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II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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STRUCTURE FORMATION IN THE UNIVERSE

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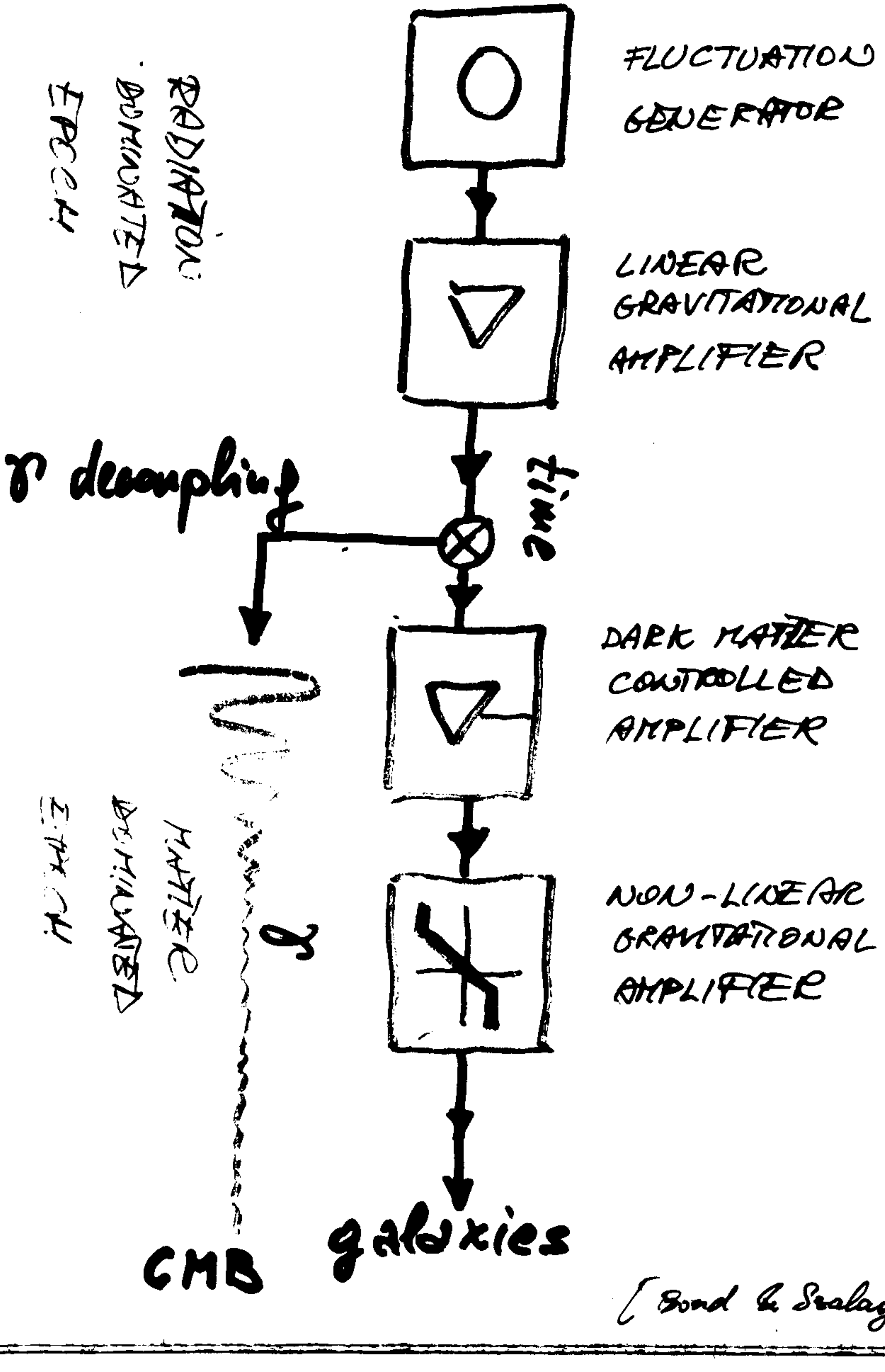
Please note: These are preliminary notes intended for internal distribution only.

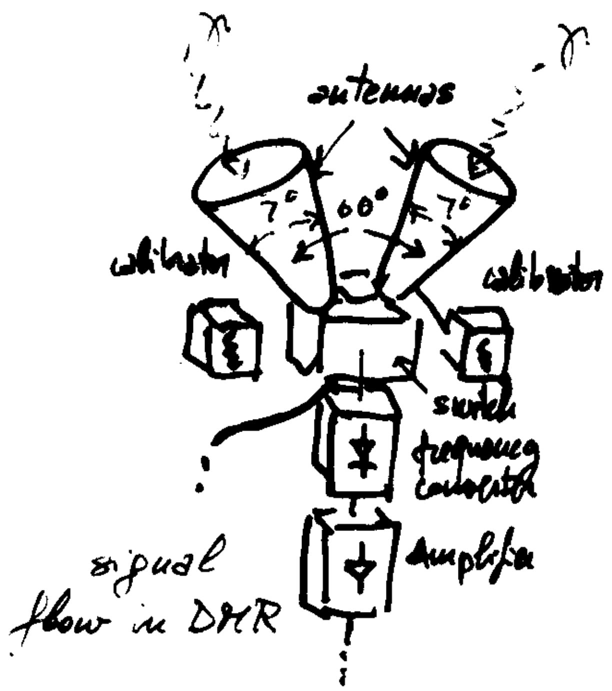
MATARRESE

GRAVITATIONAL INSTABILITY



[Bond & Szalay '83]

Differential Microwave Radiometers



DMR has 3 separate receiver boxes at 3 different wavelengths (31.5, 53, 90 GHz $\Leftrightarrow$ 9.5, 5.7, 3.3 mm) chosen to be near the minimum in Galactic emission (synchrotron, free-free & dust emission) and near the 2.73 K CMB max. Each box has two nearly-independent receivers at the same frequency. Each receiver has 2 antennas, each receiving a 7° beam, pointed 60° away. The entire spacecraft rotates at ~ 0.8 rpm.

DMR measures ΔT of regions 60° apart. The orbit and pointing of COBE result in a complete survey of the sky. Over 6 months DMR measures each portion of the sky thousand of times.

Results are now based on DMR 1st year data

Smoot et al. '92 Ap.J.
Bennett et al. '92 Ap.J.L
Kogut et al. '92
[illegible]

April 23, 1992: the COBE team announce the detection of "statistically significant structure" in the Cosmic Microwave Background (CMB) Radiation, ($T = 2.73$ K) during the APS meeting in Washington:

« We have analyzed the first year of data from the DMR [Differential Microwave Radiometers] on the COsmic Background Explorer. We observe the dipole anisotropy, Galactic emission, instrument noise and detect statistically significant structure which is well-described as a field of gaussian distributed, scale-invariant fluctuations with rms-quadrupole-normalized amplitude of $16 \pm 4\,\mu K$

$$\Downarrow$$

$$\left(\frac{\Delta T}{T}\right)_{rms} \approx 6 \times 10^{-6}$$

... The structure is consistent with a thermal spectrum at 31, 53 and 90 GHz as expected for CMB anisotropy ... The rms sky variation smoothed with a 10° FWHM Gaussian is $31 \pm 7\,\mu K$. The quadrupole amplitude is $13 \pm 4\,\mu K$. The angular auto-correlation of the signal ... gives an (angular) power-law with index $\boxed{n = 1.1 \pm 0.5}$, in accord with the Harrison-Zel'dovich (scale-invariant, $n = 1$) spectrum predicted by models of inflationary cosmology. The fluctuation amplitude is consistent with minimal predictions of theories of structure formation based on gravitational instability »

Smoot et al. '92

COBE RESULTS

- Dipole (subtraction) A dipole anisotropy ($\Delta T/T \approx 10^{-3}$) is detected by all channels, with $T = 3.36 \pm 0.1$ mK pointing in direction $l = 265.0 \pm 0.5$, $b = 48.1 \pm 0.5$

$\begin{pmatrix} l=1 \\ 180^\circ \end{pmatrix}$

kinematical effect → Motion of an observer w.r.t. CMB (blackbody) produces a dipole anisotropy

Not knowing what part of dipole is kinematical and what part primordial, the dipole is subtracted completely ($v \sim 600$ km/sec)

- Maps of the sky at 31.5, 53, 90 GHz, smoothed with a Gaussian $\exp[-\vartheta^2/2\vartheta_s^2]$, $\vartheta_s = 7^\circ$, which, convolved with $\sim 7^\circ$ FWHM Gaussian beam ⟹ smoothing angle $\sim 10^\circ$. Mean (monopole), Dipole and kinematic Quadrupole are subtracted out.

