INTERNATIONAL ATOMIC ENERGY AGENCY
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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.761/7

Workshop on Commutative Algebra and its Relation to Combinatorics and Computer Algebra

(16 - 27 May 1994)

Zero cycles and projective modules

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ZERO CYCLES AND PROJECTIVE MODULES

M. Pavaman Murthy *

(Dedicated to C.S.Seshadri on his 60th birthday)

Let A be a reduced affine k-algebra of dimension n over an algebraically closed field k. Let $F^nK_0(A)$ denote the sub-group of $K_0(A)$ generated by the images of all the residue fields of all smooth maximal ideals of height n. For a module of finite projective dimension M, let (M) denote the image of M in $K_0(A)$. For a projective A-module P of rank n, we define the nth Chern class of P to be: $C_n(P) = \sum(-1)^i(\Lambda^i P^*)$, where P^* is the dual of P. Suppose $F^nK_0(A)$ has no $(n-1)!$ torsion, then our main result (Th. 3.8) is that P has a free direct summand of rank one if and only if $C_n(P) = 0$. When characteristic of k is zero or A is normal and $n \geq 3$, it is known that $F^nK_0(A)$ is torsion-free ([Le], [Sr]). Hence our theorem is applicable in these cases. Also when A is regular, $F^nK_0(A)$ coincides with $CH^n(X)$ the Chow group of zero cycles of Spec$A$ and $C_n(P)$ coincides with the usual nth Chern class as defined by Grothendieck (see [Fu]).

When $n \leq 3$ and A is regular, this result was proved in [MKM]. When $n = 3$ and the characteristic of k is not equal to two, this result in [MKM] was extended to the singular case by M. Levine. In this paper, we extend the results of [MKM] to all dimensions. With the assumption stated above on torsion in $F^nK_0(A)$, we first show that (Cor. 3.4) if $I \subset A$ is a local complete intersection of height n, then (A/I) is zero in $K_0(A)$ if and only if I is a complete intersection. Cor. 3.4 together with a result of Mohan Kumar ([MK], Cor 1.9 here) at once gives Th. 3.8.

In §1, we give some preliminaries and generalize results in [MK2]. These results are crucially used in the rest of the paper. The basic ideas in §1 are all taken from [MK2]. In §2, we prove Th. 2.2, which strengthens a result of Boratynski [Bo] for local complete intersections with trivial co-normal bundle. Th 2.2 is one of the crucial ingredients in the proofs of the results in §3. As an amusing application of Th. 2.2, we give a new proof of a theorem of Srinivas ([Sr]) about torsion in zero cycles for normal varieties (see Th. 2.11). In §2, we also recall "Bertini's theorem" for vector bundles and record a few consequences tailored to suit our needs here. §3 contains the main results of this paper.

§4 and §5 deal with applications of results in §3 to "efficient" generation of modules. For example in §4, we show that finite A-modules are generated "efficiently" (in the sense of [EE]) if and only if $F^nK_0(A) = 0$. For a precise statement, see Th. 4.1 and Th. 4.4. In §5, for a smooth affine variety $X = $ Spec$A$ and for a finite A-module M, we attach a certain "Segre class" $s_0(M) \in CH^n(X)$ (cf. [Fu]). The cycle class $s_0(M)$ is the precise obstruction for M to be generated "efficiently" (Cor 5.3).

* *This work was partially supported by the NSF.*

The work in this paper began in 1985. The main results in this paper were announced in [Mu]. We give here proofs and extensions of those results.

Our thanks are due to N. Mohan Kumar and R.G.Swan for useful discussions which improved our exposition of Th. 2.2. Thanks are also due to S. Bloch and V. Srinivas for useful conversation on torsion in the Chow group of zero cycles. We profusely thank the referee for critically going through the paper and making several useful comments which improved the exposition considerably.

We fix some terminology and notation. Let A be a noetherian ring. We say that I is a complete intersection if I is generated by a regular sequence. We say that I is a local complete intersection of height r if $I_{\mathfrak{m}}$ is a complete intersection of height r, for all maximal ideals $\mathfrak{m}$ containing I.

It is easy to see that if I is a local complete intersection of height r and I is generated by r elements, then I is a complete intersection. For basic results on projective modules and "classical" algebraic K-theory, we refer to [Ba]. For a scheme X, we denote by $CH^p(X)$, the group of co-dimension p-cycles modulo rational equivalence, as defined in [Fu]. For intersection theory used here, we refer to [Fu].

§1

Let A be a commutative noetherian ring. We denote by $K_0(A)$, the Grothendieck group of finitely generated A-modules. We also identify $K_0(A)$ with the Grothendieck group of the category of finite A-modules of finite projective dimension. Let the Krull dimension of A be n ($\dim A = n$). We denote by $F^n K_0(A)$, the sub-group of $K_0(A)$ generated by all $(A/\mathfrak{m}) \in K_0(A)$, where $\mathfrak{m}$ is a maximal ideal of A such that $A_\mathfrak{m}$ is a regular local ring of dimension n. Here for a finite A-module M of finite projective dimension. (M) denotes its image in $K_0(A)$.

Lemma 1.1. *Let A be a noetherian ring of dimension n. Let $I \subset A$ be an ideal such that $\dim A/I < n$. Let $\eta : K_0(A) \to K_0(A/I)$ be the natural map. Then $\eta(F^n K_0(A)) = 0$.*

Proof. Let $S = 1 + I$. The map η factors as

$$K_0(A) \xrightarrow{\theta} K_0(A_S) \xrightarrow{\eta_S} K_0(A/I),$$

$\eta_S \circ \theta = \eta$, where θ and η_S are the natural maps. So it suffices to show that $\theta(F^n K_0(A)) = 0$. Let $\mathfrak{m}$ be a maximal ideal in A such that $A_\mathfrak{m}$ is a regular local ring of dimension n. We have $\theta((A/\mathfrak{m})) = ((A/\mathfrak{m})_S) \in K_0(A_S)$. If $\mathfrak{m} \not\supset I$, then $(A/\mathfrak{m})_S = 0$. If $\mathfrak{m} \supset I$, consider the maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}/I\mathfrak{m}$ in $\bar{A} = A/I\mathfrak{m}$. Since $\bar{\mathfrak{m}}A_{\bar{\mathfrak{m}}}$ is generated by n elements and $\dim\bar{A} < n$, we see by [Fo], that $\mathfrak{m}_S$ is generated by n elements. Hence $\mathfrak{m}_S$ is a complete intersection and $(A/\mathfrak{m})_S = 0$ in $K_0(A_S)$.

Let M be a finite A-module and N a sub-module of M. Recall that N is r-fold basic in M at $\mathfrak{p} \in \operatorname{Spec} A$ if $\dim_{k(\mathfrak{p})} \operatorname{Im}(N \otimes k(\mathfrak{p}) \to M \otimes k(\mathfrak{p})) \geq r$ ($k(\mathfrak{p})$ = residue field of $A_\mathfrak{p}$).

Lemma 1.2. *Let A be a commutative ring and M an A-module and N, a sub-module of M. Let P be a projective A-module of rank n and $\mathfrak{p}_i \in \operatorname{Spec} A$, $1 \leq i \leq r$. Let $f \in \operatorname{Hom}_A(P, M)$ be such that $\operatorname{Im} f + N$ is m_i-fold basic in M at $\mathfrak{p}_i$, $1 \leq i \leq r$. Then there exists a $g \in \operatorname{Hom}_A(P, N)$ such that $\operatorname{Im}(f + g)$ is $\min(m_i, n)$-fold basic in M at the $\mathfrak{p}_i$, $1 \leq i \leq r$.*

Proof. We prove the lemma by induction on r, the case $r = 0$ being trivial. We may assume that $\mathfrak{p}_r$ is minimal among the $\mathfrak{p}_i$. By induction hypothesis there exists a $g' \in \operatorname{Hom}_A(P, N)$ such that $\operatorname{Im}(f + g')$ is $\min(m_i, n)$-fold basic in M at the $\mathfrak{p}_i$, for $i < r$. Replacing f by $f + g'$, we may assume that $\operatorname{Im} f$ is $\min(m_i, n)$-fold basic in M at the $\mathfrak{p}_i$ for $i < r$. Let $e_1/1, \ldots, e_n/1 \in P_{\mathfrak{p}_r}$ be a base for $P_{\mathfrak{p}_r}$ with $e_i \in P$. Let $\dim_{k(\mathfrak{p}_r)}(\operatorname{Im} f \otimes k(\mathfrak{p}_r) \longrightarrow M \otimes k(\mathfrak{p}_r)) = l$. We may assume that $f(e_1) \otimes 1, \ldots, f(e_l) \otimes 1$ are linearly independent in $M \otimes k(\mathfrak{p}_r)$. By hypothesis we choose $t_{l+1}, \ldots, t_{\min(m_r,n)} \in N$ such that $f(e_1) \otimes 1, \ldots, f(e_l) \otimes 1, t_{l+1} \otimes 1, \ldots t_{\min(m_r,n)} \otimes 1$ are linearly independent in $M \otimes k(\mathfrak{p}_r)$. We define $h' \in \operatorname{Hom}(P_{\mathfrak{p}_r}, N_{\mathfrak{p}_r})$ by putting $h'(e_i) = 0$ for $i \leq l$ and $i > \min(m_r, n)$, $h'(e_j) = t_j$, $l + 1 \leq j \leq \min(m_r, n)$. Without loss, we may assume that $h' = h_1/1$.

$h_1 \in \mathrm{Hom}_A(P, N)$ (changing if necessary t_i, suitably). Choose $s \in \prod_{i=1}^{r-1} \mathfrak{p}_i - \mathfrak{p}_r$. It is easy to see that $g = sh_1$ has the desired properties of the lemma.

Recall that a projective module P over a ring A is *cancellative* if $P \oplus A \approx P' \oplus A$ implies that $P \approx P'$, for any A-module P'.

The following theorem, which will be frequently used in this paper is a generalization of a result of Mohan Kumar [MK2].

Theorem 1.3 cf [MK2, Th.2]. *Let A be a reduced commutative noetherian ring of dimension $n \geq 2$, and $F \subset \mathrm{Spec}A$, a closed set of dimension $\leq n-1$ with the following property (*): For any ideal $\mathfrak{a} \subset A$ with $\dim A/\mathfrak{a} \leq n-1$ and $\dim V(\mathfrak{a}) \cap F \leq n-2$, all projective $A/\mathfrak{a}$-modules of rank $\geq n-1$ are cancellative. Let I, J be ideals in A such that a) $\dim V(I) \cap F \leq n-2$. b) J is a local complete intersection ideal of height n. c) $I + J = A$. Suppose $\varphi : Q \to IJ$, $\psi : P \to J$ are surjections with P, Q projective A-modules of rank n and $(P) - (A^n) \in F^n K_0 A$. Then there exists a surjection $\varphi' : P' \to I$ with $P' \oplus P \approx A^n \oplus Q$.*

Proof. It is sufficient to show that there is a surjection $\alpha : A^n \oplus Q \to I \oplus P$. For, then we take $P' = \alpha^{-1}(I \times 0)$, $\varphi' = \alpha \mid P'$. By hypothesis c), the map $\gamma : I \oplus P \to A$, $\gamma(a, x) = a - \psi(x)$ is surjective. The projection $I \oplus P \to P$ induces an isomorphism $\ker\gamma \approx \psi^{-1}(IJ)$ and so it suffices to find a surjection: $A^{n-1} \oplus Q \to \psi^{-1}(IJ)$. We have the following commutative diagram:

$$
\text{(D)}\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & K/\eta(L) & \to & P/\eta(Q) & \xrightarrow{\lambda} & J/IJ = A/I & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & K & \hookrightarrow & P & \xrightarrow{\psi} & J & \to & 0 \\
 & & \uparrow & & \uparrow \eta & & \uparrow & & \\
0 & \to & L & \hookrightarrow & Q & \xrightarrow{\varphi} & IJ & \to & 0
\end{array}
$$

with exact horizontal and vertical rows, where $K = \ker\psi$, $L = \ker\varphi$, the maps $K \hookrightarrow P$, $L \hookrightarrow Q$ are natural inclusions. The map η exists, since Q is projective.

Since J is a local complete intersection ideal of height n and $I + J = A$, the maps φ, ψ and the inclusion $IJ \hookrightarrow J$ induce isomorphisms when tensored with A/J. Hence η induces an isomorphism $Q/JQ \xrightarrow{\sim} P/JP$, so that $P = \eta(Q) + JP$. Hence there is an $a \in A$ such that $a \equiv 1(\mathrm{mod} J)$ and $aP \subset \eta(Q)$. The surjection λ shows that $a \in I$. Let $\mathfrak{a} =$ annihilator of $K/\eta(L)$. Then $a \in \mathfrak{a}$, so that $\mathfrak{a} + J = A$ and $J/\mathfrak{a}J \approx A/\mathfrak{a}$. Also the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & K & \to & \psi^{-1}(IJ) & \xrightarrow{\psi} & IJ & \to & 0 \\
 & & \uparrow & & \uparrow \eta & & \| & & \\
0 & \to & L & \to & Q & \xrightarrow{\varphi} & IJ & \to & 0
\end{array}
$$

shows that $\psi^{-1}(IJ)/\eta(Q) \approx K/\eta(L)$.

