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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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AN INTRODUCTION TO QUANTUM GEOMETRY

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Please note: These are preliminary notes intended for internal distribution only.

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N=2 SUPERCONFORMAL SYMMETRY AND QUANTUM GEOMETRY

OUTLINE:

I) INTRODUCTION & MOTIVATION

II) THE N=2 SUPERCONFORMAL ALGEBRA
- GENERAL PROPERTIES
- EXAMPLES OF REALIZATIONS

III) FAMILIES OF N=2 THEORIES
- TRULY MARGINAL OPERATORS
- MODULI SPACES I: CFT & NLσM

IV) INTERRELATIONS BETWEEN REALIZATIONS OF N=2 SCA
- LG/CY
- LG/MIN. MODELS
- MODULI SPACES II

V) MIRROR MANIFOLDS
- MOTIVATION, CONSTRUCTION, IMPLICATIONS
- MODULI SPACES III

VI) SPACETIME TOPOLOGY CHANGE

SUMMARY OF LECTURE I

I) INTRODUCTION/MOTIVATION

II) N = 2 SUPERCONFORMAL ALGEBR

• II.1) THE ALGEBRA

$$\{T(z), G^{\pm}(z), J(z)\} \times \{\bar{T}(\bar{z}), \bar{G}^{\pm}(\bar{z}), \bar{J}(\bar{z})\}$$

- $T(z)T(\omega) \sim {}^{c/2}/_{(z-\omega)^4} + {}^{2}/_{(z-\omega)^2}\, T(\omega) + {}^{1}/_{(z-\omega)}\, \partial_\omega T(\omega)$
- $T(z)\, G^{\pm}(\omega) \sim \frac{3}{2} \cdot \frac{G^{\pm}(\omega)}{(z-\omega)^2} + \frac{\partial G^{\pm}(\omega)}{(z-\omega)}, \cdots$
- $G^{+}(z)G^{-}(\omega) \sim {}^{2c}/_{3(z-\omega)^3} + 2J(\omega)/_{(z-\omega)^2} + \frac{2T(\omega)+\partial J(\omega)}{(z-\omega)} + \cdots$

$(G^{+}G^{+} \sim G^{-}G^{-} \sim \mathcal{O}(1))$

- $J(z)G^{\pm}(\omega) \sim \pm G^{\pm}(\omega)/(z-\omega) + \cdots$
- $J(z)J(\omega) \sim {}^{c/3}/_{(z-\omega)^2} + \cdots$

SUMMARY OF LECTURE I

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II) N = 2 SUPERCONFORMAL ALGEBR

• II.1) THE ALGEBRA

$$\{T(z), G^{\pm}(z), J(z)\} \times \{\bar{T}(\bar{z}), \bar{G}^{\pm}(\bar{z}), \bar{J}(\bar{z})\}$$

- $[L_m, L_n] = (m-n)L_{m+n} + c/12\,(m^3-m)\delta_{m+n,0}$
- $[L_n, G^{\pm}_{m\pm a}] = (n/2 - (m\pm a))G^{\pm}_{m+n\pm a}$
- $\{G^+_{n+a}, G^-_{m-a}\} = 2L_{m+n} + (n-m+2a)J_{n+m} + c/3\,[(n+a)^2 - 1/4]\delta_{m+n}$
- $[L_n, J_m] = -m J_{m+n}$
- $[J_n, G^{\pm}_{m\pm a}] = \pm G^{\pm}_{m+n\pm a}$
- $[J_n, J_n] = c/3 \cdot m\, \delta_{m+n,0}$

• II.2) HIGHEST WEIGHT REPRESENTATIONS

- $\phi(z)$ SUPERCONFORMAL PRIMARY FIELD

$|\phi\rangle = \phi(0)|0\rangle$

- $L_0|\phi\rangle = h_\phi|\phi\rangle$ • $J_0|\phi\rangle = Q_\phi|\phi\rangle$
- $L_n|\phi\rangle = J_n|\phi\rangle = G^{\pm}_n|\phi\rangle = 0 \quad n>0$
- USE $L_{-n}, J_{-n}, G^{\pm}_{-n}$ TO "RAISE" TO OTHER STATES

II.3) CHIRAL PRIMARY FIELDS

ING TO RINGER

- (CHIRAL, CHIRAL) = (C,C) : $G^+_{-1/2}|\phi\rangle = \bar{G}^+_{-1/2}|\phi\rangle = 0$

