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Cohen-Macaulayness of Associated Graded Rings and Reduction Numbers of Ideals

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These are preliminary lecture notes, intended only for distribution to participants

Lecture Notes on Cohen-Macaulayness of Associated Graded Rings and Reduction Numbers of Ideals

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Introduction

These lectures are in some sense a continuation of [V3]. We focus on studying Cohen-Macaulayness and other algebraic properties of blow-up rings, such as the Rees algebra $\mathcal{R}$ and the associated graded ring G of an ideal. Four problems come to mind:

(1) Assuming that G is Cohen-Macaulay, what are its various numerical invariants, and what can be said about the Cohen-Macaulayness of $\mathcal{R}$?

(2) When is the above assumption satisfied, i.e., when is G Cohen-Macaulay?

(3) What is the canonical module of G, and when is G Gorenstein?

Finally, since the answers to these questions involve conditions on the reduction number of the ideal:

(4) How can ideals be characterized that have the required reduction number?

In section one, we address question (1), by computing the a-invariant and the Castelnuovo-Mumford regularity of G. We also mention a remarkable result by Lipman, saying that for any ideal in a regular local ring, $\mathcal{R}$ is Cohen-Macaulay if and only if G is Cohen-Macaulay. In section three, we deal with question (2), by showing that G is Cohen-Macaulay if the reduction number of the ideal is "small enough". There we intend to present a relatively short and self-contained proof of a general theorem, that contains the known results in this direction. Questions (3) and (4) are addressed in sections four and five. Among other things, we give a characterization in terms of the Hilbert-Burch matrix, for when a perfect ideal of grade two has a Cohen-Macaulay Rees algebra. Our main tool in sections three, four, and five is the theory of residual intersections, which we introduce briefly in section two.

Throughout the lectures, (R, m) will be a Noetherian local ring with infinite residue field, and I will be a proper R-ideal. The *Rees algebra* $R[It] \cong \oplus_{i \geq 0} I^i$ will usually be denoted by $\mathcal{R}$, and the *associated graded ring* $\mathrm{gr}_I(R) \cong \oplus_{i \geq 0} I^i/I^{i+1}$ by G.

Recall that the *analytic spread* $\ell(I)$ of I is defined as the Krull dimension of the fiber ring $\mathcal{R} \otimes_R R/m \cong G \otimes_R R/m$, or equivalently, as the minimal number of generators of a minimal reduction of I ([NR]). Here one says that an ideal J is a *reduction* of I, if $J \subset I$ and if $I^{r+1} = JI^r$ for some $r \geq 0$. The smallest integer r for which this equality holds is denoted by $r_J(I)$. Finally, a *minimal reduction* of I is a reduction that is minimal with respect to inclusion, and the *reduction number* $r(I)$ of I is defined as $\min\{r_J(I) | J$ a minimal reduction of $I\}$.

Furthermore, ω will stand for the canonical module of a ring (in case it exists), and μ will denote minimal number of generators. The ideal I is said to satisfy G_s if $\mu(I_p) \leq \dim R_p$ for every $p \in V(I)$ with $\dim R_p < s$, and I satisfies G_∞ (or F_1) if I is G_s for every s ([AN]).

1 Properties of Ideals Whose Associated Graded Ring is Cohen-Macaulay

In continuation of the previous lectures [V3], we are now going to investigate what more can be said about an ideal if its associated graded ring G is Cohen-Macaulay.

We begin by using ideas from [JK] to give different proofs of some results that are essentially from [AHT]: We provide formulas for the a-invariant and the regularity of G, and study how the reduction number depends on the choice of various minimal reductions. Notice that the a-invariant does not only control the transfer of Cohen-Macaulayness from G to $\mathcal{R}$ (as explained in [V3]), but that it is an interesting invariant in its own right, carrying a great deal of information about the canonical module of G. Furthermore, the regularity bounds the growth of the shifts in a minimal free resolution of G, and in particular, controls the relation type of the ideal.

Let S be a homogeneous Noetherian ring of dimension d with $A = S_0$ local, let M be the irrelevant maximal ideal of S, and let N be any graded S-ideal containing S_+. Present S as an epimorphic image of a (standard graded) polynomial ring $B = A[T_1, \ldots, T_n]$, let $F.$ be a homogeneous minimal free B-resolution of S, and write $F_i = \oplus_j B(-j)^{\beta_{ij}}$ with $\beta_{ij} \neq 0$. Further set $a_i(N, S) = \max\{j | [H^i_N(S)]_j \neq 0\}$, and notice that $-\infty \leq a_i(N, S) < \infty$.

The integer $a_d(M, S)$ is called the *a-invariant* of S, and is denoted by $a(S)$ ([GW]). Notice that by local duality, $a(S) = -\min\{j | [\omega_S]_j \neq 0\}$, in case the canonical module ω_S exists. The *Castelnuovo-Mumford regularity* $\text{reg}(S)$ of S is defined as $\max\{a_i(S_+, S) + i | i \geq 0\}$. It turns out that $\text{reg}(S) = \max\{\beta_{ij} - i | i \geq 0 \text{ and } j \text{ arbitrary}\}$ ([EG], [O]).

The link between the reduction number of an ideal and the Castelnuovo-Mumford regularity of its associated graded ring is provided by the following result of Trung, which can be proved using induction on the analytic spread:

Proposition 1.1 ([Tr]) *Let R be a Noetherian local ring with infinite residue field, let I be an R-ideal with analytic spread ℓ, let J be a minimal reduction of I, and write $G = \text{gr}_I(R)$.*

Then $a_\ell(G_+, G) + \ell \leq r_J(I) \leq \text{reg}(G)$.

Proposition 1.1 gives a formula for $a(G)$ in terms of $r_J(I)$, once we can relate $a(G)$ to $a_i(G_+, G)$. This is done in Proposition 1.3 (due to Johnston and Katz), whose proof is based on a lemma by Brodmann (which was already mentioned in the previous lectures [V3]):

Lemma 1.2 ([B]) *Let R be a Noetherian ring, let $x \in R$, let I be an R-ideal, and let E be an R-module. Then there is a long exact sequence of local cohomology,*

$$0 \to H^0_{(I,x)}(E) \to H^0_I(E) \to H^0_{I_x}(E_x) \to \ldots$$

$$\ldots \to H^i_{(I,x)}(E) \to H^i_I(E) \to H^i_{I_x}(E_x) \to \ldots .$$

Now let S be a positively graded Noetherian ring with $A = S_0$ local. For $p \in \mathrm{Spec}(A)$, write $S_p = S \otimes_A A_p$ and $d(p) = \dim S_p$, and let P be the irrelevant maximal ideal of S_p.

Proposition 1.3 (cf. [JK]) *Let S be a positively graded Noetherian ring of dimension d with $A = S_0$ local, and let t be an integer.*

(a) *If $H^i_P(S_p)$ is concentrated in degrees $\leq t$ for every $p \in Spec(A)$, then $H^i_N(S)$ is concentrated in degrees $\leq t$ for every homogeneous ideal N containing S_+.*

(b) *If $H^{d(p)}_{S_{p+}}(S_p)$ is concentrated in degrees $\leq t$ for every $p \in Spec(A)$, then $H^d_N(S)$ is concentrated in degrees $\leq t$ for every homogeneous ideal N containing S_+.*

Proof. Let m denote the maximal ideal of A.

(a): We induct on $\dim S/N$, the assertion begin trivial if $\dim S/N = 0$. So let $\dim S/N > 0$ and pick $x \in m$ such that $\dim S/(N,x) < \dim S/N$. Then by induction hypothesis, $H^i_{(N,x)}(S)$ is concentrated in degrees $\leq t$. Furthermore, $H^i_{N_x}(S_x)$ is concentrated in degrees $\leq t$, by induction hypothesis, and since local cohomology commutes with localization. Now a graded version of Lemma 1.2 implies that $H^i_N(S)$ is concentrated in degrees $\leq t$ as well.

(b): Write $I = N_0$ and induct on $\mu(I)$. If $\mu(I) = 0$, then $H^d_N(S) = H^{d(m)}_{S_{m+}}(S_m)$, and we are done. If $\mu(I) > 0$, write $I = (I', x)$ with $\mu(I') < \mu(I)$, and set $N' = (I', S_+)$. Now using the fact that local cohomology is compatible with localization and that $\dim S_x \leq d - 1$, we deduce from the induction hypothesis that $H^{d-1}_{N'_x}(S_x)$ as well as $H^d_{N'}(S)$ are concentrated in degrees $\leq t$.

Thus again by Lemma 1.2, $H^d_{(N',x)}(S) = H^d_N(S)$ is concentrated in degrees $\leq t$. ■

We are now ready to prove the first main result of this section:

Theorem 1.4 (compare to [AHT]) *Let R be a Noetherian local ring with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, write $\mathcal{A}(I) = \{p \in V(I) | \ell(I_p) = \dim R_p\}$, let J be any minimal reduction of I, and assume that $G = gr_I(R)$ is Cohen-Macaulay. Then:*

(a)
$$\begin{aligned} a(G) &= \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I)\} \\ &= \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I), \dim R_p < \ell\} \cup \{r - \ell\} \\ &= \max\{r(I_p) - \ell(I_p) | p \in V(I)\}. \end{aligned}$$

(b)
$$\begin{aligned} r &\leq r_J(I) \leq \operatorname{reg}(G) \leq a(G) + \ell \\ &= \max\{r(I_p) - \ell(I_p) + \ell | p \in \mathcal{A}(I), \dim R_p < \ell\} \cup \{r\}. \end{aligned}$$

Proof (cf. also [JK]). We wish to apply Proposition 1.3 to the ring $S = G$. Let M be the irrelevant maximal ideal of G.