Suppose now that the following condition holds:

(**) The ideal $\mathfrak{a}$ contains a non-zero- divisor and $\dim V(\mathfrak{a}) \cap F \leq n-2$.

Assuming (**) we have, $\dim A/\mathfrak{a} \leq n-1$. Since $(P)-(A^n) \in F^nK_0(A)$, it follows from Lemma 1.1 that $P/\mathfrak{a}P$ is a stably free $A/\mathfrak{a}$-module. Tensoring the exact sequence $0 \to K \to P \to J \to 0$ with $A/\mathfrak{a}$, we get that $P/\mathfrak{a}P \approx K/\mathfrak{a}K \oplus A/\mathfrak{a}$. Hence $K/\mathfrak{a}K$ is a stably free $A/\mathfrak{a}$-module of rank $n-1$. Now by assumption (**) and the hypothesis (*) of Th. 1.3, it follows that $K/\mathfrak{a}K \approx (A/\mathfrak{a})^{n-1}$. Thus, we have a composite of surjective maps:

$$A^{n-1} \longrightarrow K/\mathfrak{a}K \xrightarrow{\alpha} K/\eta(L) \approx \psi^{-1}(IJ)/\eta(Q)$$

where the surjective map α is induced by the natural map $K \to K/\eta(L)$. Let $\theta : A^{n-1} \to \psi^{-1}(IJ)$ be a lift of this last composite map. Then $(\theta, \eta) : A^{n-1} \oplus Q \to \psi^{-1}(IJ)$ is surjective and we are done with the proof of Th. 1.3. Suppose the condition (**) is not satisfied. We observe that η can in any case be replaced by $\eta' = \eta + \epsilon$ for any $\epsilon \in \mathrm{Hom}_A(Q, K)$. Hence it suffices to show that there is an $\epsilon \in \mathrm{Hom}_A(Q, K)$ such that if $\mathfrak{a}' =$ annihilator of $K/\eta'(L)$, $\eta' = \eta + \epsilon$, then $\mathfrak{a}'$ contains a non-zero-divisor and $\dim V(\mathfrak{a}') \cap F \leq n-2$.

To see this, we let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of A and $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ generic points of irreducible components of F of dimension $n-1$. Since by hypothesis on I, $\dim V(I) \cap F \leq n-2$, it follows that $I \not\subset \mathfrak{P}_i$, $1 \leq i \leq s$. Let $I \subset \mathfrak{p}_i$, $1 \leq i \leq k$, $I \not\subset \mathfrak{p}_j$, $k+1 \leq j \leq r$. Also, since $n \geq 2$, height $J = n$, we have that $J \not\subset \mathfrak{p}_i$, $1 \leq i \leq r$, $J \not\subset \mathfrak{P}_j$, $1 \leq i \leq s$.

By the diagram (D), we immediately see that if $\mathfrak{p} \in \mathrm{Spec}A$ is such that $\mathfrak{p} = \mathfrak{p}_i$, $k+1 \leq i \leq r$ or $\mathfrak{p} = \mathfrak{P}_j$, $1 \leq j \leq s$, we get the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & K_\mathfrak{p} & \to & P_\mathfrak{p} & \to & A_\mathfrak{p} & \to & 0 \\ & & \uparrow & & \uparrow \eta_\mathfrak{p} & & \| & & \\ 0 & \to & L_\mathfrak{p} & \to & Q_\mathfrak{p} & \to & A_\mathfrak{p} & \to & 0 \end{array}$$

with rows exact so that $K_\mathfrak{p} + \mathrm{Im}\eta_\mathfrak{p} = P_\mathfrak{p}$. Hence $K + \mathrm{Im}\eta$ is n-fold basic in P at such a $\mathfrak{p}$.

Similarly if $\mathfrak{p} = \mathfrak{p}_i$, $1 \leq i \leq k$, $I_{\mathfrak{p}_i} = 0$, we have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & K_\mathfrak{p} & \to & P_\mathfrak{p} & \to & A_\mathfrak{p} & \to & 0 \\ & & \uparrow & & \uparrow \eta_\mathfrak{p} & & \uparrow & & \\ 0 & \to & L_\mathfrak{p} & \to & Q_\mathfrak{p} & \to & 0 & & \end{array}$$

Hence in this case $K + \mathrm{Im}\eta$ is $n-1$-fold basic at $\mathfrak{p}$. There by Lemma 1.2, it follows that there is an $\epsilon \in \mathrm{Hom}_A(Q, K)$ such that if we set $\eta' = \eta + \epsilon$, then $\mathrm{Im}\eta'$ is n-fold basic at $\mathfrak{p}_i$, $k+1 \leq i \leq r$ and n-fold at $\mathfrak{P}_j$, $1 \leq j \leq s$. Also, $\mathrm{Im}\eta'$ is $n-1$-fold basic in P at $\mathfrak{p}_i$, $1 \leq i \leq k$. It is easily seen that this implies that $(K/\eta'(L))_{\mathfrak{p}_i} = 0$, $1 \leq i \leq r$ and $(K/\eta'(L))_{\mathfrak{P}_j} = 0$, $1 \leq j \leq s$. Thus if $\mathfrak{a}'$ is the annihilator of $K/\eta'(L)$ then $\mathfrak{a}' \not\subset \mathfrak{p}_i$, $1 \leq i \leq r$, $\mathfrak{a}' \not\subset \mathfrak{P}_j$, $1 \leq j \leq s$. Hence $\mathfrak{a}'$ contains a non-zero -divisor and $\dim V(\mathfrak{a}') \cap F \leq n-2$.This finishes the proof of Theorem 1.3.

Remark 1.4. *In Theorem 1.3, if all projective A-modules of rank $\geq n$ are cancellative, then the projective module P' in Th. 1.3 is clearly unique up to isomorphism.*

Remark 1.5. *Let A be an affine ring over a ring k. In the following cases, all projective A-modules of rank $\geq n$ are cancellative.*

a) *k, an algebraically closed field,* $\dim A = n$ *([Su])*

b) *$k = \mathbf{Z}$,* $\dim A = n$ *([MKMR])*

c) *$k = \mathbf{R}$,* $\dim A = n$, *the closure F of the set of $\mathbf{R}$ -rational points of* Spec$A$ *has dimension $\leq n-1$. [OPS, Th. 3.2].*

Thus we have:

Corollary 1.6. *Let A be a reduced affine ring of dimension n over a ring k. Let J be a local complete intersection ideal of height n and I an ideal such that $I + J = A$. Let $\psi : P \to J$, $\varphi : Q \to IJ$ be surjections, with P, Q projective A-modules of rank n and $(P) - (A^n) \in F^n K_0(A)$. Then there exists a surjection $P' \to I$ with $P' \oplus P \approx Q \oplus A^n$, in the following cases.*

a) *k is an algebraically closed field*

b) *$k = \mathbf{Z}$*

c) *$k = \mathbf{R}$, the closure F of the set of $\mathbf{R}$-rational points* Spec$A$ *has dimension $\leq n-1$ and the closure Z of the set of $\mathbf{R}$-rational points of $V(I)$ has dimension $\leq n-2$.*

Proof. The cases a), b) are immediate from Remark 1.5 and Th. 1.3, taking F in Th 1.3 to be the empty set. The case c) also follows from Remark 1.5 and Th 1.3 if we take F=closure of $\mathbf{R}$-rational points of Spec$A$. The hypothesis in c) above implies that $\dim V(I) \cap F \leq n-2$. So Th 1.3 is applicable.

Corollarry 1.7 (cf [MK2, Th.2]). *Let A be a reduced affine ring of dimension $n \geq 2$ over a noetherian ring k. Let J be a complete intersection ideal of height n and I an ideal such that $I + J = A$ and IJ is generated by n elements. Then I is generated by n elements if* a) *$k = \mathbf{Z}$ or b) an algebraically closed field or c) $k = \mathbf{R}$, the closure of $\mathbf{R}$-rational points of* Spec$A$ *have dimension $\leq n-1$ and* $\dim A/I \leq n-2$.

Proof. Immediate from Remark 1.4 and Cor. 1.6, if we take $P = Q = A^n$.

Corollary 1.8 ([MK2]). *Let A be a reduced affine algebra of dimension $n \geq 2$ over k, where $k = \mathbf{Z}$, an algebraically closed field or $k = \mathbf{R}$ and the closure of $\mathbf{R}$-rational points of* Spec$A$ *has dimension $\leq n-1$. Let I, J be local complete intersection ideals of height n with $I + J = A$. If any two of the ideals I, J and IJ are complete intersections then so is the third.*

Proof. Follows easily from Cor. 1.7. For details see [MK2, Cor. 1]

Corollary 1.9 (cf [MK2, Th.1]). *Let A be a reduced affine ring of dimension n over a ring k. Let J be a complete intersection ideal of height n. Let $\varphi : Q \to J$ be a surjection with Q a projective A-module of rank n. Then $Q \approx Q' \oplus A$, for some Q' in the following cases.*

i) $k = \mathbf{Z}$

ii) k *is an algebraically closed field.*

(iii) $k = \mathbf{R}$ *and the closure* Z *of* $\mathbf{R}$*-rational points of* Spec$A$ *has dimension* $\leq n - 1$.

Proof. We may assume $n \geq 2$. Now Cor. 1.9 is immediate from Cor. 1.6 if we take $P = A^n$, $I = A$.

§2

Let A be a noetherian ring and $I \subset A$, a local complete intersection ideal of height r. Assuming that I/I^2 is A/I-free, in this section, we show (Th. 2.2) that a certain thickening J of I is a section of a projective A-module P of rank r with $(P) - (A^r) = -(A/I)$ in $K_0(A)$. In particular, all Chern classes of P, except possibly the rth one are zero. Th. 2.2, is crucial for the rest of the paper.

For the convenience of the reader, we review a version (Th. 2.3) of Swan's Bertini's theorem for vector bundles [Sw1, Th. 1.3 and Th. 1.4] tailored to fit the needs of this paper. Th. 2.3 can easily be deduced from [Sw1, Th. 1.3 and Th. 1.4]. We record a few consequences of (2.3) which are freely used in the paper. As an amusing application of Th. 2.2, we deduce (Th. 2.11) a result of Srinivas [Sr], which states that $F^nK_0(A)$ is torsion free when A is an affine domain of dimension $n \geq 3$ over an algebraically closed field, regular in codimension one.

Lemma2.1. *Let $R = \mathbf{Z}[X_1, \ldots, X_r, Y_1, \ldots, Y_r, Z]/(\sum_{i=1}^r X_iY_i + Z(Z-1)) = \mathbf{Z}[x_1, \ldots, x_r, y_1, \ldots, y_r, z]$. For any field K, let $R_K = K \otimes R$. The natural map $\eta : K_0(R) \to K_0(R_Q)$ is an isomorphism and $K_0(R)$ is free with base (R) and $R/I_{\mathbf{Z}}$, where $I_{\mathbf{Z}} = \sum_{i=1}^r x_iR + zR$.*

Proof. It is well known ([Sw2, 17.2, 17.3]) that for any field K, $K_0(R_K) \approx \mathbf{Z} \oplus \mathbf{Z}$ with base (R_K) and (R_K/I_K), where $I_K = K \otimes I_{\mathbf{Z}}$. Also under the natural map $\eta : K_0(R) \to K_0(R_{\mathbf{Q}})$ we have, $\eta(R/I_{\mathbf{Z}}) = (R_{\mathbf{Q}}/I_{\mathbf{Q}})$ (since R is a regular ring, $R/I_{\mathbf{Z}}$ has a class in $K_0(R)$). Thus we have only to show that η is an isomorphism. We have the localization exact sequence

$$\sum_p K_0(R/p) \xrightarrow{\nu} K_0(R) \xrightarrow{\eta} K_0(R_{\mathbf{Q}}) \longrightarrow 0$$

where p runs through all primes. Since, $K_0(R/p) = \mathbf{Z} \oplus \mathbf{Z}$ with base (R/p) and $R/(I_{\mathbf{Z}}, p)$ (note that $R/p = R_{\mathbf{F}_p}$). Since p is R-regular and $R/I_{\mathbf{Z}}$-regular, it follows that, the classes defined by R/p and $R/(I_{\mathbf{Z}}, p)$ in $K_0(R)$ are zero. Hence η is an isomorphism.

