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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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INTRODUCTION TO FUNCTIONAL METHODS, GAUGE THEORIES AND QUANTIZATION

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Please note: These are preliminary notes intended for internal distribution only.

Green Functions and Scattering Amplitudes

"$\phi(x) \xrightarrow[t \to \mp\infty]{} \sqrt{Z}\, \phi_{in/out}(x)$" $\phi_{in/out}$ = free field

$$(\Box + m_{ph}^2)\, \phi_{in/out} = 0$$

(m_{ph}: physical mass)

$$\Rightarrow \phi_{in/out}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a_{in/out}(k)\, e^{-ikx} + a^\dagger_{in/out}(k)\, e^{+ikx} \right)$$

building $|in\rangle$, $|out\rangle$ Fock space

Typically $|k_1 \cdots k_n\, in\rangle = a^\dagger_{in}(k_1) \cdots a^\dagger_{in}(k_n)|0\rangle$

by en. mom. cons. $|0_{in}\rangle = |0_{out}\rangle$ also 1 particle states $|p_{in}\rangle = |p_{out}\rangle$

Normalization $\langle 0 | \phi_{in/out}(x) | p \rangle = \frac{1}{\sqrt{(2\pi)^3 2\omega_p}}\, e^{-ipx} \equiv f_p(x)$

$\langle 0 | \phi(x) | p \rangle \Rightarrow \frac{\sqrt{Z}}{\sqrt{(2\pi)^3 2\omega_p}}\, e^{-ipx}$

S-matrix $\phi_{out} = S^{-1} \phi_{in} S$

Scattering Amplitude

$$\langle f_{out} | i_{in} \rangle = \langle f_{in} | S | i_{in} \rangle$$

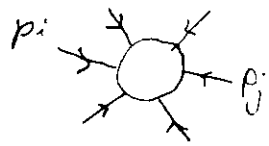
Reduction formula

$$\langle \underbrace{k_1' \dots k_n'}_{n}\ \text{out} \mid \underbrace{k_1 \dots k_m}_{m}\ \text{in} \rangle = \left(\frac{1}{\sqrt{Z}}\right)^{m+n} \int \prod_1^m d^4y_\ell \, f(y_\ell)(\Box_{y_\ell} + m_{ph}^2)$$

$$\cdot \prod_1^n d^4x_{\ell'} \, f^*(x_{\ell'})(\Box_{x_{\ell'}} + m_{ph}^2) \langle 0 | T\, \varphi(y_1) \dots \varphi(y_m)\varphi(x_1) \dots \varphi(x_n) | 0 \rangle_c$$

$$f(y) = \frac{e^{-iky}}{\sqrt{(2\pi)^3 2\omega_k}} \qquad k^2 = m^2 \quad \text{(mass shell)}$$

Define the <u>connected</u> (c) Green Functions



$$(2\pi)^4 \delta^4\Big(\sum_i^N p\Big) G^{(N)}(p_1 \dots p_N) = \int \prod_1^N d^4x_i \, e^{-ip_i x_i} \langle 0 | T\, \varphi(x_1) \dots \varphi(x_N) | 0 \rangle_c$$

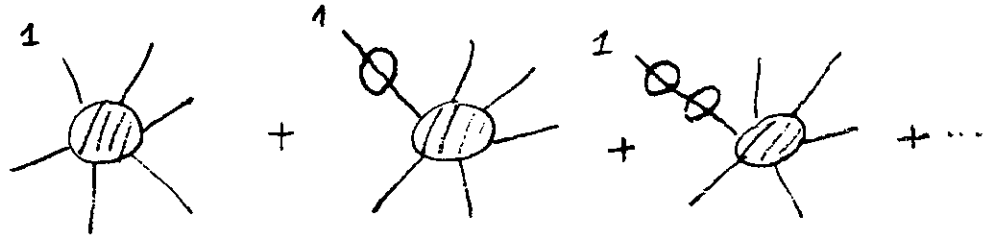
Scattering Amplitude $(N = m+n)$ (factorizing $(2\pi)^4 \delta^4(\Sigma p)$)

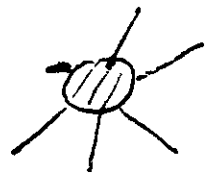
$$A(m \to n) = \prod^N \frac{1}{\sqrt{(2\pi)^3 2\omega}} \cdot M \qquad , \qquad M = \lim_{p_i^2 = m^2} \prod_1^N \frac{p_i^2 - m_{ph}^2}{i\sqrt{Z}} G^{(N)}(p_1 \dots p_N)$$

(phase space $\dfrac{d^3p}{(2\pi)^3 2\omega}$)

M is finite for $p^2 \to m_{ph}^2$, since $G^{(N)}$ has poles for $p^2 \to m_{ph}^2$

→ (4)

$G^{(N)} =$ 

$= \left(- + \text{⊖} + \text{⊖⊖} + \cdots \right)_1 \cdot$ 

$= G^{(2)}(p_1) \cdot G^{(4)\hat{1}}$

repeating for every leg

$$\Rightarrow G^{(N)}(p_1 \cdots p_N) = \left(G^{(2)}(p_1) \cdots G^{(2)}(p_N) \right) \cdot \underbrace{\tilde{\Gamma}^{(N)}(p_1 \cdots p_N)}_{\downarrow}$$

"amputated" Green function

$G^{(2)}(p)$ = complete propagator

Define the renormalized field

$$\varphi_R(x) \equiv \frac{\varphi(x)}{\sqrt{Z}} \xrightarrow[t\to\pm\infty]{} \varphi_{\substack{in\\out}}(x)$$

and renormalized Green functions

$$(2\pi)^4 \delta^4(\Sigma p)\, G^{(N)}_R(p_1 \cdots p_N) = \int \prod_1^N d^4x_i\, e^{-ip_i x_i} \langle 0|T\varphi_R(x_1)\cdots\varphi_R(x_N)|0\rangle$$

By Källen-Lehmann repr:

$$G^{(2)}_1(p) = \int d^4x\, e^{-ipx} \langle 0|T\varphi(x)\varphi(0)|0\rangle = \frac{iZ}{p^2 - m^2_{ph}} + \int_{4m^2}^{\infty} \frac{d\sigma\, \rho(\sigma)}{p^2 - \sigma}$$

$\Pi = \sum |n\rangle\langle n|$

therefore $G^{(2)}_{1R}(p) \xrightarrow[p^2\to m^2]{} \frac{i}{p^2 - m^2_{ph}}$

Thus we have the scattering amplitude

$$M = \lim_{p_i^2 \to m^2} \prod_1^N \left(G^{(2)}_R(p_i) \right)^{-1} \cdot G^{(N)}_R(p_1 \cdots p_N) = \tilde{\Gamma}^{(N)}_R$$

