INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.762 - 38

III - Revised version

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

13 June - 29 July 1994

AN INTRODUCTION TO QUANTUM GEOMETRY

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Please note: These are preliminary notes intended for internal distribution only.

SUMMARY OF LECTURE II

I) INTRODUCTION & MOTIVATION

II) THE N=2 SUPERCONFORMAL ALGEBRA

- GENERAL PROPERTIES (HIGHEST WT REPS; (ANTI) CHIRAL PRIMARIES...
- EXAMPLES (FREE THEORY; CY σ-MODEL; L-G MODELS, MIN MODELS.

III) FAMILIES OF N=2 THEORIES

- **TRULY MARGINAL OPERATORS:**

$$S_{CFT} \to S'_{CFT} = S_{CFT} + \epsilon \int d^2z\, \mathcal{O}(z,\bar{z}) \quad \left\{ \begin{array}{l} h = \bar{h} = 1 \\ \text{(NECESSARY)} \end{array} \right.$$

N=2: • $\phi \in (c,c)$, $\Phi_{c,c} \equiv (G^-_{-1/2}\, \bar{G}^-_{-1/2}\, \phi)$

• $\phi \in (a,c)$, $\Phi_{a,c} \equiv (G^+_{-1/2}\, \bar{G}^-_{-1/2}\, \phi)$

WHY? • MACROSCOPIC: $S_{SPACETIME} = \cdots + \int d^4x\, d^2\theta\, W_{SPACETIME}$

• MICROSCOPIC: WARD IDENTITIES

GEOMETRIC: S_{CFT} = NLσM ON M

• $\Phi_{c,c}$, $\Phi_{a,c}$: DEFORMATIONS OF COMPLEX STR. / KÄHLER

~ HARMONIC (2,1), (1,1) diff forms on M

~ e.g. $h_{i\bar{j}}\, \partial x^i\, \bar{\partial} x^{\bar{j}}$

- MODULI SPACES : A FIRST LOOK
  - LOCAL STRUCTURE : SPECIAL GEOMETRY
  - OUR AIM : GLOBAL STRUCTURE

IV) INTERRELATIONS BETWEEN REALIZATIONS OF $N=2$ SCA

- LANDAU-GINSBURG / MINIMAL MODELS
  - LG : $S = \int d^2z\, d^4\theta\, K(\Psi, \bar{\Psi}) + \left(\int d^2z\, d^2\theta\, \Psi^{P+2} + c.c. \right)$

    (1 CHIRAL SUPERFIELD; $W(\Psi) = \Psi^{P+2}$; TAKE IR LIMIT)
  - MIN MODEL : LEVEL P

  (NO COMPLETE PROOF; STRONG EVIDENCE)

- LANDAU-GINSBURG / CY σ-MODEL
  - LG : $S = \int d^2z\, d^4\theta\, K(\Psi_i, \bar{\Psi}_i) + \left(\int d^2z\, d^2\theta\, \sum \Psi_i^{P_i+2} + c.c. \right)$

ORBIFOLD

    (5 CHIRAL SUPERFIELDS; $W(\Psi_1, \dots, \Psi_5) = \sum \Psi_i^{P_i+2}$; TAKE IR LIMIT)
  - CY σ-MODEL: $S = \int d^2z\, g^M_{\mu\bar{\nu}}\, \partial X^\mu \bar{\partial} X^{\bar{\nu}} + \dots$ (SEE LECTURE I)

    M : $\sum z_i^{P_i+2} = 0$ IN $WCP^4(\omega_1, \dots, \omega_5)$: $d = (P_i+2)\omega_i$; $d = \sum \omega_i$

REMARKS

1) NEED NOT RESTRICT TO $W = \sum_i \Psi_i^{p_i+2}$. RATHER, W CAN BE ANY HOMOGENEOUS (QUASI) POLYNOMIAL IN $(\Psi_1, \dots, \Psi_5)$ OF DEGREE d.
- SAME ARGUMENT $\to$ CY σ-MODEL ON M: $(z_1, \dots, z_5)$: $W(z_1, \dots, z_5) = 0$ IN $W\mathbb{CP}^4_{(\omega_1, \dots, \omega_5)}$
- DIFFERENT CHOICES OF $W \leftrightarrow$ DIFFERENT COMPLEX STRUCTURES ON M.

2) NEED NOT RESTRICT TO 5 SUPERFIELDS / CY 3-FOLDS.

3) THIS ARGUMENT PICKS OUT A PARTICULAR KÄHLER STRUCTURE ON CY - (IN FACT, IT LIES OUTSIDE RANGE OF σ-MODEL PERTURBATION THEORY) - ONE OF ENHANCED DISCRETE SYMMETRIES.

4) WITTEN'S ARGUMENT LETS US MORE CLEARLY SEE FAMILY OF FIXED POINT THEORIES:

B) CY/LG CONNECTION, AGAIN (WITTEN)

- CONSIDER A 2-D $N=2$ $U(1)$ GAUGE THEORY:
(DO CASE OF QUINTIC TO BE CONCRETE) (NOT CONF. INV.!)
- FIELDS: CHIRAL SUPERFIELDS — CHARGE

CHIRAL SUPERFIELDS	CHARGE
P	-5
S_1, S_2, S_3, S_4, S_5	1

- ACTION: $S = S_{KIN} + S_W + S_{GAUGE} + S_{FAYET\text{-}ILLO}$ $\left\{ -r \int d^2y\, D \right\}$

- SUPERPOTENTIAL: $W(P, S_1, \ldots, S_5) = P \cdot G(S_1, \ldots, S_5)$

$$G(S_1, \ldots, S_5) = \sum_i S_i^5 \quad (P \cdot G \text{ is } U(1) \text{ INV})$$

- BOSONIC POTENTIAL:

$$U = |G(s_i)|^2 + |p|^2 \sum_i |\partial G/\partial s_i|^2 + \tfrac{1}{2e^2} D^2 +$$

$$2|\sigma|^2 \left(\sum |s_i|^2 + 5^2 |p|^2 \right)$$

$$D = -e^2 \left(\sum \bar{s}_i s_i - 5 \bar{p} p - r \right)$$

(SMALL LETTERS = SCALAR COMPONENTS OF CORRESPONDING SUPERFIELD;
σ ARISES FROM GAUGE FIELD (IN 4-D → 2-D).)