Theorem 2.2. *Let A be a noetherian ring and $I \subset A$ be a local complete intersection ideal of height r. Suppose I/I^2 is A/I -free with base $\bar{f}_1, \ldots, \bar{f}_r$, $f_i \in I$, $\bar{f}_i =$ class of f_i in I/I^2. Let $J = I^{(r-1)!} + \sum_{i=1}^{r-1} Af_i$. Then there exists a surjection $P \to J$, with P a projective A-module of rank r such that $(P) - (A^r) = -(A/I) \in K_0(A)$.*

Proof. Since I/I^2 is generated by $\bar{f}_1, \ldots, \bar{f}_r$, there exist $g_1, \ldots, g_r \in A$ and $h \in I$ such that $I = \sum_{i=1}^r Af_i + Ah$ and $h(h-1) + \sum_{i=1}^r f_ig_i = 0$. Let R be as in Lemma 2.1 and $\varphi : R \to A$ the ring homomorphism defined by $\varphi(x_i) = f_i$, $\varphi(y_i) = g_i$, $1 \leq i \leq r$, and $\varphi(z) = h$. Let $\mathfrak{P}$ be a prime ideal containing I and $\mathfrak{p} = \varphi^{-1}(\mathfrak{P})$. The ideal $I_{\mathbf{Z}}$ (resp. I) is a local complete intersection ideal of height r in R (resp. in A) and $x_1, \ldots, x_r$ (resp. $f_1, \ldots, f_r$) generate $(I_{\mathbf{Z}})_{\mathfrak{p}}$ (resp. $I_{\mathfrak{P}}$). Further, since $\varphi(x_i) = f_i$, it follows that $\mathrm{Tor}_i^R(R/I_{\mathbf{Z}}, A) = 0$ for $i > 0$.

Thus if $(P_i')_{0\leq i\leq r}$ is an R-projective resolution of $R/I_{\mathbf{Z}}$, then $(A\otimes_R P_i')_{0\leq i\leq r}$ is an A-projective resolution of $R/I_{\mathbf{Z}}\otimes_R A = A/I$. Hence if $K_0(\varphi): K_0(R)\to K_0(A)$ is the map induced by φ, then $K_0(\varphi)((R/I_{\mathbf{Z}})) = (A/I)$. Let $J' = I_{\mathbf{Z}}^{(r-1)!} + \sum_{i=1}^{r-1} Rx_i$.

Claim. *There exists a surjection $P'\to J'$, with P' a projective k-module of rank r such that $(P') - (R^r) = -(R/I_{\mathbf{Z}})$ in $K_0(R)$.*

Suppose that this claim has been established. Then we have a surjection $P = A\otimes_R P'\to A\otimes_R J'\to J$ and $(P)-(A^r) = K_0(\varphi)((P')-(R^r)) = K_0(\varphi)(-(R/I_{\mathbf{Z}})) = -(A/I)$.

It remains to prove the existence of P' as in the claim above. By [Bo], there exists a surjection $\theta: P'\to J'$, with P' a projective k-module of rank r. So it suffices to show that for any such surjection θ, $(P')-(R^r) = -(R/I_{\mathbf{Z}})$. The map θ give rise to a surjection

$$\theta_{\mathbf{Q}}: P'_{\mathbf{Q}} \longrightarrow J'_{\mathbf{Q}} = \mathbf{Q}\otimes J' = I_{\mathbf{Q}}^{(r-1)!} + \sum_{i=1}^{r-1} R_{\mathbf{Q}} x_i.$$

It results from the surjection $\theta_{\mathbf{Q}}$ that

$$C_r(\check{P}'_{\mathbf{Q}}) = \text{ cycle associated to } R/J'_{\mathbf{Q}} = (r-1)!\alpha,$$

where α = cycle class associated to $R/I_{\mathbf{Q}}$, in $CH(\mathrm{Spec}R_{\mathbf{Q}})$. Here, $\check{P}'_{\mathbf{Q}}$ denotes the dual of $P'_{\mathbf{Q}}$ and $C_r(\check{P}'_{\mathbf{Q}})$ denotes the rth Chern class of $\check{P}'_{\mathbf{Q}}$ with values in $CH^r(\mathrm{Spec}R_{\mathbf{Q}})$.

Write $(P'_{\mathbf{Q}}) = (R^r_{\mathbf{Q}}) + m(R_{\mathbf{Q}}/I_{\mathbf{Q}})$. Then by Riemann-Roch ([Fu]),

$$C_r(P'_{\mathbf{Q}}) = (-1)^{r-1} m(r-1)!\alpha.$$

Hence, $(r-1)!\alpha = C_r(\check{P}'_{\mathbf{Q}}) = (-1)^r C_r(P'_{\mathbf{Q}}) = (-1)^{2r-1} m(r-1)!\alpha$. Now it is well known that [Sw2, 17.2, 17.3] $CH^r(\mathrm{Spec}R_{\mathbf{Q}})\approx \mathbf{Z}$ with base α. Hence we see that $m=-1$, so that $(P'_{\mathbf{Q}}) = (R^r_{\mathbf{Q}}) - (R_{\mathbf{Q}}/I_{\mathbf{Q}})$. It is now immediate from Lemma 2.1 that $(P')-(R^r) = -(R/I_{\mathbf{Z}})$. This finishes the proof of Th. 2.2.

We now recall the following which can easily be deduced by adapting the proof of Swan's Bertini's Theorem [Sw1, Th 1.3 and Th 1.4]

Let E be a locally free $\mathcal{O}_X$-module over a scheme X. Recall that a section $s\in\Gamma(E)$ is ***unimodular*** if $Z(s)=\emptyset$ where $Z(s)$ denotes the zero-scheme of s defined by ideal $\mathrm{Im}(\check{E}\xrightarrow{s}\mathcal{O}_x)$.

(2.3) Bertini's Theorem (cf [Sw1, Th.1.3 and Th.1.4]). *Let X be a geometrically reduced scheme over an infinite field k with a locally closed embedding $X\hookrightarrow \mathbf{P}^n = \mathrm{Proj}k[T_0,\ldots,T_N]$. Let E be a locally free $\mathcal{O}_X$-module of rank r globally generated by a finite dimensional k-vector space $V\subset\Gamma(X,E)$. Let $(s,a)\in\Gamma(E(1)\oplus\mathcal{O}_X)$ be unimodular, where $E(1)=E\otimes\mathcal{O}_X(1)$. Set*

$$W = \sum_{i=0}^{N} t_i\otimes V = \mathrm{Im}(\Gamma(\mathcal{O}_{\mathbf{P}^n}(1))\otimes_k (V) \longrightarrow \Gamma(E(1)),$$

where $t_i \in \Gamma(\mathcal{O}_X(1))$ is the restriction of T_i to X. Let X_i, $1 \le i \le m$ be the irreducible components of X. Then there exists a non-empty open set $U \subset W$ such that for all $y \in U$:

i) The zero-scheme $Z(s + ay)$ is empty or is a geometrically reduced sub-scheme of X, pure of codimension r.

ii) If for some i, $Z(s + ay) \cap X_i \neq \emptyset$, then all irreducible components of $Z(s + ay) \cap X_i$ have dimension $\dim X_i - r$.

iii) $Z(s + ay)$ is smooth at all smooth points of X.

Corollary 2.4. *Let A be a geometrically reduced affine ring of dimension n over an infinite field k. Let $J \subset A$ be an ideal such that the closed set $F = V(J)$ contains the singular locus of $Y = \mathrm{Spec}A$ as well as all those irreducible components Y_i of Y with $\dim Y_i < n$. Let $I \subset A$ be an ideal such that $I + J = A$. Let $f_i \in I$, $1 \le i \le l$ such that $I = \sum_{i=1}^{l} Af_i + I^2$.*

i) For any $1 \le k \le l$, there exist $g_i \in I^2$, $1 \le i \le k$ such that if we set $I' = \sum_{i=1}^{k} A(f_i + g_i)$ then

a) $I' + J = A$

b) $\mathrm{Spec}A/I' - V(I)$ is either empty or a smooth scheme pure of dimension $n - k$

ii) Suppose further $\mathrm{Spec}A/I$ is smooth and pure of dimension $n - l$ and that I/I^2 is A/I-free of rank l. Then the ideal I' in i) is such that $\mathrm{Spec}A/I'$ is smooth and pure of dimension $n - k$.

Proof. Since $I + J = A$, there exist $h_i \in IJ$, $1 \le i \le l$ and $z \in I^2$ such that $f_i - h_i \in I^2$ and $z - 1 \in J$. Replacing f_i by $h_i + z$, we may assume that $f_i - 1 \in J$, $1 \le i \le l$. Now i) follows if we apply (2.3) with $X = \mathrm{Spec}A - V(I^2 J)$, $E = \mathcal{O}_X^{\oplus k}$, $V = \sum_{i,j} k a_i e_j$, $s = (f_1, \dots, f_k) \in \Gamma(E)$, $a = 1$, where e_j, $1 \le j \le k$ is the standard basis for E and the a_i's generate the ideal $I^2 J$. (Here, we embed $\mathrm{Spec}A$ as a closed set in $\mathrm{Spec}k[X_1, \dots, X_N] \hookrightarrow \mathbf{P}^N$, so that, $E \approx E(1)$)

ii) is an easy consequence of i) since $f_i + g_i (\mathrm{mod} I^2)$, $1 \le i \le k$ form a subset of a basis of I/I^2 and $\mathrm{Spec}A/I$ is smooth, pure of dimension $n - l$.

Corollary 2.5. *Let A be a geometrically reduced affine ring of dimension n over an infinite field k. Let $I \subset A$ be a product of distinct smooth maximal ideals of height n. Let $J \subset A$ be an ideal such that $I + J = A$. Then there exists $f_1, \dots, f_{n-1} \in I$, such that $f_i - 1 \in J$, $1 \le i \le n - 1$ and $\mathrm{Spec}A/(f_1, \dots, f_{n-1})$ is a smooth complete intersection curve (possibly reducible) over k.*

Proof. Let $J' \subset A$ be the ideal defining the closed set which is the union of non-smooth locus of $\mathrm{Spec}A$ and all the irreducible components of $\mathrm{Spec}A$ of dimension less than n. Then $I + J' = A$. Replace J by $J'J$. Now Cor. 2.5. easily follows from Cor. 2.4. with $l = n$ and $k = n - 1$.

Corollary 2.6. *Let A be as in Cor. 2.5. Let $I \subset A$ be an ideal and $f_1, \dots, f_n \in I$ such that $I = \sum_{i=1}^{n} Af_i + I^2$. Let S be an ideal in A such that $\dim A/S \le n - 1$. Then there* *exists* $g_i \in I^2$, $1 \leq i \leq n$ *such that* $\sum_{i=1}^n A(f_i + g_i) = IJ$, $SI + J = A$ *and* J *is a product of distinct smooth maximal ideals of height* n.

Proof. We have $I = \sum_{i=1}^n Af_i + \sum_{j=1}^l Ah_j$, where $I^2 = \sum_{j=1}^l Ah_j$. Since for every prime ideal $\mathfrak{p} \in Y = V(S) - V(I)$, $\dim A/\mathfrak{p} \leq n-1$, it follows from general position results in [EE] that there exist $g_i \in \sum_{j=1}^l Ah_j$, such that if $I' = \sum_{i=1}^n A(f_i + g_i)$, then $I'_\mathfrak{p} = I_\mathfrak{p} = A_\mathfrak{p}$, for all $\mathfrak{p} \in Y$. Thus replacing f_i by $f_i + g_i$, we may assume that $\sum_{i=1}^n Af_i \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in V(S) - V(I)$. Apply now (2.3) with $X = \mathrm{Spec}A - V(I^2 S)$, $E = \mathcal{O}_X^n$, $V = \sum kt_i e_j$, where e_j, $1 \leq j \leq n$ is the standard basis E and the t_i generate the ideal $I^2 S$, $s = (f_1, \ldots, f_n) \in \Gamma(E(1))$, $a = 1$ (Here we embed X as a locally closed sub-scheme of some affine space $\mathbf{A}_k^N \subset \mathbf{P}_k^N$, so that $E(1) = E$). Then by (2.3), there exist $g_i \in I^2 S$, $1 \leq i \leq n$ such that $\dim Z \cap X_j \leq \dim X_j - n \leq 0$, where Z is the sub-scheme of $\mathrm{Spec}A$ defined by $I' = \sum_{i=1}^n A(f_i + g_i)$ and X_j is any irreducible component of X. Furthermore $Z \cap X$ is geometrically reduced. Now it is easy to see that $\sum_{i=1}^n A(f_i + g_i) = IJ$, $SI + J = A$ and J is a product of distinct smooth maximal ideals and the proof of Cor 2.6 is complete.