Notice that $G^{(N)}_{1R} = Z^{-\frac{N}{2}} \cdot G^{(N)}$

$$\tilde{\Gamma}^{(N)}_R = Z^{N/2} \cdot \tilde{\Gamma}^{(N)}$$

→ (3)

$$G^{(2)}(p) = \int d^d x\, e^{-ipx} \langle 0|T\,\varphi(x)\,\varphi(0)|0\rangle$$

$$\lim_{p^2 \to m_{ph}^2} G^{(2)}(p) = \frac{iZ}{p^2 - m_{ph}^2 + i\varepsilon} + \left(\int_{4m^2}^{\infty} \frac{d\sigma\, \rho(\sigma)}{p^2 - \sigma} \right) \quad \text{Källen - Lehmann}$$

$$\text{def.} \quad G_R^{(2)}(p) = \int d^d x\, e^{-ipx} \langle 0|T\,\varphi_R(x)\,\varphi_R(0)|0\rangle$$

$$\varphi(x) = \sqrt{Z}\,\varphi_R(x)$$

"$\varphi_R \xrightarrow[x_0 \to -\infty]{} \varphi_{in}$"

$$\lim_{p^2 \to m^2} G_R^{(2)}(p) = \frac{1}{p^2 - m_{ph}^2}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi - m_0^2 \varphi^2\right) - \frac{\lambda_0}{3!}\varphi^3 = \qquad d = 6 - \varepsilon$$

$$[\lambda_0] = [\mu^{\varepsilon/2}]$$

$$= \frac{Z}{2}\left(\partial_\mu \varphi_R \partial^\mu \varphi_R - (m^2 + \delta m^2)\varphi_R^2\right) - Z_1 \lambda \mu^{\varepsilon/2} \varphi_R^3 =$$

$$= \frac{1}{2}\left(\partial_\mu \varphi_R \partial^\mu \varphi_R - m^2 \varphi_R^2\right) - \lambda \mu^{\varepsilon/2} \varphi_R^3$$

$$+ \frac{Z-1}{2} \partial_\mu \varphi_R \partial^\mu \varphi_R - \frac{Z \delta m^2}{2} \varphi_R^2 - (Z_1 - 1)\lambda \mu^{\varepsilon/2} \varphi_R^3$$

$$\lambda_0 = Z_1 Z^{-3/2} \lambda \mu^{\varepsilon/2}$$

$$m_0^2 = m^2 + \delta m^2$$

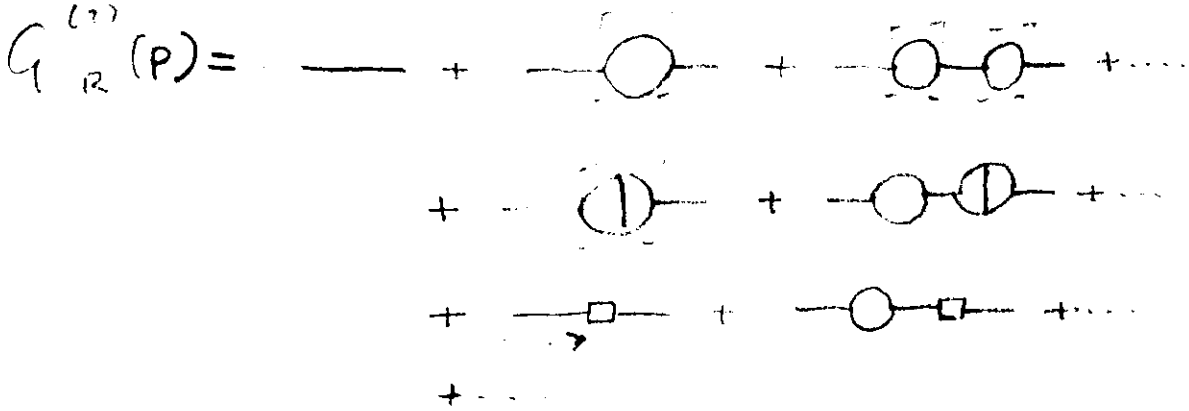
$$\left.\begin{aligned} Z - 1 &= \\ Z\delta m^2 &= \\ Z_1 - 1 &= \end{aligned}\right\} \sum_{n \geq 1} \lambda_R^n \cdot C_n \qquad \text{(pole)} \sim \frac{1}{\varepsilon^n}$$

Compute $\left.\begin{matrix} Z-1 \\ Z\delta m^2 \\ Z_1-1 \end{matrix}\right\}$ requiring cancellation of singularities $\frac{1}{\varepsilon^q}$ in the $G_R^{(n)}$.

Additional vertices: $-\square- = i\left((Z-1)p^2 - Z\delta m^2\right)$ $\quad$ $\overset{\diagup}{\square} = -i\lambda\mu^{\varepsilon/2}(Z_1-1)$

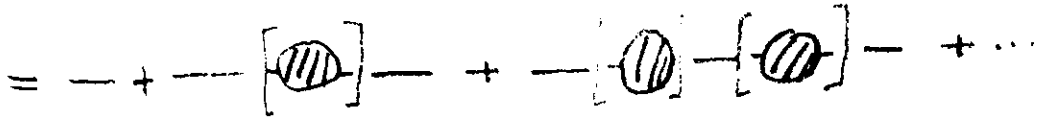
Example: compute

$G_R^{(n)}(p)$ in the theory φ^3.

$G_R^{(n)}(p) =$ 

$\mathcal{L}_{\text{ren}} = \frac{Z}{2}\partial_\mu\varphi_R\partial^\mu\varphi_R - \frac{Z\delta m^2}{2}\varphi_R^2 \equiv$

a 2-point vertex $\equiv \frac{i}{2}\varphi_R\,\square\,\varphi_R$

$=$ 

$=$ sum of all the 1PI graphs

$\equiv i\Sigma_R(p^2)$

$\xi = 1/(1+Z_R'(m_{Ph}^2))$

$m_{Ph}^2 = m^2 - \underbrace{\Sigma_n(m_{Ph}^2)}_{O(\lambda^2)}$

$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\,i\Sigma\,\frac{i}{p^2-m^2} + \cdots = \frac{i}{p^2-m^2+\Sigma_n(p^2)} \approx \frac{i\xi}{\cdot}$

At the lowest order

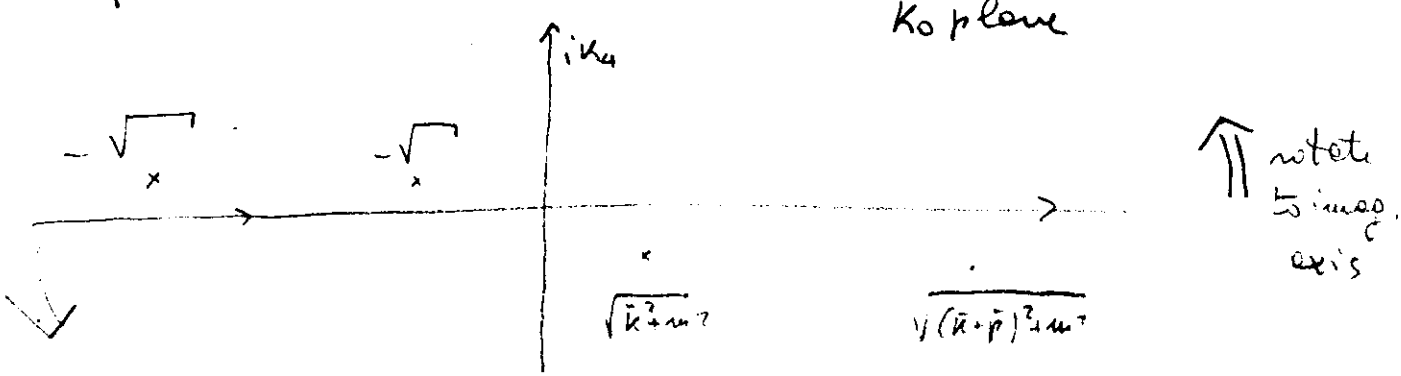
$$i\Sigma_R(p^2) = \bigcirc + \boxminus =$$

$$= \frac{(-i\lambda\mu^{\varepsilon/2})^2}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{(p+k)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} +$$

$$+ i\left((z-1)p^2 - z\delta m^2\right)$$

take p spacelike : $p_0 = 0$

Integration over k_0



"Wick rotation" $\quad k_0 \to ik_4$

$$k_0^2 - \vec{k}^2 = -(k_4^2 + \vec{k}^2) = -k_E^2$$

$$k_0^2 - (\vec{k}+\vec{p})^2 = -(k+p)_E^2$$

$$\sim\!\!\sim = -i \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2+m^2)((k+p)_E^2+m^2)}$$

