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Zero cycles splitting of projective modules and efficient generation of modules

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M. P. Murthy

§1 Introduction. Let A be a noetherian ring of dimension n, ~~and~~ Recall that a finite projective A-module is of rank r if $P_{\mathfrak{p}} \simeq A^r_{\mathfrak{p}}$, $\forall$ $\mathfrak{p} \in \operatorname{Spec} A$. There are two classical stability theorems proved by Bass and Serre (see for example Bass' fat book [Ba]).

Theorem (Serre). If rank $P > n$, then $P \simeq P' \oplus A$.

Theorem (Bass) Let P, P' be projective A-modules such that $P \oplus Q \simeq P' \oplus Q$, for some finite projective A-module Q. If rank $P = r > n$, then $P \simeq P'$.

Both these theorems are best possible. A standard example which every one gives is:

$$A = \mathbb{R}[X, Y, Z]/(X^2+Y^2+Z^2-1) = \mathbb{R}[x, y, z], \quad P = A^3/A(x, y, z).$$

Then P is a rank two indecomposible module and $P \oplus A \simeq A^2 \oplus A$, but $P \not\simeq A^2$. In some special cases these theorems can be improved.

<u>Suslin's Cancellation Theorem</u> Let A be an affine algebra of dimension n over an algebraically closed field. Let P, P', Q be projective A-modules such that $P \oplus Q \simeq P' \oplus Q$. ~~Then~~ If rank $P \geqslant n$, then $P \approx P'$.

In anology with the analytic case, many hoped that Suslin's theorem could be strengthened by replacing "rank $P \geqslant n$" by "rank $P \geqslant n/2+1$". But this turned out to be false as Mohan Kumar [MK3] showed that: for every prime p, there is a smooth affine algebra A of dimension $p+2$ and an A-module such that $P \oplus A \simeq A^{p+1}$, but $P \not\approx A^p$.

As far as cancellation of projective modules over affine algebras over algebraically closed fields is concerned, the following question is open.

<u>Open question</u>. Let A be an affine algebra of dimension n over an algebraically closed field. Let P and P' be projective A-modules of rank $n-1$ such that $P \oplus A \simeq P' \oplus A$, ~~then~~ Is $P \approx P'$?

This is open even when $n=3$ and $P=A^2$.

Now we discuss Serre's stability theorem.

It is known that Serre's theorem cannot be improved even when A is an affine k-algebra of dimension n over an algebraically closed field. Mumford's work on rational equivalence of zero cycles shows that there exist many indecomposible projective modules of rank 2 over certain smooth affine $\mathbb{C}$-algebras of dimension 2 (Cf. [MS]). For example,

$$A = \mathbb{C}[X,Y,Z]/(X^n+Y^n+Z^n-1), \quad n \geq 4.$$

Let A be a noetherian ring. We denote by $K_0(A)$, the Grothendieck group of finite projective A-modules i.e. $K_0(A)$ = Free abelian group on isomorphism classes of finite projective A-modules modulo the relations $(P \oplus Q) = (P) + (Q)$, where (P) denotes the isomorphism class of the finite projective module P.

It is known that $K_0(A)$ = Grothendieck group all finite A-modules of finite projective dimension i.e. Free abelian group on isomorphism classes of finite A-modules with relations

$(M) = (M') + (M'')$: if there is an exact sequence $0 \to M' \to M \to M'' \to 0$ of finite A-modules.

Let A be a reduced affine algebra of dimension n over an algebraically closed field and P a projective A-module of rank n. For an A-module M, put $M^* = \mathrm{Hom}_A(M, A)$.

<u>Definition</u> Let P be a projective A-module of rank n. The nth Chern class of P

$$= C_n(P) = \sum_{i=0}^{n} (-1)^i (\wedge^i P^*) \in K_0(A).$$

Remark.