Let $I \subset A$ be an ideal which is a local complete intersection ideal of height n. Since I has a finite projective dimension, A/I has a class (A/I) in $K_0 A$.

Corollary 2.7. *Let* A *be a geometrically reduced affine ring of dimension* n *over an infinite field* k. *Let* $I \subset A$ *be a local complete intersection ideal of height* n. *Then* $(A/I) \in F^n K_0(A)$.

Proof. Since $\dim A/I = 0$ and I is locally generated by n elements, it follows that I/I^2 is generated by n elements, hence by Cor. 2.6, $L = \sum_{i=1}^n Af_i = IJ$, (for some suitable $f_1, \ldots, f_n$), where J is a product of distinct smooth maximal ideals of height n and $I + J = A$. Further we may assume that $f_1, \ldots, f_n$ is a regular sequence. Hence $0 = (A/L) = (A/I) + (A/J)$, i.e. $(A/I) = -(A/J) \in F^n K_0(A)$.

Remark 2.8. *Let* A *be a geometrically reduced affine ring of dimension* n *over an infinite field* k *and* I *an ideal such that* I/I^2 *is generated by* n *elements. Let* Q *be any ideal in* A *such that* $\dim A/Q < n$. *Then it is immediate from Cor. 2.6 that there exists an ideal* J *such that*

i) IJ *is generated by* n *elements*

ii) $J + QI = A$

iii) J *is a product of distinct smooth maximal ideals.*

Corollary 2.9. *Let* A *be a geometrically reduced affine ring of dimension* n *over an infinite field. Let* $F \subset \mathrm{Spec}A$ *be a closed set such that* $\dim F \leq n-1$ *and* F *contains the non-smooth locus of* A *as well as all the irreducible components of* $\mathrm{Spec}A$ *of dimension* $< n$. *Then* $F^n K_0(A)$ *is generated by the set of all* $(A/\mathfrak{m})$, *with* $\mathfrak{m}$ *a maximal ideal in* $\mathrm{Spec}A - F$. *In fact for every* $z \in F^n K_0(A)$, $z = (A/I)$, *where* $I = \prod_{i=1}^r \mathfrak{m}_i$, $\mathfrak{m}_i \neq \mathfrak{m}_j$, *for* $i \neq j$ *and* $\mathfrak{m}_i$ *are maximal ideals in* $\mathrm{Spec}A - F$.

Proof. Follows easily from Cor 2.6.

The following fact is well known.

Lemma 2.10. *Let A be a reduced affine ring of dimension $n \geq 1$ over a field k. Suppose that*

i) k is an algebraically closed field

or

ii) $k = \mathbf{R}$ and the closure in Spec$A$ *of the set of* $\mathbf{R}$*-rational points has dimension $\leq n-1$.*

Then $F^n K_0(A)$ is divisible.

Proof.

i): It suffices to show that for any smooth maximal ideal $\mathfrak{m}$ of A of height n and an integer r, there is a $z \in F^n K_0 A$ such that $rz = (A/\mathfrak{m})$. By Cor. 2.5, there exists a complete intersection ideal L of height $n-1$ contained in $\mathfrak{m}$ such that SpecA/L is a smooth curve. Now we have a natural map $\eta : F^1 K_0(A/L) \to F^n K_0(A)$ with $(A/\mathfrak{m}) \in \text{Im}\eta$. Since k is algebraically closed $F^1 K_0(A/L) = \text{Pic}A/L$ is a divisible group and this finishes i).

ii): In view of i), it suffices to show that the natural map $\delta : F^n K_0(\mathbf{C} \otimes_{\mathbf{R}} A) \to F^n K_0(A)$ is surjective. But by hypothesis in ii) and Cor 2.9, $F^n K_0(A)$ is generated by $(A/\mathfrak{m})$, $\mathfrak{m}$ smooth maximal ideal of height n such that $A/\mathfrak{m} \approx \mathbf{C}$. Clearly such an $(A/\mathfrak{m}) \in \text{Im}\delta$. So δ is surjective.

Theorem 2.11 (cf [Sr]). *Let A be a reduced affine algebra of dimension $n \geq 2$ over an algebraically closed field k. Suppose that $n = 2$, and A is integral and regular; or that $n \geq 3$ and the singular locus of* Spec$A$ *has dimension $\leq n-2$. Then $F^n K_0(A)$ is a torsion-free divisible group.*

Proof. In view of Lemma 2.10, we have only to show that $F^n K_0(A)$ is torsion-free. Also the proof of the case $n = 2$ is contained in [BMS, Prop2.1]. So let us assume that $n \geq 3$. Let $z \in F^n K_0(A)$ be such that $rz = 0$, $r > 0$. By Cor. 2.9, choose $I_1 \subset A$, an ideal which is a product of distinct smooth maximal ideals of height n such that $z = (A/I_1)$ in $K_0(A)$, Let $f_1, \ldots, f_n \in I_1$ be such that $I_1 = \sum_{i=1}^{n} Af_i + I_1^2$. Then $0 = rz = (A/I)$, where $I = \sum_{i=1}^{n-1} Af_i + I_1^r$. Let $J = \sum_{i=1}^{n-1} Af_i + I^{(n-1)!} = \sum_{i=1}^{n-1} Af_i + I_1^{r(n-1)!}$. By Th. 2.2, there exists a surjection $P \to J$, with P a projective A-module of rank n such that $(P) - (A^n) = -(A/I) = 0$. So by [Su], $P \approx A^n$. That is, J is a complete intersection. Let $J = (g_1, \ldots, g_n)$. Let $S \subset A$ be the ideal defining the singular locus of A. Let $\bar{A} = A/S$ and let $\bar{a}$ denote the image of $a \in A$. By hypothesis dim$\bar{A} \leq n-2$. Also $(\bar{g}_1, \ldots, \bar{g}_n) \in \bar{A}^n$ is unimodular. Hence by a standard stability theorem ([Ba]), there is an $\bar{\epsilon} \in E_n(\bar{A})$ such that $(\bar{g}_1, \ldots, \bar{g}_n)\bar{\epsilon} = (1, 0, \ldots, 0)$. Lift $\bar{\epsilon}$ to $\epsilon \in E_n(A)$ and replacing $(g_1, \ldots, g_n)$ by $(g_1, \ldots, g_n)\epsilon$, we may assume $g_1 - 1 \in S$ and $g_i \in S$, for $i \geq 2$. By the very construction of the ideal J, $g_1, \ldots, g_n$ is $(n-1)$-fold basic in I/I^2 at each maximal ideal containing I. Hence we may assume (replacing if necessary g_1 by $g_1 + h$ for some $h \in \sum_{i=2}^{n} Ag_i$) that the image of g_1

in I_1/I_1^2 generates an A/I_1-free direct summand of rank 1. We can similarly replace each $g_i, 2 \le i \le n-1$, by $g_i + h_i$, where $h_i \in \sum_{j=i+1}^n Ag_j$, $2 \le i \le n-1$, and assume that the images of g_i in I_1/I_1^2, $1 \le i \le n-1$ generate A/I_1-free summand of rank $n-1$ in I_1/I_1^2.

Suppose now that $n \geq 3$. We now apply Bertini's Theorem (2.3) with $X = \mathrm{Spec}A - V(Sg_{n-1}^2 + Sg_n^2)$, $E = \mathcal{O}_X^{n-2}$, $s = (g_1, \ldots, g_{n-2})$, $a = 1$, $V = \sum_{i,j} ka_i e_j$, where the a_i generate $Sg_{n-1}^2 + Sg_n^2$ and e_j, $1 \le j \le n-2$ is the standard basis for E. Then by (2.3), there exists a $y = (y_1, \ldots, y_{n-2}) \in A^{n-2}$ with $y_i \in Sg_{n-1}^2 + Sg_n^2$ such that the zero-scheme $Z(s+y) = V(\sum_{i=1}^{n-2} A(g_i + y_i)) \cap X$ is of dimension two and smooth. Replacing g_i by $g_i + y_i$, $1 \le i \le n-2$, we may assume $V(g_1, \ldots, g_{n-2}) \cap X$ is of dimension two and smooth. Since $g_1 - 1 \in S$, it is immediate that $V(g_1, \ldots, g_{n-2}) \cap V(Sg_{n-1}^2 + Sg_n^2) = V(g_1, \ldots, g_n) = V(I_1)$. Since $g_1, \ldots, g_{n-2}$ generate a free direct summand of rank $n-2$ it is clear that $\mathrm{Spec}A/(g_1, \ldots g_{n-2})$ is of dimension two and smooth. Set $B = A/(g_1, \ldots, g_{n-2})$. Since B is a regular ring of dimension two, we have $B = \prod_{i=1}^m B_i \times \prod_{j=1}^{m'} B'_j$, B_i are regular domains of dimension 2 and B'_j are regular of dimension ≤ 1. Now $F^2K_0(B) = \prod_{i=1}^m F^2K_0(B_i)$. Since $J/(g_1, \ldots, g_{n-2})$ is a complete intersection of height 2, it follows that A/I_1 has a class in $F^2K_0(B)_{\mathrm{tor}}$. Thus $z \in \mathrm{Im}(F^2K_0(B)_{\mathrm{tor}} \to F^nK_0(A))$. But by [BMS, Prop 2.1] . $F^2K_0(B)_{\mathrm{tor}} = \prod F^2K_0(B_i)_{\mathrm{tor}} = 0$. Hence $z = 0$.

(2.12) Open question. *Let A be a reduced affine algebra of dimension $n \ge 2$ over an algebraically closed field k. Is $F^nK_0(A)$ torsion-free?*

Remark 2.13.

i) *Let A be as in (2.12). Suppose further that A is an integral domain, Then $F^nK_0(A)$ is torsion-free if* Char$k = 0$, *If* Char$k = p > 0$, *then $F^nK_0(A)_{\mathrm{tor}}$ is a p-primary group ([Le]).*

ii) *Let A be a reduced affine $\mathbf{R}$-algebra of dimension $n \ge 2$. Suppose that the closure of $\mathbf{R}$-rational points of* Spec$A$ *has dimension $\le n-1$. Then $F^nK_0(\mathbf{C} \otimes A)$ torsion free implies that $F^nK_0(A)$ is torsion-free.*

Proof. $F^nK_0(\mathbf{C} \otimes A)$ is torsion-free $\Rightarrow$ $F^nK_0(A)_{\mathrm{tor}}$ is contained in $\ker(K_0(A) \to K_0(\mathbf{C} \otimes_{\mathbf{R}} A))$. Hence $2.F^nK_0(A)_{\mathrm{tor}} = 0$. But by Lemma 2.10, $F^nK_0(A)$ and therefore $F^nK_0(A)_{\mathrm{tor}}$ is divisible. So $F^nK_0(A)_{\mathrm{tor}} = 0$.

For convenience, we collect the relevant known facts needed in the sequel about torsion in $F^nK_0(A)$ in the following theorem.

Theorem 2.14 ([Le], [Sr]). *Let A be an integral affine k-algebra of dimension $n \ge 2$. Suppose k is algebraically closed or $k = \mathbf{R}$ and the closure of $\mathbf{R}$-rational points of* Spec$A$ *is not dense in* Spec$A$.

i) *Suppose one of the following conditions hold:*

a) $n = 2$,

b) Char$k = 0$,

c) Char$k = p \geq n$,

d) $n \geq 3$ *and A is regular in codimension one.*

Then $F^n(K_0(A)$ has no $(n-1)!$ torsion.

ii) A is regular $\Rightarrow$ the natural surjective map $\varphi : CH^n(\text{Spec}A) \to F^nK_0(A)$ is an isomorphism.

Proof

i) is contained in (2.11), (2.13).

ii): We have the map $\psi : F^nK_0(A) \to CH^n(\text{Spec}A)$ given by the nth-Chern class. Since $\varphi \circ \psi$ and $\psi \circ \varphi$ are multiplication by $(-1)^{n-1}(n-1)!$ and $F^nK_0(A)$ is divisible (by Lemma 2.10) and torsion-free (by Th. 2.11), it follows that φ, ψ are isomorphisms.

§3 Main results

Let A be a reduced affine ring of dimension n over an algebraically closed field k. Let $I \subset A$ a local complete intersection ideal of height n. Suppose A is normal or Characteristic $k = 0$ or Char$k = p > n$. In this section we show that I is a complete intersection if and only if $(A/I) = 0$ in $K_0(A)$. Th. 3.3 together with Mohan Kumar's result (cf Cor 1.9) gives one of the main results of this paper that a projective A-module P of rank n has a free direct summand of rank one if and only if the top Chern class of P vanishes.

Let A be a noetherian ring of dimension n and $I \subset A$ ideal.

Definition 3.1. *An ideal $J \subset A$ is said to be residual to I if*

i) IJ is generated by n elements

ii) $I + J = A$

iii) J is a local complete intersection of height n.

