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BLACK HOLES AND GENERAL RELATIVITY

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BLACK HOLES AND GENERAL RELATIVITY: ICTP, 1994

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1 BLACK HOLES

'A stationary black hole is pure vacuum endowed only with mass, charge, and angular momentum. It is the simplest object in General Relativity'
see S Detweiler: 'Resource Letter BH-1: Black Holes' *Am Journ Phys* **49**, May 1981, pp394-400

The major issues from an astrophysical viewpoint:
1] what is the final state of massive stars?
2] what is the energy sources of qso's and agn's?
3] do Primordial black holes exist ?

The major discoveries from a general relativity viewpoint:
1] existence and implications of closed trapped surfaces (implying existence of singularities in space-time structure, Penrose 1965;)
2] existence and nature of event horizons and Killing horizons, and their bifurcations;
3] the non-trivial topologies and associated structures (e.g. wormholes) that necessarily occur in maximally extended solutions.

Major issues for other physics arise from:
1] Black body radiation emitted by black holes, arising from the synthesis of quantum field theory and curved space time geometries;
2] the associated connection of black holes to entropy and the laws of thermodynamics;
3] the question of the final state of evaporating black holes, and its implications for quantum mechanics.

References:

Gravitation C W Misner K S Thorne and J A Wheeler (Freeman, 1973)

The Large Scale Structure of Space Time S W Hawking and G F R Ellis (Cambridge University Press, 1973)

Black Holes ed B S de Witt and C de Witt (Gordon and Breach, 1973)

Black Holes White Dwarfs and Neutron Stars S L Shapiro and S A Teukolsky (Wiley-Interscience, 1983)

plus articles in

General relativity: An Einstein Centenary Survey Ed Hawking and Israel (Cambridge University Press, 1979)

Three Hundred Years of Gravity Ed S W Hawking and W Israel (Cambridge University Press, 1984).

For a very readable and clear account of the history of the development of the modern understanding of black holes, see *Black Holes and Time Warps.* K S Thorne (Norton, 1993).

2 BACKGROUND

Endpoint of Stellar evolution: when a star's nuclear fuel is used up, it cools down to a final state of 'Jupiters', white dwarfs, neutron stars [sometimes after supernova explosion], or black holes.

'Jupiters' are actually cold planets made of iron (lowest energy state of nucleus), with a maximum size about that of Jupiter.

White Dwarfs: small stars with pressure is due to relativistic electron degeneracy.

Neutron stars: highly condensed stars with pressure due to neutron degeneracy. They are produced in supernovae explosions that leave behind pulsars (spinning, magnetised neutron stars).

Mass limits:

Only if the star's mass is less than that of 1.4 suns can the attractive force of gravity be counteracted by electron degeneracy [Chandrasekhar, Maximum mass of white dwarf stars, *ApJ* **74**, 81-82 (1931); also MNRAS 91, 456 (1931) and Observatory 57, 373-377 (1934).

Similarly there is a maximum mass to neutron stars and it lies between about half a solar mass and several solar masses [Oppenheimer and Volkov, On Massive Neutron Cores, Phys Rev 55, 374-381 (1939)] (today: between 1.5 and 3 solar masses).

Wheeler: one can determine the equation of state of matter at the endpoint of thermonuclear evolution; then can understand all objects that can be made from cold dead matter. There is no third family of stable, massive cold dead objects between the white dwarfs and the neutron stars, and causality limits imply no other stable state exists even though we do not know the equation of state of nuclear matter in detail.

Massive stars do eject large amounts of matter but data suggest that most stars above 20 times the mass of the sun remain so heavy when they die that their pressure provides no protection against gravity; they are expected to collapse to black holes.

This problem arises *firstly* because of the negative specific heat of gravity (there is no maximum entropy state for a collapsing gravitating object; the highest entropy state is not a smooth distribution of matter);
secondly because of extra terms in the General relativity stellar equations (see below) compared with those in Newtonian theory, which make the problem of gravitational collapse much worse in GR than in Newtonian theory.

We first examine the case of spherical geometries, where we can identify most of the important ideas; and then the case of non-spherical geometries (axially symmetric situations, and generic asymptotically flat ones).

