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INTRODUCTION TO FUNCTIONAL METHODS, GAUGE THEORIES AND QUANTIZATION

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Please note: These are preliminary notes intended for internal distribution only.

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Gauge Theories

Covariant derivative: $\nabla_\mu(A)\psi = (\partial_\mu + \underbrace{i t_\alpha A_{\alpha\mu}}_{\hat{A}})\psi$

$\psi^\Omega = \Omega\psi$ where $\Omega = e^{i t_\alpha \omega_\alpha(x)}$ ← gauge transf.

require $\nabla(A^\Omega)\circ\psi^\Omega = \Omega\cdot\nabla(A)\psi$

$$\Rightarrow \quad \hat{A}^\Omega = \Omega\hat{A}\Omega^{-1} + \Omega\partial\Omega^{-1}$$

infinitesimal $\quad \delta\hat{A} = [\hat{\omega}, \hat{A}] - \partial\hat{\omega}$

$$\Rightarrow \delta A_\alpha = i\omega_\gamma(\theta_\gamma)_{\alpha\beta}A_\beta - \partial\omega_\alpha = -(\partial + i\theta_\beta A_\beta)\omega \equiv -\nabla(A)\omega$$

$\delta\psi = i\omega_\gamma t_\gamma\psi$

$[t_\alpha t_\beta] = i f_{\alpha\beta\gamma}t_\gamma \qquad \mathrm{Tr}(t_\alpha t_\beta) = \frac{1}{2}\delta_{\alpha\beta}$

$$f_{\alpha\beta\gamma} = i(\theta_\gamma)_{\alpha\beta}$$

$$\hat{F}_{\mu\nu} = [\nabla_\mu(A), \nabla_\nu(A)] \Rightarrow \hat{F}^\Omega = \Omega F\Omega^{-1}$$

$$\Rightarrow \mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\,\hat{F}^2 + \bar{\psi}(i\gamma\nabla(A) - m)\psi$$

<u>invariant</u>

Quantization:

$$Z = N \int \mathcal{D}A\, \mathcal{D}\bar\Psi\, \mathcal{D}\Psi\, e^{iS(A\bar\Psi\Psi)}$$

propagator from $\mathcal{L}(A) = -\frac{1}{4}(k_\mu A_\nu - k_\nu A_\mu)^2 =$

$$= -\tfrac{1}{2} k^2 A_\mu \underbrace{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}_{P^\perp_{\mu\nu}} A_\nu$$

note: $(\alpha P^\perp_{\mu\nu} + \beta P^L_{\mu\nu})^{-1} = \frac{1}{\alpha} P^\perp_{\mu\nu} + \frac{1}{\beta} P^L_{\mu\nu}$

but $\beta = 0$?

$\Rightarrow N \cdot \int \mathcal{D}A^L \mathcal{D}A^\perp e^{-\alpha A^{\perp 2}} = \mathrm{div}$

$\Rightarrow$ eliminate the redundant gauge degrees of freedom (absorb into normalization N)

write $\quad 1 = \int \mathcal{D}\Omega\, \delta(\partial A^\Omega - B) \cdot \Delta(A^\Omega) \quad (\forall B)$

let $\partial A^{\bar\Omega} = B \Rightarrow A^\Omega \cong A^{\bar\Omega} - \nabla(A^{\bar\Omega}) \cdot \omega \qquad \mathcal{D}\Omega = \mathcal{D}\omega$

$\Rightarrow \left.\frac{\delta \partial A^\Omega}{\delta \omega}\right|_{\bar\Omega} = -\partial \nabla(A^{\bar\Omega}) \Rightarrow \Delta(A^\Omega) = \det \partial\nabla(A^\Omega) =$

$$= \int \mathcal{D}\bar c\, \mathcal{D}c\, e^{i\int \bar c\, \partial \nabla(A^\Omega) c}$$