$$\int_0^1 dx \frac{1}{\left((1-x)(k_E^2+m^2) + x\left((k+p)_E^2+m^2\right)\right)^2}$$

$$(k+xp)^2 + x(1-x)p^2 + m^2$$

$$k'^2$$

$$\sim\!\sim\!\sim = -i \int_0^1 dx \quad \left(x(1-x)P_E^2 + m^2\right)^{\frac{d-4}{2}} \cdot \underbrace{\int \frac{d^d k'}{(2\pi)^d} \frac{1}{(k'^2+1)^2}}$$

$$\rightarrow \int_0^\infty d\alpha\, \alpha\, e^{-\alpha} \left(\int \frac{d^d k}{(2\pi)^d} e^{-\alpha k^2} = \alpha^{-d/2} \frac{\pi^{d/2}}{(2\pi)^d} \right) = \frac{1}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right)$$

$$\Rightarrow$$

$$i\Sigma_R = + i \frac{\lambda^2 \mu^\varepsilon}{2} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx \left(x(1-x)P_E^2 + m^2\right)^{\frac{d-4}{2}} + i\left(-(Z-1)P_E^2 - Z\delta m^2\right)$$

$$(p^2 = -P_E^2)$$

$$d = 6 - \varepsilon$$

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(-1 + \frac{\varepsilon}{2}\right) = \frac{1}{-1+\frac{\varepsilon}{2}} \Gamma\left(\frac{\varepsilon}{2}\right) \rightarrow -\frac{2}{\varepsilon}$$

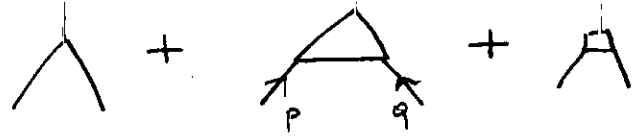
$$\mu^\varepsilon \left(x(1-x)P_E^2 + m^2\right)^{\frac{d-4}{2}} = \left(x(1-x)P_E^2 + m^2\right) \times \left(\left(x(1-x)P_E^2 + m^2\right)/\mu^2\right)^{-\varepsilon/2}$$

$$\Sigma_R = + \frac{\lambda^2}{(4\pi)^3} \cdot \frac{1}{2}\left(-\frac{2}{\varepsilon}\right) \cdot \left(\frac{1}{6} P_E^2 + m^2\right) + \lambda^2 f_R\left(p^2, m^2; \mu^2\right)$$

$$- (Z-1)P_E^2 - Z\delta m^2$$

$$\Rightarrow \quad Z = 1 - \frac{\lambda^2}{(4\pi)^3} \frac{1}{6} \frac{1}{\varepsilon} + \cdots \qquad \delta m^2 = -\frac{\lambda^2}{(4\pi)^3} m^2 \frac{1}{\varepsilon} + \cdots$$

$$\Rightarrow \Sigma_R = \lambda^2 f_R\left(p^2, m^2; \mu^2\right) = \text{finite}$$

Vertex $\Gamma_R^{(3)}$ = 

$$-i\lambda\mu^{\varepsilon/2} + (-i\lambda\mu^{\varepsilon/2})^3 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(p+k)^2-m^2} \frac{i}{(p+q+k)^2-m^2}$$

$$-i\lambda\mu^{\varepsilon/2}(Z_1-1)$$

(from the Lagrangian)

$$= -i\lambda\mu^{\varepsilon/2}\left\{1 + \underbrace{(-i\lambda\mu^{\varepsilon/2})^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{} \frac{i}{} \frac{i}{}}_{\frac{\lambda^2}{(4\pi)^3}\frac{1}{\varepsilon} + \lambda^2 g_R(p,q,m;\mu)} + (Z_1-1)\right.$$

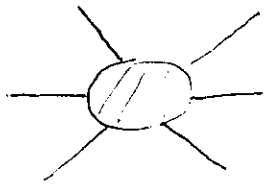
$$\Rightarrow \quad Z_1 = 1 - \frac{\lambda^2}{(4\pi)^3}\frac{1}{\varepsilon} + \dots$$

$$\Gamma_R^{(3)} = -i\lambda\left\{1 + \lambda^2 g_R(p,q,m,\mu)\right\} = \text{finite}$$

at $\varepsilon = 0$

More in general organize the computation

$G^{(6)}$ =



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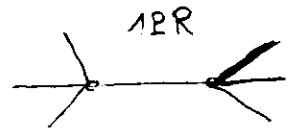
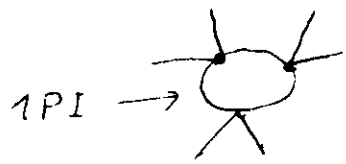


+ permutations

+



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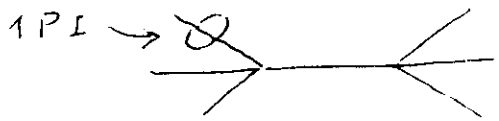

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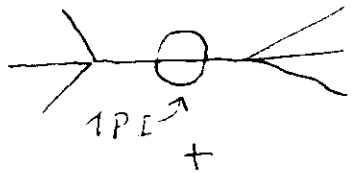

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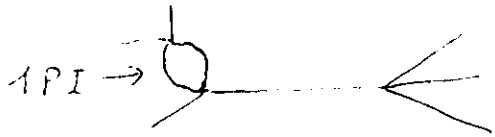
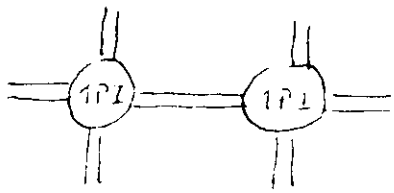


+ ...

Thus

$G^{(6)}$ =



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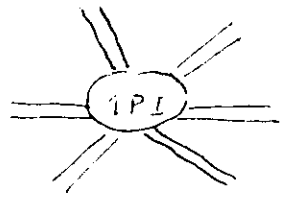


=

(+ permutations)

$$= \left(G^{(2)} G^{(2)} G^{(2)}\right) \Gamma^{(4)} \cdot G^{(2)} \cdot \Gamma^{(4)} \left(G^{(2)} G^{(2)} G^{(2)}\right) +$$

$$+ \quad G^{(2)} G^{(2)} G^{(2)} \cdot \Gamma^{(6)} \cdot G^{(2)} G^{(2)} G^{(2)}$$

defining $\Gamma^{(N)} = (-1) \tilde{\Gamma}^{(N)}$ (1PI)

(sign choice as in Tr[ ])

We wrote

$$G^{(6)} = \prod_{1}^{6} G^{(2)} \cdot \tilde{\Gamma}^{(6)}$$

thus $\tilde{\Gamma}^{(6)} = \Gamma^{(4)} \cdot G^{(2)} \cdot \Gamma^{(4)} + \Gamma^{(6)}$

$\Rightarrow$ Every Green Function can be written as a tree diagram using

$G^{(2)}$ (the <u>complete</u> propagator)

and $\Gamma^{(N)}$ the effective interaction vertices (1PI)

Compare with the Action:

$$S = \int \frac{1}{2!} \varphi (-\Box + m_0^2) \varphi - \frac{\lambda_0}{4!} \varphi^4 = \frac{1}{2!} S^{(2)} \varphi^2 + \frac{1}{4!} \cdots \varphi^4$$

define the <u>generating functional</u> $\Gamma(\varphi)$

$$\Gamma(\varphi) = \sum_N \frac{\varphi^N}{N!} \Gamma^{(N)}$$

here we mean

$$\varphi^N * \Gamma^{(N)} = \int \prod_1^N d\vec{x}_i \, \varphi(x_1) \cdots \varphi(x_N) \, \Gamma^{(N)}(x_1 \cdots x_N)$$

$$\Gamma(\varphi) = \frac{1}{2!} \varphi \cdot \Gamma^{(2)} \varphi + \frac{1}{4!} \Gamma^{(4)} \varphi^4 + \cdots + \frac{1}{N!} \Gamma^{(N)} \varphi^N + \cdots$$

$\Gamma^{(2)} = p^2 - m_0^2 + \Sigma(p^2)$ ϕ^4 theory (in Fourier transf)

$\Gamma^{(4)} = -\lambda_0 + F(p_1 p_2 p_3 p_4)$ $\quad F = O(\lambda_0^2)$

$\Gamma^{(6)} = \Gamma^{(6)}(p_1 \cdots p_6)$ $\quad \Gamma^{(6)} = O(\lambda_0^3)$

etc.