  (ANTICHIRAL, CHIRAL) (a,c) : $G^-_{-1/2}|\phi\rangle = \bar{G}^+_{-1/2}|\phi\rangle = 0$

- WE FOUND:

1) (c,c) $\leftrightarrow$ $h_\phi = Q_\phi/2$; $\bar{h}_\phi = \bar{Q}_\phi/2$

   (a,c) $\leftrightarrow$ $h_\phi = -Q_\phi/2$, $\bar{h}_\phi = \bar{Q}_\phi/2$

2) $h, \bar{h} \le c/6$ FOR SUCH STATES

3) OPERATOR ALGEBRA: CLOSED NONSINGULAR FINITE RING

   (C,C) RING (a,c) RING

II.4) FOUR EXAMPLES:

- A) ONE COMPLEX BOSON/FERMION $c = 3$
- B) NONLINEAR σ-MODEL: $S = \int d^2z\, g^M_{\mu\bar{\nu}}\, \partial X^\mu \bar{\partial} X^{\bar{\nu}} + \cdots$; $c = 3d$

  M: CALABI-YAU $\left\{\begin{array}{l} M: \text{COMPLEX, KÄHLER } (N=2) \text{ (ZUMINO)} \\ R_{i\bar{j}} = 0 \text{ (LOWEST ORDER) (CONFORMAL INV.) (CANDELAS, et al)} \end{array}\right.$

- C) LANDAU-GINSBURG MODEL:

  $$S = \int d^2z\, d^4\theta\, K(\Phi^i, \bar{\Phi}^{\bar{i}}) + \left(\int d^2z\, d^2\theta\, W(\Phi^i) + c.c.\right)$$

  $\hookrightarrow$ IR LIMIT $\quad \hookrightarrow$ CHIRAL SUPERFIELD $\bar{D}_+\Phi = 0$

- D) MINIMAL MODELS: UNITARY $c<3$, FINITE # PRIMARY FIELDS

  $c = 3P/P+2$

19 19. 19.2

III) FAMILIES OF N=2 THEORIES

III.1) MARGINAL OPERATORS

- Φ WEIGHTS $(h, \bar{h})$
  - IRRELEVANT IF $h+\bar{h} > 2$
  - RELEVANT IF $h+\bar{h} < 2$
  - MARGINAL IF $h+\bar{h} = 2$

$$S_{CFT} \to S_{CFT} + \epsilon \int d^2z\, \Phi(z,\bar{z}) \xrightarrow{\text{RG FLOW}} S'_{CFT}$$

- IRRELEVANT $\Rightarrow$ $S'_{CFT} = S_{CFT}$
- RELEVANT $\Rightarrow$ $S'_{CFT} \neq S_{CFT}$
- MARGINAL $\Rightarrow$ $S'_{CFT} \neq S_{CFT}$ BUT IS CONTINUOUSLY CONNECTED TO S_{CFT} ($\epsilon \to 0$).

- Φ IS TRULY MARGINAL IF IT STAYS MARGINAL IN PERTURBED THEORY

$\to$ TRULY MARGINAL OPERATORS GENERATE A FAMILY OF CFT's ALL CONTINUOUSLY RELATED TO ONE ANOTHER

SIMPLE EXAMPLE

$N=0$ $C=1$ CFT : BOSON ON CIRCLE OF RADIUS R

$$S_R = \int d^2z\, \partial X \bar{\partial} X \qquad X \sim X + 2\pi R$$

- CONSIDER (1,1) OPERATOR : $\vartheta = \partial X \bar{\partial} X$
- DEFORM S_R:

$$S_R \to S_R + \epsilon \int d^2z\, \vartheta(z,\bar{z})$$

$$= (1+\epsilon)\int d^2z\, \partial X \bar{\partial} X \,. \quad \text{LET } \tilde{X} = \sqrt{1+\epsilon}\, X \Rightarrow$$

$$= \int d^2z\, \partial\tilde{X}\bar{\partial}\tilde{X} \quad \text{WITH } \tilde{X} \sim \tilde{X} + 2\pi R(\sqrt{1+\epsilon})$$

- SO: MARGINAL OPERATOR ϑ CHANGES RADIUS OF TARGET SPACE CIRCLE.
- FAMILY OF CFT's GENERATED: BOSON ON CIRCLE OF ARBITRARY RADIUS
- IN FACT : $S_R \cong S_{1/R}$
- MODULI SPACE :