Let $F \subset \operatorname{Spec} A$ be a closed set of dimension $< n$ and P a projective A-module of rank n. Then by Bertini theorem (see [SW1] and [Mu2]), for a general $s \in P$, the natural map $s: P^* \to A$ is such that $\operatorname{Im} s = I$ can be chosen to be a local complete intersection. ~~a even a p~~ In fact I can be chosen to be a product of distinct smooth maximal ideals of height n, with $V(I) \cap F = \phi$. Since I is a local complete intersection, we get a Koszul resolution

$$0 \to \Lambda^n P^* \to \cdots \to \Lambda^2 P^* \to P^* \xrightarrow{s} A \to A/I \to 0 .$$

Hence we have $C_n(P) = (A/I) \in K_0(A)$.

Definition Let A be a noetherian ring of dimension n.

$F^n K_0(A)$ = sub-group of $K_0(A)$ generated by all (A/m), where m is a maximal ideal of height n such that A_m is regular.

Remark i) $C_n(P) \in F^n K_0(A)$, for a projective A-module of rank n over ~~an~~ a reduced affine Algebra of dimension n over an algebraically closed field.

ii) Suppose $P = P' \oplus A$. Then $C_n(P) = 0$. Follows easily from $\wedge^i P \simeq \wedge^i P' \oplus \wedge^{i-1} P'$.

We will be concerned with the proof and some applications of the following Theorem [Mu2].

Theorem 1 Let A be a reduced affine algebra of dimension n over an algebraically closed field k. ~~and P~~ Suppose $F^n K_0(A)$ has no $(n-1)!$ torsion (i.e. $(n-1)! \cdot x = 0$, $x \in F^n K_0(A) \Rightarrow x = 0$). Let P be a projective A-module of rank n. Then $C_n(P) = 0 \iff P \simeq P' \oplus A$, for some P'.

<u>Remarks</u> a) It is known that $F^n K_0(A)$ is a divisible group. Further $F^n K_0(A)$ is torsion-free in the following cases:

i) Char $k = 0$, $n \geq 2$ (M. Levine [Le])

ii) Char $k = p > 0$, $n = 2$ and A regular or $n \geq 3$, and A is regular in codimension one (Srinivas [Sr]).

iii) By (M. Levine [Le]), $F^n K_0(A)_{tor}$ is a p-primary group (p = char k). It is not known if $F^n K_0(A)$ is always torsion-free.

b) The theorem above is immediate for $n \leq 2$:

$n = 1$. $C_1(P) = (A) - (P^*) = 0$ in $K_0(A)$. So P^* is stably free. It is well known that rank one stably free modules are free. So P^* and hence P is free.

$n = 2$. $C_2(P) = (A) - (P^*) + (\wedge^2 P^*) = 0$ in $K_0(A)$

$\Longrightarrow (P^*) = (A \oplus \wedge^2 P^*)$. Hence by Suslin's Cancellation Theorem, $P^* \simeq A \oplus \wedge^2 P^*$.

So, $P \simeq A \oplus \wedge^2 P$.

c) For $n = 3$, this result was proved in [MKM], when A is smooth.

Idea of proof of Th. 1. Let P be a projective A-module of rank n. Let $s \in P$ be a 'general' element ~~then~~ and let $Im(P^* \xrightarrow{s} A) = I$. Then I is a product of distinct smooth maximal ideals and $(A/_I) = C_n(P) = 0$ in $K_0(A)$. On the other hand Mohan Kumar ([MK2]) if P^* maps ~~onto~~ on to a complete intersection ideal of height n, then P^* has a free direct summand of rank one. Thus it suffices to show that if $I \subset A$ is a local complete intersection of height n such that $(A/_I) = 0$ in $K_0(A)$, then I is a complete intersection. This is done with the additional hypothesis that $F^n K_0(A)$ has no $(n-1)!$ torsion. ~~To prove this~~ This last assertion is deduced by the ~~following~~ theorem stated below.

~~Theorem 2. Let A be a reduced affine k-algebra of dimension n over an algebraically closed~~

(9)

Let A be a reduced affine algebra of dimension n over an alg. closed field and I, J ideals in A.