• **NOW**: CONSIDER VACUUM CONFIGURATIONS FOR EITHER $r \gg 0$ or $r \ll 0$:

i) $r \gg 0$: $D = 0 \Rightarrow S_i$ NOT ALL ZERO
$\Rightarrow |\partial G/\partial S_i|^2 \neq 0 . \Rightarrow p = 0 . \Rightarrow \sum S_i \bar{S}_i = r .$
$\Rightarrow \sigma = 0 . \quad \Rightarrow G(S_i) = 0$
(p, σ, GAUGE FIELD MASSIVE).

• SO: $(S_1, \ldots, S_5)$ WITH $\sum S_i^5 = 0$ AND $\sum S_i \bar{S}_i = r$.
• FURTHERMORE: $U(1)$ INV $\Rightarrow (S_1, \ldots, S_5) \sim (e^{i\theta} S_1, \ldots, e^{i\theta} S_5)$

• NOTE: IN $\mathbb{CP}^4$: $(z_1, \ldots, z_5) \sim (\lambda z_1, \ldots, \lambda z_5)$. CAN CHOOSE $|\lambda|$ S.T. $\sum \bar{z}_i z_i = r$. THEN STILL HAVE $(z_1, \ldots, z_5) \sim e^{i\theta}(z_1, \ldots, z_5)$
r SETS "SIZE"
• THUS: $r \gg 0 \Rightarrow$ NLσM ON QUINTIC IN $\mathbb{CP}^4$
(IN IR LIMIT $\rightarrow$ <u>$N=2$ SCFT ON QUINTIC</u>)

ii) $r \ll 0$:
$D = 0 \Rightarrow p \neq 0 . \; p \neq 0 \Rightarrow S_i$ ALL $= 0$ $(|\partial G/\partial S_i|^2 = 0$
$\Rightarrow p = \sqrt{-r/5} \quad \Rightarrow \sigma = 0 .$

• S_i ARE MASSLESS FLUCTUATIONS ABOUT UNIQUE VACUUM STATE, (ALL OTHER FIELDS MASSIVE), W DEGENERATE. $\Rightarrow$ <u>L-G THEORY</u>

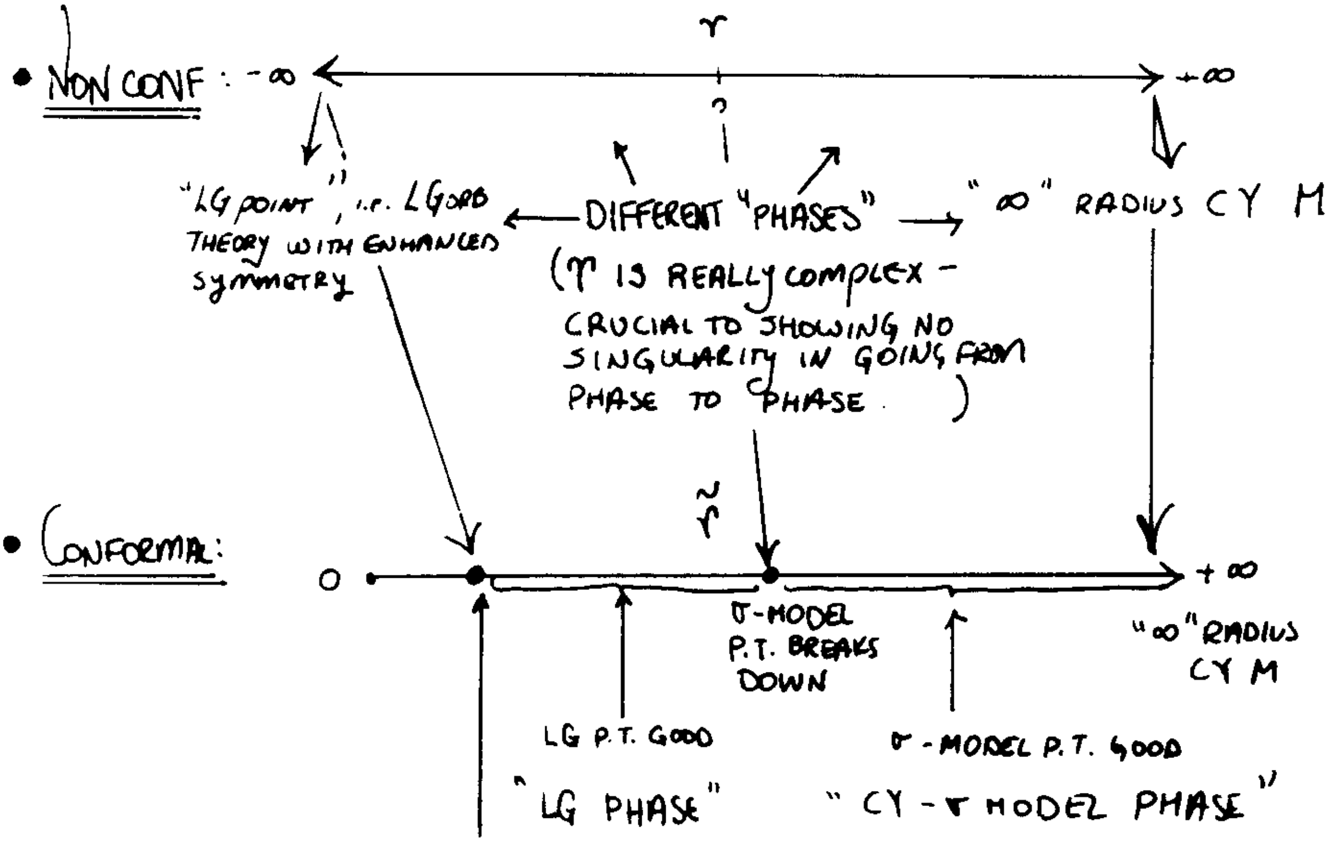
• $p \neq 0 \Rightarrow U(1) \rightarrow \mathbb{Z}_5 \Rightarrow$ LG WITH $\mathbb{Z}_5$ GAUGE GROUP
• i.e. <u>LG ORBIFOLD</u>.

