longrightarrow \mathcal{R} \rightarrow 0,\ T_i \mapsto f_i t,$$

  by looking at the structure of the ideal J.

- Through the study of the reductions of the ideal I, a far flung method that imitates Noether normalization.

These approaches emphasize the structure of the polynomial relations amongst the elements of a generating set of I and tend to mimic one another.

The equations of the Rees algebra

The first approach is greatly focused on the degrees of a generating set for the presentation ideal J and seeks to obtain those equations from the syzygies of I. The ideal J, which we refer to as the *equations* of $R[It]$, is graded

$$J = J_1 + J_2 + \cdots,$$

where J_1 is the R–module of linear forms $\sum a_i T_i$ such that $\sum a_i f_i = 0$. J_s is the module of syzygies of the s–products of the f_i. The challenge is, from a given presentation of I,

$$R^m \xrightarrow{\varphi} R^n \longrightarrow I \to 0,$$

to describe J. As a start, we have

$$J_1 = [T_1, \ldots, T_n] \cdot \varphi, \ \varphi = (a_{ij}).$$

The J_s arise by elimination from these equations of the "parameters" describing the f_i (but how?).

The module J_1 generates the ideal of definition of the symmetric algebra $S(I)$ of I, and we have a canonical surjections

$$0 \to (J_1) \longrightarrow R[T_1, \ldots, T_n] \longrightarrow S(I) \to 0,$$

$$0 \to \mathcal{A} \longrightarrow S(I) \longrightarrow R[It] \to 0,$$

where

$$J/(J_1) = \mathcal{A} = \mathcal{A}_2 + \mathcal{A}_3 + \cdots.$$

In particular the degrees of the generators of J are independent of the chosen generators of I. When $\mathcal{A} \neq 0$, some emphasis has been put on determining the first non-vanishing component $\mathcal{A}_j$ and on predicting its structure ([77], [67]).

Noteworthy is the meaning of the condition $\mathcal{A}$ = nilpotent: Let R be a local ring, of maximal ideal $\mathfrak{m}$. The elements $a_1, \ldots, a_n \in R$ are said to be *analytically independent* if any homogeneous polynomial $f(X_1, \ldots, X_n)$ for which $f(a_1, \ldots, a_n) = 0$ has all of its coefficients in $\mathfrak{m}$. If $I = (a_1, \ldots, a_n)$, this means that the ring $R[It] \otimes (R/\mathfrak{m})$ is a polynomial ring in n indeterminates over $R/\mathfrak{m}$.

Definition 2.1 The ideal I is said to be of *linear type* if $J = (J_1)$. More generally, I is said to be of *relation type* r if J can be generated by forms of degree $\leq r$.

If I is of linear type, then I is locally generated by analytically independent elements, in particular $\nu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ that contains I. There is no general theory describing the ideals of linear type, although many well circumscribed classes are known. If I is not of linear type, there is a beginning of a theory for ideals of quadratic type and for certain families of ideals whose equations are obtained from elimination and are concentrated in degrees 1 and another degree.

Example 2.2 If the ideal I is generated by a regular sequence $f_1, \ldots, f_n$, the equations of $\mathcal{R} = R[It]$ are nice:

$$\mathcal{R} \simeq R[T_1, \ldots, T_n]/I_2 \begin{pmatrix} T_1 & \cdots & T_n \\ f_1 & \cdots & f_n \end{pmatrix}.$$

In other words, J is generated by the Koszul relations of the f_i's. Knowing this description of $\mathcal{R}$ leads immediately to its canonical module,

$$\omega_{\mathcal{R}} = \omega_R t(1,t)^{g-2}\mathcal{R},$$

and, when the need arises, gives the means to test normality and other properties.

Reduction of an ideal

The other method of studying $R[It]$ is less straightforward but considerably more general. Let us recall the notion on which it is based on.

Definition 2.3 An ideal $J \subset I$ is a *reduction* of I if $JI^r = I^{r+1}$ for some integer r; the least such integer, $r_J(I)$, is the *reduction number* of I with respect to J.

Phrased otherwise, J is a reduction of I means that

$$R[Jt] \hookrightarrow R[It]$$

is a finite morphism of graded algebras, and $r_J(I)$ is the infimum of the top degree of any homogeneous set of generators of $R[It]$ as a module over $R[Jt]$. The *reduction number* of I, $rn(I)$, is the infimum of all $r_J(I)$. (If there is no confusion, it is denoted $r(I)$).

There are several equivalent ways to describe the notion of a reduction. We recall one of these. If L is an ideal of R, its integral closure $\overline{L}$ consists of the elements $z \in R$ that satisfy an equation

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0, \quad \text{with} \quad a_i \in L^i.$$

It turns out then that J is a reduction of I precisely when $I \subset \overline{J}$. Yet another way to express this is

$$\overline{L} = R \cap \bigcap L \cdot V,$$

where V runs over all the valuation rings of R (R is assumed a domain).

Example 2.4 The simplest example of a reduction is probably the case $J = (x^2, y^2) \subset I = (x^2, y^2, xy)$, when $I^2 = JI$. A significant process to produce ideals and some of its proper reductions is the following (see [66]). Let $f \in R$, where R is either a ring of polynomials or a power series ring in the indeterminates $x_1, \ldots, x_n$ over a field k of characteristic zero. Let

$$J = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

Then J is a reduction of $I = (J, f)$ (with some provisos, such as if f is a power series then it should not be a unit).

Analytic spread and reduction number

The ideal I and any of its reductions J share several properties, among which they have the same radical. One of the advantages of J is that it may have a great deal fewer generators. We indicate how this may come about, with the notion of *minimal reduction.*

Definition 2.5 Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal. The *special fiber* of the Rees algebra $R[It]$ is the ring

$$F(I) = R[It] \otimes_R R/\mathfrak{m} = \bigoplus_{s \geq 0} I^s/\mathfrak{m}I^s.$$

Its Krull dimension is called the *analytic spread* of I, and is denoted $\ell(I)$. If $I = \mathfrak{m}$, $F(\mathfrak{m})$ is the Zariski's *tangent cone* of R. $F(I)$ is also called the *fiber cone* of I.

Suppose the residue field of R, $k = R/\mathfrak{m}$, is infinite. We may then pick a Noether normalization of $F(I)$,

$$A = k[z_1, \ldots, z_\ell] \hookrightarrow F(I),$$

where $\ell = \ell(I)$, and the z_j can be chosen in degree 1. Let further $b_1, \ldots, b_s$ be a minimal set of homogeneous module generators of $F(I)$ over the algebra A:

$$F(I) = \sum_{1 \leq i \leq s} Ab_i, \ \deg(b_i) = r_i.$$

Proposition 2.6 *Let $a_1, \ldots, a_\ell$ be elements of I that are lifts of $z_1, \ldots, z_\ell$. Then $J = (a_1, \ldots, a_\ell)$ is a reduction of I and*

$$r_J(I) = \sup\{\deg(b_i) \ \ 1 \leq i \leq s\}.$$

Proof. Both assertions follow easily by lifting the equality $F(I) = \sum_i Ab_i$ to $R[It]$ and using the Nakayama Lemma. □

Proposition 2.7 *Let R be a local ring and let I be an ideal. The following inequalities hold*

$$\text{height } I \leq \ell(I) \leq \dim R.$$

Proof. The second inequality arises from the formula for the Krull dimension of $R[It]$ ([79]). The other assertion follows since any minimal prime of J is also minimal over I, but the former have, by Krull principal ideal theorem, codimension at most ℓ. □

Example 2.8 Suppose I is an ideal of $k[x_1, \ldots, x_n]$ generated by forms $f_1, \ldots, f_m$ of the same degree. With $\mathfrak{m} = (x_1, \ldots, x_n)$, we have

$$R[It] = k[f_1 t, \ldots, f_m t] \bigoplus \mathfrak{m}R[It],$$

so that the fiber cone $F(I) \simeq k[f_1, \ldots, f_m]$. Suppose that each f_i is a monomial $\mathbf{x}^{v_i}$ (where $v_i = (a_{i1}, \ldots, a_{in})$ is a vector of exponents). In this case, it is easy to see that

$$\ell(I) = \text{rank}\,[v_1, \ldots, v_m].$$

What is still missing are methods to find the reduction numbers of these ideals.

When the monomials f_i are quadratic and square-free, Villarreal ([84]) attaches to this set a graph (and conversely). Their interplay (see [70]) is useful in determining the reduction numbers. For example, if the attached graph is bipartite then the reduction number of the ideal is less than the number of indeterminates.

It is a lot more challenging to find the analytic spread of ideals generated by binomials. A tantalizing question is how toric (e.g. prime) ideals should be dealt with in these matters. Progress on this question has already been made in [16].

Question 2.9 There is one case in which it is straightforward to determine the reduction number of an ideal–when $F(I)$ is a Cohen–Macaulay. The number $r(I)$ can then be read off the Hilbert function of $F(I)$, which obviates the need to use Noether normalization. What can be done, if we only have depth $F(I) \geq \dim F(I) - 1$?

Exercise 2.10 Let $\mathfrak{p}$ be a prime ideal of a regular local ring R. Suppose that the ordinary and symbolic powers of $\mathfrak{p}$ coincide, that is $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for $n \geq 1$. Prove that $\ell(\mathfrak{p}) < \dim R$.

Reduction number one

If one wants to use ideals with low reduction as building blocks, it is of interest to catalog them.

Determinantal ideals

Let φ the an $m \times n$ generic matrix. Its determinantal ideals exhibit some of the best reduction low numbers one could expect. Here are some distinguished cases:

- If $m = n + 1$, then $I = I_n(\varphi)$ is of linear type.
- If $m = n$, then $I = I_{n-1}(\varphi)$ is of linear type ([42]).
- If φ is the generic, symmetric $n \times n$ matrix, then $I = I_{n-1}(\varphi)$ is of linear type ([46]).
- If $m \geq n + 2$, then $I = I_n(\varphi)$ has reduction number one.

Links of prime ideals

Let us indicate a method that produces plenty of ideals of reduction number 1. Let R be a Cohen–Macaulay ring and let $\mathfrak{p}$ be a prime ideal of codimension g. Let J be generated by a regular sequence of g elements contained in J and set $I = J\colon \mathfrak{p}$. Consider the following very general settings:

(L_1) $R_\mathfrak{p}$ is not a regular local ring;

(L_2) $R_\mathfrak{p}$ is a regular local ring of dimension at least 2 and two elements in the sequence $\mathbf{z}$ lie in the symbolic square $\mathfrak{p}^{(2)}$.

We then have (see [11], [10]):

Theorem 2.11 *Let R be a Cohen–Macaulay ring, $\mathfrak{p}$ a prime ideal of codimension g, and let $\mathbf{z} = (z_1, \ldots, z_g) \subset \mathfrak{p}$ be a regular sequence. Set $J = (\mathbf{z})$ and $I = J\colon \mathfrak{p}$. Suppose that $R_\mathfrak{p}$ is a Gorenstein ring. Then I is an equimultiple ideal with reduction number one, more precisely,*

$$I^2 = JI,$$

if either condition L_1 *or* L_2 *holds.*

More actors

There are several measures of 'irregularity' for ideals in local rings. The following notions will play a role in the sequel.