Γ is called the "effective Action"

it is to be used for tree diagrams, built by $\Gamma^{(2)}$ = complete propagators and effective vertices = $\Gamma^{(N)}$

The task of perturbation theory: compute Γ.

Renormalization

$$\Gamma^{(N)} = Z^{-\frac{N}{2}} \cdot \Gamma_R^{(N)}$$

Physical quantities (ex. scattering amplitudes) are expressed in terms of $\Gamma_R^{(N)}$

$$\Gamma(\varphi) = \sum \frac{1}{N!} \varphi^N \Gamma^{(N)} = (\varphi = Z^{1/2} \varphi_R) =$$

$$= \sum \frac{1}{N!} \varphi_R^N \Gamma_R^{(N)}$$

Generating Functionals for Green Functions

Define $Z^{(n)} = \langle 0 | T\, \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle$

also including disconnected

let $Z(J) = \sum_n \frac{1}{n!} J^n Z^{(n)}$ put $Z^{(0)} = 1 = Z(J=0)$

We have defined

$$G^{(n)} \equiv Z^{(n)}_{\text{connected}}$$

let $W(J) = \sum_l \frac{1}{l!} J^l G^{(l)}$ put $G^{(0)} = 0 = W(J=0)$

It can be seen that $Z(J) = e^{W(J)}$

Check: $\underbrace{\left(\sum \frac{J^l G^{(l)}}{l!}\right)\cdot\left(\sum \cdots\right)\cdots\left(\sum \cdots\right)}_{n \text{ factors}}$ $\binom{n}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_1-\cdots-s_p}{s_p} =$

$$e^{W(J)} = \sum_n \frac{1}{n!}\left[\left(\sum_l J^l \frac{G^{(l)}}{l!}\right)^n =\right]$$

$$= \sum_n \frac{1}{n!}\left[\sum_p \sum_{s_1} \cdots \sum_{s_p} \left(\frac{J}{1!}G^{(1)}\right)^{s_1} \cdots \left(\frac{J^p}{p!}G^{(p)}\right)^{s_p} \frac{n!}{s_1! \cdots s_p!}\right] =$$

$s_1 + s_2 + \cdots + s_p = n$ = number of factors

$$= 1 + \sum_p \sum_{s_1} \cdots \sum_{s_p} \frac{1}{s_1! \cdots s_p!} \left(\frac{J}{1!}G^{(1)}\right)^{s_1} \cdots \left(\frac{J^p}{p!}G^{(p)}\right)^{s_p}$$

(no constant terms)

Now, having m objects (J^m) and connecting them:

s_1 groups of 1 object
s_2 " " 2 " s
⋮
s_p " " p "

$s_1 + 2s_2 + \cdots + p s_p = m$

multiplicity $= \dfrac{m!}{s_1! \cdots s_p!} \left(\dfrac{1}{1!}\right)^{s_1} \left(\dfrac{1}{2!}\right)^{s_2} \cdots \left(\dfrac{1}{p!}\right)^{s_p} =$

$= \binom{m}{s_1} \cdot 1 \cdot \binom{m-s_1}{2s_2} \dfrac{(2s_2)!}{(2!)^{s_2} s_2!} \cdots$

$\Rightarrow Z(J) = \sum_m \dfrac{J^m}{m!} Z^{(m)} =$

$= \sum_m \dfrac{1}{m!} \sum_p \sum_{s_1} \cdots \sum_{s_p} \dfrac{m!}{s_1! \cdots s_p!} \left(\dfrac{J}{1!} G^{(1)}\right)^{s_1} \cdots \left(\dfrac{J^p}{p!} G^{(p)}\right)^{s_p} =$

($s_1 + 2s_2 + \cdots + p s_p = m$)

$= J^m Z^{(m)}$

making the sum over m: $\sum_m \delta(m - s_1 - 2s_2 - \cdots - p s_p) = 1$ eliminating the constraint

we see that

$$Z(J) = e^{W(J)}$$

In general $Z(J) = \int d\xi \, P(\xi) \, e^{J\xi} = \sum_m \dfrac{1}{m!} J^m \langle \xi^m \rangle$

$P(\xi)$ = probability distribution

$$Z^{(n)} = \langle \xi^n \rangle$$

$G^{(n)}$ = correlations example $G^{(2)} = \langle \xi^2 \rangle - \langle \xi \rangle^2$

The Green Functions can be evaluated by Path Integral.

$$\langle 0|T\,\varphi(x_1)\cdots\varphi(x_N)|0\rangle = N\int \mathcal{D}\varphi(x)\, e^{iS(\varphi)}\,\varphi(x_1)\cdots\varphi(x_N)$$

example

$$S(\varphi) = \int d^4x \left\{ \frac{1}{2}\left((\partial\varphi)^2 - m_0^2\varphi^2\right) - \frac{\lambda_0}{4!}\varphi^4 \right\}$$

$$(m_0^2 \to m_0^2 - i\varepsilon)$$

Actually the Euclidean Path Integral is better defined: $dx_0 \longrightarrow -i\,dx_4$

$$d^4x = d^3x\,dx_0 \longrightarrow -i\,d^3x\,dx_4 = -i\,d^4x_E$$

$$(\partial\varphi)^2 = \partial_\mu\varphi\,\partial^\mu\varphi = (\partial_0\varphi)^2 - (\vec{\partial}\varphi)^2 \to -\left((\partial_4\varphi)^2 + (\vec{\partial}\varphi)^2\right) = -(\partial_E\varphi)^2$$

$$e^{iS(\varphi)} \to e^{-S_E(\varphi)}$$

$$S_E(\varphi) = \int d^4x_E \left\{ \frac{1}{2}\left((\partial_E\varphi)^2 + m_0^2\varphi^2\right) + \frac{\lambda_0}{4!}\varphi^4 \right\} > 0$$

From Path Integral $\Longrightarrow$ Feynman graphs

Euclidean Path Integral $\Longleftrightarrow$ "Wick rotation"

$$K_0 \to iK_4$$

$$dK_0 \to i\,dK_4$$

this is the way loops are computed

$$Z = N\int \mathcal{D}\varphi\, e^{iS(\varphi)} = \lim_{T\to\infty}\langle 0|e^{-iHT}|0\rangle$$

→ (5)

Path integration formula

$$Z(J) = N \int \mathcal{D}\varphi \, e^{i S(\varphi) + \int_{-\infty}^{+\infty} J\varphi} = \lim_{T\to\infty} \langle 0 | e^{-i H_J \cdot T} | 0 \rangle$$

remember $$S(p,q) = \int_{-\infty}^{+\infty} dt \, (p\dot{q} - H(p,q))$$

thus $i S(\varphi) - i \int J\varphi$ is like $H(\varphi) \to H(\varphi) + J\cdot\varphi = H_J(\varphi)$

thus let us consider

$$Z(-iJ) = \sum_m \frac{1}{m!} (-iJ)^m Z^{(m)} =$$

$$= \lim_{T\to\infty} Tr\left(e^{-i H_J \cdot T}\right) = e^{-i E_0(J)\cdot T}$$

(for instance periodic b.c.s in the path integral)

$E_0(J)$ = lowest energy of H_J

$H_J |0_J\rangle = E_0(J) |0_J\rangle$ $\quad |0_J\rangle$ lowest energy state of H_J.