• THUS:

• AS $N=2$ NON CONF. THEORIES: $S(r) \begin{cases} \xrightarrow{r \gg 0} & \text{NON-CONFORMAL CY } \sigma\text{-MODEL ON } G=0 \\ \xrightarrow{r \ll 0} & \text{NON CONF. LG ORBIFOLD, SUPERPOT} = G \end{cases}$

• **REMARKS**

1) TAKE IR LIMIT OF ABOVE: RECOVER CONFORMAL LG/CY CORRESPONDENCE.
   • MORE RIGOROUS
   • $\tilde{r}$ CORRESPONDS TO (COEFF OF) MARGINAL OPERATOR ($\tilde{r}$ = IMAGE OF r AFTER IR LIMIT)



• OR: IF DISREGARD P.T.: ALL CY's, for INSTANCE.

POINT OF GVW ARGUMENT

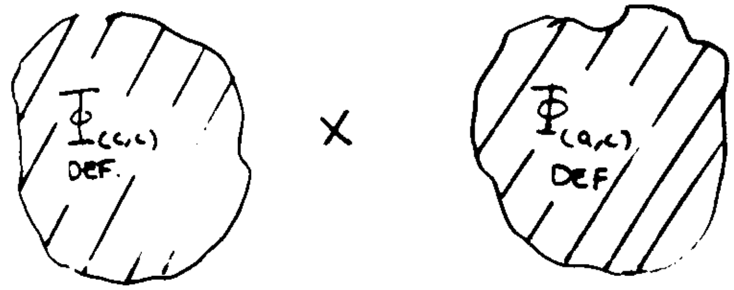
2) ARGUMENT EASILY EXTENDED TO OTHER CY/LG THEORIES. SOMETIMES CY $\leftrightarrow$ CONTINUOUS GAUGED LG THEORY.

3) (ASPINWALL, BRG, MORRISON; WITTEN) IN SLIGHTLY MORE COMPLICATED EXAMPLES CAN HAVE NOT TWO BUT MANY PHASES. SOMETIMES DIFFERENT PHASES INTERPRETABLE AS σ-MODELS ON TOPOLOGICALLY DISTINCT CY MANIFOLDS. CAN WE MOVE FROM PHASE TO PHASE, USING MARGINAL OPERATORS, WITHOUT ENCOUNTERING A PHYSICAL SINGULARITY? NEED MIRROR MANIFOLDS TO ANSWER THIS QUESTION

(4. RECALL IF $W_4 = \sum_i \Psi_i^{p_i+2}$ WE HAVE LG ORB $\cong \otimes MM_{p_i}$ ORB)

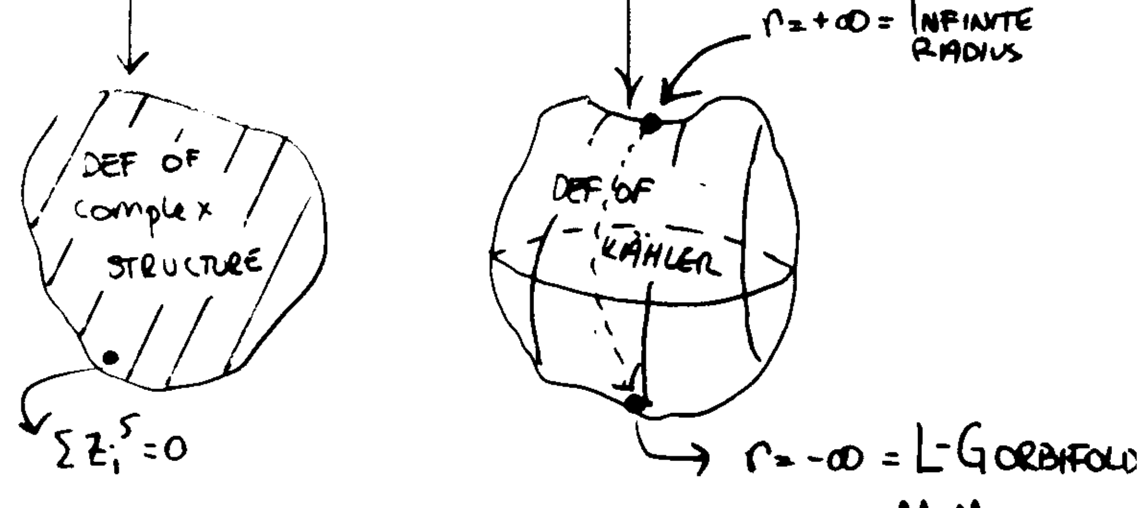
Moduli Space: II (for definiteness: Quintic)

• Abstract $N=2$ moduli space:



• Geometric Interpretation

Quintic: $h^{2,1} = 101$, $h^{1,1} = 1$ (complex dim)



Can think of whole space as σ-models of various radii or as LG theories (deformed from LG pt.) sensible: use convergence of P.T. as criteria.