- The *deviation* of I is the non–negative integer $\nu(I)$ – height I.
- Huckaba and Huneke [35] have defined the *analytic deviation* of I as $ad(I) = \ell(I) -$ height I.
- The difference between these two numbers, $\nu(I) - \ell(I)$ is the *second deviation* of I.
- The ideals of analytic deviation zero are called *equimultiple* (see [25] for a wealth of information about these ideals).

Among all measures of I defined, none play a role more central than that of the reduction number $rn(I)$ in deciding when I is explosively Cohen–Macaulay. It should come as no surprise therefore that this integer is the hardes one to determine. That we know, there are no explicit process to compute $rn(I)$ unless one resorts to 'very' generic methods à la Bertini's.

A pressing question here is, if $rn(I)$ is so significant, how do the other invariants of I relate to it? There are some intriguing relationships that however non-general occur repeatedly. For example, a basic guess for $rn(I)$ is

$$rn(I) \leq \ell(I) - \text{height}\, I + 1,$$

a value often called the *expected reduction number*. (Attend Bernd Ulrich lectures!)

Most other relationships among the actors depend on special circumstances. Here is a special one ([82]):

Theorem 2.12 *Let R be a Gorenstein local ring and let I be a Cohen–Macaulay ideal of codimension g that is of linear type in codimension $\leq g+1$. Then $\ell(I) \geq g+2$.*

Problem 2.13 Devise an algorithmic approach to compute the integral closure of an ideal of a polynomial ring. (Prizes are given!)

3 Mixing up

R be a local ring, and let I be an ideal with a reduction J. To be able to use the known properties of the Rees algebra of J as a tool to obtain properties of the Rees algebra of I, it is necessary to build structures in which the two algebras intermingle.

We are going to mention only two such structures, one introduced in [82] and another of an older vintage [78].

The Sally module

Consider the exact sequence of finitely generated $R[Jt]$–modules:

$$0 \to I \cdot R[Jt] \longrightarrow I \cdot R[It] \longrightarrow S_J(I) = \bigoplus_{n=0}^{\infty} I^{n+1}t^n/IJ^nt^n \to 0. \tag{2}$$

Definition 3.1 The *Sally module* of I with respect to J is $S_J(I)$ viewed as an $R[Jt]$-module.

A motivation for this definition is the work of Sally, particularly in [62], [63], [64], and [65] in the case of $\mathfrak{m}$–primary ideals.

To be useful, this sequence requires information about $I \cdot R[Jt]$—which is readily available in many cases—and finer properties of $S_J(I)$. Its main feature is the relationship it bears with the ring $R[Jt]$, a ring that often is simpler to study than $R[It]$, such as in the case of equimultiple ideals.

Let us begin by pointing out a simple but critical property of $S_J(I)$:

Proposition 3.2 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, let I be a $\mathfrak{m}$–primary ideal and let J be a minimal reduction of I. If $S_J(I) \neq 0$ its associated prime ideals have codimension 1, in particular its Krull dimension as an $R[Jt]$–module is d.*

Proof. We first argue that $I \cdot R[Jt]$ is a maximal Cohen–Macaulay $R[Jt]$–module. For this it suffices to consider the exact sequence

$$0 \to I \cdot R[Jt] \longrightarrow R[Jt] \longrightarrow R[Jt] \otimes_R (R/I) \to 0, \qquad (3)$$

and observe that the module on the right is a polynomial ring in d variables over R/I, and therefore Cohen–Macaulay of dimension d. Since $R[Jt]$ is a Cohen–Macaulay ring, $I \cdot R[Jt]$ has depth $d+1$.

We may assume $d \geq 1$, and that $S_J(I) \neq 0$. Let $P \subset A = R[Jt]$ be an associated prime of $S_J(I)$; it is clear that $\mathfrak{m} \subset P$. If $\mathfrak{m}A \neq P$, P has grade at least two.

Consider the homology sequence of the functor $\mathrm{Hom}_A(A/P, \cdot)$ on the sequence (3.1): we get

$$0 \to \mathrm{Hom}_A(A/P, S_J(I)) \longrightarrow \mathrm{Ext}^1_A(A/P, I \cdot R[Jt]).$$

Note that $I \cdot R[Jt]$ is a maximal Cohen–Macaulay module, hence the Ext module must vanish since P contains a regular sequence of two elements on it. □

Hilbert functions

We will indicate the usefulness of the setting of reduction modules to prove several known inequalities on the coefficients of the Hilbert polynomial of a primary ideal.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of Krull dimension d and I an $\mathfrak{m}$–primary ideal. The *Hilbert function* of I is the assignment

$$HF : n \rightsquigarrow \lambda(R/I^n).$$

For $n >> 0$ (but often not much greater than zero!) $HF(n)$ is equal to a polynomial $H(n)$ of degree d, the *Hilbert–Samuel polynomial of I*.

To study either of these functions, without loss of generality, it is convenient to assume that the residue field of R is infinite. Here we limit ourselves mostly to Hilbert polynomials leaving aside the rich area of *irregularities*, to wit the detailed comparison between the two functions.

We shall further assume that R is a Cohen–Macaulay ring. In such cases, if I is generated by a system of parameters the functions are given simply by

$$HF(n) = H(n) = \lambda(R/I) \cdot \binom{n+d-1}{d}, \ \forall n.$$

It suggests that the Hilbert function of an arbitrary ideal I be approached through the Hilbert function of one of its minimal reductions J.

We look at two exact sequences as the vehicle for this comparison:

$$0 \to IJ^{n-1}/J^n \longrightarrow I^n/J^n \longrightarrow I^n/IJ^{n-1} \to 0$$

and

$$0 \to IJ^{n-1}/J^n \longrightarrow J^{n-1}/J^n \longrightarrow J^{n-1}/IJ^{n-1} \to 0.$$

The function $\lambda(I^n/J^n)$ is our center of interest since it equals $\lambda(R/J^n) - \lambda(R/I^n)$, the first term being very well behaved. Using the other sequence we have the following expression for the Hilbert function $HF(n)$ of I:

$$\begin{aligned} HF(n) &= \lambda(R/J^n) - \lambda(IJ^{n-1}/J^n) - \lambda(I^n/IJ^{n-1}) \\ &= \lambda(R/J^n) - \lambda(J^{n-1}/J^n) + \lambda(J^{n-1}/IJ^{n-1}) - \lambda(I^n/IJ^{n-1}) \\ &= \lambda(R/J^{n-1}) + \lambda(J^{n-1}/IJ^{n-1}) - \lambda(I^n/IJ^{n-1}), \end{aligned}$$

the first 2 terms of which can be collected since the ring $\oplus(J^{n-1}/IJ^{n-1})$ is a polynomial ring in d variables with coefficients in R/I. We obtain

$$HF(n) = \lambda(R/J)\cdot\binom{n+d-2}{d} + \lambda(R/I)\cdot\binom{n+d-2}{d-1} - \lambda(I^n/IJ^{n-1}).$$

This expression turns the focus on $\lambda(I^n/IJ^{n-1})$. We note that both I^n/J^n and I^n/IJ^n are components of modules over the Rees algebra $R[Jt]$: the first comes from the quotient $R[It]/R[Jt]$, the other being a component of the Sally module $S_J(I)$. The latter has many advantages over the former, a key one being that it vanishes in cases of considerable interest.

Sometimes it will be convenient to write $H_I(\cdot)$ for the Hilbert function of I; the reader is also warned about shifts in the arguments arising from the grading of the modules. For example, the degree 1 component of $S_J(I)$ is I^2/IJ.

Proposition 3.3 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, with infinite residue field, and let I be an $\mathfrak{m}$–primary ideal. Denote by $H_I(n) = \lambda(R/I^n)$ the Hilbert function of I, and let*

$$e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}\binom{n}{1} + (-1)^d e_d \tag{4}$$

be its normalized Hilbert polynomial. Suppose J is a minimal reduction of I and let $S = S_J(I)$ be the corresponding Sally module. Then for $n \gg 0$

$$H_I(n) = e_0\binom{n+d-1}{d} + (\lambda(R/I) - e_0)\binom{n+d-2}{d-1} - \lambda(S_{n-1}). \tag{5}$$

Proof. The proof is a straightforward calculation that takes into account the equality $e_0 = \lambda(R/J)$. □

Corollary 3.4 *If $S_J(I) \neq 0$, the function $\lambda(S_n)$ has the growth of a polynomial of degree $d-1$.*

Proof. By Proposition 3.2, if $S_J(I) \neq 0$ then its Krull dimension is d. □

The following formulas can be used to establish several vanishing results on the coefficients of Hilbert functions.

Theorem 3.5 *Let $s_0, s_1, \ldots, s_{d-1}$ be the coefficients of the Hilbert polynomial of $S_J(I)$. Then*

$$\begin{aligned} s_0 &= e_1 - e_0 + \lambda(R/I) \\ s_i &= e_{i+1}, \text{ for } i \geq 1. \end{aligned}$$

Corollary 3.6 *The following hold:*

(a) $\lambda(R/I) \geq e_0 - e_1$ ([57]).

(b) *If equality above holds then* $S_J(I) = 0$ ([43, Theorem 2.1], [59]).

Proof. (a) This is clear since e_1 will be obtained by adding to $e_0 - \lambda(R/I)$ the contribution from the leading term in the Hilbert polynomial of $S_J(I)$.

(b) For equality to hold there must be no contribution from $S_J(I)$, which means that $S_J(I)$ is a module of Krull dimension $\leq d-1$. From Proposition 3.2 this implies $S_J(I) = 0$. □

Remark 3.7 Another result that can be derived from the formulas above is Narita's inequality ([55]): $e_2 \geq 0$. This results from an interpretation of $e_2(I)$ as $s_0(L)$ for some appropriate L.

Some structure

Here is an example of the use the Sally modules (see [60]; for other approaches to this kind of problem, see [24]):

Proposition 3.8 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay of dimension d and let I be an $\mathfrak{m}$–primary with a minimal reduction J such that $I^3 = JI^2$. If the trivial submodule of I^2/JI is irreducible, then* depth $\mathfrak{S} \geq d-1$.

The Valabrega–Valla module

Consider the exact sequence of finitely generated $R[Jt]$–modules:

$$0 \to J \cdot R[It] \longrightarrow J \cdot R[t] \bigcap I \cdot R[It] \longrightarrow VV_J(I) = \bigoplus_{n=0}^{\infty} (J \cap I^{n+1})/JI^n \to 0.$$

Definition 3.9 The *Valabrega–Valla module* of I with respect to J is $VV_J(I)$ viewed it an R-module.