Gravity is a manifestation of space-time curvature; we see this explicitly in the static, spherical case. We look at the Schwarzschild solution, and its use for star models, solving the field equations in the interior and exterior cases, and matching them at their boundary.

We refer to MTW [*Gravitation*, C W Misner, K S Thorne and J A Wheeler (Freeman, 1972)] and ST [*Black Holes, White Dwarfs and Neutron Stars*, S Shapiro and S Teukolsky (Wiley Interscience, 1983)] as well as to Stephani [H Stephani, *General relativity* (Cambridge University Press)].

3 SCHWARZSCHILD EXTERIOR SOLUTION

To model the field of the sun in the solar system, or of any static star, we look for a solution that is

(1) static (ignore the time changes due to the motion of the planets)

(2) spherically symmetric (ignore the rotation of the sun)

(3) vacuum - the exterior solution (later we look for the corresponding interior solution)

We choose coordinates for which the metric form is

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. Thus

$$g_{\alpha\beta} = diag\left(-A(r),\ B(r),\ r^2,\ r^2\sin^2\theta\right),$$

$$g^{\alpha\beta} = diag\left(-1/A(r),\ 1/B(r),\ 1/r^2,\ 1/(r^2\sin^2\theta)\right),$$

This is clearly

(1) static ($\frac{\partial}{\partial t}$ is a Killing vector because the time coordinate t does not appear in the metric) and

(2) spherically symmetric, as $A(r)$, $B(r)$ are independent of θ and ϕ, and $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ is the metric of a [spherically symmetric] 2-sphere (with area $4\pi r^2$).

As regards (3): the functions $A(r)$ and $B(r)$ are to be determined by the vacuum Einstein Field Equations (EFE):

$$R_{bf} = 0 \tag{2}$$

where from section 6.1, on using a coordinate basis,

$$R_{bf} = R_f{}^a{}_{ba} = \Gamma^a{}_{bf,a} - \Gamma^a{}_{af,b} + \Gamma^e{}_{bf}\Gamma^a{}_{ae} - \Gamma^e{}_{af}\Gamma^a{}_{be} \tag{3}$$

To work this out in a coordinate basis, we need the Christoffel symbols, *either* from the Christoffel relations,

$$\Gamma^\mu{}_{\alpha\beta} = g^{\mu\nu}\frac{1}{2}(g_{\nu\alpha,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}),$$

or from the Lagrangian $L = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$ for geodesics. The non-zero $\Gamma^a{}_{bc}$ for metric (1) are

$$\Gamma^0{}_{10} = \frac{1}{2}\frac{A'(r)}{A(r)} = \Gamma^0{}_{01} \tag{4}$$

$$\Gamma^1{}_{00} = \frac{1}{2}\frac{A'}{B},\quad \Gamma^1{}_{11} = \frac{1}{2}\frac{A'}{A},\quad \Gamma^1{}_{22} = -\frac{r}{B},\quad \Gamma^1{}_{33} = -\frac{r\sin^2\theta}{B} \tag{5}$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r},\quad \Gamma^2{}_{33} = -\sin\theta\cos\theta \tag{6}$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r},\quad \Gamma^3{}_{23} = \Gamma^3{}_{23} = \frac{\cos\theta}{\sin\theta}. \tag{7}$$

Hence for example, from (3) R_{00} is

$$R_{00} = R_0{}^a{}_{0a} = \Gamma^a{}_{00,a} - \Gamma^a{}_{a0,0} + \Gamma^e{}_{00}\Gamma^a{}_{ae} - \Gamma^e{}_{a0}\Gamma^a{}_{0e} \tag{8}$$

and get the explicit form by substituting for the Christoffels (4-7): with these values, carrying out the summations,

$$R_{00} = \Gamma^1{}_{00,1} - \ 0\ + \Gamma^1{}_{00}\Gamma^a{}_{a1} - \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01}$$

which is

$$R_{00} = \left(\frac{1}{2}\frac{A'}{B}\right)' + \frac{1}{2}\frac{A'}{B}\left(\frac{1}{2}\frac{A'}{A} + \frac{1}{2}\frac{B'}{B} + \frac{1}{r} + \frac{1}{r}\right) - 2\left(\frac{1}{2}\frac{A'}{A}\right)\left(\frac{1}{2}\frac{A'}{B}\right);$$

hence

$$R_{00} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{A'}{rB} = 0 \tag{9}$$