Let us thus define

$$Z(-iJ) = e^{-i W(J)} \quad \Rightarrow \quad W(J) = E_0(J)\cdot T$$

In terms of Green Functions

$$G^{(n)} = \langle 0|T\,\varphi\cdots\varphi|0\rangle$$

$$W^{(1)} = \langle 0|\varphi|0\rangle \qquad W^{(2)} = -i\,G^{(2)} = \frac{1}{p^2 - m_0^2 + \Sigma(p^2)}$$

$$W^{(n)} = i\,(-i)^n\,G^{(n)}$$

---

define $\boxed{-\Gamma(\phi) = W(J) - \phi J}$

where $\phi = \frac{\delta W}{\delta J} = \langle\varphi\rangle_J$ (before taking $J=0$)

$$\Rightarrow \quad \frac{\delta\Gamma}{\delta\phi} = J \quad \longleftrightarrow \quad J = J(\phi)$$

Comparing $W(J)$ and $\Gamma(\phi)$:

$$W(J) = -\Gamma(\phi(J)) + \phi(J)\cdot J$$

$$\frac{\delta W}{\delta J} = \phi \qquad \frac{\delta^2 W}{\delta J^2} = W^{(2)} = \frac{\delta\phi}{\delta J} = \left(\frac{\delta J}{\delta\phi}\right)^{-1} = \left(\frac{\delta^2\Gamma}{\delta\phi^2}\right)^{-1} = \frac{1}{\Gamma^{(2)}}$$

thus $\Gamma^{(2)} = p^2 - m_0^2 + \Sigma(p)$

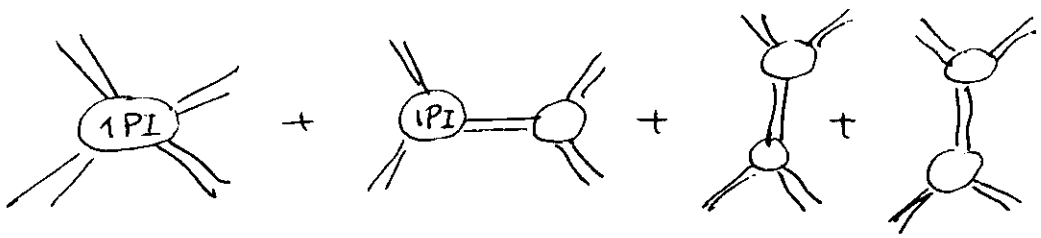
$$\frac{\delta^3 W}{\delta J^3} = W^{(3)} = \frac{\delta}{\delta J}\,\frac{1}{\Gamma^{(2)}} = -\left(\frac{1}{\Gamma^{(2)}}\right)^2 \frac{\delta\Gamma^{(2)}}{\delta\phi}\cdot\frac{\delta\phi}{\delta J} = -\left(\frac{1}{\Gamma^{(2)}}\right)^3\Gamma^{(3)} =$$

($\frac{\delta\Gamma^{(2)}}{\delta\phi} = \Gamma^{(3)}$, $\frac{\delta\phi}{\delta J} = \frac{1}{\Gamma^{(2)}}$)

$$= \quad \text{1PI}$$

$$\frac{\delta^4 W}{\delta J^4} = W^4 = \left(\frac{1}{\Gamma^{(2)}}\right)^4 \cdot \left(-\Gamma^{(4)}\right) +$$

$$+ 3\left(\frac{1}{\Gamma^{(2)}}\right)^2 \left(-\Gamma^{(3)}\right) \frac{1}{\Gamma^{(2)}} \left(-\Gamma^{(3)}\right) \left(\frac{1}{\Gamma^{(2)}}\right)^2$$



thus $\boxed{\Gamma(\phi) = \int \frac{1}{2}\left((\partial\phi)^2 - m_0^2\phi^2 + \Sigma\phi^2\right) - \frac{1}{4!}(\lambda_0 + F)\phi^4 - \frac{1}{6!}\Gamma^{(6)}\phi^6 + \dots}$

the formerly introduced effective Action

Physical meaning:

Consider a variational problem:

find $|a\rangle$ such that $\langle a|H|a\rangle$ = minimum with $\langle a|a\rangle = 1$

$$\Rightarrow \frac{\delta}{\delta a}\langle a|(H - E_0)|a\rangle = 0 \Rightarrow H|a\rangle = E_0|a\rangle$$

E_0 : multiplier

require an extra constraint $\langle a|\varphi|a\rangle = \phi$ fixed

$\Rightarrow$ another multiplier J

$$\frac{\delta}{\delta a}\langle a|\underbrace{(H + J\varphi}_{H_J} - E)|a\rangle = 0 \qquad H_J|a\rangle = E_0(J)|a\rangle$$

$$\frac{\delta E_0(J)}{\delta J} = \langle a|\varphi|a\rangle = \phi$$

$$\Rightarrow \langle a|H|a\rangle = E_0(J) - J\phi = -\Gamma(\phi)$$

Thus: $-\Gamma(\phi)$ = minimal energy for configuration such that $\langle\varphi\rangle = \phi$

$\langle 0|\varphi|0\rangle = ?$ look for the min. of $\Gamma(\phi_m)$. Then $\langle 0|\varphi|0\rangle = \phi_{min}$

Loop expansion and powers of $\hbar$

$e^{\frac{i}{\hbar} S} \Rightarrow$ propagator $= \frac{i\hbar}{p^2 - m^2 + i\varepsilon}$ Vertices $\sim \hbar^{-1}$

thus 1PI graph $\sim \hbar^{I-V} = \hbar^{L-1}$

$L=0$ leading for $\hbar \to 0$: classical theory = tree diagrams

Example :

$$Z(J) = N \int \mathcal{D}\varphi \, e^{\frac{i}{\hbar}(S\varphi - J\varphi)} \qquad \hbar \to 0 \text{ saddle point}$$

dominated by the classical eqs $\frac{\delta S}{\delta \varphi} = J \Rightarrow \varphi = \varphi_c$

put $\varphi = \varphi_c + \eta \qquad \mathcal{D}\varphi = \mathcal{D}\eta$

$$S\varphi - J\varphi = S(\varphi_c) - J\varphi_c + \eta \underbrace{\left(\frac{\delta S}{\delta \varphi} - J\right)_{\varphi_c}}_{=0} + \frac{1}{2}\frac{\delta^2 S}{\delta \varphi^2}\Big|_{\varphi_c} \eta^2 + O(\eta^3)$$

$$Z(J) = e^{\frac{i}{\hbar}\frac{S(\varphi_c) - J\varphi_c}{\hbar}} \underbrace{N \int \mathcal{D}\eta' \, e^{\frac{i}{2}\frac{\delta^2 S}{\delta \varphi^2}\Big|_{\varphi_c}\eta'^2 + O(\hbar \eta'^3)}}$$

$$\text{const.} \det\left(\frac{\delta^2 S}{\delta \varphi^2}\Big|_{\varphi_c}\right)^{-1/2}$$

more in detail : $\int \prod_n d\eta_n \, e^{-\eta_n \Lambda_{nm} \eta_m} = \text{const} \prod \lambda_i^{-1/2} = \text{const} (\det \Lambda)^{-1/2}$

λ_i = eigenvalues of Λ

$= \text{const.} \exp -\frac{1}{2} \sum \ln \lambda$

$= \text{const.} \exp -\frac{1}{2} \operatorname{tr} \ln \Lambda$

* for a complex field $\int \prod d\eta_n \, d\bar{\eta}_n \, e^{-\bar{\eta}_n \Lambda_{nm} \eta_m} = \text{const.} (\det \Lambda)^{-1}$