$VV_J(I)$ is actually a module over $R[Jt]$-module, but since it vanishes in high degrees it is more convenient to view an an R-module. Its components appear in some filtrations of $S_J(I)$. Its significance lies in the following Cohen–Macaulay criterion of [78]:

Theorem 3.10 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let I be an* m*-primary ideal with infinite residue field. Suppose J is a minimal reduction of I. Then $\mathrm{gr}_I(R)$ is Cohen–Macaulay if and only if $VV_J(I) = 0$.*

This is, in fact, a special case of a very general assertion ([78, Theorem 2.3]):

Theorem 3.11 *Let R be a Noetherian ring, I an ideal of R, and let $x_1, \ldots, x_n$ be a regular sequence in R. Then the leading forms of the x_i form a regular sequence in $\mathrm{gr}_I(R)$ if and only if for $i = 1, \ldots, n$ and all $m \geq 1$,*

$$(x_1, \ldots, x_i) \cap I^m = \sum_{j=1}^{i} I^{m-d_j} x_j,$$

where d_j is the least integer s such that $x_j \in I^s \setminus I^{s+1}$.

Cascading reductions

The simplest reduction $J \subset I$ to examine is when $I^2 = JI$. We then have that

$$I \cdot R[Jt] = I \cdot R[It],$$

and

$$0 \to I \cdot R[Jt] \longrightarrow R[Jt] \longrightarrow \mathrm{gr}_J(R) \otimes R/I \to 0. \qquad (6)$$

If J is a regular sequence, and I is a Cohen–Macaulay ideal, it follows that $I \cdot R[It]$ is a maximal Cohen–Macaulay. Taken into the sequences (14) and (15) it leads to the fact that $\mathrm{gr}_I(R)$ is Cohen–Macaulay.

When the reduction $r_J(I)$ is 2, the structure of $S_J(I)$ has to be taken into consideration (see [60] for several cases). Let us try to bridge these reductions through simpler ones ([51]).

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, let I be an $\mathfrak{m}$–primary ideal, let J be a minimal reduction and suppose $r_J(I) = 2$. Let us assume that the equality $I^3 = JI^2$ is achieved in two steps as follows: There exist $J \subset L \subset I$ with $I^2 = LI$ and $L^2 = JL$. These conditions mean that the associated Sally modules $S_J(L)$ and $S_L(I)$ vanish so that we have

$$\begin{aligned} L \cdot R[Jt] &= L \cdot R[Lt], & (7)\\ I \cdot R[Lt] &= I \cdot R[It]. & (8) \end{aligned}$$

The exact sequence (6), when applied to the pair (J, L) implies that $L \cdot R[Lt]$ is a maximal Cohen–Macaulay. It would be interesting to have a similar assertion for $I \cdot R[It]$, but to use (6) we would need to have that $\mathrm{gr}_L(R) \otimes R/I$ is Cohen–Macaulay.

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & R[Jt] & \longrightarrow & R[Lt] & \longrightarrow & C & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & L \cdot R[Jt] & \longrightarrow & L \cdot R[Lt] & \longrightarrow & S_J(L) & \to & 0. \end{array}$$

From the snake lemma there is an acyclic complex

$$0 \to K_S \longrightarrow \mathrm{gr}_J(R) \otimes R/L \longrightarrow \mathrm{gr}_L(R) \longrightarrow K_C \to 0, \tag{9}$$

where K_S and K_C are defined through the exact sequence

$$0 \to K_S \longrightarrow S_J(L) \longrightarrow C \longrightarrow K_C \to 0 \tag{10}$$

of natural inclusions.

Proposition 3.12 *If $r_J(L) = 1$ there are exact sequences of Cohen–Macaulay modules of dimension d*

$$0 \to \mathrm{gr}_J(R) \otimes R/L \longrightarrow \mathrm{gr}_L(R) \longrightarrow C \to 0, \tag{11}$$

and

$$0 \to C(+1) \longrightarrow \mathrm{gr}_J(R) \longrightarrow \mathrm{gr}_J(R) \otimes R/L \to 0. \tag{12}$$

Proof. Left as exercise.

Since we interested in $\mathrm{gr}_L(R) \otimes R/I$, the place to start is with sequence (11): we need to reduce it modulo R/I. Tensoring (12) instead with R/I over R/J, we have the exact sequence

$$\begin{array}{l} 0 \to \mathrm{Tor}_1^{R/J}(\mathrm{gr}_J(R) \otimes R/L, R/I) \to C \otimes R/I(+1) \to \\ \to \mathrm{gr}_J(R) \otimes R/I \simeq \mathrm{gr}_J(R) \otimes R/L \otimes R/I \to 0, \end{array}$$

from which we have

$$\begin{array}{rcl} C \otimes R/I & \simeq & \mathrm{Tor}_1^{R/J}(\mathrm{gr}_J(R) \otimes R/L, R/I)(-1) \\ & \simeq & \mathrm{gr}_J(R) \otimes \mathrm{Tor}_1^{R/J}(R/L, R/I)(-1). \end{array}$$

We thus have

Proposition 3.13 *$C \otimes R/I$ is a Cohen–Macaulay module of dimension d.*

One way to prove that $\text{gr}_L(R) \otimes R/I$ is Cohen–Macaulay is by showing that

$$0 \to \text{gr}_J(R) \otimes R/I \longrightarrow \text{gr}_L(R) \otimes R/I \longrightarrow C \otimes R/I \to 0 \tag{13}$$

is exact. (Tracing the Tor's carefully one is in condition to use Theorem 3.11.) For example, if L is a Gorenstein ideal (a situation that [56] often forbids to happen), the sequence (11) splits and therefore (13) will be exact. What are other more interesting cases?

4 Explosively Cohen–Macaulay ideals

We turn to one of the most active areas of investigation in the theory of Rees algebras. At the risk of offending friends and other sensitive listening devices, we propose:

Definition 4.1 Let R be a ring and let I be an ideal. We say that I is *explosively Cohen–Macaulay* if its Rees algebra $\mathcal{R} = R[It]$ is Cohen–Macaulay.

Unlike the ring $\mathcal{G}$, the ring $\mathcal{R}$ can be Cohen–Macaulay without R itself being Cohen–Macaulay. This terminology is slightly misleading because $\mathcal{R}$ can also be Cohen–Macaulay when I is far from sharing this property.

A setting for studying the relationship between the Cohen–Macaulayness of $\mathcal{R}$ and $\text{gr}_I(R)$ are the following exact sequences (first paired in [38]):

$$0 \to I \cdot R[It] \longrightarrow R[It] \longrightarrow \text{gr}_I(R) \to 0 \tag{14}$$

$$0 \to It \cdot R[It] \longrightarrow R[It] \longrightarrow R \to, \tag{15}$$

with the naive isomorphism

$$It \cdot R[It] \simeq I \cdot R[It]$$

playing a pivotal role.

We begin with the result of Huneke ([38]):

Theorem 4.2 *Let R be a Cohen–Macaulay ring and let I be an ideal containing regular elements such that $R[It]$ is Cohen–Macaulay. Then $\text{gr}_I(R)$ is Cohen–Macaulay.*

Proof. Since I contains regular elements, $\dim R[It] = \dim R + 1$. We may assume that R is a local ring. To show that the depth of $\mathrm{gr}_I(R)$ (relative to its irrelevant maximal ideal) is at least $\dim R$, we make use of the two exact sequences (14) and (15), the second of which, in this case, says that $It \cdot R[It]$ is a maximal Cohen–Macaulay module. Since it is isomorphic to $I \cdot R[It]$, the associated graded ring will be Cohen–Macaulay. □

The following elementary observation is useful here. About terminology: For a local ring the punctured spectrum has the usual meaning, the set of non-maximal prime ideals. For a graded algebra we mean the set of homogeneous, non-maximal prime ideals.

Proposition 4.3 *Let R be a Noetherian local ring and let I be an ideal of positive grade. Suppose $\mathcal{R}$ is Cohen–Macaulay on the punctured spectrum of R. If $\mathrm{gr}_I(R)$ is Cohen–Macaulay on the punctured spectrum then $\mathcal{R}$ is also Cohen–Macaulay on the punctured spectrum. In particular, if $\mathrm{gr}_I(R)$ is Cohen–Macaulay then* Proj $\mathcal{R}$ *is Cohen–Macaulay.*

Proof. We may assume that $I = (f_1, \ldots, f_n)$ with each f_i an R-regular element. Set $\mathcal{R} = R[f_1t, \ldots, f_nt]$; it suffices to prove that $\mathcal{R}_{f_it}$ is Cohen–Macaulay for each f_i. Note that f_i is a regular element of $\mathcal{R}_{f_it}$ and there is the canonical isomorphism

$$\mathcal{R}_{f_it}/(f_i)_{f_it} \simeq \mathrm{gr}_I(R)_{f_it}.$$

The assertion follows since the last ring is Cohen–Macaulay. □

Remark 4.4 These two results establish the implications in the diagram (1). None of the reverse implications hold without restrictions. The simplest situation to consider is that of a local ring $(R, \mathfrak{m})$ with infinite residue field. Let I be a primary ideal and let $a \in I$ generate a minimal reduction. It follows that a, at is a system of parameters of for the irrelevant maximal ideal of $R[It]$, which is obviously a regular sequence only if $I = (a)$. To find examples, it suffices to consider the irrelevant ideal $I = A_+$ of a 1-dimensional graded ring A, so that $\mathrm{gr}_I(A) = A$.

Cohomological criteria

The relationship between the Cohen–Macaulayness of $R[It]$ and $\mathrm{gr}_I(R)$ was shown by Trung and Ikeda ([74]) to depend on the degrees of the minimal generators of the canonical module of $\mathrm{gr}_I(R)$. It provides for a very broad setting in which to look at $\mathcal{R}$ vis-a-vis $\mathcal{G}$.

a-invariant

Let us recall the notion of the a-invariant of a graded ring introduced by Goto and Watanabe ([20]). We follow the exposition of [9].

Definition 4.5 Let
$$R = R_0 + R_1 + \cdots$$
be a graded ring of Krull dimension d with irrelevant maximal ideal $M = (\mathfrak{m}, R_+)$. ($(R_0, \mathfrak{m})$ is a local ring.) The integer

$$a(R) \quad = \quad \sup\{i \mid H^d_M(R)_i \neq 0\}, \tag{16}$$

where $H^d_M(R)$ denotes the (graded) d–dimensional local cohomology module of R with respect to M, is the *a–invariant* of R. More generally, for any finitely generated graded R–module F and for each integer $i \geq 0$, set
$$a_i(M, F) = \sup\{\, j \mid H^i_M(F)_j \neq 0 \,\}.$$

Note that since the module $H^d_M(R)$ is Artinian, $a(R)$ well defined If ω_R is the canonical module of R, by local duality, it follows that

$$a(R) \quad = \quad -\inf\{\, i \mid (\omega_R/M \cdot \omega_R)_i \neq 0 \,\}. \tag{17}$$

The depth of $\mathcal{G}$

To enhance the comparison between the properties of $R[It]$ and of $\mathrm{gr}_I(R)$, we give a result of [37] in a situation where $\mathrm{gr}_I(R)$ is not Cohen–Macaulay.