anticom

Thus $$Z = \underbrace{N\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\, e^{iS(A\bar\psi\psi)}}_{= \int \mathcal{D}A^\Omega\,\mathcal{D}\bar\psi^\Omega\,\mathcal{D}\psi^\Omega\, e^{iS(A^\Omega\bar\psi^\Omega\psi^\Omega)}} \mathcal{D}\Omega\,\delta(\partial A^\Omega - B)\,\mathcal{D}\bar c\,\mathcal{D}c\, e^{i\int \bar c\partial\nabla(A^\Omega)c} =$$

$$= \underbrace{\int \mathcal{D}\Omega\, N}_{N'} \cdot \int \mathcal{D}A'\,\mathcal{D}\bar\psi'\,\mathcal{D}\psi'\,\mathcal{D}\bar c\,\mathcal{D}c\, e^{iS(A'\bar\psi'\psi') + i\int \bar c\partial\nabla(A')c} \cdot \delta(\partial A' - B)$$

since true $\forall B$, integrate $\int \mathcal{D}B\, e^{\frac{i}{2\xi}\int B^2} = 1$

$$\Rightarrow\ Z = N\int \mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\,\mathcal{D}\bar c\,\mathcal{D}c\, e^{i\int \underbrace{\mathcal{L}(A\bar\psi\psi) + \frac{1}{2\xi}(\partial A)^2 + \bar c\partial\nabla(A)c}_{\mathcal{L}_{GF}}}$$

note: $\mathcal{L}_{GF}$ no longer gauge inv.

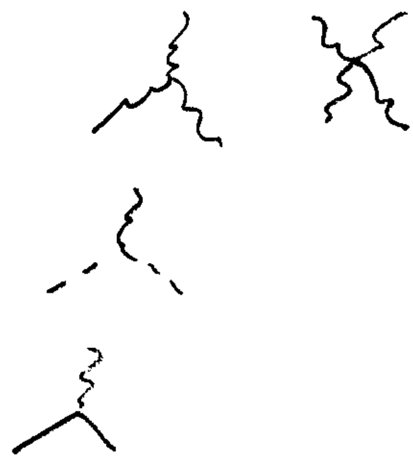
From $S_{GF}(\phi) \Longrightarrow$ F. rules from $\frac{\delta^n S(\phi)}{\delta\phi^n}$

A ~~~ A $\qquad -\frac{i}{k^2}\left(g_{\mu\nu} - (\xi-1)\frac{k_\mu k_\nu}{k^2}\right)$

c - - - $\bar c$ $\qquad \frac{i}{k^2}$

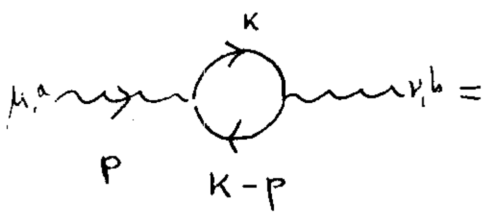
ψ —— $\bar\psi$ $\qquad \frac{i}{\gamma k - m}$

and vertices



Computation of the Gluon wave function renormalization.

a) Contribution of the Fermions



$$= (-)\,(-ig)^2 \int \frac{d^dk}{(2\pi)^4}\, \mathrm{Tr}\left(\gamma_\mu \frac{i\gamma\cdot k}{k^2}\, \gamma_\nu \frac{i\gamma(k-p)}{(k-p)^2}\right) \mathrm{tr}(t_a t_b)$$

$$= -g^2 \int \frac{d^dk'}{(2\pi)^4} \int_0^1 dx\, \frac{\mathrm{Tr}\left(\gamma_\mu(\gamma\cdot k' + x\,\gamma\cdot p)\gamma_\nu(\gamma\cdot k' - (1-x)\gamma\cdot p)\right)}{\left[k'^2 + p^2 x(1-x)\right]^2}\, \mathrm{tr}(t_a t_b) =$$

In d-dims $\quad \{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}$

$\mathrm{Tr}(\text{odd \# of } \gamma\text{'s}) = 0 \qquad \mathrm{Tr}\, I = f(d)$