N=2 SCFT

- TRULY MARGINAL OPERATORS INCLUDE

i) $\phi \in (c,c)$ WITH $h=\bar{h}=1/2$, $Q=\bar{Q}=1$

$G^{+}(\omega)\,\phi(z,\bar{z}) \sim \mathcal{O}(1)\,;\quad \bar{G}^{+}(\bar{\omega})\,\phi(z,\bar{z}) \sim \mathcal{O}(1)$

$$\hat{\phi}(z,\bar{z}) = (G^{-}_{-1/2}\,\phi)(z,\bar{z}) = \oint G^{-}(\omega)\,\phi(z,\bar{z}) \begin{cases} h=1,\ Q=0 \\ \bar{h}=1/2,\ \bar{Q}=1 \end{cases}$$

$$\Phi_{(c,c)} = (\bar{G}^{-}_{-1/2}\,\hat{\phi})(z,\bar{z}) = \oint \bar{G}^{-}(\bar{\omega})\,\hat{\phi}(z,\bar{z}) \begin{cases} h=1 & Q=0 \\ \bar{h}=1 & \bar{Q}=0 \end{cases}$$

- BY SCWI, $\Phi_{(c,c)}$ IS TRULY MARGINAL

ii) $\phi \in (a,c)$ WITH $h=\bar{h}=1/2$, $Q=-1$, $\bar{Q}=1$

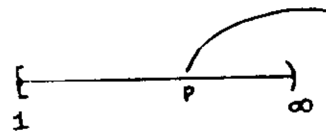
$$\Phi_{(a,c)} = \oint \bar{G}^{-}(\bar{\omega}) \oint G^{+}(\omega)\ \phi(z,\bar{z}) \begin{cases} h=\bar{h}=1 \\ Q=\bar{Q}=0 \end{cases}$$

- BY SCWI, $\Phi_{(a,c)}$ IS TRULY MARGINAL

- WE WILL FOCUS ON THESE TRULY MARGINAL OP'S AND THEIR LOWER COMPONENT SUPERPARTNERS IN (C,C), (A,C) RINGS.

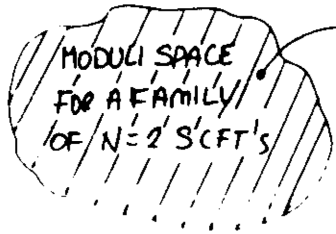
III.2) MODULI SPACES: I

- FOR BOSON ON CIRCLE:



EACH POINT CORRESPONDS TO A CFT $S_{R(p)}$

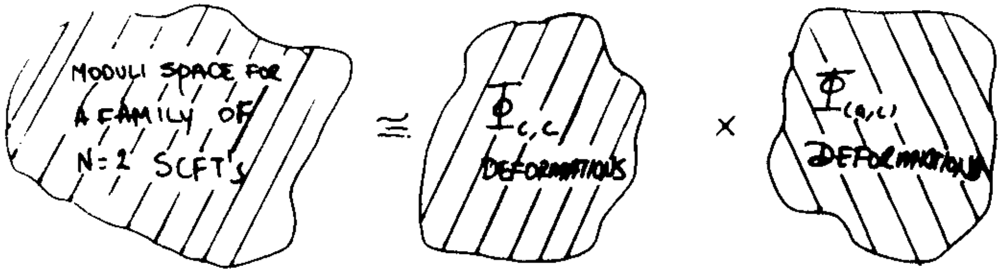
- MORE GENERALLY: MULTIDIMENSIONAL PARAMETER SPACE; "MODULI SPACE"



- EACH POINT CORRESPONDS TO 1 N=2 SCFT

- CAN DEFORM ANY SUCH SCFT TO THAT CORRESPONDING TO A DIFFERENT POINT BY USING THE TRULY MARGINAL OPS.

- FOR N=2 SCFT: AT LEAST LOCALLY CFT METRIC (ZAMOLODCHIKOV) IS BLOCK DIAGONAL BETWEEN $\Phi_{c,c}$ & $\Phi_{a,c}$:



- GOAL: FULLY UNDERSTAND THIS PICTURE, ESPECIALLY WHEN WE ARE STARTING FROM AN INITIAL $N=2$ SCFT BUILT AS A NLσM.