Theorem 4.6 *Let R be a Noetherian local ring with* depth $R \geq d$ *and let I be an ideal.*

$$If \text{ depth } \mathrm{gr}_I(R) < d, \quad then \text{ depth } R[It] = \text{depth } \mathrm{gr}_I(R) + 1.$$

Proof. This proof avoids the use of generalized depth employed in [37]. We can compute all depths with respect to the irrelevant maximal ideal $P = (\mathfrak{m}, ItR[It])$ of $R[It]$. For simplicity put $Q = ItR[It]$ and $Q(+1) = IR[It]$.

We are going to determine the depth of $R[It]$ by examining the exact sequences of local cohomology modules derived from the sequences (14) and (15). For simplicity of notation we denote $H^j_P(\bullet)$ by $H^j(\bullet)$.

Since depth $R = d$ and depth $\mathcal{G} = r < d$, we have

$$\begin{aligned} H^j(R) &= 0, \; j < d && (18) \\ H^j(\mathcal{G}) &= 0, \; j < r. && (19) \end{aligned}$$

The portions of the cohomology sequences that we are interested in are:

$$\begin{aligned} &H^{j-1}(R) \longrightarrow H^j(Q) \longrightarrow H^j(\mathcal{R}) \longrightarrow H^j(R) && (20) \\ &H^{j-1}(\mathcal{G}) \longrightarrow H^j(Q(+1)) \longrightarrow H^j(\mathcal{R}) \longrightarrow H^j(\mathcal{G}). && (21) \end{aligned}$$

Taking into these two sequences the conditions in (18) and (19) yield the exact sequences

$$H^j(Q) \simeq H^j(\mathcal{R}),\ j < d \tag{22}$$
$$0 \longrightarrow H^j(Q(+1)) \longrightarrow H^j(\mathcal{R}) \longrightarrow H^j(\mathcal{G}),\ j \leq r, \tag{23}$$

from which we claim that $H^j(\mathcal{R}) = 0$, for $j \leq r$. This will suffice to prove the assertion.

Denote the modules $H^j(Q)$ and $H^j(\mathcal{R})$ respectively by M_j and N_j. The sequences (22) and (23) then give rise to graded isomorphisms (of degree zero)

$$\begin{aligned} M_j(+1) &\simeq N_j,\ j < r \\ M_j &\simeq N_j,\ j < d \end{aligned}$$

and the monomorphism

$$0 \longrightarrow M_r(+1) \xrightarrow{\varphi} N_r. \tag{24}$$

Since $N_r \simeq M_r(+1) \simeq M_r$ as (ungraded) modules, φ is a monomorphism of isomorphic Artinian modules, and therefore must be an isomorphism. This means that we have isomorphisms of Artinian graded modules

$$M_j \simeq N_j \simeq M_j(+1),\ j \leq r,$$

with mappings of degree zero. As the graded components of these modules are zero in all degrees sufficiently high, the modules must vanish. □

The next result by Trung and Ikeda [74] is central to our understanding of the relationship between the depth properties of $\mathcal{R}$ and $\mathcal{G}$. Its proof is a model of clarity.

Theorem 4.7 (Trung-Ikeda) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay of dimension d and let I be an ideal of positive height. The following equivalence holds:*

$$\mathcal{R} \text{ is Cohen–Macaulay} \iff \begin{cases} G \text{ is Cohen–Macaulay and} \\ a(\mathcal{G}) < 0. \end{cases} \tag{25}$$

Proof. Suppose $\mathcal{G}$ is Cohen–Macaulay and $a(\mathcal{G}) < 0$. For $j = d$, the equations (20) and (21) become

$$0 \longrightarrow H^d(Q) \longrightarrow H^d(\mathcal{R}) \longrightarrow H^d(R)$$
$$0 \longrightarrow H^d(Q(+1)) \longrightarrow H^d(\mathcal{R}) \longrightarrow H^d(\mathcal{G}).$$

Because these graded modules vanish in high degrees, taken together the sequences imply that $H^d(Q)_i = H^d(\mathcal{R})_i = 0$, for $i > 0$. With however $a(\mathcal{G}) < 0$, it follows that $H^d(\mathcal{R})_0 = 0$ as well. We are then in the position to argue as in the previous proof.

Conversely, if $\mathcal{R}$ is Cohen–Macaulay then $\mathcal{G}$ is Cohen–Macaulay by Theorem 4.2. To show that $a(\mathcal{G}) < 0$, consider the exact sequence

$$0 \to H^d(\mathcal{G}) \longrightarrow H^{d+1}(I \cdot \mathcal{R}) \longrightarrow H^{d+1}(\mathcal{R}) \to 0,$$

and observe that $I \cdot \mathcal{R}$ is Cohen–Macaulay whose a–invariant must be negative (we come back more fully to this point in Theorem 5.3). □

5 The fundamental divisor of a Rees algebra

Our main purpose is to introduce a divisorial ideal associated to a Rees algebra and sketch out some of its applications ([51]). It helps to explain old puzzles while at the same time providing quite direct proofs of earlier results. The reader will note that it is a mirror image of local cohomology modules of Rees algebras. Its Noetherian character however permits a control of computation that is not always possible with Artinian modules. The properties of this divisor seem to be related to some sheaf cohomology results of Grauert–Riemenschneider ([21]), but we have not worked out this (if any) similarity.

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, with a canonical module ω. Let $I = (f_1, \ldots, f_n)$ be an ideal of positive height. Fix a presentation of $\mathcal{R}$, $B = R[T_1, \ldots, T_n] \to \mathcal{R}$. We set $\omega_B = \omega \otimes_R B(-n)$ as the canonical module of the polynomial ring B. The canonical module of $\mathcal{R}$ is the module

$$\omega_{\mathcal{R}} = \mathrm{Ext}_B^{n-1}(\mathcal{R}, \omega_B).$$

In our study, a related module plays an important role:

Definition 5.1 The *fundamental divisor* of $\mathcal{R} = R[It]$ is the module

$$\mathcal{D}(I) = \omega_{I \cdot \mathcal{R}} = \mathrm{Ext}_B^{n-1}(I \cdot \mathcal{R}, \omega_B).$$

In other words, $\mathcal{D}(I)$ is just the canonical dual of $I \cdot \mathcal{R}$. It is a rank 1 module over $\mathcal{R}$ that has the condition (S_2) of Serre and localizes to the canonical module of $K[t]$, where K is the total ring of fractions of R. This means that $\mathcal{D}(I)$ can be written as

$$\mathcal{D}(I) = W_1 t + W_2 t^2 + W_3 t^3 + \cdots,$$

where each W_i is an R–submodule of K. We fix this representation of $\mathcal{D}(I)$ from a given projective resolution of $I \cdot \mathcal{R}$ and the computation of the cohomology.

Remark 5.2 This divisorial ideal carries more information than the canonical module $\omega_{\mathcal{R}}$. Indeed, it will be seen below (see (28)) that

$$\omega_{\mathcal{R}} = W_2 t + W_3 t^2 + \cdots.$$

The next result (see [51]) recasts aspects of the characterization of the Cohen–Macaulay property of $\mathcal{R}$ in terms other than the vanishing of local cohomology.

Theorem 5.3 (Arithmetical criterion) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local with a canonical module ω and let I be an ideal of positive height. The following equivalence holds:*

$$\mathcal{R} \text{ is Cohen–Macaulay} \iff \begin{cases} \mathcal{G} \text{ is Cohen–Macaulay and} \\ W_1 \simeq \omega. \end{cases} \tag{26}$$

Before we give a proof we consider the case of 1–dimensional rings, when the assertions are stronger.

Theorem 5.4 *Let $(R, \mathfrak{m})$ be a 1–dimensional Cohen–Macaulay local ring with a canonical module ω and let I be a $\mathfrak{m}$–primary ideal. The following equivalence holds:*

$$\mathcal{R} \text{ is Cohen–Macaulay} \iff W_1 \simeq \omega. \tag{27}$$

Proof. We prove that if $W_1 \simeq \omega$, then I is a principal ideal. The other assertions have been established before or will be proved in the full theorem.

We may assume that the residue field of R is infinite. Let (a) be a reduction of I and suppose $I^{r+1} = aI^r$. We claim that $r = 0$. To this end, consider $R[at]$ whose canonical module is $at\omega R[at]$. Let

$$\mathcal{D}(I) = W_1 t + W_2 t^2 + \cdots \simeq \operatorname{Hom}_{R[at]}(I \cdot R[It], at\omega R[at]).$$

W_1 is defined by the relations

$$\begin{aligned} W_1 \cdot I &\subset a\omega \\ W_1 \cdot I^2 &\subset a^2\omega \\ &\vdots \\ W_1 \cdot I^r &\subset a^r\omega. \end{aligned}$$

The descending chain of fractional ideals of R,

$$a^r\omega : I^r \subset \cdots \subset a\omega : I,$$

implies that

$$W_1 = \lambda\omega = a^r\omega : I^r,$$

where λ is some element in the total ring of fractions of R. This equality means that

$$\omega = \operatorname{Hom}_R(I^r a^{-r} \lambda^{-1}, \omega),$$

and therefore that $I^r a^{-r}\lambda^{-1} = R$, since $\mathrm{Hom}_R(\cdot,\omega)$ is self-dualizing on the fractional ideals of R. This means that I^r is a principal ideal and I will also be principal, as R is a local ring. $\square$

Proof of Theorem 5.3. We consider the long exact sequences of graded B-modules that result from applying the functor $\mathrm{Hom}_B(\cdot,\omega_B)$ to the sequences (14) and (15). We have:

$$\begin{gathered} 0 \longrightarrow \mathrm{Ext}_B^{n-1}(\mathcal{R},\omega_B) \longrightarrow \mathrm{Ext}_B^{n-1}(It\cdot\mathcal{R},\omega_B) \longrightarrow \mathrm{Ext}_B^{n}(R,\omega_B) = \omega \\ \cdots \longrightarrow \mathrm{Ext}_B^{i}(\mathcal{R},\omega_B) \xrightarrow{\psi_i} \mathrm{Ext}_B^{i}(It\cdot\mathcal{R},\omega_B) \longrightarrow 0,\ i\geq n. \end{gathered} \tag{28}$$

and

$$\begin{gathered} 0 \longrightarrow \mathrm{Ext}_B^{n-1}(\mathcal{R},\omega_B) \longrightarrow \mathrm{Ext}_B^{n-1}(I\cdot\mathcal{R},\omega_B) \longrightarrow \mathrm{Ext}_B^{n}(\mathcal{G},\omega_B) = \omega_G \\ \cdots \longrightarrow \mathrm{Ext}_B^{i}(\mathcal{R},\omega_B) \xrightarrow{\theta_i} \mathrm{Ext}_B^{i}(I\cdot\mathcal{R},\omega_B) \longrightarrow 0,\ i\geq n. \end{gathered} \tag{29}$$

In the first of these sequences, in degree 0, we have the injection

$$0 \to W_1 \xrightarrow{\varphi} \omega \longrightarrow \square \to 0 \tag{30}$$

that is fixed and that we are going to exploit repeatedly. Suppose that $\mathcal{G}$ is Cohen–Macaulay and $W_1 \simeq \omega$. In this case, φ is an injection of modules with the (S_2) property that is an isomorphism in codimension 1. Thus φ is an isomorphism. This implies that the mappings ψ_i are (graded) isomorphisms for all $i > n$. In the other sequences meanwhile, the mappings θ_i are surjections for all $i \geq n$. In view however that $\mathrm{Ext}_B^i(I\cdot\mathcal{R},\omega_B) \simeq \mathrm{Ext}_B^i(It\cdot\mathcal{R},\omega_B)$, as ungraded modules, θ_i being surjections of isomorphic Noetherian modules must be isomorphisms. This implies that $\mathrm{Ext}_B^i(I\cdot\mathcal{R},\omega_B) \simeq \mathrm{Ext}_B^i(It\cdot\mathcal{R},\omega_B)$ as graded modules, which is a contradiction since one is obtained from the other by a non-trivial shift in the grading.