$f(4) = 4$

take $f(d) = 4$ as a normalization convention

$$\mathrm{Tr}(\cdots) = \underbrace{\mathrm{Tr}\,\gamma_\mu \gamma\cdot k' \gamma_\nu \gamma k'}_{-2k'^2\cdot g_{\mu\nu}} - \underbrace{x(1-x)}_{\frac{1}{6}}\, \underbrace{\mathrm{Tr}\,\gamma_\mu \gamma\cdot p\, \gamma_\nu \gamma p}_{8p_\mu p_\nu - 4p^2 g_{\mu\nu}}$$

$$\left(k'_\mu k'_\nu = \frac{1}{4} k'^2 g_{\mu\nu}\right)$$

$$= -g^2 i \int \frac{d^4k_E}{(2\pi)^4} \int_0^1 dx \left\{ \frac{-2k_E^2 g^E_{\mu\nu}}{\left[k_E^2 + p_E^2 x(1-x)\right]^2} - \frac{1}{6} \frac{8p_\mu p_\nu - 4p^2 g_{\mu\nu}}{\left[k_E^2 + p_E^2 x(1-x)\right]^2} \right\} \doteq$$

$k^2 g_{\mu\nu} \to k_E^2 \delta_{\mu\nu}$

$k^2 p^2 \to k_E^2 p_E^2$

$$\int d^d k_E \frac{k_E^2}{[k_E^2+H]^2} \stackrel{div}{=} \underbrace{\int d^d k_E \frac{1}{k_E^2+H}}_{-\frac{2\pi^2 H}{\varepsilon}} - H \underbrace{\int d^d k_E \frac{1}{[k_E^2+H]^2}}_{\frac{2\pi^2}{\varepsilon}} \stackrel{div}{=} \quad C2$$

$$= -\frac{4\pi^2}{\varepsilon} H$$

$$\doteq -\frac{i g^2}{16\pi^2}\left\{ 8 g_{\mu\nu} p^2 \underbrace{\int_0^1 dx\, x(1-x)}_{\frac{1}{6}} - \frac{2}{6}\left(8 p_\mu p_\nu - 4 p^2 g_{\mu\nu}\right)\right\}\frac{1}{\varepsilon}\, \mathrm{tr}(t_a t_b)$$

$$= -\frac{i g^2}{16\pi^2}\frac{8}{3}\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\frac{\delta_{ab}}{2} N_f \frac{1}{\varepsilon} = -i\Sigma \qquad (*)$$

$$\Sigma = \ldots$$

$$i\mathcal{L} = -i Z_3 \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\frac{A^a_\mu A^b_\nu}{2}\delta_{ab}$$

in order to cancel the $\frac{1}{\varepsilon}$: $\Sigma + (Z_3 - 1)(-)(p^2 g_{\mu\nu} - p_\mu p_\nu) =$

$$= \left(\frac{g^2}{16\pi^2}\frac{8}{3}\frac{1}{2}N_f\frac{1}{\varepsilon} + 1 - Z_3\right)\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)$$

(Fermion contr.)

$$\Rightarrow Z_3 = 1 - \frac{g^2}{16\pi^2}\frac{4}{3}N_f\frac{1}{\varepsilon}$$

$$Z_3^{Tot} = Z_3^{(\text{Fermion contr.})} + Z_3^{(\text{Gauge field contr.})} + Z_3^{(\text{ghost contr.})}$$

(*) Note the transversality, in agreement with the Ward Identities.

Exercise: complete the computation including the mass term

To be renormalisable, a Lagrangian must have the same form as the divergent terms

Divergent terms $\Gamma^{(N)}_{(p)} \sim \frac{P(p)}{\varepsilon^n}$

$P(p)$: a polynomial of nonnegative degree by dims

$[\Gamma^{(2)}(p)] = p^2 \Rightarrow \Gamma^{(2)}_{div} \sim \frac{\alpha p^2 + \beta M^2}{\varepsilon^n}$

$[\Gamma^{(4)}(p)] = 1 \Rightarrow \Gamma^{(4)}_{div} \sim \frac{c}{\varepsilon^n}$