In our case for instance

$$\left.\frac{\delta^2 S}{\delta \varphi^2}\right|_{\varphi_c} = -\partial^2 - m_0^2 - \frac{\lambda_0}{2}\varphi_c^2$$

$$\det\left(\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c}\right)^{-1/2} = \exp -\frac{1}{2}\,\mathrm{tr}\,\ln\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c} =$$

$$= \exp -\frac{1}{2}\underbrace{\int d^4x}_{\text{Tot}}\int\frac{d^4k}{(2\pi)^4}\ln\left(m_0^2+\frac{\lambda_0}{2}\varphi_c^2 - k^2\right)$$

$$\text{Wick rotating} = i\int\frac{d^4k_E}{(2\pi)^4}\ln\left(m_0^2 + k_E^2 + \frac{\lambda_0}{2}\varphi_c^2\right)$$

therefore $\left(\det\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi_c}\Big)^{-1/2} = \exp -\frac{i}{2}K(\varphi_c)$

$$\Rightarrow \quad W(J) = -S(\varphi_c^{(J)}) + J\varphi_c^{(J)} + \frac{\hbar}{2}K(\varphi_c) = -\Gamma(\phi) + J\phi$$

where $\phi = \frac{\delta W}{\delta J} = \varphi_c + \frac{\hbar}{2}K'\frac{\delta\varphi_c}{\delta J}$ (*)

at order $\hbar$: $\Gamma(\phi) = S(\phi) - \frac{\hbar}{2}K(\phi) =$

$$= \text{classical } S(\phi) + \text{(loop)} + \text{(loop)} + \text{(loop)} + \dots$$

$K(\phi)$ = one loop correction (quantum)

Fermions path integral

functional integration over anticomm. variables

$$\{a, b\} = 0 \quad \Rightarrow \quad a^2 = 0$$

$$\Rightarrow \quad f(a) = c_0 + c_1 a$$

requiring $\int da\, f(a+b) = \int da\, f(a)$

$$\Rightarrow \int da\, \text{const} = 0 \qquad \int da\, a = 1 \quad \text{(normalization)}$$

$$\int da\, f(a) = -c_1 \qquad \left(\text{if } da\, c_1 = -c_1\, da\right)$$

Functional int. $\psi(x)$ anticomm. funct. of x

Expanding on a basis: $\psi(x) = \sum_n \xi_n(x)\, a_n$

$\xi_n(x)$. a basis of ordinary functions

a_n : anticomm numbers $\qquad \mathcal{D}\psi = \prod_n da_n$

Typically the Fermion Action is $\int \bar{\psi} \Lambda \psi$

$$\Lambda = i\{ i\gamma(\partial + ieA) - m \} \quad \Rightarrow \text{ propagator } \frac{i}{\gamma p - m}$$

take Λ eigenfcs as a basis $\qquad \Lambda \xi_n = \lambda_n \xi_n$

Euclidean funct. integration : $\quad \gamma_0 p_0 - \vec{\gamma}\vec{p} = i(\gamma_4 p_4 + \vec{\gamma}_E \vec{p})$

$$\frac{i}{i\gamma_E p - m}$$

expand $\bar{\psi}(x) = \sum_n \bar{a}_n\, \xi_n^*(x) \qquad \bar{a}_n$ is indep. variable

$$\int \bar{\psi} \Lambda \psi = \sum_n \lambda_n \bar{a}_n a_n \qquad \mathcal{D}\psi\, \mathcal{D}\bar{\psi} = \prod_n (da_n\, d\bar{a}_n)$$

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi \exp \bar\psi \Lambda \psi =$$

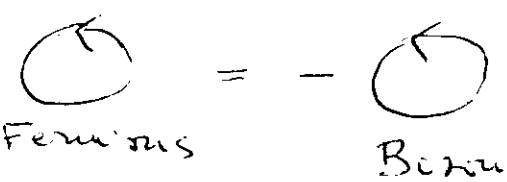
$$= \lim_{N\to\infty} \int \prod_1^N (d\alpha_n d\bar\alpha_n)\left(1 + \frac{\sum_1^N \lambda_n \bar\alpha_n \alpha_n}{1!} + \dots + \frac{\left(\sum_1^N \lambda_n \bar\alpha_n d_n\right)^N}{N!}\right) =$$

$$= \prod_1^N \lambda_n \bar\alpha_n \alpha_n$$

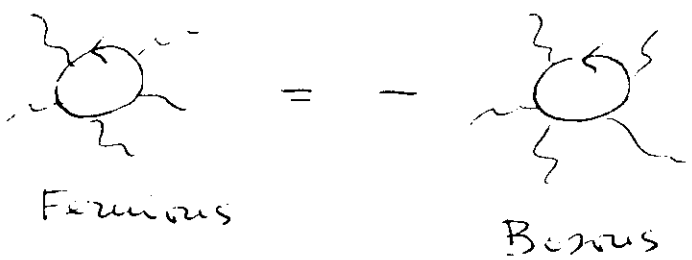
$$= \lim \prod_1^N \lambda_n = \det \Lambda = \exp \operatorname{Tr} \ln \Lambda$$

Compare Bosonic $\int \mathcal{D}\phi\mathcal{D}\bar\phi \exp \bar\phi \Lambda \phi = (\det \Lambda)^{-1} = \exp - \operatorname{Tr} \ln \Lambda$

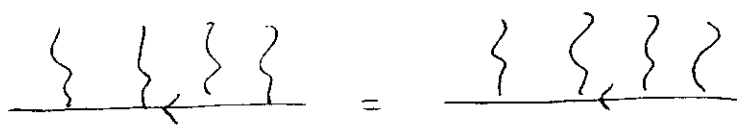
$\Rightarrow$ in the Fermionic case every loop a relative $(-)$ sign with respect to the Bosonic case



$\Lambda = \Lambda(\alpha t_n)$ by expanding in $+ t_n$:



Closed loops for Fermions: extra $(-)$ sign

For open lines 

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{\bar\psi\Lambda\psi + \bar\psi J + \bar J \psi}$$

$$(\bar\psi + \bar J\Lambda^{-1})\Lambda(\psi + \Lambda^{-1}J) - \bar J \Lambda^{-1} J$$

U.V. Divergences : from the loop integrations

$$\Gamma^{(N)}(p) \sim \int \prod_1^L d^d k_e \prod_1^I \frac{1}{(\Sigma k' + \Sigma p')^2 + m^2} \qquad d = 4 - \varepsilon$$

Euclidean

Overall divergence : rescale $k_e = Q\, k_e^0$

write $1 = \int d^d K\; \delta^{(d)}\left(K - \sum_1^L k_e\right)$ here

then rescale $K_\mu = Q\, K_\mu^0 \qquad \left((K_\mu^0)^2 = 1\right)$

$$\Rightarrow \Gamma^{(N)}(p) \sim \int \frac{dQ}{Q}\, Q^{Ld - 2I} \left(P_0(p) + P_2(p)\, Q^{-2} + \dots\right) =$$

$$= \frac{P_0(p)}{2I - Ld} + \frac{P_2(p)}{2I + 2 - Ld} + \dots$$

divergent for $Ld - 2I = n$ integer $(n = 0, 2, \dots)$

$$= \frac{1}{\varepsilon}\, \frac{P_0(p)}{L} \qquad \text{for} \quad L \cdot 4 - 2I = 0 \quad (\text{log div})$$

$$= \frac{1}{\varepsilon}\, \frac{P_2(p)}{L} \qquad \text{for} \quad L \cdot 4 - 2I = 2 \quad (\text{quadratic div})$$

φ^4 : $E + 2I = 4V \;\Rightarrow\; E = 2I - 4L + 4$

Theorem: the divergences for $\varepsilon \to 0$ are of the form

$$\Gamma^{(N)}(p) \sim \frac{P(p)}{\varepsilon^n}$$ where $P(p)$ is a polynomial in the external momenta p_μ

whose non negative degree is given by dimensionality (dimensionless)

examples ($\lambda \phi^4$ in $d=4$)

$$\left[\Gamma^{(2)}(p)\right] = p^2 \quad \text{then} \quad \Gamma^{(2)}(p) \sim \frac{\alpha p^2 + \beta M^2}{\varepsilon^n}$$

$$\left[\Gamma^{(4)}(p)\right] = 1 \quad \text{then} \quad \Gamma^{(4)}(p) \sim \frac{c}{\varepsilon^n}$$

$$\left[\Gamma^{(n>4)}(p)\right] = p^{-(n-4)} \Rightarrow \Gamma^{(n>4)} = \text{convergent}$$

here the coupling constant is dimensionless $[\lambda] = 1$

therefore the conclusion is independent of the perturbation theory order λ^l : renormalizable theory

(if $[\lambda] = p^{-}$ then unrenormalizable theory !)