(a): We first show that $a(G) \leq \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I)\}$. Let $t = \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I)\}$. By Proposition 1.3 (b), applied to $N = M$, it suffices to show that $H^{d(p)}_{G_{p+}}(G_p)$ is concentrated in degrees $\leq t$ for every $p \in \operatorname{Spec}(A) = V(I)$. However, if $p \notin \mathcal{A}(I)$, then $\ell(G_{p+}) = \ell(I_p) < \dim R_p = d(p)$, and hence $H^{d(p)}_{G_{p+}}(G_p) = 0$. On the other hand, if $p \in \mathcal{A}(I)$, then $\ell(I_p) = d(p)$, and hence by Proposition 1.1 and the very definition of t, $a_{d(p)}(G_{p+}, G_p) = a_{\ell(I_p)}(G_{p+}, G_p) \leq r(I_p) - \ell(I_p) \leq t$. This means that $H^{d(p)}_{G_{p+}}(G_p)$ is concentrated in degrees $\leq t$.

Next, notice that the inequalities

$$\begin{aligned} \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I)\} &\leq \max\{r(I_p) - \ell(I_p) | p \in \mathcal{A}(I), \dim R_p < \ell\} \\ &\qquad \cup \{r - \ell\} \\ &\leq \max\{r(I_p) - \ell(I_p) | p \in V(I)\} \end{aligned}$$

are trivially satisfied.

Thus, it suffices to show that $\max\{r(I_p) - \ell(I_p) | p \in V(I)\} \leq a(G)$. Since the a-invariant cannot increase upon localizing R (which one can see, for instance, by completing R and using local duality), we only need to check that $r - \ell \leq a(G)$. However, again using the local property of the a-invariant, we conclude from Proposition 1.3 (a), that $a_i(G_+, G) \leq a(G)$, and therefore $\text{reg}(G) = \max\{a_i(G_+, G) + i | 0 \leq i \leq \ell\} \leq a(G) + \ell$. Now by Proposition 1.1, $r \leq a(G) + \ell$.

(b): The first inequality is trivial, the second follows from Proposition 1.1, the third has just been proved above, and the last equality follows from part (a). ■

Remark 1.5 A different formula for the a-invariant of G has been shown in [SUV]: In addition to the assumptions of Theorem 1.4, write $g =$ grade I and suppose that I satisfies G_ℓ; then $a(G) = \max\{-g, r - \ell\}$.

Combining Theorem 1.4 (a) or Remark 1.5 with [TI], one obtains the results about the Rees algebra $\mathcal{R} = R[It]$ that were already discussed in the previous lectures ([V3]): Assume that I is not nilpotent. Then with the assumptions of Theorem 1.4 (a), $\mathcal{R}$ is Cohen-Macaulay if and only if $r(I_p) < \ell(I_p)$ for every $p \in \mathcal{A}(I)$ (or equivalently, for every $p \in V(I)$); on the other hand, under the assumptions of Remark 1.5, $\mathcal{R}$ is Cohen-Macaulay if and only if $g > 0$ and $r < \ell$.

Corollary 1.6 (compare to [AHT]) *In addition to the assumptions of Theorem 1.4, suppose that $r(I_p) \leq r - \ell + \ell(I_p)$ for every $p \in \mathcal{A}(I)$ with $\dim R_p < \ell$ (which holds, for instance, if $r \geq \ell - g$ and if $r(I_p) = 0$ for every $p \in \mathcal{A}(I)$ with $\dim R_p < \ell$).*

Then $a(G) = r - \ell$, $reg(G) = r$, and $r_J(I) = r$ does not depend on the choice of J.

Proof. The assertions follow from Theorem 1.4. ■

We now wish to investigate the regularity of the Rees algebra $\mathcal{R} = R[It]$.

Proposition 1.7 *Let R be a Noetherian local ring, and let I be a (proper) R-ideal.*

Then $\text{reg}(\mathcal{R}) = \text{reg}(G)$.

Proof. We look at the usual exact sequences from [H3],

$$0 \to \mathcal{R}_+ \to \mathcal{R} \to R \to 0, \tag{1.8}$$

$$0 \to \mathcal{R}_+(1) \to \mathcal{R} \to G \to 0. \tag{1.9}$$

First notice that

$$H^i_{\mathcal{R}_+}(R) = H^i_0(R) = \begin{cases} R & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases} .$$

Thus by (1.8),

$$[H^i_{\mathcal{R}_+}(\mathcal{R}_+)]_j \cong [H^i_{\mathcal{R}_+}(\mathcal{R})]_j \quad \text{for } i \geq 2 \text{ or } j \neq 0. \tag{1.10}$$

On the other hand, (1.9) gives rise to an exact sequence

$$H^i_{\mathcal{R}_+}(\mathcal{R}_+)(1) \to H^i_{\mathcal{R}_+}(\mathcal{R}) \to H^i_{\mathcal{R}_+}(G) \to H^{i+1}_{\mathcal{R}_+}(\mathcal{R}_+)(1),$$

which, when combined with (1.10), yields

$$[H^i_{\mathcal{R}_+}(\mathcal{R})]_{j+1} \to [H^i_{\mathcal{R}_+}(\mathcal{R})]_j \to [H^i_{\mathcal{R}_+}(G)]_j \to [H^{i+1}_{\mathcal{R}_+}(\mathcal{R})]_{j+1}, \tag{1.11}$$

provided that $i \geq 2$ or $j \neq -1$. Also notice that $H^i_{\mathcal{R}_+}(G) \cong H^i_{G_+}(G)$.

Now write $s = \text{reg}(\mathcal{R})$ and $t = \text{reg}(G)$, and note that $s \geq 0$, $t \geq 0$. By (1.11), if $i \geq 2$ then $H^i_{G_+}(G)$ is concentrated in degrees $\leq s - i$, and if $i \leq 1$ then $H^i_{G_+}(G)$ is concentrated in degrees $\leq \max\{s - i, -1\} = s - i$. Thus $\text{reg}(G) \leq s$. On the other hand, again by (1.11), if $i \geq 2$ then $H^i_{\mathcal{R}_+}(\mathcal{R})$ is concentrated in degrees $\leq t - i$, and if $i \leq 1$ then $H^i_{\mathcal{R}_+}(\mathcal{R})$ is concentrated in degrees $\leq \max\{t - i, -1\} = t - i$. Therefore $\text{reg}(\mathcal{R}) \leq t$. ■

Recall that $rt(I)$, the *relation type* of I, is the largest degree occurring in a homogeneous minimal generating set of the relation ideal Q, when presenting $\mathcal{R}$ as $R[T_1, \ldots, T_n]/Q$. Results similar to the next one have been shown in [AHT] (assuming that $\mathcal{R}$ is Cohen-Macaulay).

Corollary 1.12 *With the assumptions of Corollary 1.6, $\text{reg}(\mathcal{R}) = r$ and $rt(I) \leq r + 1$.*

We now turn to the aforementioned result by Lipman. A Noetherian local ring (R, m) of dimension d is *pseudo-rational* if R is Cohen-Macaulay, normal, and analytically unramified, and if for every proper birational map $f : X \to Y = \operatorname{Spec}(R)$ with closed fibre $E = f^{-1}(\{m\})$, the natural map $\delta : H^d_m(R) \to H^d_E(X, \mathcal{O}_X)$ is injective ([LT]) (one may assume X to be normal). Note that by [LT], every regular local ring is pseudo-rational.

In addition to our standard notation from the introduction, let $d = \dim R$, $\mathcal{A}(I) = \{p \in V(I) | \ell(I_p) = \dim R_p\}$, and write $X = \operatorname{Proj}(\mathcal{R})$ for the blow-up of $\operatorname{Spec}(R)$ along $V(I)$, and $E = \operatorname{Proj}(\mathcal{R} \otimes_R R/m)$ for its closed fibre. For his proof, Lipman uses the following result:

Theorem 1.13 ([S]) *If G is Cohen-Macaulay, then $H^i_E(X, \mathcal{O}_X) = 0$ for $i < d$.*

Theorem 1.14 ([L]) *Let R be a Noetherian local ring, let I be an R-ideal with* $\operatorname{ht} I > 0$, *and assume that R_p is pseudo-rational for every $p \in \mathcal{A}(I)$.*

If G is Cohen-Macaulay, then $\mathcal{R}$ is Cohen-Macaulay.

Proof. By [TI] and the first equation in Theorem 1.4 (a), $\mathcal{R}$ is Cohen-Macaulay if and only if $\mathcal{R} \otimes_R R_p$ is Cohen-Macaulay for every $p \in \mathcal{A}(I)$. Thus we may assume that R is pseudo-rational. By the normality of R, we may then suppose that $d = \dim R \geq 2$. Let M be the irrelevant maximal ideal of $\mathcal{R}$, and write $V = X \backslash E$, $U = \operatorname{Spec}(R) \backslash \{m\}$.

It suffices to prove that $[H^d_M(\mathcal{R})]_0 = 0$, which can be seen from the proof of [TI], as modified in [V3]. Indeed, suppose $[H^d_M(\mathcal{R})]_0 = 0$, then by (1.8), $H^i_M(\mathcal{R}_+) \cong H^i_M(\mathcal{R})$ for $i \leq d$, and hence (1.9) induces embeddings

$$\varphi_i : H^i_M(\mathcal{R})(1) \hookrightarrow H^i_M(R)$$

for $i \leq d$. However, when disregarding the grading, φ_i are injective endomorphisms of Artinian modules, and therefore have to be bijective. Hence there are homogeneous isomorphisms $H^i_M(\mathcal{R})(1) \cong H^i_M(\mathcal{R})$, which force $H^i_M(\mathcal{R})$ to be zero for $i \leq d$. This means that $\mathcal{R}$ is Cohen-Macaulay.