$\begin{pmatrix} \gtrsim 10^\circ \\ l < 20 \end{pmatrix}$

68% CL (?)

$$\text{Variance}^{1/2}: \quad \sigma_{SKY}(10^\circ) = \sqrt{\underbrace{\sigma_{obs}^2}_{\text{sum 2 receivers}} - \underbrace{\sigma_{DMR}^2}_{\text{difference 2 receivers}}} = 31 \pm 7\,\mu K$$

- Quadrupole All 3x2 channels give an rms quadrupole amplitude

$\begin{pmatrix} l=2 \\ 90^\circ \end{pmatrix}$

$$(Q_{rms} = 13 \pm 4\,\mu K) \Rightarrow \frac{\Delta T}{T} \approx 5 \times 10^{-6}$$

- Angular Correlation Function $C(\alpha) = \langle T(\hat{n}_1) T(\hat{n}_1 + \alpha) \rangle_{\hat{n}_1}$ rejecting pixels with $|b| < 20^\circ$, subtracting mean, dipole & quadrupole

Sachs-Wolfe effect

→ Temperature fluctuations on large angular scales arise due to:

i) our peculiar velocity w.r.t. CMB "rest frame" (mainly a dipole component)

ii) fluctuations in the gravitational potential $\phi(\vec{x},t)$ on the LSS (Sachs & Wolfe 1967):

cosmic photons from different directions, leaving the LSS from regions that have different ϕ (i.e. different density) suffer slightly different amounts of redshift, resulting in their still having blackbody spectra but with different temperatures

$$\frac{\Delta T}{T}(\vec{x}_0,\hat{\gamma}) = \frac{1}{3}\phi(\vec{x}_0 + r_0\hat{\gamma})$$

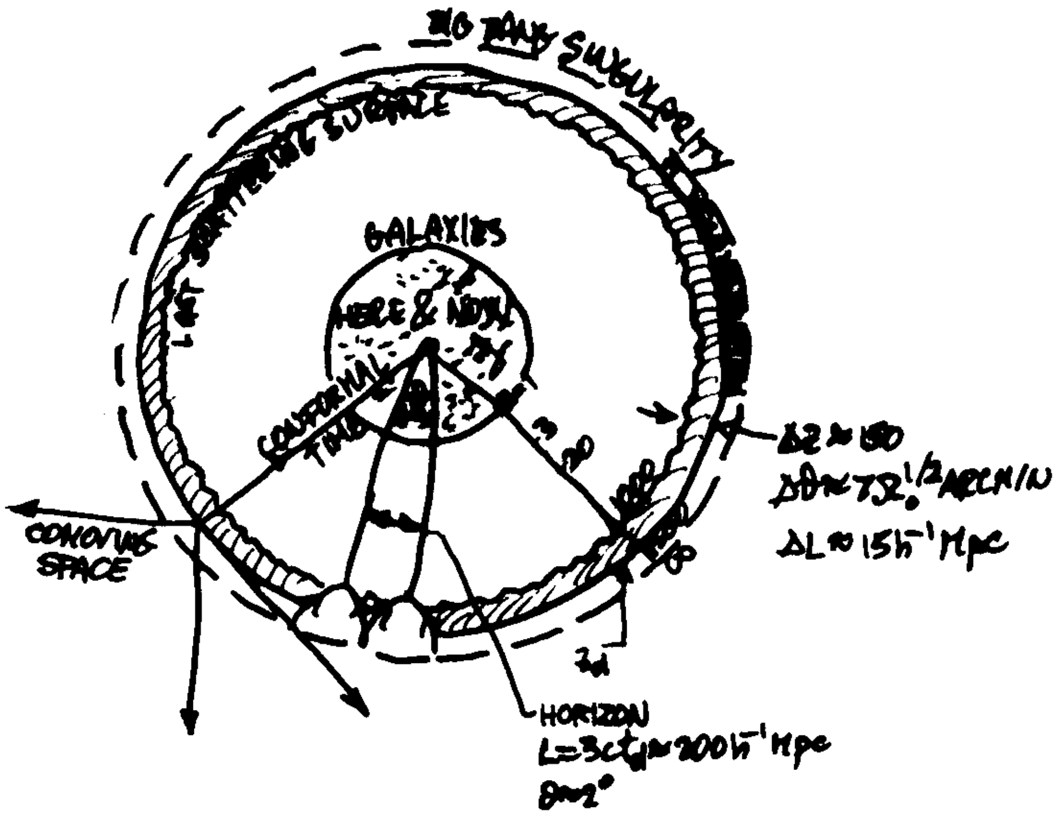
temperature anisotropy detected by an observer in $\vec{x}_0$, at rest with CMB in direction $\hat{\gamma}$

$r_0 \equiv \frac{2}{H_0}$ = Hubble radius

$$\underbrace{\nabla^2\phi}_{\text{"peculiar" gravitational potential}} = 4\pi G a^2(t)\underbrace{\delta\rho}_{\text{density fluctuation}} \Rightarrow \frac{\Delta T}{T} \sim \frac{\delta\rho}{\rho}(\ell)\left(\frac{\ell}{r_0}\right)^2$$

⇒ in Fourier space: $\phi_{\vec{k}} = \frac{3}{2}H_0^2 a_0^2 \delta_{\vec{k}}/k^2$

CMB ANISOTROPIES (LARGE SCALE)



[$h = H_0/100$ km/sec/Mpc]

MW photons stream freely to the observer ("here & now") from the last scattering surface $z \approx 1{,}100 \div 1{,}500$ (i.e. when the Universe size was $1/(1+z)$ smaller than now). Due to the non-instantaneous $(p+e \rightarrow) H$ recombination and residual ionization level, the LSS has a finite thickness $\Delta z \sim 150$, corresponding to $\Delta L \approx 15\, h^{-1}$ Mpc. The finite width of LSS smooths out fluctuations on angular scales $\Delta\vartheta \lesssim 7\, \Omega_0^{1/2}$ arcmin.

Any ambiguity (...) on the interpretation of $\Delta T/T$ is avoided if one looks at $\Delta T/T$ over $\Delta\vartheta >$ particle horizon at latest epoch of LSS (no causal processes have occurred) $\Rightarrow \Delta\vartheta > 2°$ (worst case) relation $\Delta T \leftrightarrow \delta\rho$

$$\frac{\Delta T}{T}(\vec{x}_0,\hat{\gamma}) = \frac{2}{r_0^2}\int \frac{d^3k}{(2\pi)^3}\,\frac{\delta_{\vec{k}}}{k^2}\exp\left[i\vec{k}\cdot(\vec{x}_0 + r_0\hat{\gamma})\right]$$

multipole expansion

$$\frac{\Delta T}{T}(\vec{x}_0,\hat{\gamma}) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(\vec{x}_0)\,Y_\ell^m(\gamma)$$

multipole coefficients   spherical harmonics

The rms a_ℓ^m are simply related to the rms density fluctuation

spherical Bessel fn.

$$\langle |a_\ell^m|^2\rangle = \frac{H_0^4}{2\pi}\int_0^\infty dk\,k^{-2}P_\delta(k)\,j_\ell^2(kr_0)$$

rotational invariant coefficients:

$$Q_\ell^2 \equiv \sum_{m=-\ell}^{\ell}|a_\ell^m|^2$$

$$\langle Q_\ell^2\rangle = (2\ell+1)\langle |a_\ell^m|^2\rangle$$

$\ell = 1$ dipole

$\ell = 2$ quadrupole

$\ell = 3$ octupole

....

observed angular correlation function

$$w(\alpha\,|\,\vec{x}_\theta) \equiv \int_{4\pi}\frac{d\Omega_{\gamma_1}}{4\pi}\int_{4\pi}\frac{d\Omega_{\gamma_2}}{2\pi}\,\delta_\Delta(\cos\alpha - \hat{\gamma}_1\cdot\hat{\gamma}_2)\cdot \frac{\Delta T}{T}(\vec{x}_0,\hat{\gamma}_1)\,\frac{\Delta T}{T}(\vec{x}_0,\hat{\gamma}_2)$$

suitable windfun

$$\Rightarrow\; w(\alpha\,|\,\vec{x}_\theta) = \frac{1}{4\pi}\sum_{\ell\geq 2}^{\infty} P_\ell(\cos\alpha)\,\mathcal{Q}_\ell^2(\vec{x}_\theta)\,W^2(\ell)$$

Legendre P.

multipole coeff. observ. dependent

Any theory predicts $\langle w(\alpha)\,|\,\vec{x}_\theta\rangle_{\vec{x}_\theta}$ and possibly its probability distribution (functional