$\varphi = Z^{1/2}\varphi_R$

$\lambda_0 = \lambda_R \mu^\varepsilon Z_1 Z^{-2}$

$m_0^2 = m_R^2 + \delta m^2$

$$\mathcal{L} = \frac{1}{2}\underbrace{\left(Z\, p^2 - Z(m_R^2 + \delta m^2)\right)}_{\sim \Gamma^{(2)}_{div}} \varphi_R^{\,2} - \underbrace{\frac{\lambda_R}{4!} Z_1}_{\sim \Gamma^{(4)}_{div}} \varphi_R^{\,4}$$

Of course, constraints on $\Gamma^{(N)}$ from invariances (Lorentz ...)

Gauge theories since gauge "broken"

How can be sure we do not need counterterm

like then $\mathcal{L}_{GF}$?

($\mathcal{L}_{GF}$ is a <u>very</u> special polynomial)

BRS: $\delta\lambda$ global anticomm $\Rightarrow \delta\lambda^2 = 0$

$$\delta A = -\nabla(A)(\delta\lambda\, C(x)) \qquad \text{like } \omega_\alpha(x) = \delta\lambda \cdot C_\alpha(x)$$

$$\delta\psi = i(\delta\lambda\, c_\alpha)\, t_\alpha \psi$$

$$\delta C_\alpha = -\delta\lambda \tfrac{1}{2} f_{\alpha\beta\gamma}\, c_\beta c_\gamma$$

$$\delta\bar{C}_\alpha = \delta\lambda \frac{1}{\xi}\partial A_\alpha \qquad \left(\text{if } \mathcal{L}_{GF} = \mathcal{L} + \bar{c}\,\partial\nabla c + B\partial A - \tfrac{\xi}{2}B^2 \right.$$
$$\left. \text{then } \delta\bar{c} = \delta\lambda B \quad \delta B = 0\right)$$

Properties: $\delta \int \mathcal{L}_{GF} = 0$ (note $\delta\mathcal{L} = 0$ since gauge inv)

$$\delta(\nabla c) = \delta(ct\psi) = \delta(fcc) = 0$$

like $\delta_{BRS}\varphi = [Q_{BRS}, \varphi]$ with $Q^2_{BRS} = 0$.

Introducing ad hoc sources:

$$S(K) = \int\Big\{\mathcal{L}_{GF} - K^A \cdot \nabla(A)c + K^{\psi} \cdot ict\psi + \bar{\psi}(-ict K^{\bar\psi})$$
$$-\tfrac{1}{2}K^c fcc\Big\}$$

$$\delta_{BRS} S = 0$$

$$\Rightarrow \sum_\varphi \frac{\delta S}{\delta\varphi} * \frac{\delta S}{\delta K^\varphi} + \frac{\delta S}{\delta c} * \frac{\delta S}{\delta K^c} + \frac{\delta S}{\delta\bar{c}} * \frac{1}{\xi}\partial A = 0$$

Introducing also sources for Green functions

$$Z(J_\varphi \bar{\eta} \eta) = N \int \{\mathcal{D}\varphi\} \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{i\left(S + \sum_\varphi J_\varphi \varphi + \bar{\eta} c + \bar{c}\eta\right)}$$

making a BRS transfor int. variables

$$\Rightarrow \quad \left\langle \sum_\varphi J_\varphi \delta_{BRS} \varphi + \bar{\eta}\, \delta_{BRS} c + \delta_{BRS} \bar{c}\, \eta \right\rangle = 0 \qquad (*)$$

$$\Rightarrow \quad \sum_\varphi \frac{\delta \Gamma}{\delta \varphi} \cdot \frac{\delta \Gamma}{\delta K^\varphi} + \frac{\delta \Gamma}{\delta c} \cdot \frac{\delta \Gamma}{\delta K^c} + \frac{\delta \Gamma}{\delta \bar{c}} \frac{1}{\xi} \partial A = 0 \qquad (100)$$