Since $d \geq 2$, one has $H^d_M(\mathcal{R}) \cong \oplus_j H^{d-1}(X, \mathcal{O}_X(j))$, which reduces us to showing that $H^{d-1}(X, \mathcal{O}_X) = 0$. But from Theorem 1.13 and from the long exact sequence for sheaf cohomology with support, one obtains an exact sequence

$$0 = H^{d-1}_E(X, \mathcal{O}_X) \to H^{d-1}(X, \mathcal{O}_X) \to H^{d-1}(V, \mathcal{O}_V) \xrightarrow{\lambda} H^d_E(X, \mathcal{O}_X).$$

Thus it remains to prove that λ is injective. Since $d \geq 2$, there is a commutative diagram

$$\begin{array}{ccc} H^{d-1}(U, \mathcal{O}_U) & \xrightarrow{\sim} & H^d_m(R) \\ \downarrow{\scriptstyle \mu} & \# & {\scriptstyle \delta}\downarrow \\ H^{d-1}(V, \mathcal{O}_V) & \xrightarrow{\lambda} & H^d_E(X, \mathcal{O}_X) \end{array},$$

where δ is injective by the definition of pseudo-rationality, and μ is surjective because the fibers of the map $V \to U$ have dimension $< d-1$ ([LT, p. 103]). Hence λ is injective. ■

2 Residual Intersections

We are now going to review some basic facts about residual intersections. This notion, essentially introduced by Artin and Nagata ([AN]), generalizes the concept of linkage to the case where the two "linked" ideals do not necessarily have the same height.

Definition 2.1 ([HU]) Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, let K be a proper R-ideal, and let $s \geq g$ be an integer.

(a) K is called an *s-residual intersection* of I if there exists an R-ideal $\mathfrak{a} \subset I$, such that $K = \mathfrak{a} : I$ and ht $K \geq s \geq \mu(\mathfrak{a})$.

(b) K is called a *geometric s-residual intersection* of I, if K is an s-residual intersection of I and if in addition ht $I + K > s$.

Example 2.2 ([H5]) Let R be a local Cohen-Macaulay ring, let $n > s \geq 2$, let X be an n by $n-1$ matrix of variables, let Y be the $n-s$ by $n-1$ matrix consisting of the last $n-s$ rows of X, write $S = R[X]$ (or $R[X]_{(X)}$), and consider the S-ideals $I = I_{n-1}(X)$ and $K = I_{n-s}(Y)$. Then K is a geometric s-residual intersection of I.

Example 2.3 ([H5]) Let (R, m) be a local Cohen-Macaulay ring, let I be an R-ideal satisfying G_∞ and sliding depth (cf. [V3]), let $f_1, \ldots f_n$ be a generating sequence of I, let M be the unique graded maximal ideal of the extended Rees algebra $R[It, t^{-1}]$, and let K be the kernel of the R-epimorphism from $S = R[T_1, \ldots T_n, U]_{(m, T_1, \ldots, T_n, U)}$ to $R[It, t^{-1}]_M$, mapping T_i to $f_i t$ and t^{-1} to U. Then K is a geometric n-residual intersection of the S-ideal (I, U).

Example 2.3 gives a first indication that residual intersections might play a crucial role in studying blow-up algebras. The example is, on the other hand, somewhat misleading, because the ideals we consider will usually not satisfy G_∞. In order to deal with this more general situation, we need the notion of *Artin-Nagata properties*:

Definition 2.4 ([U1]) Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, and let s be an integer.

(a) We say that I satisfies AN_s^- if for every $g \leq i \leq s$ and every geometric i-residual intersection K of I, R/K is Cohen-Macaulay.

(b) We say that I satisfies AN_s if for every $g \leq i \leq s$ and every i-residual intersection K of I, R/K is Cohen-Macaulay.

Theorem 2.5 ([HVV], cf. also [H5]) *Let R be a local Cohen-Macaulay ring, and let I be an R-ideal satisfying G_s and sliding depth.*
Then I satisfies AN_s^-.

The next result underlines the close connection between residual intersections and powers of ideals, even if the condition G_∞ does not hold:

Theorem 2.6 ([U1]) *Let R be a local Gorenstein ring of dimension d, let I be an R-ideal of grade g, let s be an integer, assume that I satisfies G_s and that depth $R/I^j \geq d - g - j + 1$ whenever $1 \leq j \leq s - g + 1$. Then:*

(a) *I satisfies AN_s.*

(b) *For every $g \leq i \leq s$ and every i-residual intersection $K = \mathfrak{a} : I$ of I, $\omega_{R/K} \cong I^{i-g+1}/\mathfrak{a}I^{i-g}$, where $\omega_{R/K} \cong I^{i-g+1} + K/K$ in case K is a geometric i-residual intersection.*

The above assumption that depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq s - g + 1$, is automatically satisfied if I is a strongly Cohen-Macaulay ideal ([V3]) satisfying G_s, as can be easily seen from the Approximation Complex ([V3]). On the other hand, recall that any perfect ideal of grade 2 and any perfect Gorenstein ideal of grade 3 is *licci* (which means, in the linkage class of a complete intersection) ([A], [G], [W]), and that in turn, any licci ideal in a local Cohen-Macaulay ring is strongly Cohen-Macaulay ([H4]). Thus Theorems 2.5 and 2.6 apply to such ideals (assuming that G_s holds and that, for the latter theorem, R is Gorenstein).

3 Conditions for the Cohen-Macaulayness of Associated Graded Rings

So far, we were mainly working under the assumption that the associated graded ring is already known to be Cohen-Macaulay. We are now going to provide sufficient conditions, again in terms of the reduction number, that would guarantee this assumption to hold.

Throughout, R will be a local Cohen-Macaulay ring of dimension d with infinite residue field, I will be a proper R-ideal with grade g, minimal number of generators n, analytic spread ℓ, and reduction number r, G and $\mathcal{R}$ will denote the associated graded ring and the Rees algebra of I.

So far, it has been shown that G is Cohen-Macaulay under any of the following assumptions:

- I satisfies G_ℓ, depth $R/I \geq d-\ell$, $\ell \leq g+1$, and $r \leq 1$ ([HH1], cf. also [V2]).
- R is Gorenstein, I is a complete intersection locally in codimension $g+1$, R/I is Cohen-Macaulay, $\ell \leq g+2$, and $r \leq 1$ ([HH2]).
- I satisfies G_ℓ and sliding depth, and $r \leq 1$ ([V2]).
- I satisfies G_ℓ and $AN^-_{\ell-2}$, depth $R/I \geq d-\ell$, $g \geq 2$, and $r \leq 1$ ([U1]).
- R is Gorenstein, I satisfies G_ℓ, depth $R/I \geq d-\ell$, depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell-g-1$, and $r \leq 1$ ([U1]).
- I satisfies G_ℓ, depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq 2$, $\ell = g+1$, and $r \leq 2$ ([GN1]).
- R is Gorenstein, I is a complete intersection locally in codimension $g+1$, R/I is Cohen-Macaulay, depth $R/I^2 \geq d-g-2$, $\ell = g+2$, and $r \leq 2$ ([GN2]).
- I satisfies G_ℓ and is perfect of grade 2, $\ell = 3$, $n \leq 4$, and $r \leq 2$ ([AH]).
- I satisfies G_ℓ and is perfect of grade 3, R/I is Gorenstein, $\ell = 4$, $n \leq 5$, and $r \leq 2$ ([AHH]).

- R is Gorenstein, I satisfies G_ℓ and is strongly Cohen-Macaulay, $n \leq \ell + 1$, and $r \leq \ell - g + 1$ ([SUV]).
- R is Gorenstein, I is a complete intersection locally in codimension $\ell - 1$, depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g + 1$, $\ell \geq g + 2$, and $r \leq \ell - g + 1$ ([T]).
- R is Gorenstein, I is a complete intersection locally in codimension $\ell - 1$, depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g - 1$, depth $R/I^{\ell - g} \geq d - \ell$, $\ell \geq g + 3$, and $r \leq \ell - g$ ([T]).
- I satisfies G_ℓ, depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq 3$, $\ell = g + 2$, and $r \leq 3$ ([AHc], using an assumption that is slightly weaker than G_ℓ).

The goal of this section is to give a self-contained proof of the following theorem, which contains the above results as special cases:

Theorem 3.1 ([JU]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal of analytic spread ℓ and reduction number r, let $k \geq 1$ be an integer with $r \leq k$, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell - \max\{2,k\}}$, and that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$.*

Then G is Cohen-Macaulay.

We are now going to discuss the assumptions of Theorem 3.1.

The condition depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$, gives a linearly decreasing bound for the depths of the powers of I so that depth $R/I^k \geq d - \ell$, where the latter inequality is necessary for G to be Cohen-Macaulay (e.g. [EH]). Also notice that if $\ell = d$, then it suffices to require depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k - 1$. Moreover, for any strongly Cohen-Macaulay ideal I satisfying G_ℓ, one has depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g + 1$. Finally, the depth assumptions in Theorem 3.1 automatically imply that $k \leq \ell - g + 1$, which can be seen by setting $j = 1$.

As to the Artin-Nagata properties, notice that these assumptions automatically hold if $\ell = g + 2$ and $k = 3$, or if I is $AN^-_{\ell-2}$. On the other hand, $AN^-_{\ell-2}$ is always satisfied if $\ell \leq g + 1$, or if I satisfies G_ℓ and sliding depth (cf. Theorem 2.5), or if R is Gorenstein, I satisfies G_ℓ, and depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g - 1$ (cf. Theorem 2.6). In particular, any reference to the Artin-Nagata property can be omitted in Theorem 3.1 if $\ell \leq g+1$, or if R is Gorenstein and $k = \ell - g + 1$.

This discussion shows that the results from the beginning of the section are indeed special cases of Theorem 3.1; it also gives the following application:

Corollary 3.2 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a strongly Cohen-Macaulay R-ideal (e.g., an R-ideal in the linkage class of a complete intersection, such as a perfect ideal of grade 2 or a perfect Gorenstein ideal of grade 3) with grade g, analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ and that $r \leq \ell - g + 1$.*
Then G is Cohen-Macaulay.

Combining Theorem 3.1 and Corollary 3.2 with [JK] (or [SUV], [AHT])(cf. [V3]), one obtains:

Corollary 3.3 *In addition to the assumptions of Theorem 3.1 or Corollary 3.2, suppose that $g \geq 2$.*
Then $\mathcal{R}$ is Cohen-Macaulay.