[illegible] $C(\alpha) \equiv T_0^2\, w(\alpha\,|\,us)$

If the density fluct. power-spectrum is scale-free $P_\delta(k) \propto k^n$ (n = spectral index)

$$\Rightarrow\; \mathcal{Q}_\ell^2 = 4\pi\, Q_{rms}^2\,\frac{(2\ell+1)}{5}\,\frac{\Gamma(\ell+(n-1)/2)\,\Gamma((9-n)/2)}{\Gamma(\ell+(5-n)/2)\,\Gamma((3+n)/2)}$$

[illegible]

Smoot et al. get the best fit

$n = 1.1 \pm 0.5$ $\qquad Q_{rms} = 16.7 \pm 4\,\mu K$

Summary of 2-year COBE-DMR data

[Bennett et al. '94]

$$\begin{cases} \Delta T_{rms}(7^\circ) = 44 \pm 7\ \mu K \\ \Delta T_{rms}(10^\circ) = 30.5 \pm 2.7\ \mu K \end{cases}$$ 53 GHz

$C(0)^{1/2} = 36 \pm 5\ \mu K$ (68% cl) cross-correlated 53 × 90 GHz

$$\begin{cases} Q_{rms-PS} = 12.4^{+5.2}_{-3.3}\ \mu K & (68\%\ cl) \\ n = 1.59^{+0.49}_{-0.55} & '' \end{cases}$$

excluding the quadrupole:

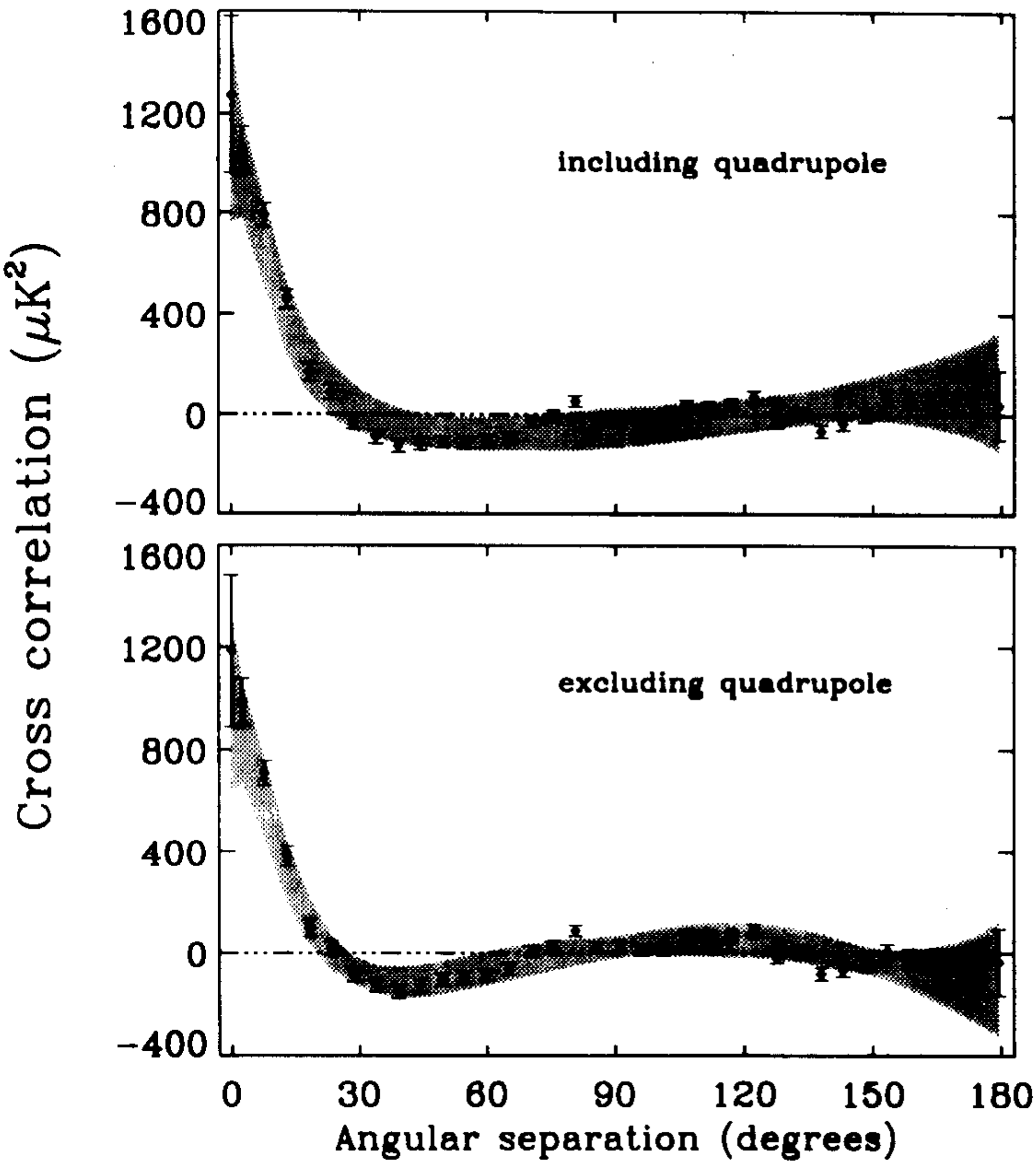
$$\begin{cases} Q_{rms-PS} = 16.0^{+7.5}_{-5.2}\ \mu K \\ n = 1.21^{+0.60}_{-0.55} \end{cases} \quad (68\%\ cl)$$

if $n = 1 \Longrightarrow Q_{rms-PS} = 18.2 \pm 1.6\ \mu K$ ($''$)

unbiased n value $\longrightarrow n = 1.1 \div 1.3$

dipole 3.363 ± 0.024 mK

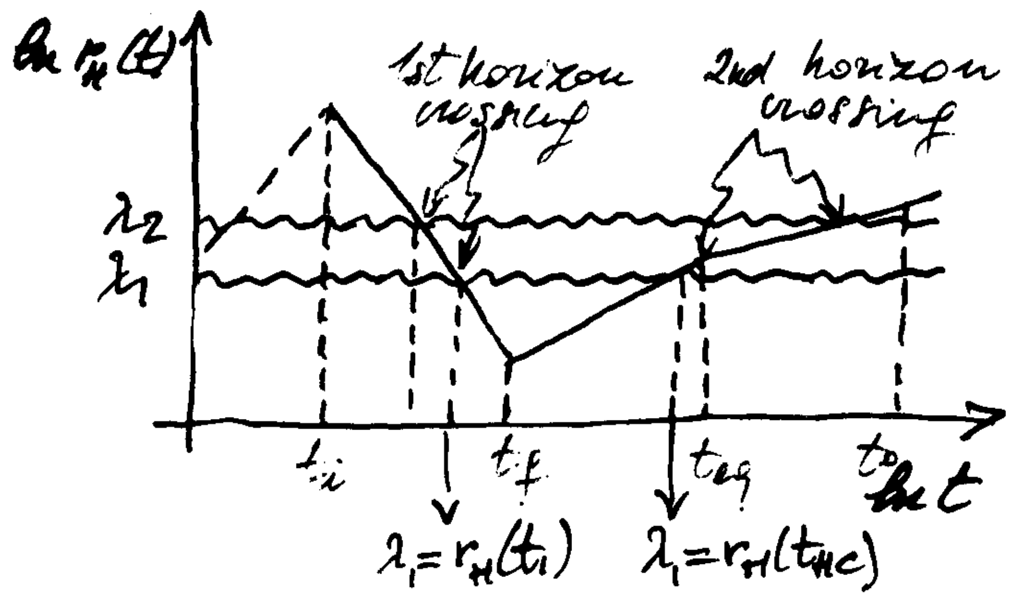
quadrupole $6 \pm 3\ \mu K$ (68% cL)



[from: Bennett et al. '94]

Power-spectrum due to the inflationary dynamics

NO-HAIR THEOREM: any perturbation preexisting inflation on scales $\lambda < r_H(t_i)$ is washed out during the first e-foldings of inflation



A perturbation of scale λ_1 is created by causal processes inside the horizon ($\lambda < r_H$) during inflation, leaves the horizon at t_1 such that $\lambda = r_H(t_1)$, keeps "frozen" outside the horizon and re-enters at time t_{HC} $r_H(t_{HC})$. After t_{HC} it will be subject to linear and then non-linear growth by self-gravity (depending on the scale...)