Example from $(*)$ take only J_A and $\eta \neq 0$

$$\Rightarrow \quad J_A * \langle \nabla(A) c \rangle + \frac{1}{\xi}\, \eta * \langle \partial A \rangle = 0$$

taking $\frac{\delta}{\delta J_A(y)}$, $\frac{\delta}{\delta \eta(z)}$ (then $J_A = \eta = 0$) and $\frac{\partial}{\partial y^\mu}$

$$\Rightarrow \quad \underbrace{\langle \bar{c}(z)\, \partial \nabla(A)\, c(y) \rangle}_{= i\,\delta(z-y)} + \frac{1}{\xi} \langle \partial A(z)\, \partial A(y) \rangle = 0$$

$$\Rightarrow \quad p^\mu \langle A_\mu A_\nu \rangle p^\nu = i\,\xi$$

since $p^\mu \langle A_\mu A_\nu \rangle_{\text{Tree}} p^\nu = i\,\xi \Rightarrow p^\mu G^{(2A)}_{\mu\nu}(\text{Loop}) = 0$

$\Rightarrow$ the loop corrections to $G^{(2A)}_{\mu\nu}$ are transverse.

From (100) $\Rightarrow$ the div. part of Γ has the same form as $\mathcal{L}_{GF}$

more precisely

$$\Gamma_{div} = -\frac{W^A}{4g}(\partial A_R - \partial A_R - Q \; f A_R A_R)^2 +$$
$$+ W^\psi \bar{\psi}_R \left(i\gamma(\partial + Q\, i t A_R) - (m + \delta m)\right)\psi_R$$
$$+ W^c \bar{c}_R \partial(\partial + Q\, i\theta A) c_R \qquad \left(W, Q = \frac{1}{\varepsilon}g\right)$$

Thus, we cancel the div terms by making counterterm appear in $\mathcal{L}_{GF}$ by a redefinition

$$\mathcal{L}_{GF} = -\frac{Z}{4}(\partial A_R - \partial A_R)^2 + \frac{Z_1 g}{2} f (\partial A_R - \partial A_R) A_R A_R$$
$$+ \frac{Z_2 g^2}{4} f A_R A_R \cdot f A_R A_R$$
$$+ Z_\psi \bar{\psi}_R (i\gamma\partial - m)\psi_R + Z_{1\psi}\, g\, \bar{\psi}_R i t \gamma A_R \psi - Z_\psi \delta m \bar{\psi}_R \psi_R$$
$$+ \tilde{Z}\, \bar{c}_R \partial^2 c_R + \tilde{Z}_1 f\, \bar{c}_R\, \partial (i\theta A_R c_R) + \frac{1}{2\xi}(\partial A_R)^2$$

$$Z_1 = Z \cdot Q \qquad \tilde{Z}_1 = \tilde{Z}\, Q \qquad Z_{1\psi} = Z_\psi \cdot Q \qquad Z_2 = Z Q^2$$

$$\Rightarrow \quad \frac{Z_1}{Z} = \frac{\tilde{Z}_1}{\tilde{Z}} = \frac{Z_{1\psi}}{Z_\psi}$$

Formally, $\mathcal{L}_{GT}$ is obtained by a redefinition

$$A = Z^{1/2} A_R \qquad \psi = Z_\psi^{1/2} \psi_R \qquad C = \tilde{Z}^{1/2} C_R$$

$$g_0 = Z_1 Z^{-3/2} g \mu^{\varepsilon/2} = Z_{1\psi} Z_\psi^{-1} Z^{-1/2} g \mu^{\varepsilon/2} = \tilde{Z}_1 \tilde{Z}^{-1} Z^{-1/2} g \mu^{\varepsilon/2}$$

to be used for computing β-function

$$\Rightarrow \beta = -\frac{g^3}{16\pi^2}\left(\frac{11}{3} N - \frac{2}{3} N_f\right)$$

$$\Rightarrow \frac{g^2(t)}{4\pi} \equiv \alpha_s(t) = \frac{\alpha_s(t_0)}{1 + 2\alpha_s^2(t_0)\frac{11N - 2N_f}{3} \cdot \ln P/p_0} \xrightarrow[p \to \infty]{} 0$$