- TO START: IF WE HAVE AN $N=2$ SCFT, REALIZED AS NLσM, CAN WE GIVE GEOMETRIC INTERPRETATION TO $\Phi_{(c,c)}$, $\Phi_{(a,c)}$ MARGINAL OPS?

- ALREADY DESCRIBED:

$$(c,c) \text{ FIELDS} \leftrightarrow \text{HARMONIC } (2,1) \text{ FORMS ON } M$$
$$(a,c) \text{ FIELDS} \leftrightarrow \text{HARMONIC } (1,1) \text{ FORMS ON } M$$

- IN PARTICULAR: (TO LOWEST ORDER)

$$(a,c) \leftrightarrow h_{i\bar{j}} \psi^i \lambda^{\bar{j}}$$

$$\Phi_{(a,c)} \sim G^{+}_{-1/2} \bar{G}_{-1/2} (h_{i\bar{j}} \psi^i \lambda^{\bar{j}})$$

$$\sim h_{i\bar{j}} \partial X^i \bar{\partial} X^{\bar{j}}$$

- THESE OPERATORS DEFORM "SIZE" OF M: (WHILE KEEPING IT RICCI FLAT!)

$J =$ KÄHLER FORM ON $M = i g_{i\bar{j}} dX^i dX^{\bar{j}}$

- (c,c) ~ HARMONIC (2,1) FORMS

$$\sim h_{ij,\bar{k}} \Omega^{ijk} \psi^{\bar{k}} \lambda_k$$

$$= h_{\bar{k}}^{\ k} \psi^{\bar{k}} \lambda_k \qquad h_{\bar{k}}^{\ k} \in H^1(M,T)$$

$$\Phi_{(c,c)} \sim G^{-}_{-1/2} \bar{G}_{-1/2} (h_{\bar{k}}^{\ k} \psi^{\bar{k}} \lambda_k)$$

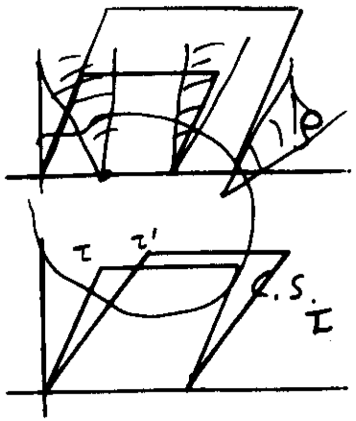
$$\sim h_{\bar{k}}^{\ k} \partial X^{\bar{k}} \bar{\partial} X^{\bar{k}'} g_{k\bar{k}'}$$

$$\sim h_{\bar{k}\bar{k}'} \partial X^{\bar{k}} \bar{\partial} X^{\bar{k}'}$$

- THESE OPERATORS DEFORM METRIC BUT SPOIL ITS (1,1) TYPE STRUCTURE (KEEPING IT RICCI FLAT). THEY ARE MOST APPROPRIATELY INTERPRETED BY PERFORMING A NONHOL. CHANGE OF COORDS TO PUT METRIC BACK IN (1,1) FORM — i.e. THESE OPERATORS CHANGE COMPLEX ST. OF M.

- $\Phi_{(a,c)}$: DEFORM KÄHLER ST. OF M ("SIZE")
  $\# = h^{1,1} = \dim H^{1,1}_{\bar{\partial}}(M)$

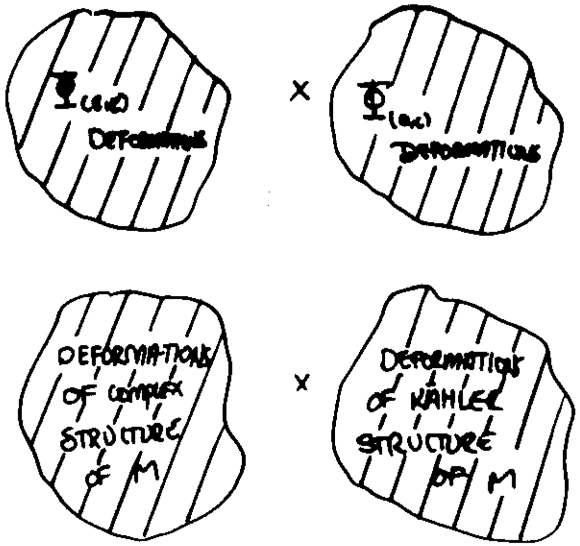
  $\Phi_{(c,c)}$: DEFORM COMPLEX ST. OF M ("SHAPE")
  $\# = h^{2,1} = \dim H^{2,1}_{\bar{\partial}}(M)$