Conversely, if $\mathcal{R}$ is Cohen–Macaulay, from (14) we have that $\mathcal{G}$ is Cohen–Macaulay, and from $\mathrm{Ext}_B^n(\mathcal{R},\omega_B) = 0$, we have an isomorphism $W_1 \simeq \omega$. $\square$

Veronese subrings

A simple application of Theorem 5.3 is to show that a common device, passing from a graded algebra to one of its Veronese subrings in order to possibly enhance Cohen–Macaulayness, will not be helpful in the setting of ideals with associated graded rings which are already Cohen–Macaulay. (Craig Huneke has informed us that J. Lipman has also observed this.)

Let $\mathcal{R} = R[It]$ be the Rees algebra of an ideal I, let $q \geq 1$ be a positive integer and denote

$$\mathcal{R}_0 = \sum_{j\geq 0} I^{jq}t^{jq},$$

the q^{th} Veronese subring of $\mathcal{R}$. Our purpose here is to prove:

Theorem 5.5 *Let R be a Cohen–Macaulay ring, let I be an ideal of positive height such that the associated graded ring $\mathcal{G} = \mathrm{gr}_I(R)$ is Cohen–Macaulay. Then $\mathcal{R}$ is Cohen–Macaulay if and only if any Veronese subring $\mathcal{R}_0$ is Cohen–Macaulay.*

Proof. Most of the assertions and clear, following from the fact that as an $\mathcal{R}_0$–module, $\mathcal{R}$ is finitely generated and contains $\mathcal{R}_0$ as a summand. As for the hypotheses, if $\mathcal{G}$ is Cohen–Macaulay, the extended Rees algebra $A = R[It, t^{-1}]$ will also be Cohen–Macaulay, and the ring $A/(t^{-q})$ with it. Since the associated graded ring $\mathcal{G}_0$ of I^q is a direct summand of the latter, $\mathcal{G}_0$ is Cohen–Macaulay.

It will suffice to show that the fundamental divisors of $\mathcal{R}$ and $\mathcal{R}_0$,

$$\begin{aligned} \mathcal{D}(I) &= W_1 t + W_2 t^2 + \cdots \\ \mathcal{D}(I^q) &= L_q t^q + L_{2q} t^{2q} + \cdots, \end{aligned}$$

relative to the respective algebras, satisfy $W_1 \simeq L_q$.

Let ω_0 denote the canonical module of $\mathcal{R}_0$. Let us calculate $\mathcal{D}(I)$ as

$$\begin{aligned} \mathcal{D}(I) &\simeq \mathrm{Hom}_{\mathcal{R}_0}(I \cdot \mathcal{R}, \omega_0) \\ &= \bigoplus_{s=1}^{q} \mathrm{Hom}_{\mathcal{R}_0}(t^{s-1} I^s \cdot \mathcal{R}_0, \omega_0) \\ &\simeq \bigoplus_{s=1}^{q} \mathrm{Hom}_{\mathcal{R}_0}(I^s \cdot \mathcal{R}_0, \omega_0)(s-1). \end{aligned}$$

The degrees have been kept track of, permitting us to match the components of degree 1, respectively W_1 on the left and L_q on the right. The remaining assertion will then follow from Theorem 5.3.
□

Symbolic powers

Proposition 5.6 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module ω, and let I be an ideal which is generically a complete intersection. Suppose that for each prime ideal $\mathfrak{p} \supset I$, with* $\mathrm{height}(\mathfrak{p}/I) \geq 1$, $\ell(I_\mathfrak{p}) <$ height $\mathfrak{p}$. *Then $W_1 \simeq \omega$.*

Proof. We claim that the mapping labeled φ above is an isomorphism:

$$0 \to W_1 \xrightarrow{\varphi} \omega \longrightarrow C \to 0.$$

We must show that $C = 0$. By induction on the dimension of R, we may assume that C is a module of finite length.

If $\mathfrak{m}$ is a minimal prime of I, this ideal is a complete intersection. Suppose then that I is not $\mathfrak{m}$–primary. By assumption $\ell(I) <$ height $\mathfrak{m}$, so that height $\mathfrak{m}\mathcal{R} \geq 2$. We may thus find $a, b \in \mathfrak{m}$ so that height $(a, b)\mathcal{R} = 2$. Since $\mathcal{B}(I)$ is an (S_2)–module over $\mathcal{R}$, a, b must be a regular sequence on $\mathcal{B}(I)$. In particular, a, b is a regular sequence on W_1, which is clearly impossible if C is a nonzero module of finite length. □

The following is an application to the symbolic powers of a prime ideal (see [82] for the Gorenstein case, and [4] for the general case):

Corollary 5.7 *Let R be a Cohen–Macaulay ring and let $\mathfrak{p}$ be a prime ideal of positive height such that $R_\mathfrak{p}$ is a regular local ring. Suppose that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for $n \geq 1$. Then $R[\mathfrak{p}t]$ is Cohen–Macaulay if and only if $\mathrm{gr}_\mathfrak{p}(R)$ is Cohen–Macaulay.*

Proof. The condition on the equality of the ordinary and symbolic powers of $\mathfrak{p}$ implies the condition on the local analytic spread of $\mathfrak{p}$. In turn, this condition is preserved after we localize R at any prime ideal and complete. □

For these ideals one can weaken the hypothesis that $\mathcal{G}$ be Cohen–Macaulay in a number of ways. Here is a result from [53]:

Theorem 5.8 *Let R be a Gorenstein local ring of dimension d and let I be an unmixed ideal of codimension $g \geq 1$, that is generically a complete intersection and is such that $I^{(n)} = I^n$ for $n \geq 1$. Then $\mathcal{R}$ is Cohen–Macaulay if and only if $\mathcal{G}$ satisfies (S_r) for $r = \lceil \frac{d+1}{2} \rceil$.*

A first step in the proof consists in the following calculation ([53]):

Proposition 5.9 *Let R be a Gorenstein local ring and let I be an unmixed ideal of codimension $g \geq 1$, that is generically a complete intersection and is such that $I^{(n)} = I^n$ for $n \geq 1$. Then the canonical module of $\mathcal{R} = R[It]$ has the expected form, that is $\omega_{\mathcal{R}} \simeq (t(1,t)^{g-2})$.*

Question 5.10 Which toric prime ideals $\mathfrak{p}$ have the property that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for $n \geq 1$? Particularly interesting are those of codimension 2 and dimension 4.

Equimultiple ideals

One landmark result in the relationship between $\mathcal{R}$ and $\mathrm{gr}_I(R)$ was discovered by Goto–Shimoda [19] (later extended in [22]).

Theorem 5.11 (Goto-Shimoda) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay ring of dimension $d > 1$ with infinite residue field, and let I be an equimultiple ideal of codimension $g \geq 2$. Then*

$$\mathcal{R} \text{ is Cohen–Macaulay} \iff \begin{cases} \mathcal{G} \text{ is Cohen–Macaulay and} \\ r(I) < g. \end{cases} \quad (31)$$

Proof. We may assume that R is a complete local ring, and therefore there is a canonical module ω. Let J be a minimal reduction of I. Since J is generated by a regular sequence, the Rees algebra $\mathcal{R}_0 = R[Jt]$ is determinantal and its canonical module is (see [8], [33]):

$$\omega_0 = \omega \cdot t(1,t)^{g-2} = \omega \cdot t + \cdots + \omega \cdot t^{g-1} + J\omega \cdot t^g + \cdots .$$

We can calculate $\mathcal{D}(I)$ as

$$\mathcal{D}(I) = \mathrm{Hom}_{\mathcal{R}_0}(I\mathcal{R}, \omega_0) = W_1 t + W_2 t^2 + \cdots ,$$

where W_1 must satisfy the equations

$$\begin{array}{rcl} I \cdot W_1 & \subset & \omega \\ & \vdots & \\ I^{g-1} \cdot W_1 & \subset & \omega \\ I^g \cdot W_1 & \subset & J \cdot \omega \\ & \vdots & \end{array}$$

Note that since ω has (S_2) and height $I > 1$, W_1 can be identified to a subideal of ω and coincides with ω in codimension 1.

Suppose $\mathcal{R}$ is Cohen–Macaulay, so that $W_1 \simeq \omega$. But $W_1 \subset \omega$ and both fractionary ideals are (S_2) and thus they must coincide since they are equal in codimension 1. From the equation $I^g \cdot \omega \subset J \cdot \omega$, it follows that I^g is contained in the annihilator of $\omega / J \cdot \omega$. But this is the canonical module of R/J, and therefore $I^g \subset J$. Since $\mathcal{G}$ is Cohen–Macaulay, by Theorem 3.11 we must have $I^g = J \cdot I^{g-1}$.

For the converse, the equations give that $W_1 = \omega$, so we may apply Theorem 5.3. □

Conjecture 5.12 If $(R, \mathfrak{m})$ is a regular local ring, for any nonzero ideal I the degree 1 component of $\mathcal{D}(I)$ is always principal.

If I is an $\mathfrak{m}$–primary ideal, the conjecture just rephrases a basic form of the theorem of Briançon–Skoda (see a discussion of this theorem and its role in the Cohen–Macaulayness Rees algebras in [3], [4]). It would be a far–reaching generalization of one of the main results of [48]:

Theorem 5.13 *Let R be a regular local and let I be an ideal. Then $\mathcal{R} = R[It]$ is Cohen–Macaulay if and only if $\mathcal{G} = \mathrm{gr}_I(R)$ is Cohen–Macaulay.*

Problem 5.14 Let I be an ideal of the Cohen–Macaulay ring R, and let x be an indeterminate over R. Study the relationship between $\mathcal{D}(I)$ and $\mathcal{D}(I, x)$.