We now turn to the proof of Theorem 3.1. The statement of this theorem was somewhat inspired by [T]; for its proof however, we are going to follow [JU].

We first need to recall a technical result about residual intersections (generalizing [H5] and [HVV]).

Lemma 3.4 ([U1]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let $\mathfrak{a} \subset I$ be (not necessarily distinct) R-ideals with $\mu(\mathfrak{a}) \leq s \leq$ ht $\mathfrak{a} : I$, and assume that I satisfies G_s.*

(a) *There exists a generating sequence $a_1, \ldots, a_s$ of $\mathfrak{a}$ such that for every $0 \leq i \leq s-1$ and every subset $\{\nu_1, \ldots, \nu_i\}$* of $\{1, \ldots, s\}$, ht $(a_{\nu_1}, \ldots, a_{\nu_i}) : I \geq i$ *and* ht $I + (a_{\nu_1}, \ldots, a_{\nu_i}) : I \geq i+1$.

(b) *Assume that I satisfies AN_{s-2}^-. Then any sequence $a_1, \ldots, a_s$ as in* (a) *forms an unconditioned d-sequence.*

(c) *Assume that I satisfies AN_t^- for some t, write $\mathfrak{a}_i = (a_1, \ldots, a_i)$, $K_i = \mathfrak{a}_i : I$, and let " $^-$ " denote images in R/K_i. Then for* $0 \leq i \leq \min\{s, t+1\}$:

(i) $K_i = \mathfrak{a}_i : (a_{i+1})$ *and* $\mathfrak{a}_i = I \cap K_i$, *if* $i \leq s - 1$.

(ii) depth $R/\mathfrak{a}_i = d - i$.

(iii) K_i *is unmixed of height* i.

(iv) $\bar{a}_{i+1}$ *is regular on* $\bar{R}$, *if* $i \leq s - 1$.

(v) ([JU]) $\bar{I}$ *satisfies* G_{s-i} *and* AN^-_{t-i}.

Lemma 3.5 *Let* R *be a local Cohen-Macaulay ring of dimension* d *with infinite residue field, let* $\mathfrak{a} \subset I$ *be* R*-ideals with* $\mu(\mathfrak{a}) \leq s \leq \operatorname{ht} \mathfrak{a} : I$, *let* t *and* k *be integers, assume that* I *satisfies* G_s *and* AN^-_{s-2} *locally in codimension* $s-1$, *that* I *satisfies* AN^-_t, *and that* depth $R/I^j \geq d - s + k - j$ *whenever* $1 \leq j \leq k$. *Let* $\mathfrak{a}_i$ *be the ideals as defined in Lemma 3.4 (a). Then*

(a) depth $R/\mathfrak{a}_i I^j \geq \min\{d - i, d - s + k - j\}$ *whenever* $0 \leq i \leq s$ *and* $\max\{0, i - t - 1\} \leq j \leq k$.

(b) $[\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever* $0 \leq i \leq s - 1$ *and* $\max\{1, i - t\} \leq j \leq k$.

Proof. Let $0 \leq i \leq s - 1$. We first show that if (a) holds for i, then so does (b). However, it suffices to check the equality in (b) locally at every $p \in \operatorname{Ass}(R/\mathfrak{a}_i I^{j-1})$. Now by (a), $\dim R_p \leq \max\{i, s - k + j - 1\} \leq s - 1$, since $i \leq s - 1$ and $j \leq k$. But then $I_p = \mathfrak{a}_p$. On the other hand by Lemma 3.4 (b), $a_1, \ldots, a_s$ from a d-sequence in R_p and the assertion follows from ([H2]).

Thus it suffices to prove (a), which we are going to do by induction on i, $0 \leq i \leq s$. The assertion being trivial for $i = 0$, we may assume that $0 \leq i \leq s - 1$, and that (a) and hence (b) hold for i. We need to verify (a) for $i + 1$. But for $j = 0$ (which can only occur if $i + 1 \leq t + 1$), our assertion follows from Lemma 3.4 (c.ii). Thus we may suppose that $j \geq 1$. But then by part (b) for i,

$$\begin{aligned} \mathfrak{a}_i I^j \cap a_{i+1} I^j &= a_{i+1}[(\mathfrak{a}_i I^j : (a_{i+1})) \cap I^j] \subset a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^j] \\ &= a_{i+1}\mathfrak{a}_i I^{j-1} \subset \mathfrak{a}_i I^j \cap a_{i+1} I^j. \end{aligned}$$

Hence, writing $\mathfrak{a}_{i+1} = \mathfrak{a}_i + (a_{i+1})$, we obtain an exact sequence

$$0 \to a_{i+1}\mathfrak{a}_i I^{j-1} \to \mathfrak{a}_i I^j \oplus a_{i+1} I^j \to \mathfrak{a}_{i+1} I^j \to 0. \quad (3.6)$$

On the other hand, by part (b) for $i = 0$, $[0 : (a_{i+1})] \cap \mathfrak{a}_i I^{j-1} \subset [0 : (a_{i+1})] \cap I^j = 0$, and therefore $a_{i+1}\mathfrak{a}_i I^{j-1} \cong \mathfrak{a}_i I^{j-1}$, $a_{i+1}I^j \cong I^j$. Now (3.6) and part (a) for i yield the required depth estimate for $R/\mathfrak{a}_{i+1}I^j$. ∎

Remark 3.7 ([U1]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with analytic spread ℓ satisfying G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell - 1$, and let J be a minimal reduction of I. Then* ht $J : I \geq \ell$.

Proof. Since for every $p \in V(I)$ with $\dim R_p \leq \ell - 1$, I_p satisfies G_ℓ and $AN^-_{\ell-2}$, it follows that I_p is of linear type ([U1]), and hence, $I_p = J_p$. (Alternatively, one could use Lemma 3.4 (b)). ∎

Lemma 3.8 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ and reduction number r, let k be an integer with $r \leq k$, let $t \geq \ell - k - 1$ be an integer, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell - 1$, that I satisfies AN^-_t, and that* depth $R/I^j \geq d - \ell + k - j$ *whenever $1 \leq j \leq k$. Let J be a minimal reduction of I with $r_J(I) = r$, write $G = \mathrm{gr}_I(R)$, for $a \in I$ let a' denote the image of a in $[G]_1$, and for $\mathfrak{a} = J$, let $a_1, \ldots, a_\ell$ and $\mathfrak{a}_i$ be as defined in Lemma 3.4 (a). Then:*

(a) *$\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$ whenever $0 \leq i \leq \ell - 1$ and $j \geq \max\{1, i - t\}$, or $i = \ell$ and $j \geq r + 1$.*

(b) *$a'_1, \ldots, a'_g$ form a G-regular sequence, and $[(a'_1, \ldots, a'_i) :_G (a'_{i+1})]_j = [(a'_1, \ldots, a'_i)]_j$ whenever $g \leq i \leq \ell - 2$ and $j \geq \max\{1, i - t\}$, or $i = \ell - 1$ and $j \geq \max\{1, \ell - t - 1, r - 1\}$.*

Proof. (a): If $i = \ell$, our claim is clear since $j \geq r + 1$ and therefore $I^j = JI^{j-1} = \mathfrak{a}_\ell I^{j-1}$. Furthermore, if $1 \leq j \leq k$, then the assertion follows from Lemma 3.5 (b), with $s = \ell$. Thus we may assume that $j \geq k + 1$. In this case, we are going to prove by decreasing induction on i, $0 \leq i \leq \ell$, that $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$. For $i = \ell$ this is clear. Thus let $0 \leq i \leq \ell - 1$; we need to show the equality $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$. However, since $i - t \leq k$ and since $\mathfrak{a}_i \cap I^\nu = \mathfrak{a}_i I^{\nu-1}$ is already known for $\nu = 1$ and for $\max\{1, i - t\} \leq \nu \leq k$, we may suppose that $j \geq \max\{2, i - t + 1\}$ and that by induction on j, $\mathfrak{a}_i \cap I^{j-1} = \mathfrak{a}_i I^{j-2}$.

Furthermore, by induction on i, $\mathfrak{a}_{i+1} \cap I^j = \mathfrak{a}_{i+1} I^{j-1}$. But then, using our induction hypotheses and Lemma 3.5 (b), we obtain

$$\begin{aligned}
\mathfrak{a}_i \cap I^j \subset \mathfrak{a}_i \;\cap\; \mathfrak{a}_{i+1} \cap I^j = \mathfrak{a}_i \cap \mathfrak{a}_{i+1} I^{j-1} &= \mathfrak{a}_i \cap [\mathfrak{a}_i I^{j-1} + a_{i+1} I^{j-1}] \\
&= \mathfrak{a}_i I^{j-1} + \mathfrak{a}_i \cap a_{i+1} I^{j-1} = \mathfrak{a}_i I^{j-1} + a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^{j-1}] \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^{\max\{1, i-t\}} \cap I^{j-1}] \\
&\subset \mathfrak{a}_i I^{j-1} + a_{i+1}(\mathfrak{a}_i \cap I^{j-1}) \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1} \mathfrak{a}_i I^{j-2} = \mathfrak{a}_i I^{j-1}.
\end{aligned}$$

(b): We may assume that $t \geq g-1$. Since $a_1, \ldots, a_g$ form an R-regular sequence, part (a) and [VV] already imply that $a'_1, \ldots, a'_g$ form a G-regular sequence.

So, let $u \in [(a'_1, \ldots, a'_i) : (a'_{i+1})]_j$. Picking an element $x \in I^j$ with $x + I^{j+1} = u$, we have $a_{i+1} x \in \mathfrak{a}_i + I^{j+2}$, and therefore by part (a), $a_{i+1} x \in \mathfrak{a}_{i+1} \cap (\mathfrak{a}_i + I^{j+2}) = \mathfrak{a}_i + \mathfrak{a}_{i+1} \cap I^{j+2} = \mathfrak{a}_i + \mathfrak{a}_{i+1} I^{j+1} = \mathfrak{a}_i + a_{i+1} I^{j+1}$. Thus $a_{i+1}(x-y) \in \mathfrak{a}_i$ for some $y \in I^{j+1}$. Since $x - y + I^{j+1} = x + I^{j+1} = u$, we may replace x be $x-y$ to assume that $a_{i+1} x \in \mathfrak{a}_i$. But then by part (a) and Lemma 3.5 (b), $x \in [\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$, which implies $u \in (a'_1, \ldots, a'_i)$. ■

A more special version of Lemma 3.8 (b) has been proved in [T].