Linear Perturbation Theory

- First-order ("linear") perturbations of the RW metric and of its source (stress-energy tensor).

- Linear theory applies before the action of gravitational instability has lead to non-linear behaviour (for those quantities which are interested by this phenomenon, and/or on suitably large scales ($\gtrsim 8h^{-1}$Mpc).
Cosmological perturbations can be classified as: (take $K=0$ for simplicity)

SCALARS: i.e. linear superpositions of

$$Q_{\vec{k}}^{(0)}: \quad \nabla^2 Q_{\vec{k}}^{(0)} + k^2 Q_{\vec{k}}^{(0)} = 0$$ scalar harmonics

VECTORS: linear superpositions of

$$Q_{\vec{k}}^{(1)\alpha}: \quad \nabla^2 Q_{\vec{k}}^{(1)\alpha} + k^2 Q_{\vec{k}}^{(1)\alpha} = 0;$$ vector harmonics

$$\vec{\nabla}\cdot\vec{Q}_{k}^{(1)} = 0$$

TENSORS: linear superpositions of

$$Q_{\vec{k}}^{(2)\alpha\beta}: \quad \nabla^2 Q_{\vec{k}}^{(2)\alpha\beta} + k^2 Q_{\vec{k}}^{(2)\alpha\beta} = 0;$$ tensor harmonics

$$\nabla_\alpha Q_{\vec{k}}^{(2)\alpha\beta} = 0; \quad Q_{\vec{k}}^{(2)\alpha\beta} = Q_{\vec{k}}^{(2)\beta\alpha}; \quad Q_{\vec{k}}^{(2)\alpha}{}_{\alpha} = 0$$

(rank 2)

Perturbation modes

$$ds^2 = a^2(\tau)\left[\tilde{g}_{00}d\tau^2 + 2\tilde{g}_{0\alpha}d\tau dx^\alpha + \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta\right]$$

$d\tau = dt/a(t)$ the "conformal" time

RW: $ds^2 = a^2(\tau)\left[d\tau^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta\right]$
$(k=0)$

DEFINITIONS (perfect fluid)

GEOMETRY

$$\delta\tilde{g}_{00} = 2A(\tau)Q^{(0)}$$

$$\delta\tilde{g}_{0\alpha} = \left[B^{(0)}(\tau)Q^{(0)}_\alpha + B^{(1)}(\tau)Q^{(1)}_\alpha\right]$$

$$\rightsquigarrow Q^{(0)}_\alpha = -\frac{1}{k}\partial_\alpha Q^{(0)}$$

$$\delta\tilde{g}_{\alpha\beta} = -\left[2H_L(\tau)Q^{(0)}\delta_{\alpha\beta} + 2H_T^{(0)}Q^{(0)}_{\alpha\beta} + 2H_T^{(1)}Q^{(1)}_{\alpha\beta} + 2H_T^{(2)}Q^{(2)}_{\alpha\beta}\right]$$

$$\begin{cases} Q^{(0)}_{\alpha\beta} = \frac{1}{k^2}Q^{(0)}_{,\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}Q^{(0)} \\ Q^{(1)}_{\alpha\beta} = -\frac{1}{2k}\left(Q^{(1)}_{\alpha,\beta} + Q^{(1)}_{\beta,\alpha}\right) \end{cases}$$

MATTER

$$v^\alpha = v^{(0)}(\tau)Q^{(0)\alpha} + v^{(1)}(\tau)Q^{(1)\alpha}$$

$$\frac{\delta\rho}{\rho} = \delta(\tau)Q^{(0)}$$

$$\delta T^\alpha{}_\beta = p\left[\pi_L(\tau)Q^{(0)}\delta^\alpha_\beta + \text{imperfections}\right]$$

entropy perturb. $\eta(\tau) = \pi_L - \frac{c_s^2}{w}\delta$ $c_s^2 \to$ speed of sound $= \left.\frac{dp}{d\rho}\right|_\sigma$; $w \to p/\rho$

SCALARS

4 geometrical modes $A, B^{(0)}, H_L, H_T^{(0)}$

3 matter " $v^{(0)}, \delta, \pi_L$ ← density fluctuations

VECTORS

2 geometrical modes $B^{(1)}, H_T^{(1)}$

1 matter mode $v^{(1)}$ ← vorticity

TENSORS

1 geometrical mode $H_T^{(2)}$ ← gravitational waves

GAUGE TRANSFORMATIONS

$$\begin{cases} \delta\tau = T(\tau)\mathcal{Q}^{(0)} \\ \delta x^{\alpha} = L^{(0)}(\tau)\mathcal{Q}^{(0)\alpha} + L^{(1)}(\tau)\mathcal{Q}^{(1)\alpha} \end{cases}$$

[e.g. Bardeen '80]

⇒ map allotting physical to background space

⇒ # INDEPENDENT MODES

SCALARS $= 7 - 2 = 5$

VECTORS $= 3 - 1 = 2$

TENSORS $= 1 - 0 = 1$

gauge choice

gauge choices (scalar case)

synchronous gauge $A = B^{(0)} = 0$

longitudinal gauge $H_T^{(0)} = B^{(0)} = 0$

comoving proper time g. $A = v^{(0)} = 0$

" time-orthogonal g. $B^{(0)} = v^{(0)} = 0$

gauge-indep. quantities (scalar)

$$\Phi_H = H_L + \frac{1}{3} H_T^{(0)} + \frac{1}{k}\frac{a'}{a} B^{(0)} - \frac{1}{k^2}\frac{a'}{a} H_T^{\prime(0)}$$

$$' \equiv \frac{d}{d\eta}$$

$$v_s^{(0)} = v^{(0)} - \frac{1}{k} H_T^{\prime(0)}$$

$$\epsilon_m \equiv \delta + 3(1+w)\frac{1}{k}\frac{a'}{a}\left(v^{(0)} - B^{(0)}\right)$$

$$\eta = \frac{1}{w}\left(w \Pi_L - c_s^2 \delta\right)$$

vector $\Psi = B^{(1)} - \frac{1}{k} H_T^{(1)\prime}$

$v_s^{(1)} = v^{(1)} - \frac{1}{k} H_T^{(1)\prime}$ ← shear

$v_c = v^{(1)} - B^{(1)} = v_s^{(1)} - \Psi$ ← vorticity

tensor already g.i.

$k^2\Phi_H = a^2\rho\epsilon_m\, 4\pi G$ $\qquad (8\pi G = 1)$ $\qquad$ ④

$\boxed{\text{perfect fluid} \quad \pi_T^{(0)} = 0}$

$$v_S'^{(0)} + \frac{a'}{a} v_S^{(0)} = -k\Phi_H + k(1+w)^{-1}(c_s^2\epsilon_m + w\eta)$$

$$\boxed{(\rho a^3\epsilon_m)'' + (1+3c_s^2)\frac{a'}{a}(\rho a^3\epsilon_m)' + [k^2c_s^2 - 4\pi G(p+\rho)a^2] \times (\rho a^3\epsilon_m) = -k^2\rho a^3\eta w \quad (+\ \pi_T^{(0)} \ldots)}$$

η, π_T = source functions

Vector

$$\begin{cases} k^2\Psi = 4\pi G\, a^2(p+\rho)v_c \\ v_c' + \frac{a'}{a}v_c - 3\frac{a'}{a}c_s^2 v_c = \left(-k\frac{w}{1+w}\pi_T^{(1)}\right) = 0 \end{cases} \Rightarrow \boxed{a^4(\rho+p)v_c = \text{const}}$$

Tensor

$$H_T^{(2)''} + 2\frac{a'}{a}H_T^{(1)'} + k^2 H_T^{(2)} = \left(a^2 p\, \pi_T^{(2)}\right) = 0$$

Solutions $\qquad j_n(z) \equiv \sqrt{\pi/2z}\, J_{n+1/2}(z)$

$a \propto \tau^\beta \qquad \beta = 2/(3w+1) \qquad y_n(z) = \sqrt{\pi/2z}\, N_{n+1/2}(z)$

$x \equiv k\tau \quad ; \; f \equiv x^{\beta-2}\epsilon_m$

$\pi_T = \eta = 0$

$$f'' + \frac{2}{x}f' + [c_s^2 - \beta(\beta+1)x^{-2}]f = 0$$

$$f = a\, j_\beta(c_s x) + b\, y_\beta(c_s x)$$

$$\Rightarrow \epsilon_m \simeq cx^2 + dx^{-2\beta+1} \qquad c_s x \ll 1$$

$$\boxed{H_T^{(2)\prime\prime} + \frac{2\beta}{\tau} H_T^{(2)\prime} + k^2 H_T^{(2)} = 0}$$