Remark 3.2. *Let A be a geometrically reduced affine ring of dimension n over an infinite field k. Let $I \subset A$ be an ideal such that I/I^2 is generated by n elements. By Remark 2.8, there exist ideals $J \subset A$, which are residual to I. Furthermore we can choose J to be a product of smooth maximal ideals of height n.*

The following theorem is crucial to subsequent results in the paper.

Theorem 3.3. *Let A be a reduced affine ring of dimension n over a field k. Let $I \subset A$ be an ideal such that I/I^2 is generated by n elements. Suppose one of the following conditions holds:*

a) k is an algebraically closed field.

or

b) $k = \mathbf{R}$ and the closure in Spec$A$ *(resp. in* SpecA/I*) of* $\mathbf{R}$*-rational points of* Spec$A$ *(resp. of* SpecA/I*) has dimension $\leq n-1$ (resp. $\leq n-2$).*

Then given any ideal J residual to I, there exists a surjection $\varphi : P \to I$ such that

i) $z = (P) - (A^n) \in F^n K_0(A)$

ii) $(n-1)!z = (A/J)$.

Proof. First observe that if I is a local complete intersection of height n and J is residual to I, then IJ is generated by a regular sequence of length n. Hence $(A/J) = -(A/I)$ in $K_0(A)$ and ii) becomes $(n-1)!z = -(A/I)$.

Suppose that the theorem is true for J. Let $\varphi' : P' \to J$ be a surjection with P', a projective A-module of rank n such that $z' = (P') - (A^n) \in F^n K_0(A)$ and $(n-1)!z' = -(A/J)$. Since IJ is generated by n elements, there is a surjection $A^n \to IJ$. So by Cor 1.6 a) and c), we get a surjection $P \to I$ with $P \oplus P' \approx A^{2n}$. Hence

$$z = (P) - (A^n) = (A^n) - (P') = -z' \in F^n K_0(A).$$

Moreover $(n-1)!z = -(n-1)!z' = (A/J)$. Thus we may replace I by J and assume that I is a local complete intersection of height n. In the case when k is algebraically closed, using Remark 3.2, we may again replace I by an ideal which is residual to I and which is a product of smooth maximal ideals of height n. In the case $k = \mathbf{R}$, let S be an ideal defining the closure of $\mathbf{R}$-rational points of Spec$A$. By hypothesis, dim$A/S < n$. So by Remarks 2.8 and 3.2, there exists an ideal J' such that

i) J' is residual to I

ii) J' is a product of distinct smooth maximal ideals.

Since $J' + S = A$, it follows that if $\mathfrak{m}$ is a prime ideal containing J', then $A/\mathfrak{m} \approx \mathbf{C}$. Thus in the case $k = \mathbf{R}$, we may replace I by J' and assume that I is a product of distinct smooth maximal ideals $\mathfrak{m}_i$ with $A/\mathfrak{m}_i \approx \mathbf{C}$.

By Cor. 2.5, there exist $f_1, \ldots, f_{n-1} \in I$ such that $C = \text{Spec}A/(f_1, \ldots, f_{n-1})$ is a smooth complete intersection curve. Further in the case $k = \mathbf{R}$, (taking $J = S$ in Cor. 2.5) we may assume that C has no $\mathbf{R}$-rational points. In any case, Pic$C$ is a divisible group. Let $\bar{A} = A/(f_1, \ldots, f_{n-1})$, $\bar{I} = I/(f_1, \ldots, f_{n-1})$, etc. Then $\bar{I}$ is an invertible ideal in $\bar{A}$ and so by the divisibility of Pic$A$, $\bar{I}^{-1} \approx \bar{J}^{(n-1)!}$, for some invertible ideal $\bar{J} \subset \bar{A}$. Without loss, we may assume that $\bar{J} + \bar{I} = \bar{A}$ and that $\bar{J}$ is a product of distinct maximal ideals of $\bar{A}$. Hence, $\bar{I}\bar{J}^{(n-1)!} = \bar{A}\bar{f}_n$, where $\bar{f}_n$ is a non-zero divisor in $\bar{A}$. Let $\eta : A \to \bar{A}$ be the canonical map and $f_n \in I$ such that $\eta(f_n) = \bar{f}_n$ and let $J = \eta^{-1}(\bar{J})$. We also have $\eta^{-1}(\bar{I}) = I$ and $\eta^{-1}(\bar{J}^{(n-1)!}) = \sum_{i=1}^{n-1} Af_i + J^{(n-1)!}$. Hence $\sum_{i=1}^{n} Af_i = I \cdot J'$, where $J' = \sum_{i=1}^{n-1} Af_i + J^{(n-1)!}$. Also J is a local complete intersection ideal of height n and the images of f_i, $1 \le i \le n-1$ in J/J^2 form a part of the base for J/J^2. Furthermore, J' is residual to I. Hence it suffices to prove the Theorem for J'. Now by Th 2.2, there exists a projective A-module P of rank n and a surjective $P \to J'$ such that $z = (P) - (A^n) = -(A/J)$. Now $(A/J) \in F^nK_0A$ by Cor. 2.7. It is easy to see that $(n-1)!z = -(n-1)!(A/J) = -(A/J')$. This proves the Theorem for J' and completes the Proof of Th. 3.3.

Corollary 3.4. *Let A be a reduced affine algebra of dimension n over a field k. Suppose that the following conditions hold:*

a) $F^nK_0(A)$ has no $(n-1)!$ torsion (i.e. $(n-1)! \cdot x = 0$, $x \in F^nK_0(A) \Rightarrow x = 0$).

b) k is algebraically closed or $k = \mathbf{R}$ and the closure of $\mathbf{R}$-rational points in Spec$A$ has dimension $\le n-1$

Let $I \subset A$ be a local complete intersection ideal of height n. Then I is a complete intersection if and only if $A/I = 0$ in $K_0(A)$.

Proof. For $n \le 1$, Cor. 3.4 is trivial. So, we may assume $n \ge 2$. Since height $I = n$, the hypothesis b in Cor 3.4 implies that the hypotheses of Th. 3.3 are satisfied for the ideal I. Hence there exists a surjection $P \to I$, with P a projective A-module of rank n and

$z = (P) - (A^n) \in F^nK_0(A)$ and $(n-1)!z = -(A/I) = 0$. Since $F^nK_0(A)$ has no $(n-1)!$ torsion, we have $z = 0$. Hence $(P) = (A^n)$ in $K_0(A)$. So by Suslin's Cancellation theorem [Su], P is free. Hence I is generated by n elements.

Let A be a geometrically reduced affine k-algebra, over an infinite field k. Let dimension $A = n$. Let P be a projective A-module of rank n and $P^* = \mathrm{Hom}_A(P, A)$.

Definition (3.5). *The nth Chern class $C_n(P)$ of P is:* $C_n(P) = \sum_{i=0}^{n}(-1)^i(\Lambda^i P^*) \in K_0(A)$.

Remark (3.6).

i) Let A and P be as above. Let (the surjective map) $s : P^* \twoheadrightarrow I$ be a "section" of P, such that I is a local complete intersection ideal of height n. (By (2.3), there always exist an s such that $s(P^*) = I$ is a product of distinct smooth maximal ideals.) Then the Koszul Complex associated to s gives a projective resolution of A/I. Hence by (2.7)

$$C_n(P) = \sum(-1)^i(\Lambda^i P^*) = (A/I) \in F^nK_0(A).$$

ii) When A is regular and k is an algebraically closed field $CH^n(\mathrm{Spec}A) \xrightarrow{\sim} F^nK_0(A)$ is an isomorphism (Th 2.14 (ii)) and our definition of $C_n(P)$ coincides with the usual nth Chern class with values in $CH^n(\mathrm{Spec}A)$ (cf [Fu])

iii) Let P be a projective A-module of rank n with A as in i).
Then $C_n(P^*) = (-1)^n C_n(P)$.

Proof. We have $\Lambda^i P \approx \mathrm{Hom}_A(\Lambda^{n-i}P, \Lambda^n P) \approx \Lambda^{n-i}P^* \otimes \Lambda^n P$.
Hence $C_n(P^*) = \sum(-1)^i(\Lambda^i P) = \sum(-1)^i\Lambda^{n-i}P^* \otimes \Lambda^n P = (-1)^n(\Lambda^n P) \cdot C_n(P)$, where the 'dot' in the last expression denotes the multiplication in $K_0(A)$. Now if Q is any projective A-module of rank r, then $(Q) \cdot z = rz$, for $z \in F^nK_0(A)$. Hence $C_n(P^*) = (-1)^n C_n(P)$.

Let A and P be as in Remark (3.6) i). If $P \approx P' \oplus A$, then it is easy to see that $C_n(P) = 0$. The following is the converse.

Theorem 3.7. *Let A be a reduced affine k-algebra of dimension n, where k is an algebraically closed field or $k = \mathbf{R}$ and the set of $\mathbf{R}$-rational points in $\mathrm{Spec}A$ has dimension $\leq n-1$. Suppose $F^nK_0(A)$ has no $(n-1)!$-torsion. Let P be a projective A-module of rank n. Then $C_n(P) = 0 \Leftrightarrow P \approx P' \oplus A$ for some P'.*

Proof. We have only to show that $C_n(P) = 0$ implies that P has a free direct summand of rank one. Let (the surjective map) $s : P^* \twoheadrightarrow I$ be a generic section of P, so that I is a product of distinct smooth maximal ideals. Now $0 = C_n(P) = (A/I) \in F^nK_0(A)$. Hence by Cor. 3.4, I is a complete intersection. Hence by Cor. (1.9) P^* and therefore P has a free direct summand of rank 1.

We also give the following version of Th. (3.7), which follows at once from Th. (2.11), Th. (2.13) and Th. (3.7).

Theorem 3.8. *Let A be a reduced affine k-algebra of dimension n over a field k. Suppose one of the following conditions hold.*

1) *k is algebraically closed and the dimension of the singular locus of* Spec$A$ *is $\leq n-2$*
2) *k is algebraically closed, A is an integral domain and* Char$k = p \geq n$
3) *$k = \mathbf{R}$, A is an integral domain and the $\mathbf{R}$-rational points of* Spec$A$ *is not dense in* Spec$A$.

   Let P be a projective A-module of rank n. Then $C_n(P) = 0 \Leftrightarrow P \approx P' \oplus A$ for some P'.

Corollary 3.9. *Let A be a reduced affine k-algebra of dimension n over a field k. Suppose k is algebraically closed or $k = \mathbf{R}$ and the closure of $\mathbf{R}$-rational points of* Spec$A$ *has dimension $\leq n-1$. Then following two conditions are equivalent*

i) $F^n K_0(A) = 0$

ii) *Every projective A-module of rank $\geq n$ has a free direct summand of rank one.*

Proof. Immediate from Th. (3.7).

Examples 3.10

i) Let X = Spec$A$ be a smooth integral affine scheme of dimension n , which is birationally uni-ruled – i.e the quotient field K of A is contained in $L(t)$, where L is a function field over k of trancendence degree $n-1$. Then $F^n K_0(A) = 0$. This can be seen as follows:

Let V be a smooth affine variety of dimension $n-1$ with function field L. Since $L(t)$ is a finite extension of K, it follows that there are affine open sets V' and U' of $V \times \mathbf{A}^1$ and X respectively, and a finite surjective map $f : V' \to U'$. Every closed point of V' lies on a rational curve. Hence by Luroth's theorem, it follows that every closed point of U' lies on a closed rational curve of X. Thus the class of any point of $U'$is zero in $CH^n(X) = F^n K_0(A)$. Hence by Cor.2.9, we have $F^n K_0(A) = 0$.

ii) $A = \mathbf{R}[x_0, \ldots, x_n]/(\sum_{i=0}^n x_i^2)$ or $\mathbf{R}[x_0, \ldots, x_n]/(\sum_{i=0}^n x_i^2 + 1)$. Then Spec$A$ has at most one $\mathbf{R}$-rational point. Further $F^n K_0(A) = 0$. Hence projective A-modules of rank $\geq n$ have uni-modular elements. This is easily seen as follows:

In view of Remark 2.13(ii), we may replace $\mathbf{R}$ by $\mathbf{C}$ and thus it suffices to show that $F^n K_0(\mathbf{C} \otimes_{\mathbf{R}} A) = 0$. In the case when $A = \mathbf{R}[x_0, \cdots, x_n]/(\sum_{i=0}^n x_i^2 + 1)$, Spec$(\mathbf{C} \otimes_{\mathbf{R}} A)$ is a smooth affine quadric and hence is a rational variety. So, $F^n K_0(\mathbf{C} \otimes_{\mathbf{R}} A) = 0$, by 3.10(i). Suppose now that $A = \mathbf{R}[x_0, \cdots, x_n]/(\sum_{i=0}^n x_i^2)$, then Spec$(\mathbf{C} \otimes_{\mathbf{R}} A)$ is the affine cone of a smooth projective quadric. Hence every smooth point of Spec$(\mathbf{C} \otimes_{\mathbf{R}} A)$ lies on a smooth affine quadric not passing through the singular point. Thus if $\mathfrak{m}$ is a smooth maximal ideal of $\mathbf{C} \otimes_{\mathbf{R}} A$ and $\mathfrak{p}$ is the ideal of a smooth affine quadric passing through $\mathfrak{m}$, then $\mathfrak{m} \supset \mathfrak{p}$. Further, since the quadric defined by $\mathfrak{p}$ does not pass through the singular point, we have the natural map $K_0((\mathbf{C} \otimes_{\mathbf{R}} A)/\mathfrak{p}) \to K_0(\mathbf{C} \otimes_{\mathbf{R}} A)$. Further $F^{n-1} K_0((\mathbf{C} \otimes_{\mathbf{R}} A)/\mathfrak{p}) = 0$, by

3.10(i). Thus the class of $(\mathbf{C} \otimes_{\mathbf{R}} A)/\mathfrak{m}$ is zero in $K_0((\mathbf{C} \otimes_{\mathbf{R}} A)/\mathfrak{p})$ and hence afortiori zero in $K_0(\mathbf{C} \otimes_{\mathbf{R}} A)$. Hence $F^n K_0(\mathbf{C} \otimes_{\mathbf{R}} A) = 0$.