To formulate the next proposition, which provides a crucial step in the proof of Theorem 3.1, we need to recall that for a graded module $M = \oplus_j M_j$, the truncated submodule $\oplus_{j \geq i} M_j$ is denoted by $[M]_{\geq i}$.

Proposition 3.9 *Let S be a homogeneous Noetherian ring of dimension d with S_0 local, write $I = S_+$, let $b_1, \ldots, b_\ell$ be linear forms in S, set $\mathfrak{b}_i = (b_1, \ldots, b_i)$ for $-1 \leq i \leq \ell$ (where $(\emptyset) = 0$), $J = \mathfrak{b}_\ell$, and let g be an integer with $0 \leq g \leq \ell$. Further let $H^\bullet(-)$ denote local cohomology with support in the irrelevant maximal ideal of S.*

Assume that $I^{k+1} \subset J$ (i.e., J is a reduction of I with $r_J(I) \leq k$), that $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell - 1$, that depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d-i-1$ *for $g-1 \leq i \leq \ell-1$, and that* depth $[S/J]_j \geq d-\ell$ *for $\ell-g+1 \leq j \leq k$.*

Then S is a Cohen-Macaulay ring. Furthermore, $\operatorname{socle}(H^d(S))$ *is concentrated in degrees at least* $-g$ *and at most* $\max\{-g, k-\ell\}$.

Proof. To simplify notation we factor out $\mathfrak{b}_g$ and assume $g = 0$ (notice that $b_1, \ldots, b_g$ form an S-regular sequence and that $[S/\mathfrak{b}_{g-1}]_0 = S_0 = [S/\mathfrak{b}_{-1}]_0$). Now $[\mathfrak{b}_i : (b_{i+1})]_{\geq i+1} = [\mathfrak{b}_i]_{\geq i+1}$ whenever $0 \leq i \leq \ell - 1$, depth $[S/\mathfrak{b}_i]_{i+1} \geq d - i - 1$ whenever $-1 \leq i \leq \ell - 1$, and depth $[S/J]_j \geq d - \ell$ whenever $\ell + 1 \leq j \leq k$.

For $0 \leq i \leq \ell$ consider the graded S-modules $M_{(i)} = [S/\mathfrak{b}_i]_{\geq i+1} = I^{i+1}/\mathfrak{b}_i I^i$, and $N_{(i)} = I^i/\mathfrak{b}_{i-1}I^{i-1} + b_i I^i$ (where $I^{-1} = I^0 = S$). Notice that $[N_{(i)}]_{\geq i+1} = M_{(i)}$ and $[N_{(i)}]_i = [S/\mathfrak{b}_{i-1}]_i$, which yields exact sequences

$$0 \to M_{(i)} \to N_{(i)} \to [S/\mathfrak{b}_{i-1}]_i \to 0. \tag{3.10}$$

On the other hand, if $0 \leq i \leq \ell - 1$, then $N_{(i+1)} = M_{(i)}/b_{i+1}M_{(i)}$, and since $[\mathfrak{b}_i : (b_{i+1})]_{\geq i+1} = [\mathfrak{b}_i]_{\geq i+1}$ it follows that $0 :_{M_{(i)}} (b_{i+1}) = 0$. Thus, in the range $0 \leq i \leq \ell - 1$, we have the exact sequences

$$0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0. \tag{3.11}$$

Also notice that $N_{(0)} = S$. Hence it suffices to prove by decreasing induction on i, $0 \leq i \leq \ell$, that $\operatorname{depth}_S N_{(i)} \geq d - i$, and that $\operatorname{socle}(H^{d-i}(N_{(i)}))$ is concentrated in degrees at least i and at most $\max\{i, k - \ell + i\}$.

If $i = \ell$, then $N_{(\ell)} = [S/\mathfrak{b}_{\ell-1}]_\ell \oplus \bigoplus_{j=\ell+1}^{k} [S/J]_j$ has depth at least $d - \ell$ as an S_0-module, and hence as an S-module as well. Furthermore, $H^{d-\ell}(N_{(\ell)})$ is concentrated in degrees at least ℓ and at most $\max\{\ell, k\}$ (see, e.g., [GH, 2.2]).

So let $0 \leq i \leq \ell - 1$ and suppose that our assertions hold for $i+1$. Applying $H^\bullet(-)$ to the exact sequence (3.11), we obtain homogeneous embeddings

$$H^j(M_{(i)})(-1) \hookrightarrow H^j(M_{(i)}) \tag{3.12}$$

in the range $j \leq d - i - 1$, and an exact sequence

$$H^{d-i-1}(M_{(i)}) \to H^{d-i-1}(N_{(i+1)}) \to H^{d-i}(M_{(i)})(-1) \xrightarrow{b_{i+1}} H^{d-i}(M_{(i)}). \tag{3.13}$$

Now, $H^j(M_{(i)})$ being Artinian (and hence zero in large degrees), (3.12) implies that $H^j(M_{(i)}) = 0$ for $j \leq d - i - 1$. Therefore $\operatorname{depth}_S M_{(i)} \geq d - i$. Thus by (3.13),

$$\operatorname{socle}(H^{d-i}(M_{(i)}) \cong \operatorname{socle}(H^{d-i-1}(N_{(i+1)}))(1). \tag{3.14}$$

On the other hand, $\operatorname{depth}_S[S/\mathfrak{b}_{i-1}]_i = \operatorname{depth}_{S_0}[S/\mathfrak{b}_{i-1}]_i \geq d-i$. Hence (3.10) implies that $\operatorname{depth}_S N_{(i)} \geq d-i$ and that there is an exact sequence

$$0 \to H^{d-i}(M_{(i)}) \to H^{d-i}(N_{(i)}) \to H^{d-i}([S/\mathfrak{b}_{i-1}]_i).$$

Now taking socles and using (3.14), we conclude from our induction hypothesis that $\operatorname{socle}(H^{d-i}(N_{(i)}))$ is concentrated in degrees at least i and at most $\max\{i, k-\ell+i\}$. ∎

We are now ready to prove a special case of Theorem 3.1.

Proposition 3.15 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell-1$, that* depth $R/I^j \geq d-g-j+1$ *for* $1 \leq j \leq \ell-g+1$, *and that* $r \leq \ell-g+1$.

Then $G = \operatorname{gr}_I(R)$ is Cohen-Macaulay. Furthermore, ω_G (in case it exists) is generated in degrees $\min\{g, \ell-r\}$ *and* g.

Proof. Let J be a minimal reduction of I with $r_J(I) = r$, let $a_1, \ldots, a_\ell$ and $\mathfrak{a}_i$ be as defined in Lemma 3.8, and let a_i' denote the image of a_i in $[G]_1$. We wish to apply Proposition 3.9 with $k = \ell-g+1$ to the ring $S = G$ and the linear forms $b_i = a_i'$, $1 \leq i \leq \ell$. From Lemma 3.8 (b) for $t = g-1$ we already know that $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell-1$. Thus it suffices to verify that depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d-i-1$ for $g-1 \leq i \leq \ell-1$, and that depth $[S/J]_{\ell-g+1} \geq d-\ell$.

Since $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell-1$, there are exact sequences

$$(3.16) \qquad 0 \to [S/\mathfrak{b}_i]_j \to [S/\mathfrak{b}_i]_{j+1} \to [S/\mathfrak{b}_{i+1}]_{j+1} \to 0$$

whenever $0 \leq i \leq \ell-1$ and $j \geq i-g+1$. On the other hand by our assumption, depth $[S/\mathfrak{b}_0]_j =$ depth $[S]_j \geq d-g-j$ for $j \leq \ell-g$. Hence using (3.16), we can see by induction on i that depth $[S/\mathfrak{b}_i]_j \geq d-g-j$ whenever $0 \leq i \leq \ell-1$ and $i-g+1 \leq j \leq \ell-g$. In particular, depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d-i-1$ for $0 \leq i \leq \ell-1$. As to $[S/J]_{\ell-g+1}$, notice that this module is $I^{\ell-g+1}/JI^{\ell-g} + I^{\ell-g+2} = I^{\ell-g+1}/JI^{\ell-g} + JI^{\ell-g+1} = I^{\ell-g+1}/JI^{\ell-g}$, which has the required depth by Lemma 3.5 (a).

Now by Proposition 3.9, G is Cohen-Macaulay. Furthermore, if $r = \ell - g + 1$, then the assertion concerning ω_G follows immediately, whereas if $r \leq \ell - g$, we have to repeat the above argument with $k = \ell - g$. ∎

We will need the following special case of [HH1, 2.9], which we will prove using an argument from [V2]:

Proposition 3.17 *Let R be a local Cohen-Macaulay ring of dimension d, let I be an R-ideal with $I^2 = aI$ for some $a \in I$, assume that $I_p = 0$ for every associated prime p of R containing I, and that depth $R/I \geq d - 1$.*
Then $G = \mathrm{gr}_I(R)$ is Cohen-Macaulay.

Proof. The R-homomorphism from the polynomial ring $R[T]$ to the Rees algebra $\mathcal{R} = R[It]$ sending T to at, induces a homomorphism of $R[T]$-modules $\varphi : IR[T] \rightarrow IR[It]$. Now φ is surjective because $I^2 = aI$, and φ is injective because $I_p = 0$ for every $p \in V(I) \cap \mathrm{Ass}(R)$ and hence $[0 : (a^j)] \cap I = 0$ for every $j \geq 1$. Thus $IR[It] \cong IR[T]$ has depth at least $d + 1$. Now a depth chase using the two exact sequences

$$0 \rightarrow I\mathcal{R}(-1) \rightarrow \mathcal{R} \rightarrow R \rightarrow 0,$$

$$0 \rightarrow I\mathcal{R} \rightarrow \mathcal{R} \rightarrow G \rightarrow 0$$

shows that G has depth at least d. ∎

We are now ready to complete the proof of Theorem 3.1. The main idea is to deduce this theorem from Proposition 3.15, by factoring out a suitable link of the ideal and thereby lowering the analytic deviation (this method has been employed by other authors, e.g., [GN2] and [T], or earlier, but in a different context, [H1], [H5], [HVV]).