<u>Definition</u>: J is residual to I, if

i) J is a local complete intersection of height n

ii) $I+J=A$ and IJ is generated by n elements.

<u>Remark</u> If there is an ideal J residual to I, then

$\frac{I}{I^2} = IJ \otimes A/_I$ is generated by n elements.

Conversely if $I/_{I^2}$ is generated by n elements, say $f_1, \ldots, f_n \in I$ generates $I/_{I^2}$. Then by Bertini's theorem, there exist ('general') $h_i \in I^2$, $1 \le i \le n$, such that if we put ~~such that~~ $f_i' = f_i + h_i$, then

$(f_1', \ldots, f_n') = IJ$, $I+J=A$ and

J is a product of distinct smooth maximal ideals of height n. Moreover if $F \subset \operatorname{Spec} A$ such ~~s.t.~~ that $\dim F \le n-1$, then we can chose

J such that $V(J) \cap F = \emptyset$. Thus J is residual to I.

~~Theorem Let P be a~~

~~Let~~

<u>Theorem</u> Let $I \subset A$ ideal such that I/I^2 is generated by n elements. Then there exists a projective A-module P of rank n and a surjection $P \twoheadrightarrow I$ such that

i) $z = (P) - (A^n) \in F^n K_0(A)$.

ii) Further given any ideal J residual to I, we can choose P such that

$(n-1)!\, z = (A/J)$.

Corollary Suppose $F^n K_0(A)$ has no $(n-1)!$ torsion. Let $I \subset A$ be an ideal which is a local complete intersection of height-n. Then I is a complete intersection if and only if $(A/I) = 0$ in $K_0(A)$.

~~Pfrool~~
Proof By Th. 2, $\exists$ a surjection $P \twoheadrightarrow I$ with $(n-1)!\, z = (A/J)$, where $z = (P) - (A^n)$ and J an ideal residual to I. Then IJ is generated by n elements and IJ is a local complete intersection of height-n. ~~It is~~ Now ~~Easy~~ IJ is generated by n elements and IJ is a local complete intersection of height n. Then IJ is generated by a regular sequence of length n. So, ~~A~~ $0 = (A/IJ) = (A/I) + (A/J)$.

So, $(n-1)!\, z = -(A/I) = 0$. So $z = 0$ by hypothesis. Hence $(P) = (A^n)$. So by

Suslin's Cancellation Theorem. $P \simeq A^n$ and I is generated by n elts. So I is a Complete intersection.

Proof that Th.2 $\Rightarrow$ Th.1. ~~Proof of Th~~. ~~By & Since by~~ By hypothesis $F^n K_0(A)$ has no $(n-1)!$ torsion, Chose a general $s \in P$ so that $\mathrm{Im}(P^* \xrightarrow{s} A) = I$ is a product of distinct smooth maximal ideals. Now $C_n(P) = (A/I) = 0$. So I is a Complete intersection of height n. So I is a Complete intersection and $P^* \xrightarrow{s} I \to 0$ exact. So by Mohan Kumar's theorem, P^* and hence P has a free direct summand of rank one.

~~Corollary Let $X = \mathrm{Spec}\, A$ smooth integral affine ~~variety~~ scheme over k. Suppose X is unirulled. Then~~

Corollary: Let A be a reduced affine algebra of dimension n over an algebraically closed field. Then the following are equivalent.

1) Every projective A-module of rank $\geq n$ has a free direct summand of rank one

2) $F^n K_0(A) = 0$.

3) Let $I \subset A$ ideal. ~~And that~~ Then I/I^2 is generated by n elements $\iff$ I is generated by n elements.

4). Let M be a finite A-module. Then M is generated by $\sup\{\mu_{\mathfrak{p}}(M) + \dim A/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} A, \dim A/\mathfrak{p} < n\}$ elements.