TWO "PHASES":

- Upper hemisphere: Kähler mod of quintic
- Lower hemisphere: mod space of LG orbifold.

V) MIRROR MANIFOLDS

V.1) MOTIVATION AND STRATEGY

- HAVE SEEN: CORRESPONDENCE BETWEEN
  - ABSTRACT PROPERTIES OF $N=2$ SCFT's
  - GEOMETRIC PROPERTIES OF CY's USED TO REALISE $N=2$ VIA NLσ MODELS

- IN PARTICULAR:

  WILL RETURN TO RING STRUCTURE

  - ABSTRACT SCFT: (c,c) RING $\left[h=\bar{h}=\frac{1}{2}, Q=\bar{Q}=1\right]$ (a,c) RING $\left[h=\bar{h}=\frac{1}{2}; Q=\bar{Q}=1\right]$

    $\to$ MARGINAL OPERATORS $\Phi_{c,c}$; $\Phi_{a,c}$

  - GEOMETRIC:

    [illegible] FORM. $\leftrightarrow$ HARMONIC FORMS $h_{\bar{i}}{}^{j}\, dx^{\bar{i}} \otimes \frac{\partial}{\partial x^j}$; $(H^1(M,T))$ HARMONIC FORMS $h_{\bar{i}j}\, dx^{\bar{i}} dx^{j}$ $(H^1(M,T^*))$

    $\to (g_{\bar{j}j} h_{\bar{i}}{}^{j})\psi^{\bar{i}}\lambda^{\bar{j}}$ $h_{\bar{i}j}\psi^{\bar{i}}\lambda^{j}$

    $\to$ MARGINAL OPS: (TO LOWEST ORDER)

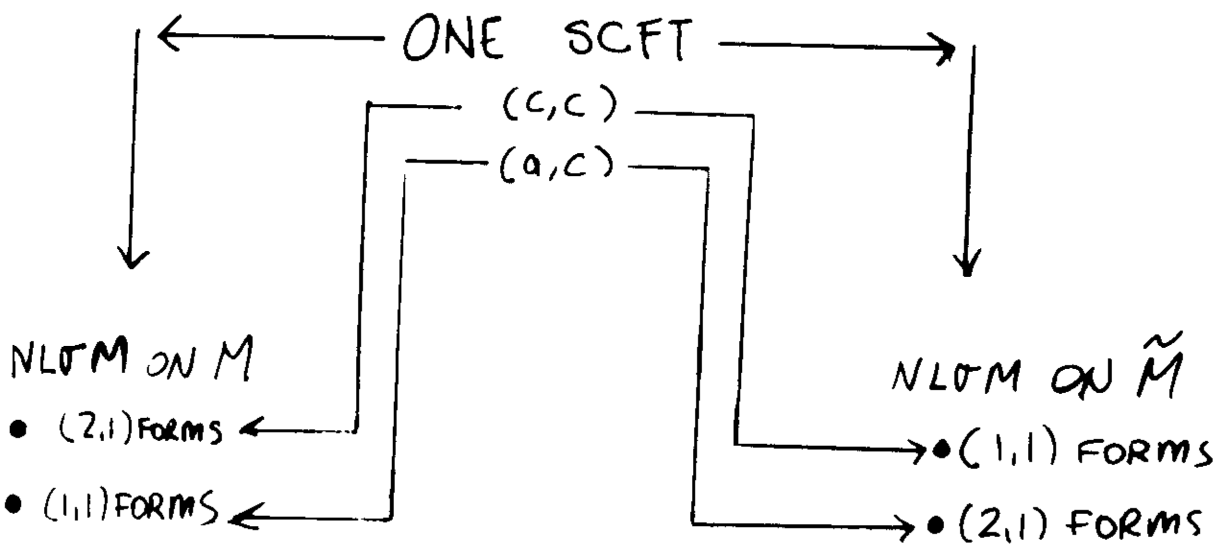
    $$(g_{\bar{j}j} h_{\bar{i}}{}^{j})\bar{\partial} x^{\bar{i}}\partial x^{\bar{j}}; \quad h_{\bar{i}j}\bar{\partial} x^{\bar{i}}\partial x^{j}$$

- ASYMMETRY: CFT: $Q \leftrightarrow -Q$
  GEOMETRY: MORE PRONOUNCED DIFFERENCE

• RESOLUTIONS:

1) THAT'S JUST THE WAY IT IS

2) (LVW, DIXON) CONJECTURE: MAYBE THERE IS ANOTHER HALF TO THE PICTURE



• CAN THIS PICTURE BE REALIZED?