The Proof of Theorem 3.1: Write g = grade I and $\delta = \delta(I) = \ell - g + 1 - k$, and recall that $\delta \geq 0$. We are going to induct on δ, the case $\delta = 0$ being covered by Proposition 3.15. Thus we may assume that $\delta \geq 1$ and that the assertion holds for smaller values of δ. Now $\ell \geq g + k \geq g + 1$.

We adopt the notation of Lemma 3.8. By that lemma, $\mathfrak{a}_g \cap I^j = \mathfrak{a}_g I^{j-1}$ for $j \geq 1$, and, equivalently, $a'_1, \ldots a'_g$ from a G-regular sequence. Thus we do not change our assumptions and the conclusion if we factor out $\mathfrak{a}_g$ to assume that $g = 0$ (thereby d and ℓ decrease by g, whereas k may be taken to remain unchanged). Now $\ell \geq k \geq 1$, and in particular, I satisfies G_1 and therefore

$I_p = 0$ for every $p \in V(I) \cap \mathrm{Ass}(R)$. Thus if $\ell = 1$, then our assertion follows from Proposition 3.17.

Hence we may assume that $\ell \geq \max\{2, k\}$, in which case I satisfies AN_0^-. Write $K = 0 : I$ and let " $^-$ " denote images in $\bar{R} = R/K$. Now $\bar{R}$ is Cohen-Macaulay since I satisfies AN_0^-, and by Lemma 3.4 (c) for instance, $I \cap K = 0$, grade $\bar{I} \geq 1$, $\bar{I}$ still satisfies G_ℓ and $AN_{\ell-2}^-$ locally in codimension $\ell - 1$, and $\bar{I}$ is $AN_{\ell-\max\{2,k\}}^-$. Furthermore, $\dim \bar{R} = \dim R = d$; and since $I \cap K = 0$, we have $\ell(\bar{I}) = \ell(I) = \ell$ and thus k may be taken to remain unchanged. Therefore $\delta(\bar{I}) = \ell - \text{grade } \bar{I} + 1 - k < \ell + 1 - k = \delta(I)$. Again, as $I \cap K = 0$, we have an exact sequence

$$0 \to K \to \mathrm{gr}_I(R) \to \mathrm{gr}_{\bar{I}}(\bar{R}) \to 0, \tag{3.18}$$

where depth $K = d$ since depth $\bar{R} = d$. Now by (3.18), depth $\bar{R}/\bar{I} \geq \min\{\text{depth } K - 1, \text{depth } R/I\} \geq \min\{d-1, d-\ell+k-1\} = d-\ell+k-1$, where the latter equality holds because $\ell \geq k$. Furthermore, again by (3.18), $\bar{I}^{j-1}/\bar{I}^j \cong I^{j-1}/I^j$ for $j \geq 2$, and we conclude that depth $\bar{R}/\bar{I}^j \geq d-\ell+k-j$ whenever $j \leq k$. Thus we may apply our induction hypothesis to conclude that $\mathrm{gr}_{\bar{I}}(\bar{R})$ is Cohen-Macaulay, and hence by (3.18), $\mathrm{gr}_I(R)$ has the same property. ■

Having completed the proof of Theorem 3.1, we are now going to draw further conclusions, mainly from Lemmas 3.5 (a) and 3.8 (b) (cf. [JU] for most of this material). The next proposition says that it suffices to check one of the assumptions of Theorem 3.1 locally in codimension ℓ (generalizing results from [HH1], [HH2], [V2], [U1], [SUV]).

Proposition 3.19 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with* grade *g and analytic spread ℓ, let $k \geq 1$ be an integer, and assume that I satisfies G_ℓ and $AN_{\ell-2}^-$, and that* depth *$R/I^j \geq d - \ell + k - j$ whenever $1 \leq j \leq k$. The following are equivalent:*

(a) *$r(I_p) \leq k$ for every $p \in V(I)$ with $\dim R_p = \ell < \mu(I_p)$.*

(b) *$r(I) \leq k$.*

We are now going to investigate the degrees of the syzygies and the number of defining equations of the Rees algebra $\mathcal{R}$ and of the associated graded ring G.

Proposition 3.20 *If in addition to the assumptions of Theorem 3.1, I satisfies $AN^-_{\ell-2}$, then*

$$\operatorname{reg}(\mathcal{R}) = \operatorname{reg}(G) = r.$$

In particular, $rt(I) \le r+1$, and $r_J(I) = r$ for every minimal reduction J of I.

Proof. By [Tr, 3.3], in conjunction with Lemma 3.8 (b), $\operatorname{reg}(G) \le r$. Now the rest follows from Propositions 1.1 and 1.7. ∎

When studying the defining equations of $\mathcal{R}$ and G, it suffices to consider the ideals $\mathcal{A}$ and $\mathcal{B}$ of the symmetric algebras $S(I)$ and $S(I/I^2)$ that fit into the natural exact sequences

$$0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0, \tag{3.21}$$

and

$$0 \to \mathcal{B} \to S(I/I^2) \to G \to 0. \tag{3.22}$$

Proposition 3.23 *Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with* grade *g, analytic spread ℓ, minimal number of generators n, and reduction number r, assume that I satisfies G_ℓ, that the Koszul homology modules $H_j(I)$ are Cohen-Macaulay whenever $0 \le j \le \ell-g$, and that $r \le \ell - g + 1$. Further let $J = (a_1, \ldots, a_\ell)$ be a minimal reduction of I, let $B = R[T_1, \ldots, T_\ell]$ be a polynomial ring, and consider $S(I)$ and R as B-modules via the R-algebra homomorphisms mapping T_i to $a_i \in I = [S(I)]_1$ and to 0, respectively.*

Then $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\ge \ell-g+2}$ and $\mathcal{B} \otimes_B R \cong [S(I/J+I^2)]_{\ge \ell-g+2}$. In particular, the ideals $\mathcal{A}$ and $\mathcal{B}$ are minimally generated by $\binom{n-g+1}{n-\ell-1}$ forms of degree $\ell - g + 2$.

Proof. We first show that $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\ge \ell-g+2}$.

From our assumption on the Koszul homology modules we know that I satisfies $AN^-_{\ell-2}$ (Theorem 2.6 (a)), and that the graded pieces $[\mathcal{M}]_j$ of the $\mathcal{M}$-complex are acyclic for $0 \le j \le \ell - g$ ([V3]). By the acyclicity of these complexes, $S_j(I) \cong I^j$ ([HSV, the proof of 4.6]) and depth $R/I^j \ge d-g-j+1$ in the range $1 \le j \le \ell - g + 1$. Thus $[\mathcal{A}]_j = 0$ whenever $j \le \ell - g + 1$, and furthermore by Proposition 3.20, Lemma 3.8 applies to the ideal J.

After changing the generators of $a_1, \ldots, a_\ell$ of J if needed, we conclude from that lemma that in G, $[(a'_1, \ldots, a'_i) : (a'_{i+1})]_{\geq \ell-g+1} = [(a'_1, \ldots, a'_i)]_{\geq \ell-g+1}$ whenever $0 \leq i \leq \ell - 1$. Thus in $\mathcal{R}$, $[(a_1t, \ldots, a_it) : (a_{i+1}t)]_{\geq \ell-g+1} = [(a_1t, \ldots, a_it)]_{\geq \ell-g+1}$ whenever $0 \leq i \leq \ell - 1$, as can be seen from [AH, the proof of 6.5]. Now let $H.$ denote Koszul homology with values in $\mathcal{R}$. Using the long exact sequence

$$H_1(a_1t, \ldots, a_it) \rightarrow H_1(a_1t, \ldots, a_{i+1}t) \rightarrow$$
$$H_0(a_1t, \ldots, a_it)(-1) \xrightarrow{a_{i+1}t} H_0(a_1t, \ldots, a_it)$$

and induction on i, one sees that $[H_1(a_1t, \ldots a_it)]_{\geq \ell-g+2} = 0$ whenever $0 \leq i \leq \ell$ (cf. also [AH, 4.4]). In particular, $[\mathrm{Tor}_1^B(\mathcal{R}, R)]_{\geq \ell-g+2} = [H_1(a_1t, \ldots, a_\ell t)]_{\geq \ell-g+2} = 0$.

On the other hand, we had seen that $[\mathcal{A}]_j = 0$ whenever $j \leq \ell - g + 1$. Thus applying $- \otimes_B R$ to (3.21) yields an exact sequence

$$0 \rightarrow \mathcal{A} \otimes_B R \rightarrow S(I/J) \rightarrow \mathcal{R} \otimes_B R \rightarrow 0.$$

Since $[\mathcal{R} \otimes_B B/B_+]_{\geq \ell-g+2} = 0$, we conclude that $[\mathcal{A} \otimes_B R]_{\geq \ell-g+2} \cong [S(I/J)]_{\geq \ell-g+2}$, and therefore $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\geq \ell-g+2}$.

To prove the remaining assertions of the proposition, notice that upon applying $- \otimes_R R/I$ to the latter isomorphism, one obtains a commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes_B R/I & \xrightarrow{\sim} & [S(I/J + I^2)]_{\geq \ell-g+2} \\ \varphi \searrow & & \nearrow \psi \\ & \mathcal{B} \otimes_B R & \end{array}$$

where φ is surjective. Thus ψ is an isomorphism as well. Finally, notice that both $S(I)$-modules $[S(I/J)]_{\geq \ell-g+2}$ and $[S(I/J + I^2)]_{\geq \ell-g+2}$ are minimally generated by $\binom{n-g+1}{n-\ell-1}$ homogeneous elements of degree $\ell - g + 2$. ■

4 Gorenstein Property and Canonical Module

The relationship between the Gorenstein property of the Rees algebra $\mathcal{R}$ and of the associated graded ring G is well understood:

Theorem 4.1 ([I]) *Let R be a local Gorenstein ring and let I be an R-ideal with* grade $I \geq 2$.