6 Approximation complexes

We now introduce a family of extensions of Koszul complexes that lead to many classes of ideals which are explosively Cohen–Macaulay ([28] has an extensive discussion). They are intimately related–in the same manner that Koszul complexes are related to regular sequences–to the following property.

Ideals generated by d–sequences

Definition 6.1 Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a sequence of elements in a ring R generating the ideal I. $\mathbf{x}$ is called a *d–sequence* if $(x_1, \ldots, x_i) \colon x_{i+1}x_k = (x_1, \ldots, x_i) \colon x_k$ for $i = 0, \ldots, n-1$ and $k \geq i+1$.

Its significance was recognized early ([39], [80]):

Theorem 6.2 *Every ideal generated by a d–sequence is of linear type.*

The sequences can also be defined with respect to a module E. The natural habitat of these sequences is a family of differential graded modules whose construction we recall (see [28] and [29]).

Construction of the complexes

Consider a double Koszul complex in the following sense. Let F and G be R–modules and suppose there are two mappings:

$$\begin{array}{ccc} F & \xrightarrow{\psi} & G \\ \varphi \downarrow & & \\ R & & \end{array}$$

The algebra $\wedge F \otimes S(G)$ is a double complex with differentials

$$d_\varphi = \partial \colon \wedge^r F \otimes S_t(G) \longrightarrow \wedge^{r-1} F \otimes S_t(G),$$

$$\partial(e_1 \wedge \cdots \wedge e_r \otimes g) = \sum (-1)^{i-1} \varphi(e_i)(e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r) \otimes g,$$

which is the Koszul complex associated to the mapping φ and coefficients in $S(G)$, while

$$d_\psi = \partial' \colon \wedge^r F \otimes S_t(G) \longrightarrow \wedge^{r-1} F \otimes S_{t+1}(G),$$

$$\partial\prime(e_1 \wedge \cdots \wedge e_r \otimes g) = \sum (-1)^{i-1} (e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_r) \otimes \psi(e_i) \cdot g,$$

defines the Koszul mapping of ψ. It checks easily that

$$\partial \cdot \partial' + \partial' \cdot \partial = 0.$$

This skew–commutativity is nice for the following reason. Each differential will then induce on the cycles of the other a differential graded structure.

The resulting double complex will be denoted by $\mathcal{L} = \mathcal{L}(\varphi, \psi)$. Of significance here is the case where $\varphi\colon F = R^n \to R$ is the mapping associated to a sequence $\mathbf{x} = \{x_1, \ldots, x_n\}$ of elements of R.

Definition 6.3 The complex $\mathcal{L} = \mathcal{L}(\mathbf{x}) = \mathcal{L}(\varphi,$ identity),

$$\begin{array}{ccc} R^n & \stackrel{\text{identity}}{\longrightarrow} & R^n \\ \varphi \downarrow & & \\ R & & \end{array}$$

will be called the double Koszul complex of $\mathbf{x}$.

The complex $\mathcal{L}(\partial)$ is the Koszul complex associated to the sequence $\mathbf{x}$ in the polynomial ring $S = S(R^n) = R[T_1, \ldots, T_n]$. $\mathcal{L}(\partial')$, on the other hand, is also an ordinary Koszul complex but constructed over the sequence $\mathbf{T} = T_1, \ldots, T_n$. Thus we have a grading

$$\mathcal{L}(\partial') = \sum \mathcal{L}_t$$

by subcomplexes of R–modules,

$$\mathcal{L}_t = \sum_{r+s=t} \wedge^r R^n \otimes S_s(R^n),$$

and the $\mathcal{L}_t$ are exact for $t > 0$. Attaching coefficients from an R–module E extends the construction to the complex $\mathcal{L}(\mathbf{x}, E) = E \otimes \wedge R^n \otimes S(R^n)$. If we denote by $Z_\bullet = Z_\bullet(\mathbf{x}; E)$ the module of cycles of $K(\mathbf{x}; E)$ and by $H_\bullet = H_\bullet(\mathbf{x}; E)$ its homology, the (skew–) commutativity of ∂ and ∂' yield several new complexes among which we single out:

- $\mathcal{Z}$–complex : $\mathcal{Z}_\bullet = \mathcal{Z}_\bullet(\mathbf{x}; E) = \{Z_\bullet \otimes S, \partial'\}$
- $\mathcal{M}$–complex : $\mathcal{M}_\bullet = \mathcal{M}(\mathbf{x}; E = \{H_\bullet \otimes S, \partial'\}$

The properties of these complexes, particularly their acyclicity, are treated in detail in [28] and [29].

The $\mathcal{M}$–complex, as well as the $\mathcal{Z}$–complex, are graded complexes over the polynomial ring $S = R[T_1, \ldots, T_n]$. The rth homogeneous component $\mathcal{M}_r$ of $\mathcal{M}_\bullet$ is a complex of finitely generated R–modules

$$0 \to H_n \otimes S_{r-n} \longrightarrow \cdots \longrightarrow H_1 \otimes S_{r-1} \longrightarrow H_0 \otimes S_r \to 0.$$

For certain uses however we shall view them as defined over S:

$$0 \to H_n \otimes S(-n) \longrightarrow \cdots \longrightarrow H_1 \otimes S(-1) \longrightarrow H_0 \otimes S \to 0. \tag{32}$$

One key aspect of these complexes lies on the fact that $H_0(\mathcal{M}(\mathbf{x}; E))$ maps onto $\mathrm{gr}_{(\mathbf{x})}(E)$, the associated graded module of E with respect to $(\mathbf{x})$, often allowing to predict arithmetical properties of $\mathrm{gr}_{(\mathbf{x})}(E)$ and of its torsion free version.

The modules we want to apply this construction to are either the ideal I (for $\mathcal{Z}$–complex), or its conormal module I/I^2 (for the $\mathcal{M}$–complex).

The relationship between d–sequences and the acyclicity of $\mathcal{M}_\bullet$ is ([28, Theorem 12.9]):

Theorem 6.4 *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Let I be an ideal of R. The following conditions are equivalent:*

(a) *$\mathcal{M}(I)$ is acyclic.*

(b) *I is generated by a d–sequence.*

Strongly Cohen–Macaulay ideals

To be useful these complexes require that its coefficient modules, say H_i, have good depth properties. We shall define two classes of ideals that meet these requirements.

Definition 6.5 Let I be an ideal of the ring R of dimension d. Let $\mathbb{K}$ be the Koszul complex on the set $\mathbf{a} = \{a_1, \ldots, a_n\}$ of generators of I, and let k be a positive integer. I has *sliding depth* condition SD_k if the homology modules of $\mathbb{K}$ satisfy

$$\operatorname{depth} H_i(\mathbb{K}) \geq d - n + i + k, \text{ for all } i.$$

The unspecified sliding depth is the case $k = 0$. There is a more stringent condition on the depth of the Koszul homology modules ([40], [41]):

Definition 6.6 I is *strongly Cohen–Macaulay* if the Koszul homology modules of I with respect to one (and then to any) generating set are Cohen–Macaulay.

Example 6.7 Let I be the ideal generated by the minors of order 2 of the generic symmetric matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

I satisfies sliding depth but it is not strongly Cohen–Macaulay.

The broadest class of examples of strongly Cohen–Macaulay ideals arises from a theorem of Huneke ([40]):

Theorem 6.8 *Let R be a Cohen–Macaulay local ring and suppose L is an ideal of grade n. Let $\mathbf{x} = x_1, \ldots, x_n$ and $\mathbf{y} = y_1, \ldots, y_n$ be two regular sequences in L and set $I = (\mathbf{x})\colon L$ and $J = (\mathbf{y})\colon L$. If I has sliding depth then J has also sliding depth.*

Corollary 6.9 *Let I be a Cohen–Macaulay ideal in the linkage class of a complete intersection. Then I is strongly Cohen–Macaulay.*

Corollary 6.10 *Let R be a regular local ring and let I be either a Cohen–Macaulay ideal of codimension two or a Gorenstein ideal of codimension three. Then I is strongly Cohen–Macaulay.*

Summary of results

Before we collect several results on these complexes, we recall a condition on the Fitting ideals of a module.

Definition 6.11 Let φ be a matrix with entries in R, defining the R–module E. For an integer k, φ (or E) satisfies the condition $\mathcal{F}_k$ if:

$$\text{height } I_t(\varphi) \geq \text{rank}(\varphi) - t + 1 + k, \quad 1 \leq t \leq \text{rank}(\varphi).$$

Remark 6.12 If E is an ideal, $\mathcal{F}_1$ is the condition G_∞ of [5]. There are constraints as to which $\mathcal{F}_k$ condition a module may support; for instance, for an ideal the condition $\mathcal{F}_2$ would contradict Krull's principal ideal theorem. More discriminating is the condition G_s of [5] requiring $\mathcal{F}_1$ but only in codimension at most $s - 1$.

The following statement encapsulates some of the most important aspects of these complexes. We first record

Theorem 6.13 *Let R be a local ring with infinite residue field, and let E be a finitely generated R–module.*

1. *The following are equivalent:*

   (a) $\mathcal{Z}(E)$ *is acyclic.*

   (b) S_+ *is generated by a d–sequence of linear forms of* $S(E)$.

2. *If* $\mathcal{Z}(E)$ *is acyclic, the Betti numbers of* $S(E)$ *as a module over* $B = S(R^n)$ *are given by*

$$\beta_i^B(S(E)) = \sum_j \beta_j^R(Z_{i-j}(E)).$$

3. *If R is Cohen–Macaulay and E has rank e, the following conditions are equivalent:*

   (a) $\mathcal{Z}(E)$ *is acyclic and* $S(E)$ *is Cohen–Macaulay.*

   (b) *E satisfies* $\mathcal{F}_0$ *and*

$$\operatorname{depth} Z_i(E) \geq d - n + i + e,\ i \geq 0.$$

4. *Moreover, if R is Cohen–Macaulay with canonical module* ω_R *then*

   (a)

$$\omega_S/S_+\omega_S = \bigoplus_{i=0}^{\ell} \operatorname{Ext}_R^{\ell-i}(Z_i(E),\ \omega_R).$$

   (b) $S(E)$ *is Gorenstein if and only if* $\operatorname{Hom}_R(Z_{n-e}(E), \omega_R) = R$ *and*

$$\operatorname{depth} Z_i(E) \geq d - n + i + e + 1,\ i \leq n - e - 1.$$

We apply this to the conormal module I/I^2 of an ideal I:

Corollary 6.14 *Let R be a Cohen–Macaulay ring and let I be an ideal generated by a d–sequence. The following conditions are equivalent:*

(a) *The Rees algebra* $\mathcal{R}$ *is Cohen–Macaulay.*

(b) *The associated graded ring* $\mathcal{G}$ *is Cohen–Macaulay.*

(c) *I satisfies sliding depth.*

Corollary 6.15 *Let R be a Gorenstein ring and let I be an ideal generated by a d–sequence. The following conditions are equivalent:*