• YES: (BRG, PLESSER) FOR ∃ PARTICULAR CLASS OF N=2 THEORIES / CY σ-MODELS, CAN REALIZE BY DIRECT CONSTRUCTION

• NB: CANDELAS, LYNKER, SCHIMMRIGK; BATYREV
BERGLUND, HUBSCH;

STRATEGY OF CONSTRUCTION

$$W = \sum \Psi_i^{P_i+2}$$

$$\begin{tikzcd}
\text{FERMAT LG ORBIFOLD ("LG POINT")} \arrow[r, phantom, "\simeq"] \arrow[d, "\substack{\text{GEPNER} \\ \text{* GVW} \\ \text{MARTINEC} \\ \text{* WITTEN}}"'] & \otimes\,\text{MIN MODELS ORBIFOLD} \arrow[r, "\sigma"] \arrow[dr, "Q \to -Q"] & \sigma\left(\frac{LG}{ORB} \otimes \frac{\otimes MM_i}{ORB}\right) \overset{Q\to -Q}{\simeq} \left(\frac{LG}{ORB} \otimes \frac{MM}{ORB}\right) \arrow[d] \\
\sigma\text{-MODEL ON CY } M \arrow[rr, "\sigma"] & & \sigma\text{-MODEL ON CY } \sigma(M) \equiv \tilde{M}
\end{tikzcd}$$

- CAN SUCH AN OPERATION σ BE FOUND ?

  YES: σ = MORE ORBIFOLDING

- IN PARTICULAR:

$$\begin{tikzcd}
& (c,c) \arrow[dl] \arrow[dr] & & & (a,c) \arrow[dl] \arrow[dr] & \\
(2,1) \text{ FORMS } M & & (1,1) \text{ FORMS } \tilde{M} & (1,1) \text{ FORMS } M & & (2,1) \text{ FORMS } \tilde{M}
\end{tikzcd}$$

$$\Rightarrow \chi_M = 2\left(\#\,(1,1)\text{ HARM. FORMS} = h^{1,1}_M - \#\text{ HARM. } (2,1)\text{ FORMS} = h^{2,1}_M\right)$$

$$\chi_{\tilde{M}} = 2\left(h^{1,1}_{\tilde{M}} - h^{2,1}_{\tilde{M}}\right) = 2\left(h^{2,1}_M - h^{1,1}_M\right) = -\chi_M$$

$\Rightarrow$ "MIRRORS"

• ROUGH ARGUMENT

1. MM_p HAS $\mathbb{Z}_{p+2} \times \bar{\mathbb{Z}}_{p+2}$ DISCRETE SYMMETRY

$$\mathbb{Z}_{p+2}^{DIAG} \subset \mathbb{Z}_{p+2} \times \bar{\mathbb{Z}}_{p+2}$$

$$MM_p / \mathbb{Z}_{p+2}^{DIAG} \overset{Q \to -Q}{\cong} MM_p$$

2. $$\left(\bigotimes_{i=1}^{5} MM_{p_i} \right) / \mathbb{Z}_{p_i+2}^{DIAG} \overset{Q_{TOT} \to -Q_{TOT}}{\cong} \bigotimes_{i=1}^{5} MM_{p_i}$$

3. HAVE SEEN: CY $\sum z_i^{p_i+2} = 0 \longleftrightarrow$ LG: $W = \sum \Psi_i^{p_i+2}$ ORBIFOLDED

$$\begin{array}{c} \updownarrow \\ \otimes MM_{p_i} \text{ ORBIFOLDED} \\ \updownarrow \\ \text{PROJECTION TO INTEGRAL } U(1) \text{ CHARGES} \end{array}$$

NOTATION: $(P_1, P_2, \ldots, P_5) = \otimes MM_{P_i}$ ORBIFOLDED $\longleftrightarrow$ PROJECTION TO INTEGRAL $U(1)$ CHARGES

$$+ \left[\bigotimes_{i=1}^{5} MM_{p_i} / \mathbb{Z}_{p_1+2} \times \cdots \times \mathbb{Z}_{p_5+2} \right]_{U(1)} \overset{Q_{TOT} \to -Q_{TOT}}{\cong} \cancel{\mathcal{O}} \quad (P_1, \ldots, P_5)$$

U(1) PROJECTION

$$\sum z_i^{p_i+2} = 0 \ : \ M$$

$$(P_1, \ldots, P_5)/H$$

(H = MAXIMAL SUBGROUP of $\mathbb{Z}_{p_1+2} \times \cdots \mathbb{Z}_{p_5+2}$ PRESERVING (C,C) FIELD OF $h = 3/2$ $Q = 3$, $\bar{h} = 0$ $\bar{Q} = 0$)

(HOLO (3,0) FORM ON M)

$$\updownarrow$$

$$\left(\sum z_i^{p_i+2} = 0 \right) / H \ : \ \tilde{M}$$

V.2) DETAILS

1. MINIMAL MODEL AUTOMORPHISM

A) CHARACTERS: $c = 3p/p+2 \quad p=1,2,\ldots$

HW STATES:

• NS: $\Phi_{\ell,m}$ $\quad h_{\ell,m} = \frac{\ell(\ell+2)}{4(p+2)} - \frac{m^2}{4(p+2)}$, $Q_{\ell,m} = m/p+2$

$\ell = 0,1,\ldots,p$
$m = -\ell,-\ell+2,\ldots,\ell$

• SIMILARLY FOR ANTIHOL. SECTOR

• $m=\ell \Rightarrow h_{\ell,\ell} = \ell/2(p+2) = \frac{1}{2}Q_{\ell,\ell}$

$\Rightarrow \Phi_{\ell,\ell}$ CHIRAL PRIMARY

• $m=-\ell \Rightarrow \Phi_{\ell,-\ell}$ ANTICHIRAL PRIMARY

• R: $G_0^+\Psi^+_{\ell,m} = 0 \qquad G_0^-\Psi^-_{\ell,m} = 0$

$\Psi^{\pm}_{\ell,m}$ $\quad h^{\pm}_{\ell,m} = \frac{\ell(\ell+2)}{4(p+2)} - \frac{(m\pm 1)^2}{4(p+2)} + \frac{1}{8}$