Then $\mathcal{R}$ is Gorenstein if and only if G is Gorenstein and $a(G) = -2$.

Combining this result with Theorem 1.4 (a), one obtains (cf. also [AHT]):

Theorem 4.2 *Let R be a local Gorenstein ring and let I be an R-ideal with* grade $I \geq 2$.

Then $\mathcal{R}$ is Gorenstein if and only if G is Gorenstein, $r(I_p) = \ell(I_p) - 2$ for some $p \in V(I)$, and $r(I_p) \leq \ell(I_p) - 2$ for every $p \in V(I)$ with $\dim R_p = \ell(I_p)$.

In the light of Theorems 4.1 and 4.2 we may restrict our attention to the Gorenstein property of the associated graded ring G. One can expect that this property corresponds to the reduction number of the ideal being "very small". As a first illustration of this theme, we mention the following observation:

Proposition 4.3 ([SUV]) *Let R be a Noetherian local ring with infinite residue class field, let I be an R-ideal with analytic spread ℓ and reduction number r, and assume that I_p is a complete intersection of* grade $g > 0$ *for some minimal prime p of I.*

If G is Gorenstein, then $a(G) = -g$, $\mathcal{R}$ is Cohen-Macaulay, and $r \leq \ell - g$.

Proof. In the light of [TI] (cf. also [V3]) and Theorem 1.4 (a), we only need to prove that $a(G) = -g$.

To do so, we may assume that R is complete. Let ω_G stand for the canonical module of G. By assumption, $\omega_G \cong G(a)$, where $a = a(G)$ (cf. [HIO]). Note that $\omega_G \otimes R_p = \omega_{G \otimes R_p}$, because G is equidimensional. Thus $\omega_{G \otimes R_p} \cong (G \otimes R_p)(a)$. But on the other hand, $G \otimes R_p$ is a (standard graded) polynomial ring in g variables, and hence $\omega_{G \otimes R_p} \cong (G \otimes R_p)(-g)$. Therefore $a = -g$. ■

Results of this type become considerably more subtle, if one allows the complete intersection locus of I to be empty:

Theorem 4.4 ([HHR]) *Let (R, m) be a regular local ring of dimension $d \geq 2$ with infinite residue field, and let I be an m-primary ideal with reduction number r.*

If G is Gorenstein, then $r \leq d - 2$.

Proof. Adopt the notation of Lemma 3.8 and notice that $a'_1, \ldots, a'_d$ form a G-regular sequence.

Since grade $G_+ > 0$, we have that $I^j : m \subset I^j : I = I^{j-1}$ for $j > 0$. Thus $I^j : m \subset J \subset I^j + J$ for $j >> 0$, whereas $I : m \not\subset I = I + J$. Therefore $k = \max\{j | I^j : m \not\subset I^j + J\}$ is a well-defined positive integer. We claim that $G/(a'_1, \ldots, a'_d)$ has a nontrivial socle element in degree $k - 1$. So let $x \in (I^k : m) \backslash (I^k + J)$. Then $x \in I^{k-1} \backslash (I^k + J)$, and hence x gives rise to a nonzero element x' of degree $k - 1$ in $G/(a'_1, \ldots, a'_d)$. To see that x' is in the socle of the latter ring, notice that $mx \in I^k$ since $x \in I^k : m$, and that $Ix \subset I^{k+1} : m \subset I^{k+1} + J$ by the maximality of k, which gives $Ix \subset I^{k+1} + J \cap I^k = I^{k+1} + JI^{k-1}$ ([VV]). Thus x' is in the socle of $G/(a'_1, \ldots a'_d)$.

Since x' is nontrivial and has degree $k - 1$, and since G is Gorenstein, we conclude that the socle of $G/(a'_1, \ldots, a'_d)$ is concentrated in degree $k - 1$, or equivalently, that $r = k - 1$.

Finally, one invokes a result by J. Lipman, saying that $I^d : m \subset J$ since R is regular and $d \geq 2$. But then $k \leq d - 1$, and hence $r \leq d - 2$. ∎

We now turn to the case of ideals that are generically complete intersections. We are going to provide conditions (a necessary one and a sufficient one) for G to be Gorenstein, generalizing results from [HSV2], [GN2], [GN3], [SUV], [T], [HHR].

The following theorem says that in the presence of a Gorenstein associated graded ring, one can improve the bound on the reduction number given in Proposition 4.3, if (and only if) sufficiently many powers of the ideal have "good" depth:

Theorem 4.5 ([JU]) *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with* grade g, *analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ, that I is unmixed and generically a complete intersection, and that G is Gorenstein. Let $0 \leq k \leq \ell - g$ be an integer. The following are equivalent:*

(a) depth $R/I^j \geq d - g - j + 1$ *for* $1 \leq j \leq \ell - g - k$

(b) $r \leq k$ *and* I *satisfies* $AN^-_{\ell-k-1}$.

With the next two results we are going to prove a converse of Theorem 4.5, to the effect that the above conditions (a) and (b) imply the Gorensteinness of G (under some additional assumptions).

Proposition 4.6 ([JU]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field and canonical module ω, let I be an R-ideal with* grade g, *analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell-1$, that I satisfies AN^-_g, that* depth $R/I^j \geq d-g-j+1$ *for* $1 \leq j \leq \ell - g$, *and that* $r \leq \ell - g$.

Then $\omega_G \cong \mathrm{gr}_I(\omega)(-g)$.

Proof. Theorem 3.1 already shows that G is Cohen-Macaulay. But then by [HSV2, 2.4], our assertion is equivalent to saying that ω_G is generated in degree g, which we may check after reducing modulo a G-regular sequence in R. Now since G is Cohen-Macaulay, there is a sequence $x_1, \ldots, x_{d-\ell}$ in R which is regular on R and G; furthermore, factoring out these elements does not change our assumptions (as for the Artin-Nagata property, see [U1, 1.13]). But after doing so, $d = \ell$, hence depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell - g + 1$. In this case our assertion follows from Proposition 3.15. ■

Theorem 4.7 ([JU]) *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with* grade g, *analytic spread ℓ, and reduction number r, let k, $0 \leq k \leq \ell - g$, be an integer with $r \leq k$, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$, that* depth $R/I^j \geq d - \ell + k - j + 1$ *for* $1 \leq j \leq k$, *and that* depth $R/I^j \geq d-g-j+1$ *for* $1 \leq j \leq \ell - g - k$.

Then G is Gorenstein.

Proof. Adopt the notation of Lemma 3.8, write $K_i = \mathfrak{a}_i : I$ for $0 \leq i \leq \ell$, and $K_{-1} = 0$. Our assumptions together with Theorem 2.6 (b), Lemma 3.4 (c.iii, iv), and Lemma 3.8 (a) imply that there are natural isomorphisms

$$\omega_{R/K_{i+1}} \cong I\omega_{R/(K_i, a_{i+1})} \quad \text{for } g-1 \leq i \leq \ell - k - 2, \tag{4.8}$$

and that

(4.9) $R/(K_i, a_{i+1})$ is Gorenstein locally in codimension one for $i \leq \min\{\ell - k - 1, \ell - 3\}$.

We now replace the assumption of R being Gorenstein by the weaker condition that R is Cohen-Macaulay and that (4.8) and (4.9) hold. With this new assumption, we are going to show that $\omega_G \cong \mathrm{gr}_I(\omega_R)(-g)$. We use increasing induction on $\delta = \delta(I) = \ell - g - k \geq 0$, the cases $\delta = 0$ and $\delta = \ell - g$ being covered by Proposition 4.6.

So let $0 < \delta < \ell - g$. As in the proof of Theorem 3.1, we replace R by $R/(a_1, \ldots, a_g) = R/(K_{g-1}, a_g)$, thus reducing to the case $g = 0$. Write $\omega = \omega_R$, $K = 0 : I$, and let " $^-$ " denote images in $\bar{R} = R/K$. Since $\ell - g > k > 0$, we may use the property $AN^-_{\ell-\max\{2,k\}}$, (4.8), and (4.9), to conclude that $\bar{R}$ is Cohen-Macaulay with $\omega_{\bar{R}} \cong I\omega$ and that R is Gorenstein locally in codimension one. Furthermore, by the same arguments as in the proof of Theorem 3.1, all our assumptions are preserved as we pass from I and R to $\bar{I}$ and $\bar{R}$ (including (4.8) and (4.9), whereas $\bar{R}$ need no longer be Gorenstein), depth $\bar{R}/\bar{I} = d - 1 = \dim \bar{R}/\bar{I}$, and $\delta(\bar{I}) < \delta(I)$. Thus Theorem 3.1 and our induction hypothesis imply that $\mathrm{gr}_{\bar{I}}(\bar{R})$ is Cohen-Macaulay with canonical module $\mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1)$, where $\mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1) \cong \mathrm{gr}_I(I\omega)(-1)$.

Furthermore, by Theorem 3.1, $G = \mathrm{gr}_I(R)$ is Cohen-Macaulay, and from (3.18) we have an exact sequence

$$(4.10) \qquad 0 \to K \to \mathrm{gr}_I(R) \to \mathrm{gr}_{\bar{I}}(\bar{R}) \to 0$$

of maximal Cohen-Macaulay G-modules. Here $\mathrm{Hom}_G(K, \omega_G) \cong \mathrm{Hom}_R(K, \omega) \cong \omega \otimes_R R/I$, where the last isomorphism follows from the fact that $\bar{R} \cong R/K$ and $\omega_{\bar{R}} \cong I\omega$. Since the canonical module of $\mathrm{gr}_{\bar{I}}(\bar{R})$ is $\mathrm{gr}_I(I\omega)(-1)$, we may now dualize (4.10) into ω_G to obtain an exact sequence

$$(4.11) \qquad 0 \to \mathrm{gr}_I(I\omega)(-1) \to \omega_G \to \omega \otimes_R R/I \to 0$$

of graded maximal Cohen-Macaulay G-modules.