(a) *The associated graded ring* $\mathcal{G}$ *is Gorenstein.*

(b) *I is strongly Cohen–Macaulay.*

Fundamental divisor

Theorem 6.16 *Let R be a Cohen–Macaulay local ring with a canonical module ω and let I be an ideal of height $g \geq 2$ that is generated by a d–sequence. Then the first component of the fundamental divisor of I is ω.*

We first need:

Proposition 6.17 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with infinite residue field and let I be an ideal of height $g \geq 2$. If $W_1 = W_2$ then $W_1 = \omega$.*

Proof. Let $a \in I$ be a regular element and choose $b \in I$ satisfying the following two requirements: (i) b is regular on $R/(a)$, and (ii) a is a minimal generator of I and its initial form $b^* \in \mathcal{G}_1$ does not belong to any minimal prime of $\mathcal{G}$. We claim that the ideal $(a, bt)\mathcal{R}$ has height 2. If P is a prime ideal of height 1 containing a, bt it cannot contain I, since b^*, the image of bt in $\mathcal{R}/I\mathcal{R} = \mathcal{G}$ does not lie in any minimal prime of $\mathcal{G}$. This shows that $\mathcal{R}_P$ is a localization of a polynomial ring $R_c[t]$, and in this case (a, bt) has obviously height 2. We then have that a, bt is a regular sequence on the $\mathcal{R}$–module $\mathcal{D}(I)$ which has the property (S_2). As in the previous proof, if W_1 is not isomorphic to ω, we may assume that cokernel φ is a module of finite length. If $W_1 = W_2$, since a is regular on $\mathcal{D}(I)/bt \cdot \mathcal{D}(I)$, this implies that a is regular on $W_2/bW_1 = W_1/bW_1$. But this is a contradiction since W_1 has depth 1. □

Proof of Theorem 6.16. From (29), we have the exact sequence

$$0 \to W_2t + W_3t^2 + \cdots \longrightarrow W_1t + W_2t^2 + \cdots \longrightarrow \omega_{\mathcal{G}}, \qquad (33)$$

showing that W_1/W_2 embeds in the degree 1 component of $\omega_{\mathcal{G}}$.

In order to apply Proposition 6.17, it will be enough to show that ω_G is generated by elements of degree $g \geq 2$, or higher. For this purpose we use the acyclicity of the approximation complex (32). Let $\nu(I) = n = q + g$, so that the approximation complex of I provides a complex over $\mathcal{G}$

$$0 \to H_q \otimes S(-q) \longrightarrow \cdots \longrightarrow H_1 \otimes S(-1) \longrightarrow H_0 \otimes S \longrightarrow \mathcal{G} \to 0. \qquad (34)$$

In this complex, we may assume that S is actually a polynomial ring $S = A[T_1, \ldots, T_n]$, where A is a Cohen–Macaulay ring of dimension $d - g$, by simply taking $A = R$ modulo a regular sequence of g elements contained in I. Since the canonical module of S is $\omega_S = \omega_A \otimes S(-n)$, we can express $\omega_{\mathcal{G}}$ as

$$\omega_{\mathcal{G}} = \mathrm{Ext}^q_S(\mathcal{G}, \omega_S).$$

Applying $\mathrm{Hom}_S(\cdot, \omega_S)$ to the complex (34), it is easy to see that the module $\mathrm{Ext}^q_S(\mathcal{G}, \omega_S)$ is contained in a short exact sequence of modules derived entirely from submodules of

$$\mathrm{Ext}^i_S(H_j \otimes_A S(-j), \omega_S) = \mathrm{Ext}^i_S(H_j \otimes_A S(-j), \omega_A \otimes S(-n)) = \mathrm{Ext}^i_A(H_j, \omega_A) \otimes_A S(-n + j),$$

all of which are generated by elements of degree at least $n - j \geq g \geq 2$. □

7 Depth and cohomology of Proj

Let $A = A_0 + A_1 + \cdots$ be a Noetherian ring and let F be a graded A–module. For a given ideal $\mathfrak{p} = \mathfrak{p}_0 + A_+$, $\mathfrak{p}_0 \subset A_0$ of A we study briefly the relation between the local cohomology of F with regards to $\mathfrak{p}$ and to A_+. From the point of view of Rees algebras, this is justified because the vanishing of one cohomology reflects depth and the other reflects reduction number.

We begin by recalling a result of Brodmann [7, Lemma 3.9], and for its relative inaccessibility we sketch a proof.

Lemma 7.1 *Let R be a commutative Noetherian ring, I an ideal, $x \in R$ and F an R–module. There exists a natural exact sequence*

$$\begin{aligned} 0 \to H^0_{(I,x)}(F) \to H^0_I(F) \to H^0_{I_x}(F) \to H^1_{(I,x)}(F) \to \cdots \\ \cdots \to H^i_{(I,x)}(F) \to H^i_I(F) \to H^i_{I_x}(F) \to H^{i+1}_{(I,x)}(F) \to \cdots. \end{aligned}$$

Proof. Let $\mathbb{E}$

$$0 \to F \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots,$$

be an injective resolution of F. Each E_i is a direct sum of injective modules of the form $E(R/\mathfrak{q})$, the injective envelope of $R/\mathfrak{q}$ for some prime $\mathfrak{q}$. As a consequence, there is direct sum decomposition

$$0 \to \Gamma_{xR}(E_i) \longrightarrow E_i \longrightarrow (E_i)_x \to 0.$$

Thus for each $i \geq 0$, we have the exact sequence

$$0 \to \Gamma_I(\Gamma_{xR}(E_i)) \longrightarrow \Gamma(E_i) \longrightarrow \Gamma((E_i)_x) \to 0.$$

Since $\Gamma_I \cdot \Gamma_{xR} = \Gamma_{(I,x)}$, these exact sequences give rise to the short exact sequence of complexes

$$0 \to \Gamma_{(I,x)}(\mathbb{E}) \longrightarrow \Gamma_I(\mathbb{E}) \longrightarrow \Gamma_I((\mathbb{E})_x) \to 0.$$

The assertion of the lemma is the long exact cohomology sequence that results. □

Local reduction numbers

We now give a characterization of the Cohen–Macaulayness of a Rees algebra $R[It]$ that involve the reduction numbers of the localizations $I_\mathfrak{p}$. In fact, only finitely many come to play. We give the theorem of [45] (see also [4], [67]).

Theorem 7.2 *Let R be a Cohen–Macaulay local ring and let I be an ideal of positive height. The following equivalence holds:*

$$\mathcal{R} \text{ is Cohen–Macaulay} \iff \begin{cases} \mathcal{G} \text{ is Cohen–Macaulay and for each prime ideal } \mathfrak{p} \\ \text{such that } \ell(I_\mathfrak{p}) = \text{height}\, \mathfrak{p},\ r(I_\mathfrak{p}) < \text{height}\, \mathfrak{p}. \end{cases}$$

Proof. The point of the argument is to see how the assertions impact on the invariant $a(\mathcal{B})$. To this purpose, we are going to relate $a(\mathcal{B})$ to the cohomology of Proj $\mathcal{B}$.

Denote by N the irrelevant ideal of $\mathcal{B}$, $N = \mathcal{B}_+$, and let $M = (\mathfrak{m}, N)$ be the maximal irrelevant ideal of $\mathcal{B}$. We argue by induction on dim $R = d$ and on dim R/I that $H^d_M(\mathcal{B})$ vanishes in non-negative degrees if and only if $H^d_N(\mathcal{B})$ does so. If dim $R/I = 0$, I is an $\mathfrak{m}$–primary ideal and $\sqrt{N} = M$, and the functors Γ_M and Γ_N are equivalent. Suppose then dim $R/I > 0$, and pick $x \in \mathfrak{m}$ such that height $(N, x) =$ height $N + 1$.

By Lemma 7.1, we have the exact sequence

$$H^{d-1}_{N_x}(\mathcal{B}) \longrightarrow H^d_{(N,x)}(\mathcal{B}) \longrightarrow H^d_N(\mathcal{B}) \longrightarrow H^d_{N_x}(\mathcal{B}) \to 0,$$

where $H^d_{N_x}(\mathcal{B}) = 0$ since dim $R_x <$ dim $R = d$, and $H^{d-1}_{N_x}(\mathcal{B})$ is zero in non-negative degrees by induction. If (N, x) is not M–primary, we replace N by (N, x) and find an element $y \in \mathfrak{m}$ for which height $(N, x, y) =$ height $N + 2$.

If $J = (a_1, \ldots, a_d)$ is a reduction of I, $\sqrt{(a_1^*, \ldots, a_d^*)} = \sqrt{N = \mathcal{B}_+}$, so that in computing the local cohomology modules $H^i_N(\mathcal{B})$ we may use the ideal $(a_1^*, \ldots, a_d^*)$. This means that as in the assertion of the theorem, we may assume that $\ell(I) = d$.

We are now ready for the proof of the theorem. Again we may assume that the assertions are valid on the punctured spectrum of R. Suppose $\mathcal{R}$ is Cohen–Macaulay but $rn(I) \geq d$. Let $J = (a_1, \ldots, a_d)$, and denote by $\mathbb{K}$ the Koszul complex defined by the 1–forms $a_1^*, \ldots, a_d^*$ of $\mathcal{B}$:

$$0 \to K_d = G(-d) \longrightarrow K_{d-1} = G(-d+1)^d \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \to 0.$$

Under these conditions a calculation of Trung ([73, Proposition 3.2]) establishes:

Proposition 7.3 *Let $(R, \mathfrak{m})$ be a local ring dimension d and let $I \subset R$ be an ideal. Suppose J is a reduction of I generated by d elements. If $r_J(I) = r$, then*

$$a_d(\mathcal{B}_+, \mathcal{B}) + d \leq r \leq \max\{a_i(\mathcal{B}_+, \mathcal{B}) + i \mid i = 0 \ldots d\}.$$

This proposition implies that if $r < d$, then $a(\mathcal{B}) < 0$. The converse makes use of the transfer of the vanishing of cohomology sketched above (see [45]). □

8 Ideals with good reductions

Let R be a Cohen–Macaulay local ring, let I be an ideal and let J be a minimal reduction of I. If the reduction number $r_J(I)$ is low–practically meaning 1 or 2–it is often to possible determine the fundamental divisor of I from that of J. We examine two cases in this section, and a process under which they arise.

We explore briefly a merging of the techniques of the approximation complexes–which tend to work best for ideals without proper reductions–with the general method of minimal reductions. Obviously it will require special ideals.

Ideals with sliding depth

The template for ideals with sliding depth are the ideals in the linkage class of complete intersections. A derivative source arises from minimal reductions (see [76], [82], [83, Chapter 5] for more details).

Residual intersections

We first need to frame the notion of sliding depth in a natural setting. It will be the program of residual intersection initiated by Artin and Nagata (see [5]) and carried out to full development by Huneke and Ulrich in several works.