$\ell = 0,1,\ldots,p$
$m = -\ell,-\ell+2,\ldots,\ell$ $\quad Q^{\pm}_{\ell,m} = \frac{m\pm 1}{p+2} \mp \frac{1}{2}$

• SIMILARLY FOR ANTIHOL. SECTOR

(• RAMOND GROUND STATES: $h = c/24 = p/8(p+2)$
$\Psi^+_{\ell,\ell}$; $\Psi^-_{\ell,-\ell}$ WITH $\Psi^-_{\ell-p,-(\ell-p)} \simeq \Psi^+_{\ell,\ell}$

- PROVES USEFUL: SEPARATE REPRESENTATIONS INTO TWO PIECES VIA "FERMION # MOD 2" (GEPNER)
  $G^{\pm}$ FERMION # = 1
  → DIVIDE STATES IN THE REP BY # of $G^{\pm}$ ACTIONS: EVEN vs ODD: INDEX "S"

- $$\left.\begin{matrix}\Phi_{\ell,m} \\ \psi^{\pm}_{\ell,m}\end{matrix}\right\} \Phi_{\ell,m,s} \begin{cases} s=0,2 & NS \\ s=\pm 1 & R \\ s \equiv s \pmod 4 \end{cases}$$

- $$\chi^{\ell}_{ms} = \mathrm{Tr}_{H^{\ell}_{ms}}\, e^{2\pi i \tau (L_0 - c/24)} = \sum (\text{STRING FCNS})(\theta\text{-FCNS}) \quad (\text{Qiu})$$
  $$\left(\chi^{\ell}_{ms} \text{ INV UNDER } \begin{matrix} m \to m+2(p+2) \\ s \to s+4 \end{matrix}\right)$$

- MODULAR TRANSFORMATION PROPERTIES (Qiu)

$$\chi^{\ell}_{ms}(\tau+1) = e^{i\pi\frac{\ell(\ell+2)}{2(p+2)}}\, e^{i\pi\left(\frac{-m^2}{2(p+2)}+\frac{s^2}{4}\right)} \chi^{\ell}_{ms}(\tau)$$

$$\chi^{\ell}_{ms}(-1/\tau) = \frac{1}{\sqrt{2}(p+2)} \sum_{\substack{\ell' m' s' \\ \ell'+m'+s' = 0 \bmod 2}} \sin\left(\pi\frac{(\ell+1)(\ell'+1)}{p+2}\right) e^{i\pi\left(\frac{mm'}{p+2} - \frac{ss'}{2}\right)} \chi^{\ell'}_{m's'}(\tau)$$

FACTORIZATION: ℓ → LEVEL p AFFINE SU(2); m → LEVEL $p+2$ θ FCNS; s → LEVEL 2 θ FCN

$$\Rightarrow Z = \frac{1}{2}\sum A_{\ell,\bar\ell}\, B^{m\bar m}\, C^{s\bar s}\, \chi^{\ell}_{ms}\, \chi^{*\,\bar\ell}_{\bar m \bar s}$$

$$= \frac{1}{2}\sum A_{\ell\bar\ell}\, \chi^{\ell}_{ms}\, \chi^{*\bar\ell}_{ms}$$

- "A-INVARIANT" : $A_{\ell\bar\ell} = \delta_{\ell\bar\ell}$

$$Z = \frac{1}{2}\sum \chi^{\ell}_{ms}\, \chi^{*\ell}_{ms}$$

NB: EQUIVALENTLY: $Z = \frac{1}{2}\sum \chi^{\ell}_{ms}\, \chi^{*\ell}_{-m-s}$ (CHANGE U(1) SIGN CONJUGATION IN ANTIHOL. SECTOR)

B) SYMMETRIES

N=2 MIN MODEL AT LEVEL P $\quad \mathbb{Z}_{P+2} \times \bar{\mathbb{Z}}_{P+2}$ symmetry

$$\mathbb{Z}^{DIAG}_{P+2}\quad \Phi^{\ell,\bar\ell}_{ms;\bar m\bar s} = e^{2\pi i (m+\bar m)/P+2}\, \Phi^{\ell,\bar\ell}_{ms;\bar m\bar s}$$

$$= e^{2\pi i (J_0 + \bar J_0)}\, \Phi^{\ell,\bar\ell}_{ms;\bar m\bar s}$$

C) ORBIFOLD:

$$Z[x,y] = {}^{e^{2\pi i (J_0+\bar J_0) x}}\underset{e^{2\pi i (J_0+\bar J_0) y}}{\square} \qquad x,y \in \mathbb{Z} \bmod (P+2)$$

$$= \sum_{\ell, m', s} e^{-2\pi i x \frac{[2m' - 2y]}{2(P+2)}}\, \chi^{\ell}_{m's}\, \chi^{*\ell}_{m'-2y,\,s}$$

• HOW GET THIS? $Z[x,0] = \mathrm{Tr}_{\mathcal{H}}\, e^{2\pi i x(J_0+\bar{J}_0)}\, e^{2\pi i \tau (L_0 - c/24)}\, e^{2\pi i \bar{\tau}(\bar{L}_0 - c/24)}$

• STATES IN $\chi^{\ell}_{ms}\chi^{*\ell}_{ms}$ HAVE $J_0+\bar{J}_0 = \text{INTEGER} + \frac{m+m}{p+2}$

$\Rightarrow Z[x,0] = \sum e^{2\pi i x m/p+2}\chi^{\ell}_{ms}\chi^{*\ell}_{ms}$

• THEN MODULAR TRANSFORM TO GET $Z[x,y]$ $\left({}^{x}\square_{y} \to {}^{ax+by}\square_{cx+dy} \quad \tau \to a\tau+b/c\tau+d \right)$