$k\tau \gg 1 \qquad H_T^{(2)} \propto \exp(\pm i k \tau)$

$k\tau \ll 1 \qquad H_T^{(2)} \propto \begin{cases} \text{const} \\ \tau^{1-2\beta} \end{cases}$ decaying mode

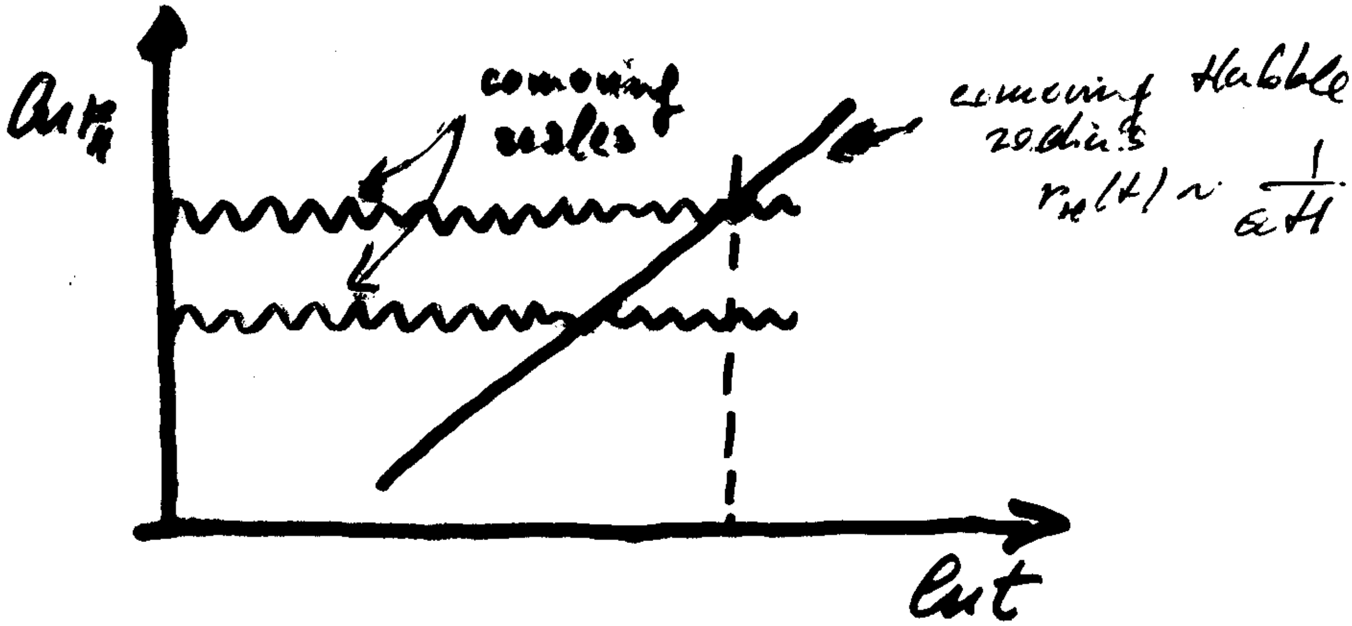
~~$2\beta = 1$~~

e.g. $w = 0 \quad \beta = 2 \quad \rightarrow \quad \tau^{-3} \propto t^{-1}$

$w = 1/3 \quad \beta = 1 \quad \rightarrow \quad \tau^{-1} \propto t^{-1/2}$

$(2\beta = 1 \iff w = 1)$

Slope of perturbations



$P(k,t) \equiv \langle \delta_k^2 \rangle$ = power-spectrum

constant time: $P(k,t) \propto k^{n}$ primordial spectral index

horizon crossing: $P(k, t_H(k)) \propto k^{2\alpha-3}$

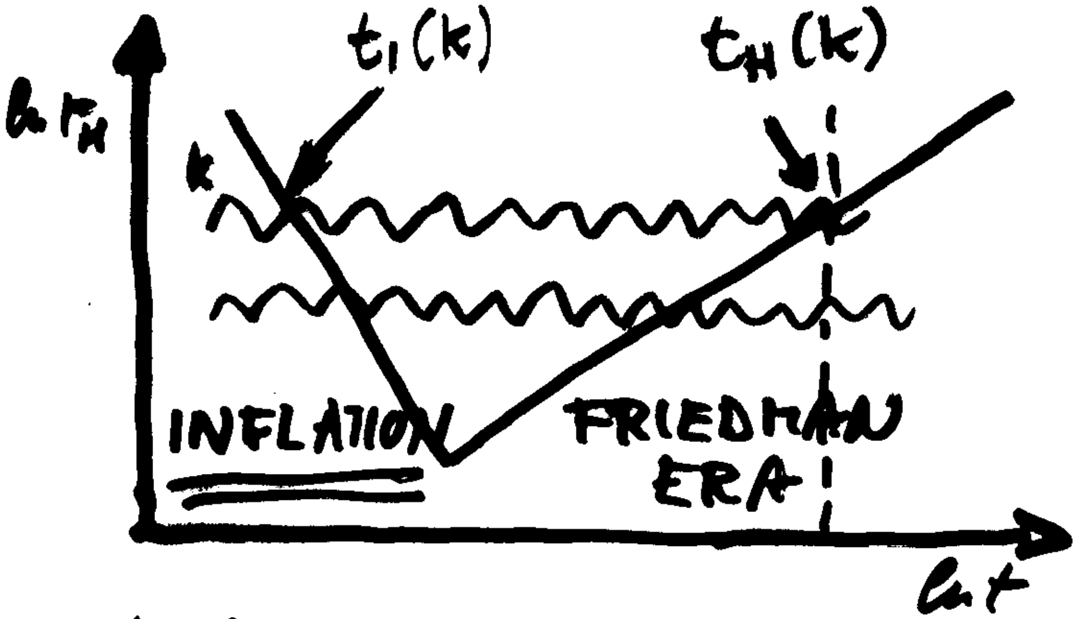
(before action of transfer function)

$$n = 2\alpha + 1$$

COBE: $n = 1.1 \pm 0.5$

$\begin{cases} n = 1 \\ \alpha = 0 \end{cases} \equiv$ "Zel'dovich spectrum"

... with inflation



$$k^{3/2}\, P^{1/2}(k, t_H(k)) \approx \left.\frac{H^2}{\dot{\varphi}}\right|_{t_1(k)}$$

$\varphi \equiv$ INFLATON

$$\begin{cases} \dot{\varphi} \simeq -\dfrac{V'}{3H} \\ H^2 \simeq \dfrac{8\pi G}{3} V \\ \dot{H} = -4\pi G\, \dot{\varphi}^2 \end{cases}$$

INFLATON DYNAMICS (SLOW ROLLING)

RED spectra : "tilt"

$$n < 1 \Leftrightarrow \alpha < 0$$

more power on large scales

common from inflation :

$\Rightarrow \dot{H} < 0$ & $\dot{\varphi}$ driven by V' (slope)

e.g. exponential potential
Lucchin & Matarrese '85

$$\begin{cases} V(\varphi) = M^4 \exp(-\lambda\varphi/\sigma) \\ \sigma \equiv m_P/\sqrt{8\pi} \; , \; \lambda < \sqrt{2} \\ \alpha = -\frac{\lambda^2}{2}\left(1 - \frac{\lambda^2}{2}\right) < 0 \end{cases}$$

power-law inflation

exponential-like potentials are common in

extended inflation La & Steinhardt '89 ...

natural inflation Adams et al. '93 , ...
(actually $\cos(k\varphi)$)