PICTORIALLY: MODULI SPACE

N=2 SCFT (GENERALLY):

GEOMETRIC INTERPRETATION ON N=2 NLσM:



- THIS GEOMETRIC INTERPRETATION IS DEFICIENT. WE WILL RETURN TO THIS

- NOTE: (AS A PRELUDE) ASYMMETRY

  $\Phi_{(c,c)}$, $\Phi_{(a,c)}$ DIFFER BY SIGN OF $U(1)$ CHARGE

  Complex St, Kähler St DIFFER SIGNIFICANTLY.

IV) INTERRELATIONS BETWEEN VARIOUS N=2 SCFT's

- NLσM ON CY TARGET
- LANDAU-GINSBURG (L-G) MODELS
- MINIMAL MODELS (MM)

RELATIONS?

ZAMOLOCHIKOV IN [illegible]

1) L-G/MM

CONSIDER L-G THEORY WITH SINGLE CHIRAL SUPERFIELD Ψ

$$S = \int d^2z\, d^4\theta\, K(\Psi, \bar{\Psi}) + \left(\int d^2z\, d^2\theta\, W(\Psi) + c.c.\right)$$

TAKE $W(\Psi) = \Psi^{P+2}$. LVW CALCULATE CENTRAL CHARGE C_P AT IR FIXED POINT:

$$C_P = 3P/P+2$$

• RECALL: $c<3$ $N=2$ THEORIES ARE THE MINIMAL MODELS!

$\Rightarrow$ L-G FOR $W = \Psi^{p+2} \cong$ LEVEL p $N=2$ M-M

(MORE PRECISELY: A-INVARIANT LEVEL p M-M)

• FOR LATER USE NOTE:

$$MM_{p_1} \otimes MM_{p_2} \otimes \cdots \otimes MM_{p_r} \cong$$

IR FIXED POINT OF $\left\{ \int d^2z\, d^4\theta\, K(\Psi_i, \bar{\Psi}_i) + \int d^2z\, d^2\theta \left(\sum \Psi_i^{p_i+2}\right) + c.c. \right\}$

TOR, MARTINEC
NKAR;
, VAFA,
;
RTINEC;
TTEN

B) L-G / CY σ-MODEL

• CONSTRUCTING (SOME) CY MANIFOLDS:

HYPERSURFACES IN $WCP^4_{(\omega_1,\dots,\omega_5)}$:

• $WCP^4_{(\omega_1,\dots,\omega_5)}$: $[z_1, z_2, z_3, z_4, z_5] \sim [\lambda^{\omega_1} z_1, \lambda^{\omega_2} z_2, \dots, \lambda^{\omega_5} z_5]$

• $\dim_{\mathbb{C}} WCP^4 = 4$: PLACE 1 CONSTRAINT TO GET A 3 COMPLEX DIM SPACE (CENTRAL CHARGE = 9)

• EQN: $\sum_i z_i^{p_i+2} = 0$ SUCH THAT:

• MAKES SENSE UNDER ALL $z_i \to \lambda^{\omega_i} z_i$
i.e. $\omega_i (p_i+2)$ ALL EQUAL

• $d = \omega_i (p_i+2) = \sum_i \omega_i$

(THIS ENSURES M IS CY)

• e.g. $CP^4_{(1,1,1,1,1)}$: $\sum z_i^5 = 0$ "QUINTIC" CY.

• CONSIDER NOW :

L-G THEORY: 5 CHIRAL SUPERFIELDS $(\Psi_1,\dots,\Psi_5)$; $W(\Psi_1,\dots,\Psi_5) = \sum \Psi_i^{P_i+2}$

WITH P_i's AS FOR C-Y

• WHAT IS CENTRAL CHARGE?

$$C = \sum_{i=1}^{5} 3P_i / P_i+2 \qquad (P_i+2)\omega_i = d \Rightarrow$$

$$= \sum 3(d/\omega_i - 2)/(d/\omega_i) = 3\cdot\sum(1 - 2\omega_i/d)$$

$$= 3\cdot 5 - 3\cdot 2\cdot \sum\omega_i/d = 3\cdot 5 - 3\cdot 2 = \boxed{9}$$

GEOMETRICAL CY CONDITION $\leftrightarrow$ CFT CONDITION

COULD IT BE THAT :

• NLσM : $\int d^2z\, g_{\mu\bar\nu}\,\partial X^\mu \bar\partial X^{\bar\nu} + \cdots$ ON $M = \sum z_i^{P_i+2} = 0$ IN $WCP^4_{(\omega_1 \dots \omega_5)}$

• LG : $\int d^2z\, d^4\theta\, K(\Psi^i, \Psi^{\bar i}) + \left(\int d^2z\, d^2\theta\, (\sum \Psi_i^{P_i+2}) + c.c.\right)$

IN IR LIMIT ARE ISOMORPHIC CFT's ?