• NOW: BUILD NEW THEORY $Z^{NEW} \equiv Z/\mathbb{Z}_{p+2}$

• $Z^{NEW} = \frac{1}{p+2}\sum_{x,y=0,1,\dots,p+1} Z[x,y] = \frac{1}{p+2}\sum_{x,y}\sum_{m} e^{2\pi i x\frac{[m+\tilde{m}]}{2(p+2)}}\chi^{\ell}_{ms}\chi^{*\ell}_{\tilde{m}s}$

$\tilde{m} = m-2y$

• SUM ON x: $m+\tilde{m} \equiv 0\ (2(p+2))$

$2m-2y \equiv 0 \Rightarrow \tilde{m} = m-2y \equiv -m\ (2(p+2))$

$\Rightarrow Z^{NEW} = \frac{1}{p+2}\sum_{y}\sum_{m}\chi^{\ell}_{ms}\chi^{*\ell}_{-ms}$

• $s \to -s$ (NS: $s=0,2$ NOTHING; R: INTERCHANGE ψ^+ & ψ^-)

$= \sum_{m}\chi^{\ell}_{ms}\chi^{*\ell}_{-ms} = \sum\chi^{\ell}_{ms}\chi^{*\ell}_{-m-s}$

$\Rightarrow Z^{NEW} \overset{Q\to -Q}{\cong} Z^{ORIGINAL}$

• So:

HAVE EXPLAINED PRECISE MEANING

PROJECTION TO INTEGRAL U(1) CHARGES

$$\text{CY}:\ \sum z_i^{p_i+2}=0 \longleftrightarrow \left[\bigotimes_{i=1}^{5} MM_{p_i}\right]_{\text{ORBIFOLDED}}$$

$$\Big\updownarrow\ Q\to -Q \quad \text{NOW}$$

$$\left[\bigotimes_{i=1}^{5} \frac{MM_{p_i}}{\mathbb{Z}_{p_i+2}}\right]_{\text{ORBIFOLDED}}$$

$$\cong \quad \text{(FINAL STEP)} \to \text{DIRECT CALCULATION}$$

$$\left[\bigotimes_{i=1}^{5} MM_{p_i}\right]_{\text{ORBIFOLDED}} \Big/ H \qquad H \subset \prod \mathbb{Z}_{p_i+2}$$

H AS DEFINED

• Thus:

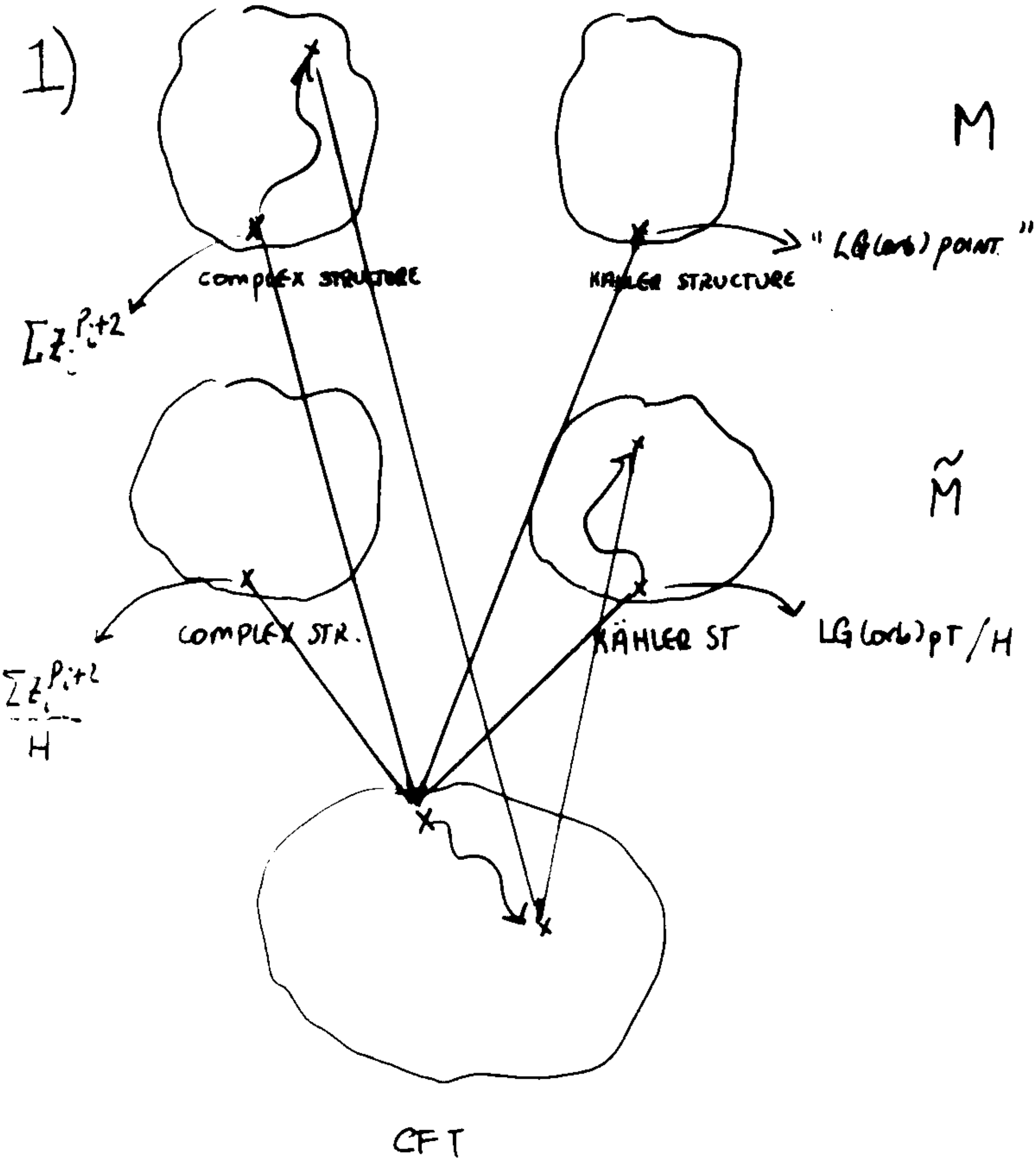
$$\begin{tikzcd}
\left[\otimes MM_{p_i}\right]_{\text{ORB}} = (P_1,\dots,P_5) \arrow[r] \arrow[d] \arrow[dr, dashed] & (P_1,\dots,P_5)/H \overset{Q\to -Q}{\cong} (P_1,\dots,P_5) \arrow[d] \\
\sum z_i^{p_i+2}=0 \arrow[r] & \sum z_i^{p_i+2}=0/H
\end{tikzcd}$$