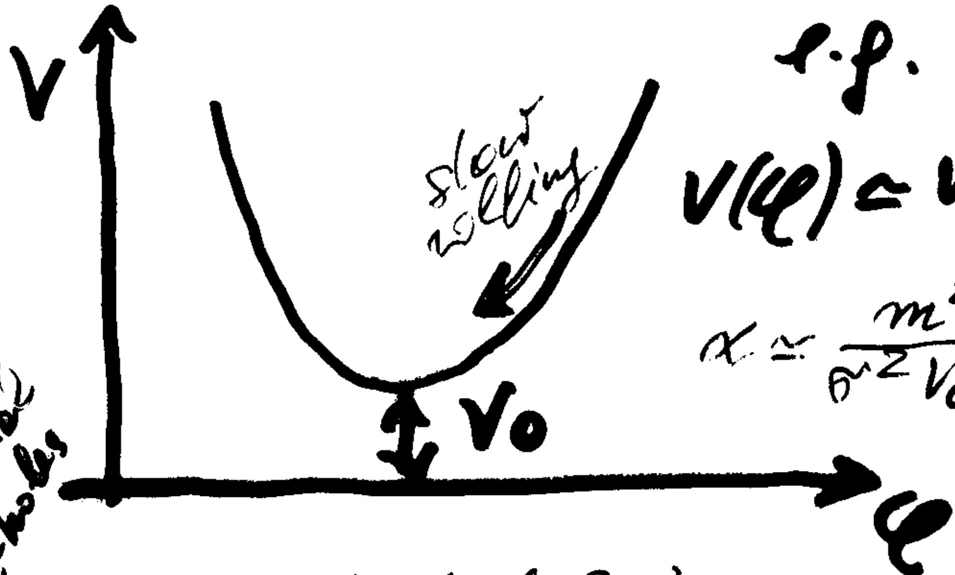
more power on small scales

BLUE spectra

$$n>1 \longleftrightarrow \alpha>0$$

$\dot H > 0$?? FORBIDDEN WITH MINIMALLY COUPLED INFLATON

$\dot H < 0$ but $\dot\varphi$ decreases more rapidly than H^2, i.e. $\dot\varphi$ driven by an excess vacuum energy



e.g.

$$V(\varphi) \simeq V_0 + \frac{m^2}{2}\varphi^2$$

$$\alpha \simeq \frac{m^2}{\pi^2 V_0} > 0$$

primordial black-holes

Lidsey et al. '93 } examples of α = const
Carr et al. '93
Mollerach, Lucchin & S.M. '93

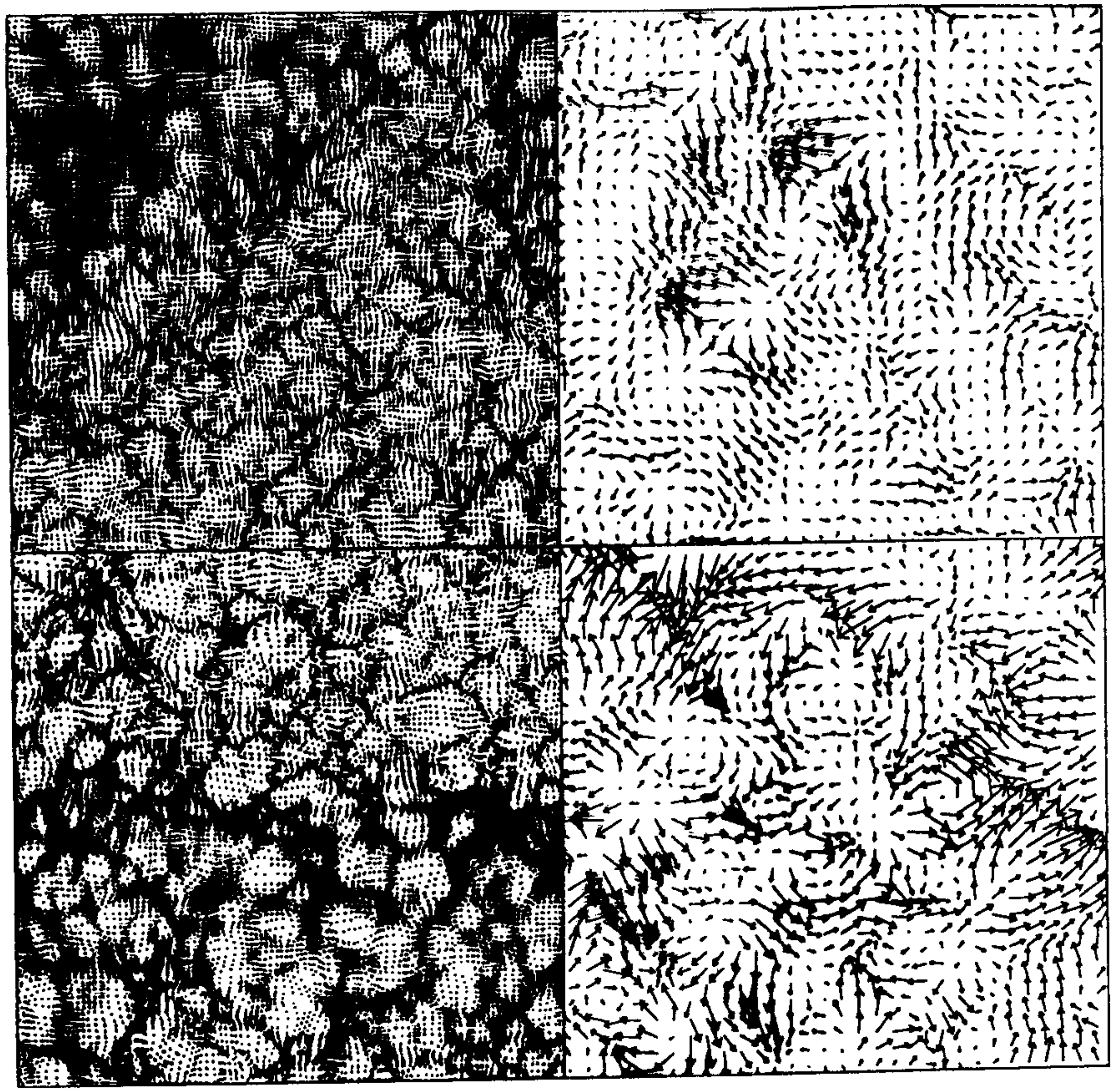
related to hybrid inflation
Linde et al. '93
Wands et al. '93

CFA voids $\rightarrow n \simeq 1.25$ (Blumenthal et al. '93)

CDM (COBE normalized)

n=0.6
b=2.5
TCDM

n=0.8
b=1.5
TCDM



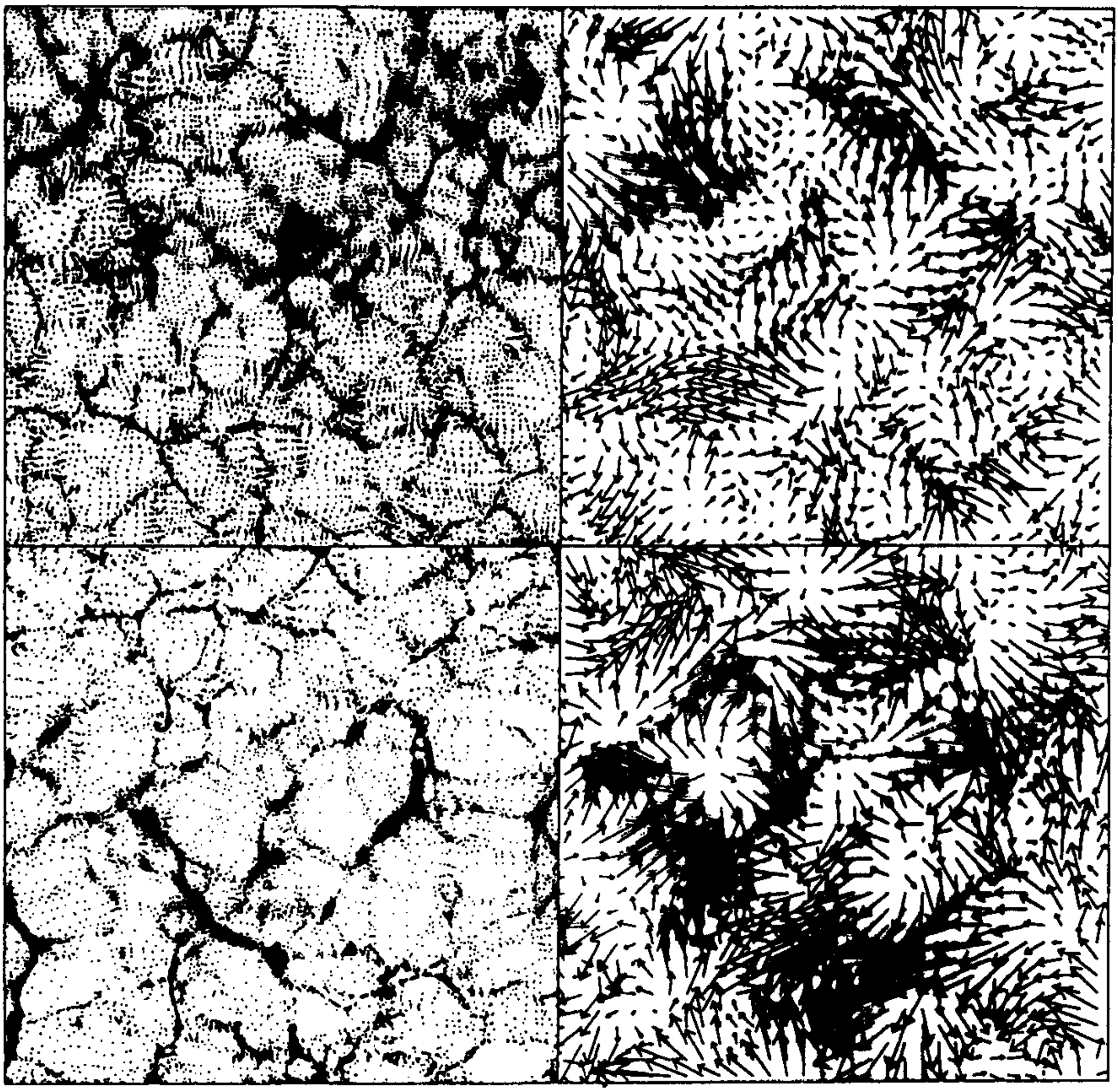
[from: Moscardini et al. '94]

Fig. 1a

CDM (COBE normalised)

$n=1$
$b=1$
SCDM

$n=1.2$
$b=0.5$
BCDM



[from Moscardini et al. '94]

Fig 1b

Gravitational Waves from Inflation

(original idea due to Starobinsky 79 (!))