We next discuss some applications of Th. 3.7. We first apply Cor. 1.6. and Th. 3.3 to prove:

Theorem 3.11. *Let A be a reduced affine algebra of dimension n over an algebraically closed field. Suppose $F^n K_0(A)$ has no $(n-1)!$ torsion. Let P be a projective A-module of rank n. Then there exists projective A-module P' of rank $n-1$ such that $(P) = (P' \oplus A) + \frac{(-1)^{n-1}}{(n-1)!} C_n(P)$ in $K_0(A)$.*

Remark 3.12. *Since $F^n K_0(A)$ is divisible and $C_n(P) \in F^n K_0(A)$, the hypothesis that $F^n K_0(A)$ has no $(n-1)!$ torsion implies that $C_n(P)$ is uniquely divisible by $(n-1)!$. Hence the expression above makes sense.*

Proof of (3.11). Choose a generic section $s \in P^*$ so that we have a surjective map $s : P \to I$, with I, a product of distinct smooth maximal ideals. By Th. 3.3, there exists a projective A-module Q of rank n and a surjective $f : Q \to I$ such that $(Q) = (A^n) - \frac{1}{(n-1)!}(A/I)$. Since $(A/I) = C_n(P^*) = (-1)^n C_n(P)$, we have $(Q) = A^n + \frac{(-1)^{n-1}}{(n-1)!} C_n(P)$. By applying Cor. (1.6) (with "$I = A$" in Cor. 1.6) , we see that there exists a projective A-module P_1 of rank n and a surjection $P_1 \to A$, where $(P) + (A^n) = (Q) + (P_1)$. Let $P_1 = P' \oplus A$. Writing (Q) in terms of $C_n(P)$, we have $(P) = (P' \oplus A) + \frac{(-1)^{n-1}}{(n-1)!} C_n(P)$.

Let $X = \mathrm{Spec} A$ be a smooth affine variety of dimension n over an algebraically closed field. Let $C : K_0(A) = K_0(X) \to 1 + \sum_{p=0}^{n} CH^p(X)$ denote the total Chern class map.

Let $\pi : 1 + \sum_{p=0}^{n} CH^p(X) \to 1 + \sum_{p=0}^{n-1} CH^p(X)$ be the map $\pi(1 + c_1 + \cdots + c_{n-1} + c_n) = 1 + c_1 + \cdots + c_{n-1}$ (note that $1 + \sum_{p=0}^{n-1} CH^p(X)$ is not a group under multiplication). We also denote by $F^p K_0(A)$ = sub-group of $K_0(A)$ generated by classes of modules M with $\mathrm{Codim\,supp} M \geq p$.

Corollary 3.13. *Let $X = \mathrm{Spec} A$ be a smooth affine variety of dimension n over an algebraically closed field. Given $c_p \in CH^p(X)$, $1 \leq p \leq n-1$, there exists a projective A-module P of rank $n-1$ with $C(P) = 1 + \sum_{i=1}^{n-1} c_i$ if and only if $1 + \sum_{i=1}^{n-1} c_i \in \pi(C(K_0(A)))$.*

Proof. We only have to show that if for some $z \in K_0(A)$, $C(z) = 1 + \sum_{i=1}^{n} c_i$, then there is a projective A-module P of rank $n-1$ such that $C(P) = 1 + \sum_{i=1}^{n-1} c_i$. We may assume that $z = (P') - (A^n)$, with P' a projective A-module of rank n. Then by Th. 3.11, there exists a projective A-module P of rank $n-1$ such that $P' = (P \oplus A) + \frac{(-1)^{n-1}}{(n-1)!} c_n$. Now $C(P) = C(P') C(\frac{(-1)^n}{(n-1)!} c_n)$. By Riemann-Roch [Fu], $C(\frac{(-1)^n}{(n-1)!} c_n) = 1 - c_n$. Hence

$$C(P) = C(P')(1 - c_n) = 1 + \sum_{i=1}^{n-1} c_i.$$

We recover the following result proved in [MKM]:

Corollary 3.14. *Let $X = \mathrm{Spec}A$ be a smooth 3-fold over an algebraically closed field.*

i) Given $c_i \in CH^i(X)$, $1 \le i \le 3$, there exists a projective A-module P of rank 3 with $C_i(P) = c_i$, $1 \le i \le 3$.

ii) Given $c_i \in CH^i(X)$, $i = 1, 2$, there exists a projective A-module P of rank 2 with $C_i(P) = c_i$, $i = 1, 2$.

Proof. In view of Cor. 3.13, it suffices to prove i) or equivalently, we have to show that the total Chern class map $C : K_0(A) \to 1 + \sum_{p=1}^{3} CH^p(X)$ is surjective.

Since C induces maps $F^iK_0(A)/F^{i+1}K_0(A) \xrightarrow{C_i} CH^i(X)$ which are isomorphisms for $1 \le i \le 3$ (by Riemann Roch [Fu] and the fact that $F^3K_0(A)$ is a torsion free divisible group [cf Remark 3.6 (ii)]). Hence $C : F^1K_0(A) \to 1 + \sum_{p=1}^{3} CH^p(X)$ is an isomorphism. This finishes the proof of Cor. 3.14.

Let $X = \mathrm{Spec}A$ be a smooth affine variety of dimension n over an algebraically closed field. For $z \in K_0(A)$, let $C(z)$ denote its total Chern class. Recall that pth Segre class $s_p(z)$ and the total Segre class $s(z)$ [Fu, §3.2] are defined by the equation $C(z)^{-1} = s(z) = \sum_p s_p(z)$, $s_0(z) = 1$, $s_p(z) \in CH^p(X)$. Let P be a projective A-module of rank r. It is well known that (cf [Ba] or [Fo]) P is always generated by $r + n$ elements. We have the following:

Corollary 3.15. *Let $X = \mathrm{Spec}A$ be a smooth affine variety of dimension n and P a projective A-module of rank r. Then P is generated by $r + n - 1$ elements if and only if $s_n(P) = 0$.*

Proof. Since P is generated by $r + n$ elements and stably free modules of rank $\ge n$ are free over A, it follows that P is generated by $r + n - 1$ elements if and only if there is a surjection $f : A^{n+r} \to P$ such that $\ker f$ contains an unimodular element. Now $C(\ker f) = s(P)$. Now Cor. 3.15. follows from Th. 3.8.

Since for $L \in \mathrm{Pic}A$, $s_n(L) = (-1)^n C_1(L)^n$, we have

Corollary 3.16. *With A as in Cor.1.5, $L \in \mathrm{Pic}A$ is generated by n elements if and only if $C_1(L)^n = 0$.*

Remark. *When $n = 3$ and A is regular, (3.15) and (3.16) were proved in [MKM].*

Corollary 3.17. *Let $X \subset \mathbf{A}_k^n$ be a closed smooth sub-variety of dimension d over an algebraically closed field. Let $I(X)$ denote the prime ideal of X in $k[T_1, \ldots, T_n]$. Then $I(X)$ is generated by $n - 1$ elements if and only if $C_d(\Omega_{X/k}) = 0$.*

Proof. By [BMS, Th. 1.11], $I(X)$ is generated by $n - 1$ elements if and only if $\Omega_{X/k}$ has a free direct summand of rank one. Hence Corollary 3.17 is immediate from Th. 3.8.

Let X be a smooth affine variety of dimension $n \ge 1$ over an algebraically closed field k. Suppose X is birationally uni-ruled or $k = \bar{\mathbf{F}}_p$ and $n \ge 2$, then it is known that

$CH^n(X) = 0$ (when X is birationally uni-ruled, this is an easy exercise. When $k = \bar{\mathbf{F}}_p$ and $n \geq 2$, see [MKMR, Th. 3.6]. Now Cor. 3.17. immediately gives

Corollary 3.18. *Let $X \subset \mathbf{A}_k^n$ be a smooth affine variety of dimension d. Let $I(X)$ be the prime ideal of X in $k[T_1, \ldots, T_n]$. Suppose $d \geq 1$ and X is birationally uni-ruled or $d \geq 2$ and $k = \bar{\mathbf{F}}_p$. Then $I(X)$ is generated by $n - 1$ elements.*

§4 Efficient generation of modules.

Here, we further explore applications of results in §3. Let A be a reduced affine k-algebra of dimension n over an algebraically closed field k. Let M be a finite A-module. For the rest of the paper we try to stick to the following notation.

For $\mathfrak{p} \in \mathrm{Spec}A$, let $\mu_{\mathfrak{p}}(M)$ denote the number of elements in a minimal set of generators for $M_{\mathfrak{p}}$. Set

$$\nu(\mathfrak{p}, M) = \mu_{\mathfrak{p}}(M) + \dim A/\mathfrak{p}, \ \mathfrak{p} \in \mathrm{Spec}A$$

$$\eta(M) = \sup\{\nu(\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{supp}(M)\}$$

$$\delta(M) = \sup\{\nu(\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{supp}(M), \ \dim A/\mathfrak{p} < n\}$$

By [Fo], M is generated by $\eta(M)$ elements. It is easy to see that in general M is not generated by $\delta(M)$ elements. When A is a polynomial ring, by [MK1] or [Sa], M is generated by $\delta(M)$ elements.

In this section we give an analogue of Th. 3.3 for arbitrary finite A-modules (Th. 4.1). We then characterize affine k-algebras A for which finite A-modules are generated by $\delta(M)$ elements (Th. 4.4). For example an equivalent condition is $F^nK_0(A) = 0$. When A is regular and $n = 3$, this is a result in [MKM].

Theorem 4.1. *Let A be a reduced affine k-algebra of dimension n over an algebraically closed field k. Let M be a finite A-module. Then there exists a surjection $P \to M$, with P a projective A-module of rank $\delta(M)$ such that $(P) - (A^{\delta(M)}) \in F^nK_0(A)$.*

Following the method of [MK1] and [Sa], we reduce the Th. 4.1 to the case when $M = I$ is an ideal not contained in any minimal prime and then appeal to Th. 3.3.

Unless otherwise stated, A will denote a *reduced affine k-algebra of dimension n over an algebraically closed field k.*

Lemma 4.2. *Let $0 = \bigcap_{i=1}^r \mathfrak{p}_i$, where $\mathfrak{p}_i$ are the minimal primes of A. Let $I_1 = \bigcap_{i=1}^l \mathfrak{p}_i$, $I_2 = \bigcap_{i=l+1}^r \mathfrak{p}_i$, $1 \le l \le r$. Let $\eta_i : A \to A/I_i$ denote the natural surjection, $i = 1, 2$. Then the natural map*

$$(K_0(\eta_1), K_0(\eta_2)) : K_0(A) \longrightarrow K_0(A/I_1) \times K_0(A/I_2)$$

induces a surjection $\varphi : F^nK_0(A) \to F^nK_0(A/I_1) \times F^nK_0(A/I_2)$. In particular, the natural map $F^nK_0(A) \to F^nK_0(A/I_1)$ is surjective.