Definition 8.1 Let R be a Noetherian ring, I be an ideal and s be an integer $s \geq$ height I.

(a) An *s–residual intersection* of I is an ideal J such that height $J \geq s$ and $J = \mathfrak{a}: I$, for some s–generated ideal $\mathfrak{a} \subset I$.

(b) A *geometric s–residual intersection* is an s–residual intersection J of I such that height $(I + J) \geq s + 1$.

Remark 8.2 These definitions mean the following. Let $\mathfrak{a} = (a_1, \ldots, a_s) \subset I$, $J = \mathfrak{a}: I$. Then J is an s-residual intersection of I if for all prime ideals $\mathfrak{p}$ with dim $R_\mathfrak{p} \leq s - 1$ we have $I_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$. A geometric s–residual intersection requires that in addition for all $\mathfrak{p} \in V(I)$ with dim $R_\mathfrak{p} = s$ the equality $\mathfrak{a}_\mathfrak{p} = I_\mathfrak{p}$ also holds. The case where $s =$ height I is the notion of linkage.

Remark 8.3 We will be particularly interested in residual intersections that arise in the following fashion. Let I be an ideal of R of height g and let $\mathbf{x} = \{x_1, \ldots, x_s\}$ be a sequence of elements of I satisfying:

(1) height $(\mathbf{x}): I \geq s \geq g$.

(2) For all primes $\mathfrak{p} \supset I$ with height $\mathfrak{p} \leq s$, one has

  (i) $(\mathbf{x})_\mathfrak{p} = I_\mathfrak{p}$;

  (ii) $\nu((\mathbf{x})_\mathfrak{p}) \leq$ height $\mathfrak{p}$.

These sequences have the following additional properties:

(a) height $(\mathbf{x}) =$ height I;

(b) $\nu((\mathbf{x})_\mathfrak{p}) \leq$ height $\mathfrak{p}$ for all primes $(\mathbf{x}) \subset \mathfrak{p}$.

To prove (a), let $\mathfrak{p}$ be a minimal prime of $(\mathbf{x})$. Suppose $I \not\subset \mathfrak{p}$; then $((\mathbf{x}): I)_\mathfrak{p} = (\mathbf{x})_\mathfrak{p}$. It will follow from (1) that height $\mathfrak{p} \geq s \geq$ height I.

To verify (b), if height $\mathfrak{p} \geq s$, the assertion is trivial; meanwhile, if height $\mathfrak{p} < s$, the proof of (a) shows that $\mathfrak{p} \supset I$ and (2) applies.

Definition 8.4 The ideal I is said to be *residually Cohen–Macaulay* if for any sequence $\mathbf{x} \subset I$ with the properties (1) and (2) of the previous remark, it holds that:

(a) $R/((\mathbf{x}): I)$ is Cohen–Macaulay of dimension $d - s$;

(b) $((\mathbf{x}): I) \cap I = (\mathbf{x})$;

(c) height $((\mathbf{x}): I + I) >$ height $(\mathbf{x}): I$.

The next two statements spell out the significance of these rather technical definitions (see [34], [83, Chapter 4]).

Theorem 8.5 *Let R be a Cohen–Macaulay local ring and I be an ideal. If I has sliding depth then it is a residually Cohen–Macaulay ideal.*

Theorem 8.6 *Let R be a Cohen–Macaulay local ring and let I be an ideal satisfying the condition $\mathcal{F}_1$. The following conditions are equivalent:*

(a) *I satisfies the sliding depth condition.*

(b) *I is residually Cohen–Macaulay.*

(c) *I can be generated by a d–sequence $\{x_1, \ldots, x_n\}$ such that*

$$(x_1, \ldots, x_{i+1})/(x_1, \ldots, x_i)$$

is Cohen–Macaulay module of dimension $d - i$, for $i = 0, \ldots, n - 1$.

Koszul homology of reductions

Theorem 8.7 *Let R be a Cohen–Macaulay local ring of dimension d, let I be an ideal, let J be a reduction of I with $\nu(J) = s \leq d$, and assume that I satisfies $\mathcal{F}_1$, locally in codimension $\leq s - 1$. If I has sliding depth, then J has sliding depth, and in particular J is of linear type and $R[Jt]$ is Cohen–Macaulay.*

Proof. We may assume that $g =$ height $J < s \leq d$. Further notice that $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq s - 1$.

At this juncture, it is no restriction to assume that the residue field of R is infinite. Now let $a_1, \ldots, a_s$ be a generating sequence of J, and for $g \leq i \leq s - 1$ write $L_i = (a_1, \ldots, a_i)$ and $K_i = L_i: J$. Since J satisfies $\mathcal{F}_1$, we may choose $a_1, \ldots, a_s$ in such a way that $J_{\mathfrak{p}} = (L_i)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq i - 1$ and all $\mathfrak{p} \in V(J)$ with $\dim R_i \leq i$. In other words, height $K_i \geq i$ and height$(J + K_i) \geq i + 1$. We claim that for $g \leq i \leq s - 1$, R/K_i is Cohen–Macaulay of dimension $d - i$ and $J \cap K_i = L_i$. Once this is shown, the theorem will follow from Theorem 8.6.

To prove the claim notice that since $i \leq s-1$, we have $I_{\mathfrak{p}} = J_{\mathfrak{p}} = (L_i)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq i-1$ and all $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq i$. Thus height $L_i : I \geq i$ and height$(I + (L_i : I)) \geq i+1$. However, $\nu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq i \leq s-1$, and I is assumed to have sliding depth. In this situation, Theorem 8.5 implies that $R/L_i : I$ is a Cohen–Macaulay ideal of dimension $d-i$ and $I \cap (L_i : I) = L_i$.

Now it suffices to prove that $L_i : I = K_i$. The inclusion $L_i : I \subset L_i : J = K_i$ being trivial, we only need to show the asserted equality at every associated prime $\mathfrak{p}$ of $L_i : I$. Since the latter ideal is Cohen–Macaulay of height i, we know that $\dim R_{\mathfrak{p}} = i \leq s-1$. Thus $I_{\mathfrak{p}} = J_{\mathfrak{p}}$, and $(L_i : I)_{\mathfrak{p}} = (K_i)_{\mathfrak{p}}$.

Finally, that $R[Jt]$ is Cohen–Macaulay follows from Theorem 6.13. □

Corollary 8.8 *Let I and J be as in* Theorem 8.7. *Let $\mathbb{K}$ be the Koszul complex on a minimal set of generators of J. Then I annihilates $H_i(\mathbb{K})$ for $i > 0$.*

Proof. Since $I \cdot H_i(\mathbb{K}) \hookrightarrow H_i(\mathbb{K})$, and this is a module of depth $\geq d - s + i$, it suffices to check the prime ideals $\mathfrak{p}$ of codimension at most $s - i \leq s - 1$. But in this range we have $I_{\mathfrak{p}} = J_{\mathfrak{p}}$. □

Corollary 8.9 *Let R be a Cohen–Macaulay local ring of dimension d, with infinite residue field, and let I be an ideal with sliding depth. If I is of linear type in codimension h then*

$$\ell(I) \geq \inf\{h+1, \nu(I)\}.$$

Ideals of reduction number one

An immediate consequence is:

Corollary 8.10 *Let R, I and J be as above, and assume that* depth $R/I \geq \dim R - s$. *Then $I \cdot R[Jt]$ is a maximal Cohen–Macaulay module. If the reduction number $r_J(I) = 1$ then* $\text{gr}_I(R)$ *is Cohen–Macaulay.*

We make some observations about what is required for the equation $r_J(I) = 1$ to hold.

Theorem 8.11 *Let I and J be ideals as in* Theorem 8.7, *and suppose every associated prime ideal of I has codimension at most s. If for each prime ideal $\mathfrak{p}$ of codimension s the equality $I_{\mathfrak{p}}^2 = (JI)_{\mathfrak{p}}$ holds then $I^2 = JI$.*

Proof. It will be enough to show that the associated prime ideals of JI have codimension at most s. From the proofs of Corollaries 8.8 and 8.10 we use the exact sequence

$$0 \to H_1(J) \longrightarrow (R/I)^{\ell} \longrightarrow J/JI \to 0,$$

which will be combined with the sequence

$$0 \to J/JI \longrightarrow R/JI \longrightarrow R/J \to 0.$$

Since depth $H_1(J) \geq d-s+1$, it follows from the first sequence and the condition that dim $R_{\mathfrak{p}} \leq s$ for every associated prime $\mathfrak{p}$ of I, that any prime in Ass(J/JI) has codimension at most s. The claim now follows from the second sequence, since the associated prime ideals of R/J have codimension at most s.

Alternatively, we can argue as follows to show the vanishing of the Sally module $S_J(I)$: In the exact sequence (2), as in Proposition 3.2, $I \cdot R[Jt]$ being a maximal Cohen–Macaulay module (and therefore an unmixed ideal of codimension one) implies that $S_J(I)$ either vanishes or has Krull dimension d. By induction on dim R we may assume that $I^2 = JI$ holds on the punctured spectrum of the local ring $(R, \mathbf{m})$. This means that $S_J(I)$ is annihilated by some power of $\mathbf{m}$ so that the dimension of $S_J(I)$ is at most $\nu(J) = s < d$, which is a contradiction unless $r_J(I) = 1$. □

These methods come in full fruition in the following ([82]):

Theorem 8.12 *Let R be a Cohen–Macaulay local ring and let I and J be ideals as in* Theorem 8.7. *Suppose that* depth $R/I \geq$ dim $R - s$. *If I has codimension at least two and $r_J(I) = 1$ then $R[It]$ is a Cohen–Macaulay algebra.*

Fundamental divisor

Let I be an ideal with sliding depth that satisfies $\mathcal{F}_1$. The canonical module

$$\omega_{\mathcal{R}} = C_1 t + C_2 t^2 + \cdots$$

has the property that $C_1 \simeq \omega$. This implies that in the representation of

$$\mathcal{D}(I) = W_1 t + W_2 t^2 + \cdots,$$

we also have $W_1 \simeq \omega$. Indeed, from the sequence (14), we have the exact sequence

$$0 \to \omega_{\mathcal{R}} \longrightarrow L \longrightarrow \omega_{\mathcal{G}},$$

we have since I annihilates $\mathcal{G}$, $I \cdot W_1 \subset C_1$, which means that $W_1 \subset C_1$, and therefore $W_1 \simeq \omega$.

Suppose now that N is an ideal $I \subset N$ and $N^2 = I \cdot N$. From the

$$0 \to N \cdot R[It] \longrightarrow R[It] \longrightarrow U \to 0,$$

$$0 \to \omega_{R[It]} \longrightarrow \omega_{N \cdot R[It]} \longrightarrow \omega_U,$$

from which we obtain that $C_1 = D_1$, the first component of the canonical module of $N \cdot R[It]$, since D_1 is conducted into C_1 by an ideal of height at least two. On the other hand, we have $N \cdot R[It] = N \cdot R[Nt]$, by the hypothesis on the reduction number.

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