— Dimensional analysis —

Of course there are also subdivergences

example subdivergent

iterative pattern : we assume subdivergencies have already taken care by at the previous orders.

In <u>renormalizable</u> theories the divergent terms in Γ have the <u>same form</u> as the classical Lagrangian :

$$\Gamma_c^{(2)} = p^2 - m_0^2 \qquad \Gamma_c^{(4)} = -\lambda_0$$

⇒ Take in the classical Lagrangian <u>all</u> the terms $\cdot \Gamma_c^{(n)}$ whose dimensionality is non negative :

$$\mathcal{L} = \frac{1}{2}\left((\partial\varphi)^2 - m_0^2\varphi^2\right) - \frac{\lambda_0}{4!}\varphi^4 + \cancel{\frac{g_0}{6!}\varphi^6}$$

because $[g_0] = p^{-2}$

Make redefinitions:

$$\varphi = Z^{1/2}\varphi_R \qquad \lambda_0 = \lambda_R \mu^\varepsilon Z_1 Z^{-2} \qquad m_0^2 = m_R^2 + \delta m^2$$

(for $d = 4-\varepsilon$ $[\lambda_0] = p^\varepsilon$ $[\lambda_R] = 1$)

Rewrite thus the <u>new</u> Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left\{(\partial\varphi_R)^2 - m_R^2\varphi_R^2\right\} - \frac{\lambda_R\mu^\varepsilon}{4!}\varphi_R^4$$

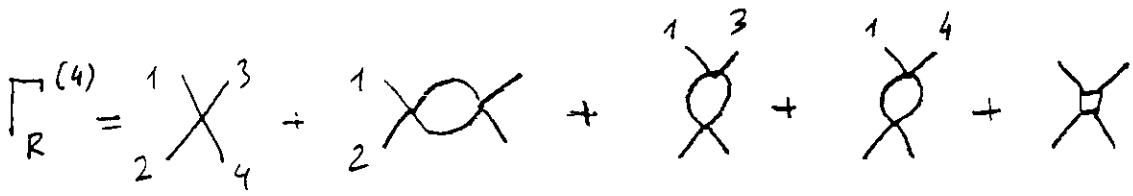
$$+ \frac{1}{2}(Z-1)\left\{(\partial\varphi_R)^2 - m_R^2\varphi_R^2\right\} - \frac{Z\delta m^2}{2}\varphi_R^2 - \frac{\lambda_R\mu^\varepsilon}{4!}(Z_1-1)\varphi_R^4$$

expand $\left.\begin{matrix} Z-1 \\ Z_1-1 \\ Z\delta m^2 \end{matrix}\right\} = \sum_{n\geq 1}^{\infty} \lambda_R^n a_n$

<u>use</u> <u>perturbation expansion in λ_R</u>

fix a_n requiring cancellation of poles in ε

$\mathcal{L} = \frac{Z}{2}\{(\partial\varphi_R)^2 - (m^2+\delta m^2)\varphi_R^2\} - Z_1 \frac{\lambda\mu^\varepsilon}{4!}\varphi_R^4 \qquad d = 4-\varepsilon$ $\qquad \varphi^4_1$

$\Gamma_R^{(4)} =$ 

$$= -i\lambda\mu^\varepsilon + (-i\lambda\mu^\varepsilon)^2 \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(p_1+p_2+k)^2-m^2+i\varepsilon} + \text{two more}$$

$$- i\lambda\mu^\varepsilon (Z_1 - 1)$$

$d = 4-\varepsilon$

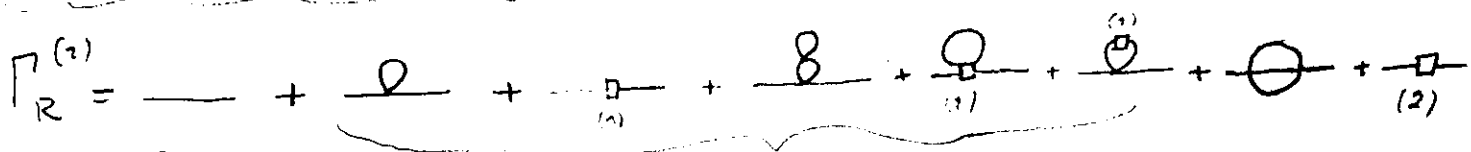
$$= -i \int_0^1 \left(x(1-x)(p_1+p_2)_E^2 + m^2\right)^{\frac{d-4}{2}} \cdot \frac{1}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) =$$

$\Gamma\left(\frac{\varepsilon}{2}\right) \sim \frac{2}{\varepsilon}$

$$= -\frac{i}{(4\pi)^2}\cdot\frac{2}{\varepsilon} + \text{finite}$$

$$\Rightarrow \Gamma_R^{(4)} = -i\lambda\mu^\varepsilon \left(1 - 3\frac{\lambda}{(4\pi)^2}\frac{1}{\varepsilon} + \lambda^2\left(\phi_R(p_1+p_2)^2, m^2; \mu^2\right) + \phi_R^2\left((p_1+p_3)^2, m^2; \mu^2\right) + \phi_R\left((p_1+p_4)^2, m^2; \mu^2\right) + Z_1 - 1\right)$$

$$\Rightarrow Z_1 = 1 + 3\frac{\lambda}{(4\pi)^2}\frac{1}{\varepsilon}$$

$\Gamma_R^{(2)} =$ 

p^2-m^2 $\qquad$ p^2-independent $\qquad$ $(Z-1)(p^2-m^2)$

$$\Rightarrow Z = 1 + O(\lambda^2)$$

Renormalization group:

$$\Gamma^{(n)}(p, \lambda_0, m_0^2) = Z^{-n/2}\, \Gamma_R^{(n)}(p, \lambda_R, m_R^2; \mu)$$

$$\mu \frac{d}{d\mu}\left(Z^{-n/2}\, \Gamma_R^{(n)}(p, \lambda_R, m_R^2; \mu)\right)\Big|_B = 0$$

$$\Rightarrow \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda_R}{\partial \mu}\Big|_B \frac{\partial}{\partial \lambda_R} + \mu \frac{\partial m_R^2}{\partial \mu}\Big|_B \frac{\partial}{\partial m_R^2} - \frac{n}{2}\mu \frac{\partial \ln Z}{\partial \mu}\Big|_B\right)\Gamma_R^{(n)} = 0$$

combine it with $\left[\Gamma_R^{(n)}\right] = p^{\underbrace{4 - n + \frac{\varepsilon}{2}(n-2)}_{d_n}}$ (φ^4 in 4 dims)

i.e. $\left(\mu \frac{\partial}{\partial \mu} + p \frac{\partial}{\partial p} + 2 m_R^2 \frac{\partial}{\partial m_R^2} - \left(4 - n + \frac{\varepsilon}{2}\cdot(n-2)\right)\right)\Gamma_R^{(n)} = 0$