Each polarization state of the graviton behaves like a (minimally coupled) free, massless scalar field ⇒ during inflation it fluctuates with the same amplitude

A stochastic background of gravitational waves is produced which keeps constant (in amplitude) outside the horizon (just like the Newtonian potential for scalar perturbations)

Inside the horizon gravitational waves are red-shifted and diluted just like free photons (because there is <u>no</u> <u>gravitational instability for tensor modes</u>)

⇒ the largest abundance of gravitational waves is to be found on the largest observable scales

⇒ Sachs-Wolfe effect

(...)

gravitational-wave "bias"

$b =$ normalization

\# Every inflation model simultaneously provides both density (scalar mode) and gravitational (tensor mode) waves, in a well-defined mixture

$$\left.\begin{array}{l} \dfrac{\delta\rho}{\rho} \sim \dfrac{H^2}{\dot{\varphi}} \\[2ex] \delta h \sim \dfrac{H}{m_P} \end{array}\right\} \begin{array}{l} S/T \sim \dfrac{2}{5}\sqrt{\dfrac{1}{6}+\dfrac{1}{3\epsilon}} \\[2ex] \epsilon \equiv \dot{\varphi}^2/V(\varphi) < 1 \end{array}$$

\# In **de Sitter inflation** (e^{Ht}), leading to the scale-invariant ($n=1$) spectrum, $\epsilon \ll 1$, gravitational waves are generally sub-dominant

\# In **power-law inflation** (t^p, $p>1$), leading to tilted spectra ($n<1$), and arising from an exponential potential, gravitational waves are relevant

$$S/T = \frac{\sqrt{2p}}{5}$$

Davis et al. '92
Lucchin, S.M. Mollerach '92
Dolgov & Silk '93
Coles & Lidsey '93
Salopek '92
Liddle & Lyth '92

The gravitational-wave contribution to large-scale $\Delta T/T$ can be substantial (and even dominant) for tilted models ($n<1$).

The g.w. effect leads to a decrease of the mass-fluctuation level for $n<1$, i.e. to an increase of the "bias factor", b, which helps in reconciling CDM with small-scale observations (Davis & Peebles '85) but causes troubles on $\gtrsim 10^2$ Mpc scales (e.g. bulk motions)

Given the $n<1$ model (?) of inflation COBE would imply g.w. detection.

How to distinguish $\Delta T/T$ from g.w. or $\delta\rho/\rho$?

more detailed calculations:
Crittenden et al. '93
Lidsey, Turner & White '93

LINK TO OBSERVATIONS

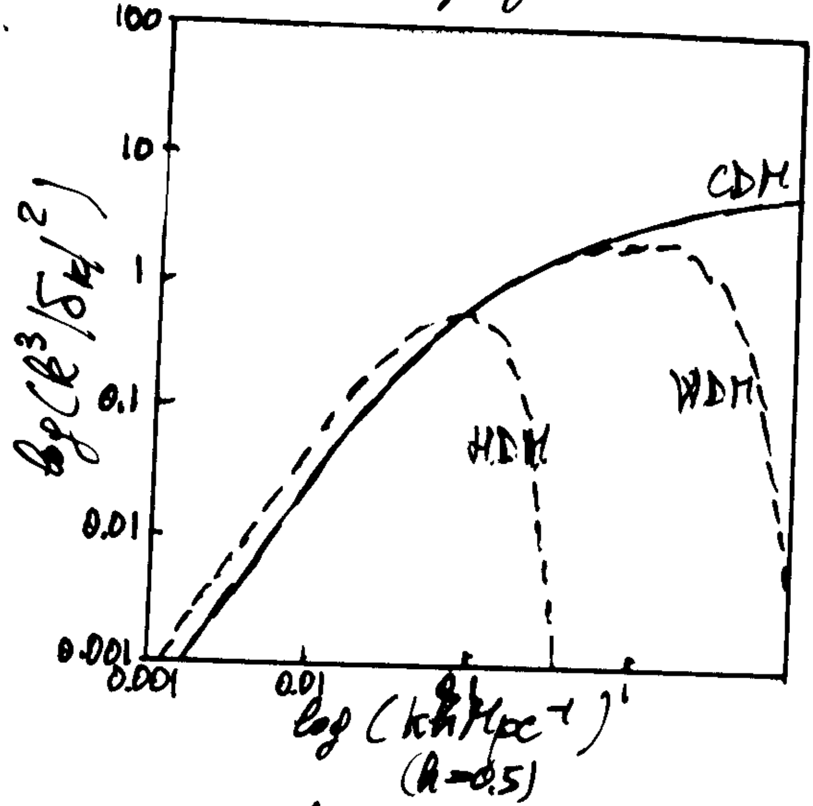
• DENSITY FLUCTUATIONS	• GRAVITATIONAL WAVES
$\Delta T/T$ COBE, SP, $<1^\circ$,...	$\Delta T/T$ COBE
GALAXY FORMATION	STABILITY ms PSR
CLUSTERING	NUCLEOSYNTHESIS
PECULIAR MOTIONS	LIGO, ...

Origin of structures

$P_\delta(k) = A k T^2(k)$ linear power-spectrum of density fluctuations

normalization $\propto 1/k$

transfer function depends on the type of dark matter



CDM model: $P_\delta(k) \sim \begin{cases} k & \lambda \gg \lambda_{eq} \\ k^{-3} & \lambda \ll \lambda_{eq} \end{cases}$

$\lambda_{eq} \approx 13(\Omega_0 h^2)^{-1}$ Mpc $\equiv$ horizon scale at the equivalence

$\Rightarrow$ with CDM structures form from the small to the large scale BOTTOM $\Rightarrow$ UP by the gravitational instability mechanism. small-scale structures merge into larger ones (.. galaxies $\to$ galaxy clusters $\to$ superclusters)

Dynamics of a self-gravitating collisionless fluid [Newtonian approx.]

$(\Omega_0 = 1)$

$$* \quad \boxed{\frac{d\vec{v}}{dt} + 2\frac{\dot{a}}{a}\vec{v} = -\frac{1}{a^2}\vec{\nabla}\phi}$$

Euler eq.s

$\vec{v}(\vec{x},t) = \dot{\vec{x}}$ comoving peculiar velocity field

$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_x$ total time derivative

$a(t) = a_{in}\left(\frac{t}{t_{in}}\right)^{2/3}$ scale-factor (matter dominance, $\Omega_0 = 1$)

$$* \quad \boxed{\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho + \rho\vec{\nabla}\cdot\vec{v} = 0}$$

Continuity eq.

ρ mass-density

$$* \quad \boxed{\nabla^2\phi = \frac{3}{2}\dot{a}^2\delta}$$

Poisson eq.