Similarly we can obtain all the non-zero Ricci tensor components, giving the full set of EFE. The further non-trivial equations (as well as (9)) are

$$R_{11} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{B'}{rB} = 0 \tag{10}$$

$$R_{22} = -\frac{1}{B} + 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0 \tag{11}$$

$$R_{33} = +R_{22}\sin^2\theta = 0 \tag{12}$$

Now $\{(B/A) \times R_{00} + R_{11}\}$ gives

$$\frac{1}{r}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0 \;\Leftrightarrow\; A'B + B'A = 0 \;\Leftrightarrow\; (AB)' = 0$$

so

$$A(r)\,B(r) = const$$

Use the boundary conditions of asymptotic flatness: $r \to \infty$, $A(r) \to c^2$ (we introduce the speed of light c - which we usually take as 1 – explicitly here), $B(r) \to 1$, so $A(r)B(r) = c^2$ (determine the constant by its limiting value at infinity) Hence

$$B(r) = \frac{c^2}{A(r)} \tag{13}$$

Substitute into R_{22}: we get

$$-\frac{A}{c^2} + 1 - \frac{rA}{2c^2}\left(\frac{A'}{A} + \frac{A'}{A}\right) = 0$$

that is

$$rA' + A = c^2 \Leftrightarrow \frac{d}{dr}(rA) = c^2$$

which integrates to give

$$rA = c^2(r+k), \quad k = const \Leftrightarrow A(r) = c^2(1+\frac{k}{r}), \quad B(r) = (1+\frac{k}{r})^{-1}. \tag{14}$$

Hence we have,

$$ds^2 = -c^2(1+\frac{k}{r})dt^2 + (1+\frac{k}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{15}$$

NB: we must ensure that *all* the EFE are satisfied. We have solved two of the three non-trivial equations; now check that the last one is true (substitute into

R_{00} or R_{11} to check that both (9) and (10) are valid).

To determine the value of the constant k: compare with weak field:

$$g_{00} = -(1 + \frac{2\Phi}{c^2}), \quad \Phi = -\frac{MG}{r} \tag{16}$$

so

$$1 + \frac{k}{r} = 1 - \frac{2MG}{c^2 r} \quad \Leftrightarrow \quad k = -\frac{2MG}{c^2} \tag{17}$$

Define

$$m \equiv MG/c^2 > 0 \tag{18}$$

(mass in geometrical units, giving the one essential constant of the solution); then, setting $c = 1$,

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{19}$$

This is the Schwarzschild (Exterior) solution [K Schwarzschild 'On the gravitational field of a point mass according to the Einsteinian theory'. Sitz Preuss Akad Wiss Phys Mat Kl 189, 1898-196 (1916)].

Note 1: This is an *exact* solution of the EFE

Note 2: This solution is valid for $r > r_S$ where r_S is the coordinate radius of the massive object; we require that

$$r_S > r_G \equiv 2m \equiv 2MG/c^2 \tag{20}$$

where r_G is the gravitational radius or *Schwarzschild radius* of the object (if $r_S < r_G$, we would have a black hole; see later). This is the mass in geometrical units. Thus the Schwarzschild geometry predicts that for each star there is a critical circumference that depends on the stars mass.

For the earth: $2MG_{earth}/c^2 \simeq 8.8mm$, for the sun: $2MG_{sun}/c^2 \simeq 2.96km$, so in both cases $r_S >> r_G$. We see here a fundamental feature of the solution: that there is a characteristic length associated with the Schwarzschild solution. By contrast, there is no such length associated with the Newtonian equivalent solution ($\Phi(r) = -MG/r$, where $\Phi(r)$ is arbitrary by an additive constant, so e.g. $\Phi = 1$ has no special significance).