Proof. First, we show that $K_0(\eta_i)(F^nK_0(A)) \subset F^nK_0(A/I_i)$. Say $i = 1$. Let $\mathfrak{m} \subset A$ be a maximal ideal such that $A_{\mathfrak{m}}$ is regular of dimension n. If $\mathfrak{m} \not\supset I_1$, then $\mathrm{Tor}_i(A/\mathfrak{m}, A/I_1) = 0$, for $i \ge 0$. Let (P_i), $0 \le i \le n$ be a projective resolution of $A/\mathfrak{m}$. Then $K_0(\eta_1)((A/\mathfrak{m})) = \sum(-1)^i(\bar{P}_i) = 0$, $(\bar{P}_i = A/I_1 \otimes P_i)$. If $\mathfrak{m} \supset I_1$, then $I_1A_{\mathfrak{m}} = 0$. So, $\mathrm{Tor}_i^A(A/\mathfrak{m}, A/I_1) = \mathrm{Tor}_i^A(A/\mathfrak{m}, A/I_1)_{\mathfrak{m}} = 0$ for $i \ge 1$. Hence $K_0(\eta_1)((A/\mathfrak{m})) = (A/\mathfrak{m})$.

Thus (η_1, η_2) induces a map $\varphi : F^nK_0(A) \to F^nK_0(A/I_1) \times F^nK_0(A/I_2)$. We now show that φ is surjective. Since $\dim A/(I_1+I_2) < n$, by Cor. 2.9, $F^nK_0(A/I_1)$ is generated by all $(A/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of A such that $(A/I_1)_\mathfrak{m}$ is regular of dimension n and $\mathfrak{m} \not\supset I_2$. We, then have $A_\mathfrak{m} = (A/I_1)_\mathfrak{m}$ and $\varphi((A/\mathfrak{m})) = ((A/\mathfrak{m}), 0) \in F^nK_0(A/I_1) \times F^nK_0(A/I_2)$. This shows that $F^nK_0(A/I_1)\times 0$ is in $\mathrm{Im}\varphi$. Similarly $0\times F^nK_0(A/I_2) \in \mathrm{Im}\varphi$ and φ is surjective.

Lemma 4.3. *With the notation as in Theorem 4.1, let $N \subset M$ be a sub-module and $I \subset A$ ideal such that $IN = 0$. Let P be a projective A-module of rank $\delta(M)$ and $f \in \mathrm{Hom}_A(P, M)$, such that $M = f(P) + N$. Suppose that $\eta(M/IM) \leq \delta(M)$ and P/IP is A/I-free. Then there is a $g \in \mathrm{Hom}_A(P, N)$ such that $f + g : P \to M$ is surjective.*

Proof. Let "bar" denote going modulo I. Let $e_1, \ldots, e_\delta$ be a base for $\bar{P}$ ($\delta = \delta(M)$) and $\bar{f}(e_i) = \bar{x}_i \in \bar{M}$, $1 \leq i \leq \delta$. Thus we have, $\sum_{i=1}^{\delta} A\bar{x}_i + \bar{N} = \bar{M}$.

Since, by hypothesis, $\delta \geq \eta(\bar{M})$, by a standard stability argument of Eisenbud-Evans ([EE] or see the proof of Proposition 1 and Corollary 1 in [MK1]), there exist $\bar{y}_i \in \bar{N}$, $1 \leq i \leq \delta$, such that $\bar{M} = \sum_{i=1}^{\delta} A(\bar{x}_i + \bar{y}_i)$.

Let $\bar{g} : \bar{P} \to \bar{N}$ be defined by $\bar{g}(e_i) = \bar{y}_i$. Lift $\bar{g}$ to $g : P \to N$. Let $f' = f + g$. We have $M = f'(P) + IM$. Hence there is an $a \in I$ such that $(1 + a)M \subset f'(P)$. Since $aN = 0$, we have $N \subset f'(P)$. Now $M = f'(P) + N$ implies that $f'(P) = M$. This finishes the proof of Lemma 4.3.

Proof of Theorem 4.1. We first make few preliminary remarks. If $\eta(M) \leq \delta(M)$, then by Foster's theorem [Fo], M is generated by $\delta(M)$ elements. So we may take $P = A^{\delta(M)}$. Hence we have only to consider the case when $\eta(M) > \delta(M)$. We may also assume that $M \neq 0$.

Thus there is a minimal prime $\mathfrak{p}$ such that $\dim A/\mathfrak{p} = n$ and $\eta(M) = \mu_\mathfrak{p}(M) + n > \delta(M)$, $\mu_\mathfrak{p}(M) > 0$. Let $\mathfrak{P}$ be a prime ideal of height one containing $\mathfrak{p}$. Then $\delta(M) \geq \mu_\mathfrak{P}(M) + (n-1) \geq \mu_\mathfrak{p}(M) + (n-1) = \eta(M) - 1$. Thus $\delta(M) = \eta(M) - 1 \geq n$. Hence from now on, we assume that $\eta(M) > \delta(M) \geq n$ and $\delta(M) = \eta(M) - 1$.

Lemmas 4.2 and 4.3 allow us to adapt the method of proof of Theorem 1 in [MK1] to reduce the theorem to the case when M is an ideal in A. Let $S = A - \bigcup_{i=1}^{r} \mathfrak{p}_i$, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ = set of all minimal primes $\mathfrak{p}$ such that $\dim A/\mathfrak{p} = n$. Let $Z(M) = \ker(M \to M_S)$. We have $Z(A) = \bigcap_{i=1}^{r} \mathfrak{p}_i$, $Z(A)M \subset Z(M)$ and $Z(M/Z(M)) = 0$.

Step I. *Th. 4.1 is valid for the A-module $M/Z(M)$ $\Rightarrow$ Th. 4.1 is valid for M.*

Proof. Let $\delta(M) = \delta$. Then $\delta(M/Z(M)) \leq \delta$ and so by assumption there is a projective A-module P of rank δ and a surjection $\bar{f} : P \to M/Z(M)$ such that $(P) - (A^\delta) \in F^nK_0(A)$. Lift $\bar{f}$ to $f : P \to M$. We have, $M = f(P) + Z(M)$.

There is an $s \in S$ such that $s \cdot Z(M) = 0$. Let $\bar{A} = A/As$. Then $\dim\bar{A} < n$. Put $I = As$ and let "bar" denote going modulo I. By Lemma 1.1, $\bar{P}$ is stably free. Since rank

$\bar{P} \geq n$, by Suslin's Cancellation theorem, $\bar{P} \approx \bar{A}^\delta$. Also, $\eta(\bar{M}) \leq \delta(M)$. Hence by Lemma 4.3, $(f+g) : P \to M$ is surjective for some $g \in \text{Hom}(P, Z(M))$.

Step II. *It is sufficient to prove Th. 4.1 for A such that* $\dim A/\mathfrak{p} = n$ *for all minimal primes* $\mathfrak{p}$ *and for M such that the* $\mu_\mathfrak{p}(M)$ *are equal for all minimal primes* $\mathfrak{p}$

Let $d = \max\{\mu_\mathfrak{p}(M) \mid \mathfrak{p}$ minimal prime of with $\dim A/\mathfrak{p} = n\}$. By our assumptions in the beginning of the proof of Th. 4.1, we have $\eta(M) = d + n$ and $\delta(M) = d + n - 1$. Let I be the intersection of those minimal primes such that $\mu_\mathfrak{p}(M) = d$, and $\dim A/\mathfrak{p} = n$. Let J be the intersection of those minimal primes $\mathfrak{p}$ such that either $\dim A/\mathfrak{p} < n$ or $\dim A/\mathfrak{p} = n$ and $\mu_\mathfrak{p}(M) < d$. We have $0 = I \cap J$. By the hypothesis of Step II, Th. 4.1 is valid for the ring A/I and the A/I-module M/IM. Since anyway $\delta(M) \geq \delta(M/IM)$, there is a projective A/I-module P' of rank $\delta(M)$ and a surjective map $P' \to M/IM$ such that $(P') - (A/I)^{\delta(M)} \in F^n K_0(A/I)$.

Now by Lemma 4.2, the natural map $F^n K_0(A) \to F^n K_0(A/I) \times F^n K_0(A/J)$ is surjective. Also, $\delta(M) \geq n$ implies that the cancellation for projective modules holds for rank $\geq \delta(M)$ for rings A, A/I and A/J. Hence it follows that there is a projective A-module P of rank $\delta(M)$ such that $A/I \otimes P = P'$, $A/J \otimes P$ is A/J-free and $(P) - A^{\delta(M)}) \in F^n K_0(A)$. Let $\tilde{f}$ denote the composite of surjective maps $\tilde{f} : P \to P' \to M/IM$. Lift $\tilde{f}$ to $f : P \to M$. We then have $M = f(P) + IM$. Further $J \cdot (IM) = 0$, $\eta(M/JM) \leq \delta(M)$ and P/JP is A/J-free.

Hence by Lemma 4.3, it follows that there is a $g \in \text{Hom}(P, IM)$ such that $(f+g) : P \to M$ is surjective. This finishes the proof of Step II.

Step III. *It is sufficient to prove Th. 4.1 for ideals in A which are not contained in any minimal prime of B.*

Proof. By previous steps we may assume A is equi-dimensional and $\mu_\mathfrak{p}(M) = d \geq 1$ for all minimal primes $\mathfrak{p}$. Also, we may assume $Z(M) = 0$. Thus if $d = 1$, one easily sees that M is isomorphic to an ideal in A not contained in any minimal prime of $\mathfrak{p}$ (for details see the Proof of Step 4 on page 229 of [MK1]). If $d \geq 2$, by standard stability arguments, there is an $x \in M$ such that

i) $\mu_\mathfrak{p}(M/Ax) = \mu_\mathfrak{p}(M) - 1 = d - 1$ for all minimal primes $\mathfrak{p}$ of A

ii) x is basic at all primes of height ≤ 1

iii) $\delta(M/Ax) = \delta(M) - 1$.

(For details see Proof of "Step 4", p. 228-229 of [MK1]) Thus by induction on d we are reduced to the case when $M \approx I$, where I is an ideal in A, not containing any minimal prime of A.

Conclusion of the Proof of Th. 4.1 Now A is equidimensional, $M \approx I$ and I is not contained in any minimal prime of $\mathfrak{p}$. Also $\eta(I) = 1 + n$ and $\delta(I) = n$. Since $\eta(I/I^2) \leq \delta(I) = n$ it follows that I/I^2 is generated by n elements. Now Theorem 4.1 follows from Th. 3.3.

Theorem 4.4. *Let A be a reduced affine k-algebra of dimension n over an algebraically closed field k. The following conditions are equivalent:*

1) Every finite A-module M is generated by

$$\delta(M) = \sup\{\mu_{\mathfrak{p}}(M) + \dim A/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}(M),\ \dim A/\mathfrak{p} < n\}$$

elements.

2) Every local complete intersection ideal is generated by n elements.

3) Every local complete intersection ideal of height n is a complete intersection.

4) Every smooth maximal ideal is a complete intersection.

5) $F^n K_0(A) = 0$

6) Every projective A-module of rank n has a free direct summand of rank one.

Proof. 1) $\Rightarrow$ 2) : Let $I \subset A$ be a local complete intersection ideal of height r. Then $\delta(I) = n$.

2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) : trivial

5) $\Rightarrow$ 6) : Immediate from Th. 3.7.

6) $\Rightarrow$ 5) : Let $\mathfrak{m}$ be a smooth maximal ideal of height n. By Th. 3.3 there is a surjection $P \to \mathfrak{m}$ with P a projective A-module of rank n. So $C_n(P^*) \approx (A/\mathfrak{m})$ in $K_0(A)$, $P^* = \operatorname{Hom}(P, A)$. Now P^* has a free direct summand of rank one implies $C_n(P^*) = 0$. So $F^n K_0(A) = 0$.

5) $\Rightarrow$ 1) : Let M be a finite A-module. By Th. 4.1, there exists a surjection $P \to M$ with P a projective A-module of rank $\delta(M)$ such that $(P) - (A^{\delta(M)}) \in F^n K_0(A) = 0$. Now if $\eta(M) = \delta(M)$, then M is generated by $\delta(M)$ elements. So we may assume $\eta(M) > \delta(M)$. As we have seen in the preliminary remarks in the beginning of the proof of Th. 4.1, $\eta(M) > \delta(M) \Rightarrow \delta(M) \geq n$. Hence $(P) = (A^{\delta(M)})$ in $K_0(A) \Rightarrow P \approx A^{\delta(M)}$, by Suslin's Cancellation Theorem. Hence M is generated by $\delta(M)$ elements and the Proof of Th. 4.4 is complete.

§5 Generators for modules and Segre classes

Let $X = \text{Spec}A$ be a smooth affine n-dimensional variety over an algebraically closed field k. Let M be a finite A-module. We attach to M, a certain zero-cycle $s_0(M)$ in $F^nK_0(A) = CH^n(X)$ and show that $s_0(M) = 0$ if and only if M is generated by $\delta(M)$ elements. The invariant $s_0(M)$ is essentially the zero dimensional "Segre class" of M (taking values in the Chow ring of X) defined in [Fu, Example 4.17]

Let I, J be arbitrary ideals in A such that $I + J = A$. Using the invariant $s_0()$, we show that if any two of the ideals I, J and IJ are generated by n elements, then so is the third. This generalizes a result of [MK2] (see Cor.1.8). In this section we freely use intersection theory for singular schemes as developed in [Fu]. We have closely followed the notation in [Fu].