VW;
IRTWGC;
ITEN

• YES!

• ARGUMENT # 1 (GVW)

CONSIDER LG THEORY

$$\int K(\Psi^i, \Psi^{\bar i})\, d^2z\, d^4\theta + \left(\int d^2z\, d^2\theta\, \sum \Psi_i^{P_i+2} + c.c.\right)$$

• RECALL : W NOT RENORMALIZED ALONG RG FLOWS
K IS IRRELEVANT OPERATOR
($d^4\theta$ picks out wt $(h+1, \bar h+1)$ component
if h = dim of lowest component of K
$h, \bar h \geq 0 \Rightarrow h+1+\bar h+1 \geq 2$)

• SO : DEFORM $K \to \varepsilon K$ SHOULD YIELD SAME FIXED POINT THEORY

• IN FACT : BE BOLD : $\varepsilon \to 0$

• $\int [D\Psi_1] \cdots [D\Psi_5]\, e^{i\left(\int d^2z\, d^2\theta\, \sum \Psi_i^{P_i+2} + c.c.\right)}$

• CHANGE VARIABLES : $\xi_1 = \Psi_1^{P_1+2}$; $\xi_i^{P_i+2} = \Psi_i^{P_i+2}/\Psi_1^{P_1+2}$ $i = 2,3,4,5$
$(\Psi_1 \neq 0)$

• By direct calculation:

$$[\mathcal{D}\Psi_1]\cdots[\mathcal{D}\Psi_5] = [\mathcal{D}\xi_1]\cdots[\mathcal{D}\xi_5] \quad (\text{Use } d=\sum \omega_i)$$

• Theory $\rightarrow$ $\int [\mathcal{D}\xi_1]\cdots[\mathcal{D}\xi_5]\, e^{i\left(\int d^2z\, d^2\theta\, \xi_1 (1+\sum \xi_i^{P_i+2}) + c.c.\right)}$

• Consider factor:

$$\int [\mathcal{D}\xi_{1,F}][\mathcal{D}\xi_2]\cdots[\mathcal{D}\xi_5]\, e^{i\left(\int d^2z\, \xi_{1,F} (1+\sum \xi_{i,s}^{P_i+2}) + c.c.\right)}$$

F-comp scalar comp

$$\propto \delta\left(1+\sum \xi_{i,s}^{P_i+2}\right)$$

• $\Rightarrow$ Scalar part of L-G superfields live on

$$1+\sum \xi_{i,s}^{P_i+2} = 0$$

• But, CY : $\sum z_i^{P_i+2} = 0 \Rightarrow [z_1,\dots,z_5] \sim [\lambda^{\omega_1} z_1,\dots,\lambda^{\omega_5} z_5]$

with $\lambda^{\omega_1} = z_1^{-1}$

$$\Rightarrow \quad 1+\sum y_i^{P_i+2} = 0 \quad \left(y_i^{P_i+2} = z_i^{P_i+2}/z_1^{P_1+2}\right)$$

So: LG fields actually live on CY!

Remarks

1.) Nonlinear change of variables not 1:1.
To remedy:
Identify $(\Psi_1,\dots,\Psi_5) \sim (\alpha^{\omega_1}\Psi_1,\dots,\alpha^{\omega_5}\Psi_5)$
$\alpha^d = 1$.

So: Really: $\underset{W}{\text{LG}}$ orbifold $\cong$ $\underset{W=0}{\text{CY } \sigma\text{-model}}$

2) Specifying eqn $\sum z_i^{P_i+2} = 0$ fixes complex structure of CY. What about Kähler structure? (Vafa) (enhanced symmetry point)

LG orbifold (W) $\cong$ CY σ-model
W=0 at a specific Kähler structure

3.) Argument involves "uncomfortable" manipulations $\rightarrow$ Witten has given a new, more powerful argument.