Proof. ~~1) $\iff$ 2)~~ 1) $\Rightarrow$ 2): Let $m \subset A$ smooth maximal ideal. then there is a surjection $P^\vee \twoheadrightarrow m$.
~~2) $\Rightarrow$ 1): Follows from Th.1.~~
But 1) $\Rightarrow$ $C_n(P) = 0$. So $C_n(P) = (A/m) = 0$.
So, $F^n K_0(A) = 0$. Now 2) $\Rightarrow$ 1) is immediate from Th.1. So 1) $\iff$ 2).
2) $\Rightarrow$ 3): Immediate from Th.2.

3) $\Longrightarrow$ 4): follows from [MK1].

4) $\Longrightarrow$ 2): ~~It follows 4) imply~~ It is easy to see that 4) ~~implies~~ that any local complete intersection ideal is generated by $n = \dim A$ elements. So every smooth maximal ideal of height n is generated by n elements. Hence $F^n K_0(A) = 0$.

Corollary 2. Let $X = \operatorname{Spec} A$ be a smooth n-dimensional affine variety defined over an algebraically closed field. Suppose X is uni-ruled.

~~Then every projective~~

~~Then $F^n K_0(A) = 0$.~~

Then every projective A-module of rank $\geq n$ has a free direct summand of rank one.

We recall that X is uni-ruled if the quotient field $R(X)$ of A is contained $L(t)$, where L is a finitely generated field of transcendence degree $n-1$.

In other words, there is a dominant rational map $f: V \times \mathbb{A}^1 \dashrightarrow X$, where V is a variety of dimension $n-1$.

Proof of Cor. 2. By Cor. 1, it suffices to show that $F^n K_0(A) = 0$. As above, let ~~$f: V \times \mathbb{A}^1$~~ let $f: V \times \mathbb{A}^1 \dashrightarrow X$ be a dominant rational map. Then there affine open sets W and U of $V \times \mathbb{A}^1$ and X respectively such that f induces a finite surjective map $W \twoheadrightarrow U$. Since every point of W lies on a rational curve, it follows that every point of U lies on a rational curve, by Lüroth theorem.
By Bertini theorem, it is easy to see that $F^n K_0(A)$ is generated $\{(A/m) \mid m \in U \text{ maximal ideal}\}$.
So it suffices to show that $(A/m) = 0$ in $K_0(A)$ for $m \in U$. Now m lies on a rational curve $C = V(\wp)$. ~~St.~~ Let R be the ~~integral~~ normalization of $A/\wp$. Since C is rational $R \cong k[t, 1/g]$, for some $g \in k[t]$.
The composite map $A \to A/\wp \hookrightarrow R$, being finite induces a map $\varphi: K_0(R) \to K_0(A)$. If m' is a maximal ideal of R ~~by~~ lying over $m/\wp$, then $\varphi((R/m')) = (A/m) \in K_0(A)$. Since R is a P.I.D., $(R/m') = 0$. Hence $(A/m) = 0$.

An obstruction for smooth prime ideals in $R[x_1, \ldots, x_n]$ to be $(n-1)$-generated

Let $X \subset \mathbb{A}^n$ be a smooth affine variety of dimension $d \geq 1$. Let $I \subset A = k[x_1, \ldots, x_n]$ be the prime ideal of X and $R = A/I$ the coordinate ring of X. As a consequence of Th. 1, we show that I is generated by $n-1$ elements if and only if $C_d(\Omega_{R/k}) = 0$, Here $\Omega_{R/k}$ denotes the module of differentials of R over k.

We first need a theorem from [BMS]

Theorem [BMS] Let $X \subset \mathbb{A}^n_k$ be a smooth affine variety of dimension d over an algebraically closed field k. Let $I \subset A = k[x_1, \ldots, x_n]$ ~~do~~ be the prime ideal of X. Then I is generated by $(n-1)$ elements if and only if $\Omega_{R/k}$ has a free direct summand of rank one (Here $R = A/I$).