On the other hand, there is the natural exact exact

$$(4.12) \qquad 0 \to \mathrm{gr}_I(I\omega)(-1) \to \mathrm{gr}_I(\omega) \to \omega \otimes_R R/I \to 0.$$

When comparing (4.11) and (4.12), we obtain a commutative diagram of homogeneous maps,

$$(4.13) \qquad \begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{gr}_I(I\omega)(-1) & \longrightarrow & \mathrm{gr}_I(\omega) & \longrightarrow & \omega \otimes_R R/I & \longrightarrow 0 \\ & \| & \# & \downarrow \varphi_1 & \# & \downarrow \varphi_2 & \\ 0 \longrightarrow & \mathrm{gr}_I(I\omega)(-1) & \longrightarrow & \omega_G & \longrightarrow & \omega \otimes_R R/I & \longrightarrow 0 \end{array}$$

where φ_2 is induced by φ_1, and φ_1 lifts the identity. Such a lifting exists, because $\mathrm{Ext}^1_G(\omega \otimes_R R/I, \omega_G) = 0$.

We will be done once we have shown that φ_1 is an isomorphism, or equivalently, that φ_2 is an epimorphism. However, since depth $\omega \otimes R/I =$ depth $R = \dim R$, and since R is Gorenstein locally in codimension one, it follows that the natural map $R \to \mathrm{End}_R(\omega \otimes R/I)$ is surjective. Thus φ_2 is multiplication by some element $x \in R$. If x is a unit, then we are done. Otherwise choose a minimal prime p of the R-ideal (K, x). Now $\dim R_p \leq 1$, hence $\ell(I_p) \leq 1 < \ell$ and therefore $\delta(I_p) < \delta(I)$. Thus, again, by induction hypothesis, $\omega_G \otimes_R R_p \cong \mathrm{gr}_I(\omega) \otimes_R R_p$, which implies that $\varphi_1 \otimes_R R_p$ arises from multiplication by some $y \in R_p$. We need to conclude that y is a unit. However, multiplication by y induces an automorphism on $\mathrm{gr}_I(I\omega)(-1) \otimes_R R_p$, as can be seen from (4.13), and $\mathrm{gr}_I(I\omega)(-1) \otimes_R R_p \neq 0$ because $K \subset p$. This forces y to be a unit in R_p. ■

Although the associated graded ring cannot be Gorenstein if the reduction number is too large, one might still be able to compute its canonical module:

Theorem 4.14 ([U2]) *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with grade $g \geq 2$, analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ, that depth $R/I^j \geq d - g - j + 1$ whenever $1 \leq j \leq \ell - g + 1$, and that $r \leq \ell - g + 1$. Further let J be a minimal reduction of I, write $K = J : I$, $\mathcal{R} = R[It]$, $T = R[It, t^{-1}]$, and $G = gr_I(R)$. Then:*

(a) $\omega_{\mathcal{R}} \cong ((1,t)^{g-3}t, Kt^{g-1})\mathcal{R} = \bigoplus_{i=1}^{g-2} Rt^i \oplus \bigoplus_{i \geq g-1} I^{i-g+1}Kt^i$.

(b) $\omega_T \cong (t^{g-2}, Kt^{g-1})T = \bigoplus_{i \leq g-2} Rt^i \oplus \bigoplus_{i \geq g-1} I^{i-g+1}Kt^i$.

(c) $\omega_G \cong (R/K \oplus \bigoplus_{i \geq 0} I^iK/I^{i+1}K)(-g+1)$.

5 A Criterion for the Expected Reduction Number

So far we have seen that a great deal can be said about blow-up algebras, if the reduction number of the ideal is at most $\ell - g + 1$. In the present section now we intend to provide a "concrete" characterization (in the sense of [V1, 3.1.3] or [AH, 8.2]), for when an ideal has this *expected reduction number* $\ell - g + 1$. The criterion we have in mind should be stated in terms of the syzygies of the ideal, i.e., in terms of the linear relations on the Rees algebra, whereas a priori, the reduction number only corresponds to relations of higher degrees. The results in this section are from [U2].

Definition 5.1 Let R be a Noetherian local ring, let I be an R-ideal, and let s be a positive integer. For a generating sequence $f_1, \ldots, f_n$ of I, let X be an n by n matrix of indeterminates, and write $[a_1, \ldots, a_n] = [f_1, \ldots, f_n]X$.

We say that I is *s-balanced* if for some $n \geq s$ and some generating sequence $f_1, \ldots, f_n$ of I, $(a_{i_1}, \ldots, a_{i_s})R(X) : IR(X)$ gives the same $R(X)$-ideal for every choice of the subset $\{i_1, \ldots i_s\} \subset \{1, \ldots, n\}$.

Remark 5.2 The above definition does not depend on the choice of n or of $f_1, \ldots, f_n$.

We are now ready to state the main result of this section:

Theorem 5.3 *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade $g > 0$, and let s be a positive integer. Assume that I satisfies G_s and that* depth $R/I^j \geq d - g - j + 1$ *whenever $1 \leq j \leq s - g + 1$. The following are equivalent:*

(a) *I is s-balanced.*

(b) *$r(I) \leq \ell(I) - g + 1$ and $\ell(I) \leq s$.*

(c) *Either $r(I) = 0$ and $\mu(I) \leq s$, or else, $r(I) = \ell(I) - g + 1$ and $\ell(I) = s$.*

The implication (c) $\Rightarrow$ (a) in the above theorem can be easily deduced from Theorem 4.14 (c), which says that $J : I = K = \mathrm{ann}_R([\omega_G]_{g-1})$ gives the same ideal for every minimal reduction J of I. The proof of the implication (a) $\Rightarrow$ (b) however, is more involved.

Notice that if an ideal satisfies one of the equivalent assumptions in Theorem 5.3, then its associated graded ring is Cohen-Macaulay, and Propositions 3.20, 3.23 and Theorem 4.14 apply. One can think of various instances where the condition of being s-balanced takes a more concrete form:

Corollary 5.4 *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade $g > 0$, let $n = \mu(I)$, assume that I satisfies G_{n-1} and that* depth $R/I^j \geq d - g - j + 1$ *whenever $1 \leq g \leq n - g$. The following are equivalent:*

(a) *For some matrix φ with n rows presenting I, $I_1(\varphi)$ is generated by the entries of the last row of φ.*

(b) $r(I) = \ell(I) - g + 1$ *and* $\ell(I) = n - 1$.

If I is a perfect ideal with $g = 2$, $\ell(I) = 3$, and $\mu(I) = 4$, or if I is a perfect Gorenstein ideal with $g = 3$, $\ell(I) = 4$, and $\mu(I) = 5$, then Corollary 5.4 would also follow from [V1], [AH], [AHH]. The case of arbitrary strongly Cohen-Macaulay ideals was later treated in [SUV].

Results similar to our next one have been first considered in [CPV] for equimultiple ideals (cf. also [CP]).

Corollary 5.5 *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade $g > 0$, and let s be an integer. Assume that I satisfies G_s, but not G_{s+1}, and that* depth $R/I^j \geq d - g - j + 1$ *whenever $1 \leq j \leq s - g + 1$.*

If there exists a prime ideal K of height at least s such that $KI \subset (f_1, \ldots, f_s) \subset I$, then $r(I) = \ell(I) - g + 1$ and $\ell(I) = s$.

A typical case where the above corollary might apply is when $s = d$ and K is the maximal ideal of R.

Corollary 5.6 *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of grade two, let s be a positive integer, and assume that I satisfies G_s. The following are equivalent:*

(a) *For some n by $n-1$ matrix φ presenting I, $I_{n-s}(\varphi)$ is generated by the minors of the $n-s$ by $n-1$ matrix consisting of the last $n-s$ rows of φ.*

(b) $r(I) < \ell(I) \leq s$.

(c) $R[It]$ *is Cohen-Macaulay and* $\ell(I) \leq s$.

Proof. (a) $\Leftrightarrow$ (b): One can see, using Example 2.2 and a deformation argument, that (a) holds if and only if I is s-balanced. Now the asserted equivalence follows from Theorem 5.3.

(b) $\Rightarrow$ (c): One uses Corollary 3.3.

(c) $\Rightarrow$ (b): Since $R[It]$ is Cohen-Macaulay, $r(I) < \ell(I)$ ([JK], [SUV], or [AHT]). ■

Corollary 5.7 *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of grade two with $\ell = \ell(I)$, and assume that I satisfies G_ℓ. The following are equivalent:*

(a) *For some n by $n-1$ matrix φ presenting I, $I_{n-\ell}(\varphi)$ is generated by the maximal minors of the $n-\ell$ by $n-1$ matrix consisting of the last $n-\ell$ rows of φ.*

(b) $r(I) < \ell$.

(c) $R[It]$ *is Cohen-Macaulay.*

Notice that if I satisfies one of the equivalent conditions of Corollary 5.7, then by part (a) of that corollary, the ℓ-th Fitting ideal of I, $\text{Fitt}_\ell(I)$, is either R or else a perfect ideal of grade ℓ with $\mu(\text{Fitt}_\ell(I)) = \binom{\mu(I)-1}{\ell-1}$. Likewise, in the next corollary, $\text{Fitt}_\ell(I)$ is necessarily a complete intersection of grade ℓ.

Corollary 5.8 *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of grade three with R/I Gorenstein, set $n = \mu(I)$, $\ell = \ell(I)$, and assume that R satisfies G_ℓ. The following are equivalent:*

(a) *$\ell = n-1$ and for some alternating n by n matrix φ presenting I, $I_1(\varphi)$ is generated by the entries of the last row of φ.*

(b) $r(I) = \ell - 2$.

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