(p: common scale)

$$\Rightarrow \left(p \frac{\partial}{\partial p} - \beta \frac{\partial}{\partial \lambda_R} + \underbrace{\left(2 m_R^2 - \mu \frac{\partial m_R^2}{\partial \mu}\Big|_B\right)}_{\eta(\lambda_R, m_R^2)} \frac{\partial}{\partial m_R^2} - \underbrace{\left(4 - n + \frac{\varepsilon}{2}(n-2) - \frac{n}{2}\mu \frac{\partial \ln Z}{\partial \mu}\Big|_B\right)}_{d_n - n\gamma}\right)\Gamma_R^{(n)} = 0$$

$$\beta = \mu \frac{\partial}{\partial \mu}\lambda_R\Big|_B = -\varepsilon \lambda_R + \lambda_0 \mu^{-\varepsilon} \underbrace{\mu \frac{\partial}{\partial \mu}\left(Z_1^{-1} Z^2\right)\Big|_B}_{\mu \frac{\partial}{\partial \mu}\left(1 - \frac{\lambda_0}{(2\pi)^4}\mu^{-\varepsilon}\frac{3\pi^2}{\varepsilon} + \cdots\right)} = -\varepsilon \lambda_R + \frac{\lambda_R^2}{(2\pi)^4}\, 3\pi^2 + \cdots$$

β-functions:

$\lambda\varphi^3$ theory

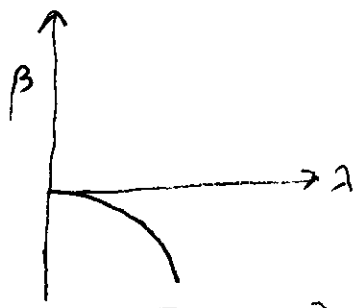
$$\lambda = Z_1^{-1} Z^{+3/2} \mu^{-\varepsilon/2} \lambda_0$$

$$\beta = \mu \frac{\partial}{\partial \mu} \lambda \Big|_{\lambda_0} = \lambda_0 \cdot \mu \frac{\partial}{\partial \mu} \left(\mu^{-\varepsilon/2} \cdot \left(1 + \frac{\lambda^2}{(4\pi)^3} \frac{1}{\varepsilon}\right) \cdot \left(1 - \frac{3}{2} \frac{\lambda^2}{(4\pi)^3} \frac{1}{6} \frac{1}{\varepsilon}\right) \right)_{\lambda_0}$$

$$= -\frac{\varepsilon}{2} \lambda + \lambda_0 \mu^{\varepsilon/2} \underbrace{\mu \frac{\partial}{\partial \mu} \left(\frac{\lambda^2}{(4\pi)^3} \frac{1}{\varepsilon} \right)_{\lambda_0}} \left(1 - \frac{1}{4}\right) + \cdots$$

$$\frac{\lambda_0^2}{(4\pi)^3} \frac{1}{\varepsilon} \mu \frac{\partial}{\partial \mu} \mu^{\varepsilon} + \text{higher orders}$$

$$= -\frac{\varepsilon}{2} \lambda - \frac{3}{4} \frac{\lambda^2}{(4\pi)^3} \xrightarrow{\varepsilon \to 0} -\frac{3}{4} \frac{\lambda^2}{(4\pi)^4}$$



$\lambda\varphi^4$ theory

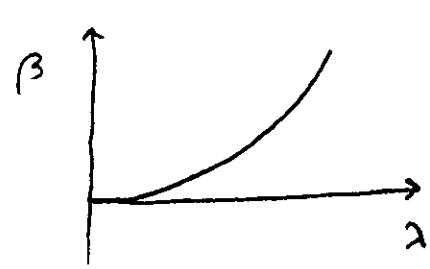
$$\lambda = Z_1^{-1} Z^{-2} \mu^{-\varepsilon} \lambda_0$$

$$\beta = \mu \frac{\partial}{\partial \mu} \lambda = \lambda_0 \mu \frac{\partial}{\partial \mu} \left(\mu^{-\varepsilon} \cdot \left(1 - 3 \frac{\lambda}{(4\pi)^2} \frac{1}{\varepsilon}\right) \left(1 + O(\lambda^2)\right) \right)_{\lambda_0}$$

$$= -\varepsilon \lambda + \lambda_0 \mu^{-\varepsilon} \underbrace{\mu \frac{\partial}{\partial \mu} \left(\frac{\lambda}{(4\pi)^2} \frac{1}{\varepsilon} \right)_{\lambda_0}} \cdot (-3)$$

$$\frac{\lambda_0}{(4\pi)^2} \frac{1}{\varepsilon} \mu \frac{\partial}{\partial \mu} \mu^{-\varepsilon} + \text{higher orders}$$

$$= -\varepsilon \lambda + \frac{3\lambda^2}{(4\pi)^2} \to + \frac{3\lambda^2}{(4\pi)^2}$$



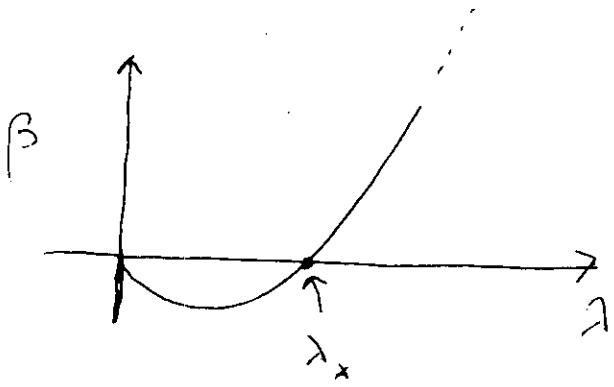
"Solution" of the renorm. group eq

$$\Gamma_R^{(n)}(p) = \left(\frac{p}{p_0}\right)^{d_n - n\gamma^*} \cdot \Gamma_R^{(n)}(p_0, \lambda^*, m^{2*})$$

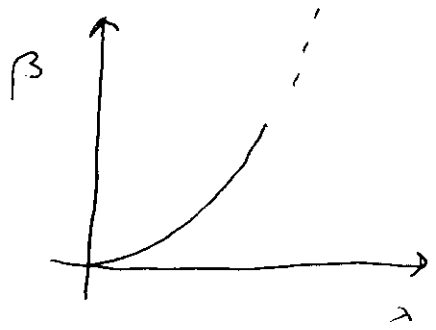
$$\frac{d\lambda(t)}{dt} = \beta(\lambda(t), m^2(t)) \qquad \beta(\lambda^*, m^*) = 0$$

$$\frac{d m^2(t)}{dt} = \gamma(\lambda(t), m^2(t)) \qquad \gamma(\lambda^*, m^*) = 0$$

$$t = \ln p/p_0$$



for $d = 4$



$d=4 \qquad \beta = \frac{3\lambda^2}{16\pi^2} \Rightarrow \frac{\lambda^R(t)}{(2\pi)^4} \sim -\frac{1}{3\pi^2 t}$ for $t \to -\infty$ $(\lambda^2 > 0)$

$$\frac{\lambda(t)}{(2\pi)^4} = \frac{\lambda(p_0)/(2\pi)^4}{1 - \frac{\lambda(p_0)}{(2\pi)^4} 3\pi^2 \ln \frac{p}{p_0}}$$

Infrared behavior $\frac{p}{p_0} \to 0$ $(t \to -\infty)$ controlled

provided $m_R^2 = m^{2*}$ $\longleftrightarrow$ fine tuning of a parameter

corresponding to $T = T_c$

in phase transitions.