$Q\to -Q$ RELATIVE TO

$$\overset{\shortparallel}{M} \qquad\qquad \overset{\shortparallel}{\tilde{M}} \qquad \text{MIRROR PAIR}$$

V.3) Remarks

1)



2) Construction works
- in any # dimensions
- using any SU(2) affine inv
- non hypersurface examples

3) MIRROR SYMMETRY IS *NOT* THE TRIVIAL CFT STATEMENT THAT WE CAN REPACE $Q \to -Q$.
THIS TRIVIAL CFT STATEMENT HAS A TRIVIAL GEOMETRIC COUNTERPART: $\underline{\underline{(M, T_M) \leftrightarrow (M, T_M^*)}}$

- RATHER: MIRROR SYMMETRY SAYS THAT $\exists$ 2 CY's $M, \tilde{M}$ S.T. CORRESPONDING CFT's DIFFER BY $Q \to -Q$.

  THEN THE TRIVIAL CFT STATEMENT ABOVE IS USEFUL TO CONCLUDE $M, \tilde{M} \to$ SAME $N=2$ SCFT.

4) OTHER CONSTRUCTIONS?

4.4) EXAMPLES

- QUINTIC IN $\mathbb{CP}^4$: $M = \sum Z_i^5 = 0$ in $\mathbb{CP}^4$ $\begin{cases} h_M^{1,1} = 1 \\ h_M^{2,1} = 101 \end{cases}$

  $H \subset (\mathbb{Z}_5)^5$; $H = (\mathbb{Z}_5)^3$

  $\tilde{M} = \sum Z_i^5 = 0 \,/\, (\mathbb{Z}_5)^3$ $\begin{cases} h_{\tilde{M}}^{1,1} = 101 \\ h_{\tilde{M}}^{2,1} = \underline{1} \end{cases}$

Symmetries	$h^{2,1}$	$h^{1,1}$	χ
	101	1	-200
[0,0,0,1,4]	49	5	-88
[0,1,2,3,4]	21	1	-40
[0,1,1,4,4] [0,1,2,3,4]	21	17	-8
[0,1,1,4,4]	17	21	8
[0,1,3,1,0] [0,1,1,0,3]	1	21	40
[0,1,4,0,0] [0,3,0,1,1]	5	49	88
[0,1,0,0,4] [0,0,1,0,4] [0,0,0,1,4]	1	101	200

Table 2

Orbifolds of $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ in $\mathbb{CP}^4$

NOTATION:

[0,0,0,1,4] MEANS: $(z_1, \dots, z_5) \longmapsto (z_1, z_2, z_3, \alpha z_4, \alpha^4 z_5)$

$\alpha^5 = 1$

M	$h^{2,1}$	$h^{1,1}$	S	M'
$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$	101	1	$\mathbb{Z}_5^5$	$M/(\mathbb{Z}_5^3)$
$z_1^5 + z_2^{10} + z_3^{10} + z_4^{10} + z_5^2 = 0$	145	1	$\mathbb{Z}_5 \times \mathbb{Z}_{10}^3 \times \mathbb{Z}_2$	$M/(\mathbb{Z}_{10}^2)$
$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ $z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0$	35	8	$\mathbb{Z}_3 \times \mathbb{Z}_9^3$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_9)$
$z_1^3 + z_2^4 + z_3^5 + z_4^6 + z_5^{20} = 0$	47	23	$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{20}$	$M/(\mathbb{Z}_2)$
$z_1^3 + z_2^4 + z_3^4 + z_4^{12} + z_5^{12} = 0$	89	5	$\mathbb{Z}_3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{12}^2$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_4^2)$
$z_1^4 + z_2^6 + z_3^{21} + z_4^{28} + z_5^2 = 0$	52	28	$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{21} \times \mathbb{Z}_{28}$	$M/(\mathbb{Z}_2)$
$z_1^5 + z_2^6 + z_3^{10} + z_4^{30} + z_5^2 = 0$	91	7	$\mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{30}$	$M/(\mathbb{Z}_2 \times \mathbb{Z}_{10})$

Table 1

Mirrors M' of Some Minimal Model Compactifications M

- WE THEREFORE HAVE EXPLICIT EXAMPLES OF

PHYSICALLY IDENTICAL IN STRING THEORY

ANY OBSERVABLE

FORMULATION #1

FORMULATION #2

$M_4 \times M_{CY}$ $M_4 \times \tilde{M}_{CY}$

MATHEMATICALLY DISTINCT SPACETIMES

- LET'S STUDY THIS