Let A be a regular domain which is an affine k-algebra of dimension n over an algebraically closed field k. Let $X = \text{Spec}A$. Let M be a finite A-module. Let $S(M)$ denote the symmetric algebra of M. The following lemma is easy and we omit the proof.

Lemma 5.1.

i) $\dim S(M) = \sup\{\mu_\mathfrak{p}(M) + \dim A/\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}A\}$. *In particular if* $\text{supp}M = \text{Spec}A$, *then* $\dim S(M) = \eta(M)$ *(see §4 for the definition of* $\eta(M)$*).*

ii) Let $\mathbf{P}(M) = \text{Proj}S(M)$. *Then* $\dim\mathbf{P}(M) = \sup\{\mu_\mathfrak{p}(M) + \dim A/\mathfrak{p} - 1 \mid \mathfrak{p} \in \text{supp}M\}$. *In particular if* $\text{supp}(M) = \text{Spec}A$, *then* $\dim\mathbf{P}(M) = \eta(M) - 1$.

With the notation as above, suppose that $\eta(M) > \delta(M)$ (here $\delta(M)$ is as in §4). Then as in §4, we have $\eta(M) = r + n$, $\delta(M) = r + n - 1$, where r =rank of $M = \dim_K K \otimes_A M$, where K is the field of fractions of A. Further since $\eta(M) > \delta(M) \Rightarrow \text{supp}M = \text{Spec}A$, we have $r > 0$, Thus by Lemma 5.1, we see that $\eta(M) > \delta(M) \Rightarrow \dim\mathbf{P}(M) = \delta(M)$.

Since $X = \text{Spec}A$ is smooth, we identify

$$F^nK_0(A) = CH^n(X).$$

Recall that $CH^n(X)$ is a torsion-free divisible group. Let $\pi : \mathbf{P}(M) \to X$ be the structural morphism.

Theorem 5.2. *Let* $X = \text{Spec}A$ *be regular of dimension* n. *Let* M *be a finite* A*-module. Put* $\eta(M) = \eta$ *and* $\delta(M) = \delta$. *Then either*

i) $\eta = \delta$ *and* M *is generated by* δ *elements.*

or

ii) $\delta = \eta - 1$, $\dim\mathbf{P}(M) = \delta$ *and there exists a projective* A*-module* P *of rank* δ *and a surjection* $P \to M$ *such that* $(P) - (A^\delta) \in F^nK_0(A)$.

Further P *is unique up to isomorphism and*

$$(*) \quad (P) - (A^\delta) = \frac{1}{(n-1)!}\pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap [\mathbf{P}(M)]) \in CH^n(X) = F^nK_0(A).$$

Proof. In view of Theorem 4.1, we need only to show the uniqueness of P and the formula (*). Now let P be any projective A-module of rank δ such that there is a surjective map $f : P \to M$ and $(P) - (A^\delta) = z \in F^n K_0(A)$. To simplify notation, for a finite A-module N, we shall also write N for the coherent sheaf $\tilde{N}$ associated to N. Thus $\pi^*(N)$, for example would mean the sheaf $\pi^*(\tilde{N})$ on $\mathbf{P}(M)$. With this convention, we have the surjection $\pi^*(f) : \pi^*(P) \to \pi^*(M)$. Now composing $\pi^*(f)$ with the canonical surjection $\pi^*(M) \to \mathcal{O}_{\mathbf{P}(M)}(1)$, we get a surjection $\pi^*(P) \to \mathcal{O}_{\mathbf{P}(M)}(1)$. Hence we have a surjection

$$\pi^*(P)(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(M)}$$

Hence $c_\delta(\pi^*(P)(-1)) \cap [\mathbf{P}(M)] = 0$.This gives,

$$\sum_{i=0}^{\delta} (-1)^i c_{\delta-i}(\pi^*(P)) \cap c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^i \cap [\mathbf{P}(M)] = 0$$

Applying π_* and using the projection formula we get

$$\sum_{i=0}^{\delta} (-1)^i c_{\delta-i}(P) \cap \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1)^i) \cap \mathbf{P}(M)) = 0$$

Since $(P) - (A^\delta) = z \in F^n K_0(A) = CH^n(X)$, we have $C_0(P) = 1,\ C_i(P) = 0,\ i < n$ and $C_n(P) = (-1)^{n-1}(n-1)!z$. Hence we have,

$$(-1)^\delta \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap \mathbf{P}(M)) + (-1)^{\delta-n} C_n(P) \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^{\delta-n} \cap [\mathbf{P}(M)]) = 0$$

We claim that

$$C_n(P)\pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^{\delta-n} \cap [\mathbf{P}(M)]) = C_n(P)$$

Put $\alpha = (c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^{\delta-n} \cap [\mathbf{P}(M)]) = x_0 + x_1 + \ldots + x_\delta,\ x_i \in CH_i(\mathbf{P}(M))$.

Then $C_n(P) \cdot \pi_*(\alpha) = C_n(P) \cdot \pi_*(x_0)$. Suppose $\pi_*(x_0) = m \cdot [X]$. To compute m, we can pass to an open set U of X and assume that M is free of rank r. Replacing X by U, we have, $\mathbf{P}(M) = X \times \mathbf{P}^{r-1}$ and $\delta = r + n - 1$. In this case $\alpha = X \times t$ where $t \in \mathbf{P}^{r-1}$ is a point. Now it is clear that $m = 1$.

Hence, $C_n(P)\pi_*(\alpha) = C_n(P)$. Thus we have,

$$C_n(P) = (-1)^{n+1}\pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap [\mathbf{P}(M)]).$$

Since $(P) - (A^\delta) = z \in F^n K_0(A)$, by the Riemann-Roch Theorem,

$$C_n(P) = (-1)^{n-1}(n-1)!z$$

. Hence

$$z = \frac{(-1)^{n-1}}{(n-1)!} C_n(P) = \frac{(-1)^{n-1}}{(n-1)!} (-1)^n \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap [\mathbf{P}(M)])$$

$$= -\frac{1}{(n-1)!} \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap [\mathbf{P}(M)])$$

Since $\delta \geq n$ and P was any projective A-module of rank δ surjecting onto M with the property $(P) - (A^\delta) \in F^n K_0(A)$, the above calculation together with [Su] establishes the uniqueness of P up to isomorphism.

For a finite A-module M, we define class of zero cycle $s_0(M)$ in $CH^n(X)$ as follows:

$$s_0(M) = 0, \quad \text{if} \quad \eta(M) = \delta(M)$$

$$s_0(M) = \pi_*(c_1(\mathcal{O}_{\mathbf{P}(M)}(1))^\delta \cap [\mathbf{P}(M)]), \quad \text{if} \quad \eta(M) > \delta(M)$$

(here $\delta = \delta(M)$)

Corollary 5.3. *Let A and M be as in Th. 5.2. Then M is generated by $\delta(M)$ elements if and only if $s_0(M) = 0$.*

In case $M = I$ is an ideal in A, it is more convenient to describe $s_0(I)$ in a slightly different way. Observe that either $\eta(I) \leq \delta(I)$ or $\eta(I) > \delta(I)$ and in the later case $\eta(I) = n + 1$ and $\delta(I) = n$. Let $\pi : \tilde{X} \to X$ be the blow up of the ideal I. Let P be a projective A-module of rank n and $f : P \to I$ a surjection with $z = (P) - (A^n) \in F^n K_0(A)$.

This gives rise to a surjection $\pi^*(P) \to I\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(1)$, where $\mathcal{O}_{\tilde{X}}(1) = \mathcal{O}_{\tilde{X}}(-E)$, $E =$ exceptional divisor.

Thus we get a surjection

$$\pi^*(P)(-1) \longrightarrow \mathcal{O}_{\tilde{X}}$$

As in Th. 5.2, $c_n(\pi^*(P)(-1)) = 0$, i.e. $\sum_{i=0}^{n} (-1)^i c_{n-i}(\pi^*(P)) \cap (c_1(\mathcal{O}_{\tilde{X}}(1))^i \cap [\tilde{X}])$. Applying π_*, using the projection formula and noting that $C_i(P) = 0,\ 0 < i < n$, we get

$$(-1)^n \pi_*(c_1(\mathcal{O}_{\tilde{X}}(1)^n) \cap [\tilde{X}]) + C_n(P) = 0$$

$$\text{i.e. } C_n(P) = (-1)^{n-1} \pi_*(c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}])$$

As before, since $(P) - (A^n) = z \in F^n K_0(A)$, we have $C_n(P) = (-1)^{n-1}(n-1)! z$. Hence $z = \frac{(-1)^{n-1}}{(n-1)!} C_n(P) = \frac{1}{(n-1)!} \pi_*(c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}])$.

As in Th. 5.2, the above calculations and Th. 3.3, give

Theorem 5.4. *Let A be a regular integral affine k-algebra of dimension n over an algebraically closed field k. Let $I \subset A$ be an ideal such that I/I^2 is generated by n elements. Then there exists a projective A-module P of rank n and a surjection $P \to I$ such that $z = (P) - (A^n) \in F^n K_0(A)$. Further P with these properties is unique up to isomorphism and $(P) - (A^n) = \frac{1}{(n-1)!}\pi_*(c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}])$.*

For an ideal $I \subset A$, define

$$s_0(I) = \pi_*(c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}]) \in CH^n(X).$$

Corollary 5.5. *Suppose I/I^2 is generated by n elements. Then I is generated by n elements if and only if $s_0(I) = 0$.*

Lemma 5.6. *Let A be as in Theorem 5.4. Let I_1, I_2 be ideals in A such that $I_1 + I_2 = A$. Then $s_0(I_1 I_2) = s_0(I_1) + s_0(I_2)$.*

Proof. Let $\pi_i : \tilde{X}_i \to X$ be the blow up of the ideal I_i, $i = 1, 2$ and let $\pi : \tilde{X} \to X$ be the blow up of the ideal $I = I_1 I_2$. We have the commutative diagram

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & p_2 \swarrow & | & \searrow p_1 & \\ \tilde{X}_2 & & |\pi & & \tilde{X}_1 \\ & \pi_2 \searrow & \downarrow & \swarrow \pi_1 & \\ & & X & & \end{array}$$

Let $L_i = \mathcal{O}_{\tilde{X}_i}(1)$, $i = 1, 2$. Then $\mathcal{O}_{\tilde{X}}(1) = p_1^*(L_1) \otimes p_2^*(L_2)$. Further since $V(I_1) \cap V(I_2) = \emptyset$, we see easily (using for example [Fu,2.5]) that

$$c_1(p_1^*(L_1))^i \cap c_1(p_2^*(L_2)^j \cap [\tilde{X}] = 0, \text{ for } i, j > 0$$

Hence $c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}] = c_1(p_1^*(L_1))^n \cap [\tilde{X}] + c_1(p_2^*(L_2))^n \cap [\tilde{X}]$. So, $\pi_*(c_1(\mathcal{O}_{\tilde{X}}(1))^n \cap [\tilde{X}])$

$$= \pi_{1*} p_{1*}(c_1(p_1^*(L_1)^n \cap [\tilde{X}]) + \pi_{2*} p_{2*}(c_1(p_2^*(L_2)^n \cap [\tilde{X}])$$

$$= \pi_{1*}(c_1(L_1)^n \cap [\tilde{X}_1]) + \pi_{2*}(c_1(L_2)^n \cap [\tilde{X}_2])$$

$$= s_0(I_1) + s_0(I_2)$$

Theorem 5.7. *Let A be a regular integral affine algebra of dimension n over an algebraically closed field. Let I_1, I_2 be ideals in A such that $I_1 + I_2 = A$. Suppose any two of the ideals I_1, I_2 and $I = I_1 I_2$ are generated by n elements then so is the third.*

Proof. Since $I/I^2 \approx I_1/I_1^2 \oplus I_2/I_2^2$, it is easy to see that if any two of the modules I_1/I_1^2, I_2/I_2^2 and I/I^2 are generated by n elements then so is the third. Now the theorem is immediate from Cor. 5.5 and Lemma 5.6.

Remark 5.8.

i) Lyubeznik [Ly] has used Cor. 5.5 to show that any positive dimensional sub-variety of X is set -theoretically generated by n elements.

ii) If A is not regular, we do not know if the projective module P in Th. 5.2 and Th. 5.4 is unique up to isomorphism.

iii) We do not know if Th. 5.7 is valid if we drop the assumption that A is regular or that the ground field is algebraically closed.

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