Proof. Suppose I is generated by $n-1$ elements. Then So ~~is~~ I/I^2 is generated by $n-1$ elements. So

Hence $I/I^2 \oplus Q \simeq R^{n-1}$, for some R-module Q.
But we have $I/I^2 \oplus \Omega_{R/k} \simeq R^n$: So

$I/I^2 \oplus Q \oplus R \simeq I/I^2 \oplus \Omega_{R/k}$. Hence

by Suslin's Cancellation Theorem, $\Omega_{R/k} \simeq Q \oplus R$.

Conversely suppose $\Omega_{R/k} \simeq Q \oplus R$. Then

$I/I^2 \oplus Q \oplus R \simeq R^{n-1} \oplus R$. But $n-1 \geq d$, so

again by Suslin's Cancellation Theorem,

$I/I^2 \oplus Q \simeq R^{n-1}$ and I/I^2 is generated by

$n-1$ elements. Now we use the following

theorem of Mohan Kumar ([MK1]).

<u>Theorem ([MK1])</u>. Let $I \subset A = R[x_1, .., x_n]$ (R any field) ideal such that I/I^2 is r-generated and $r \geq \dim A/I + 2$, then I is generated by r elements.

In our case I/I^2 is generated by $n-1$ elements, $\dim A/I = d$. So if $n-1 \geq d+2$ i.e. if $d \leq n-3$, then I is generated by $n-1$ element

In case $d = n-1$, I is principal and there is nothing to prove. So we have to consider the case $d = n-2$ i.e. height $I = 2$. We ~~first~~ first show that $\omega_R = \mathrm{Ext}^1_A(I, A)$ is generated by $n-2$ elements. Now ω_R is an invertible ideal of R and $\hat{\Lambda}^2 I/I^2 \simeq \omega_R^{-1} = \mathrm{Hom}_R(\omega_R, R)$. Also I has projective dimension ≤ 1. So, we have an exact sequence

$$0 \to A^{m-1} \to A^m \to I \to 0$$

(Here we are using the fact that all projective A-modules are free (or stably free).). Tensoring with R, we get an exact sequence:

~~we get~~ $$0 \to L \to R^{m-1} \to R^m \to I/I^2 \to 0$$

Since I/I^2 is a projective R-module of rank 2, it follows that, $(I/I^2) = (R \oplus L)$ in $K_0(R)$, ~~with~~ with L a projective R-module of rank one. Taking determinants, we see that $L \simeq \Lambda^2 I/I^2 \simeq \omega_R^{-1}$. Hence I/I^2 is stably isomorphic to $R \oplus \omega_R^{-1}$.

Since I/I^2 is generated by $n-1$ elements, we have $I/I^2 \oplus Q \simeq R^{n-1}$. Hence $R \oplus \omega_R^{-1} \oplus Q$ and R^{n-1} are stably isomorphic. Since $\dim R = n-2$, by Bass' cancellation theorem, $R \oplus \omega_R^{-1} \oplus Q \simeq R^{n-1}$. Now by Suslin's cancellation theorem $\omega_R^{-1} \oplus Q \simeq R^{n-2}$ i.e. $\omega_R \oplus Q^* \simeq R^{n-2}$ and ω is generated by $n-2$ elements. Thus $Ext_R^1(I, R) = \omega_R$ is generated by $n-2$ elements. Hence by the following ~~Result~~ lemma in [Mu 3], it follows that I is generated by $n-1$ elements. (since we get an exact sequence $0 \to A^{n-2} \to P \to I \to 0$, with P projective and hence $P \simeq A^{n-1}$).

~~Theorem~~ Lemma [Mu 3]. Let A be a noetherian ring and M an A-module of projective dimension ≤ 1. Suppose $Ext_A^1(M, A)$ is generated by r elements. Then there is an exact sequence

$$0 \to A^r \to P \to M \to 0 \quad \text{with } P \text{ projective.}$$

Now Th. 1 and the theorem above

~~[illegible]~~ immediately give

Corollary. With notation as in the theorem above, I is generated by $n-1$ elements $\iff C^d(-\Omega_{R/k}) = 0$. ~~For example~~ In particular the prime ideal of any uni-ruled variety in $\mathbb{A}^n$ is generated by $n-1$ elements.

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