$\delta \equiv \rho/\rho_b - 1$ density fluctuation

ϕ "peculiar" gravitational potential

→ N-body methods numerically solve this [illegible] set of equations useful rescalings:

time $\quad t \longrightarrow \tau \equiv a(t) - a_{in}$

velocity $\quad \vec{v} \longrightarrow \vec{u} \equiv \dfrac{d\vec{x}}{d\tau} = \vec{v}/\dot{a}$

density $\quad \rho \longrightarrow \eta \equiv 1+\delta$ (comoving density)

gravitat. potential $\quad \phi \longrightarrow \varphi \equiv \dfrac{3t_{in}^2}{2a_{in}^3}\phi$

(total derivative $\dfrac{d}{d\tau} = \dfrac{\partial}{\partial\tau} + \vec{u}\cdot\vec{\nabla}_x$)

lead to: **FLUID LIMIT:**

$$\frac{d\vec{u}}{d\tau} + \frac{3}{2a}\vec{u} = -\frac{3}{2a}\vec{\nabla}\varphi \qquad \text{Euler}$$

$$\frac{d\eta}{d\tau} + \eta\,\vec{\nabla}\cdot\vec{u} = 0 \qquad \text{Continuity}$$

$$\nabla^2\varphi = \frac{\delta}{a} \qquad \text{Poisson}$$

easily generalizable to open universe (e.g. Gramann '92)

ZEL'DOVICH APPROXIMATION

[Zel'dovich 1970]

Neglecting the terms $(\vec{u}\cdot\vec{\nabla})\vec{u}$ and $\vec{\nabla}\cdot(\delta\vec{u})$ one gets the linear approximation

$$* \begin{cases} \vec{u}_{LIN}(\vec{x},\tau) = -\vec{\nabla}\varphi_{LIN}(\vec{x},\tau) \\ \varphi_{LIN}(\vec{x},\tau) = \varphi_0(\vec{x}) \\ \eta_{LIN}(\vec{x},\tau) = 1 + a\nabla^2\varphi_0(\vec{x}) \end{cases}$$

LINEAR THEORY

[growing mode only]

Extrapolating * beyond **LIN** one gets the Zel'dovich ansatz:

$$\boxed{\begin{aligned} &\frac{d\vec{u}}{d\tau} = 0 \\ &\frac{d\eta}{d\tau} + \eta\vec{\nabla}\cdot\vec{u} = 0 \end{aligned}}$$

<u>Zel'dovich approximation</u>

particles move along straight lines with constant speed $\vec{u}$ determined by initial peculiar gravitational potential in $\vec{q}$

trajectory

$$\vec{x}(\vec{q},\tau) = \vec{q} - \tau \vec{\nabla}_q \varphi_0(\vec{q})$$

velocity

$$\vec{u}(\vec{x},\tau) = \vec{u}_0(\vec{q}) = -\vec{\nabla}_q \varphi_0(\vec{q})$$

velocity potential, satisfies

$$\boxed{\frac{\partial \Phi}{\partial \tau} + \frac{1}{2}(\vec{\nabla}_x \Phi)^2 = 0}$$

Zel'dovich - Bernoulli eq.

solved by $\Phi(\vec{x},\tau) = \Phi_0(\vec{q}) + \dfrac{(\vec{x}-\vec{q})^2}{2\tau}$

density: let $\alpha_i(\vec{q})$ be the eigenvalues of the deformation tensor $D_{0,ij}(\vec{q}) = \dfrac{\partial^2 \Phi_0(\vec{q})}{\partial q_i \partial q_j}$, then, from $\eta(\vec{x},\tau)d^3x = \eta_0(\vec{q})d^3q$,

$$\eta(\vec{q},\tau) = \frac{\eta_0(\vec{q})}{(1+\tau\alpha_1(\vec{q}))(1+\tau\alpha_2(\vec{q}))(1+\tau\alpha_3(\vec{q}))}$$

Lagrangian expression

a singularity (caustic) forms in Lagrangian space at every point $\vec{q}$, where at least one eigenvalue (say α_1) is negative

$\Rightarrow$ shell-crossing

- a Lagrangian region where only one $\alpha_i < 0$ collapses along one axis to a **pancake**
- one for which two $\alpha_i < 0$ and one ≥ 0 collapses to a **filament**
- one where all $\alpha_i < 0$ collapses to a **knot**

ZEL is a fairly good approximation up to the time of first shell-crossing. After this time the mutual attraction between neighbouring particles cannot be neglected anymore. Subsequent evolution is dominated by merging, cannibalism, fragmentation, ..., not described by ZEL

NOTE if one uses $\vec{u} = -\vec{\nabla}\varphi$ into Poisson $\Rightarrow$

$$\eta_{DYN} = 1 - a\vec{\nabla}\cdot\vec{u} \quad (\text{like in } LIN)$$

if replaced into the continuity eq. $\Rightarrow$

$$\eta_{DYN}(\vec{q},\tau) = \frac{\eta_0(\vec{q})}{1-\tau\delta_+(\vec{q})} = \frac{\eta_0(\vec{q})}{1+\tau(\alpha_1(\vec{q})+\alpha_2(\vec{q})+\alpha_3(\vec{q}))}$$

$\hookrightarrow \bar{\delta}_c / a_c$

ZEL is exact up to shell-crossing for one-dimensional (i.e. plane-symmetric) initial perturbations : $\varphi_0(\vec{q}) \equiv \varphi_0(z)$

self-similar solution

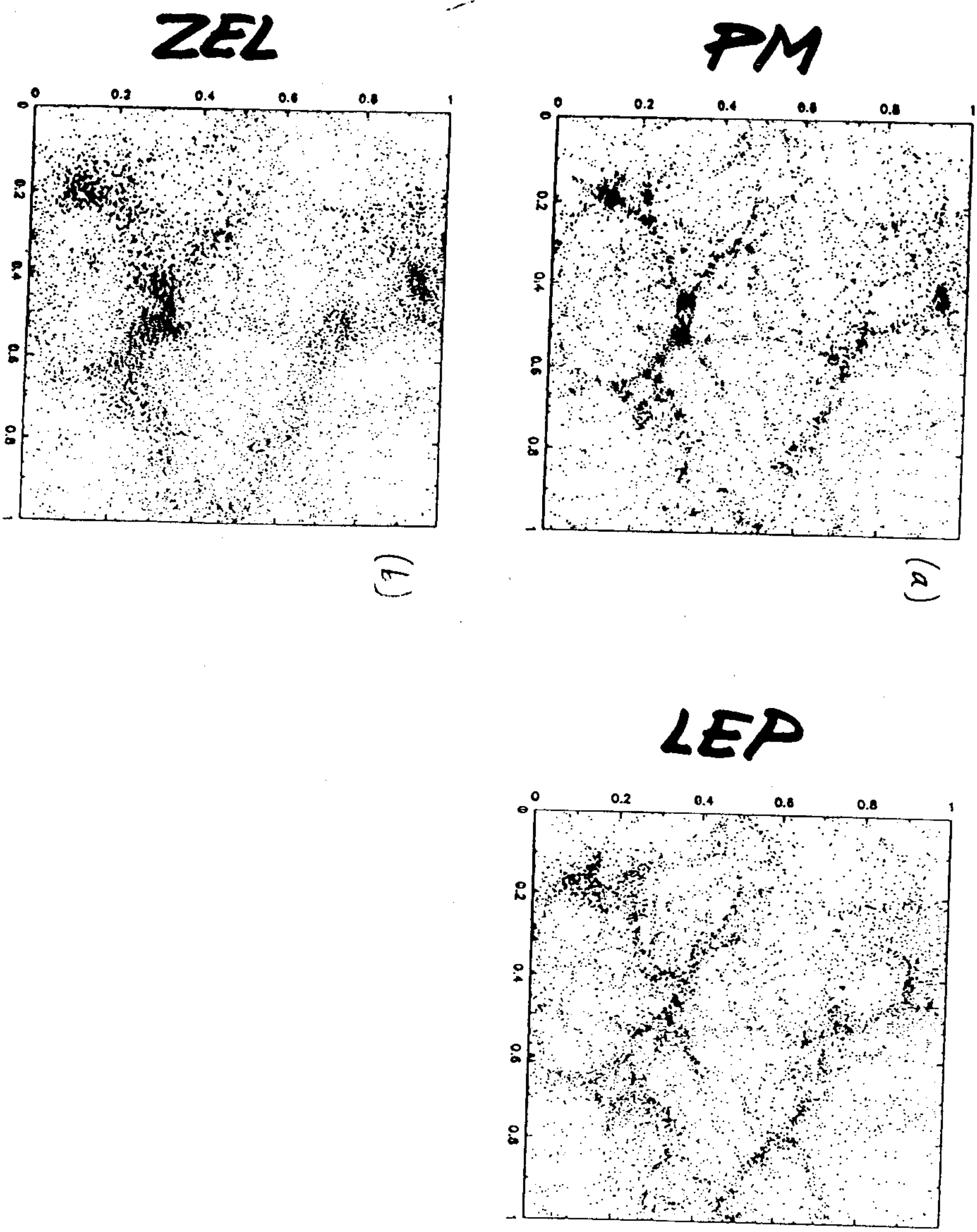


Figure 3

BSI '92