Note 3: Considering time-dependent spherical vacuum solutions, where now we havemetric (1) but with $A = A(r,t)$, $B = B(r,t)$, one can prove a Uniqueness theorem, namely : **Birkhoff's theorem**: The Einstein vacuum field equations then demand that in fact A and B are functions of only one variable. Consequently, A spherical vacuum solution is either static or spatially homogeneous (there is necessarily another independent Killing vector field, apart from those generating the spherical symmetry). Choosing the solution $A = A(r)$, $B = B(r)$ which corresponds to $r > 2m$, this shows that

> The Schwarzschild solution is the valid exterior solution for every spherical object, no matter how it is evolving: it can be static, collapsing, expanding, pulsating: always - provided it is spherically symmetric - the exterior solution is the Schwarzschild solution.

This shows the importance of the Schwarzschild geometry: it describes the exterior of any star that is spherical including not only static stars but also imploding, exploding, and pulsating ones. Thus for example the spacetime geometry outside an imploding spherical star is the same as that outside a static star [nb: there is no analogous theorem in the case of rotating stars]. This essentially follows from the fact the General Relativity does not allow monopole gravitational radiation: so spherical pulsations cannot radiate mass away.

Note 4: Relation to flat space time:

(a) If $m = 0$ we have just flat space-time, thus as $m \to 0$ we get Minkowski space as the (local) limit [globally there will be problems see later];

(b) For $r >> r_G$ we have the weak field case; the metric goes to perturbed flat space-time at large distances in the form

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 + \frac{2m}{r})dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (21)$$

(locally – but not globally – the spacetime is arbitrarily close to flat space time as we approach infinity).

We have a *Strong field* when we can't use this approximation, say $r < 10r_G$. So we have a weak field when $r_S >> r_G$. NB: the exterior fields of the Earth and of the Sun are weak fields!

3.1 Geometry of the solution

There are unique timelines - integral curves of the static timelike Killing vector $\partial/\partial t$ (which is timelike everywhere). These are orthogonal to preferred spacelike sections (surfaces of constant coordinate time t).

The Coordinates:

(1) t: - Time coordinate, but not proper time along the (preferred) static observer's world lines $\{r, \theta, \phi\ const \Leftrightarrow t \text{ only varies }\}$ In fact, from the metric we see that along those lines, where $dr = d\theta = d\phi = 0$, proper time τ (given by

$ds^2 = -d\tau^2$) is related to coordinate time t by

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 \quad \Leftrightarrow \quad \tau = \int_{t_1}^{t_2} (1 - \frac{2m}{r})^{1/2} dt < t_2 - t_1 \tag{22}$$

The ratio between proper time and coordinate time is thus position- dependent.

(2) r: - Distance, but not proper distance along the (unique) radial geodesics orthogonal to the spheres $\{r = const\}$ in the unique time-surfaces $t = const$ On these curves, $dt = d\theta = d\phi = 0$, so from the metric, proper distance R is related to coordinate distance r by

$$dR^2 = (1 - \frac{2m}{r})^{-1} dr^2 \quad \Leftrightarrow \quad R = \int_{r_1}^{r_2} (1 - \frac{2m}{r})^{-1/2} dr > r_2 - r_1 \tag{23}$$

Areas: r is in fact an *area coordinate*, because the surface area of the (uniquely defined) 2-spheres $t = const$, $r = const$ is just $4\pi r^2$ Equivalently, the circumference of these spheres is $c = 2\pi r$. We can thus characterise the geometry of the constant- time surfaces by the relation $c(R)$ of circumference to proper distance, given by the equation above. We can represent this by a curved surface in Euclidean space that is bent so as to correctly reproduce this relation [Stephani: Figure 10.1] This is an imbedding diagram (see Thorne): imagine imbedding the equatorial sheet ($\Theta = \pi/2$) of the 3-space in a fictitious flat 3-d hyperspace; then the sheet can maintain its curved geometry (the correct relation of circumference to radial distance) by bending downward like a bowl. Thus we can correctly express the circumference to radial distance relation in such an imbedding diagram (where the remaining coordinates are r and ϕ).

Together (1) and (2) show that 'gravity causes a warping of both time and space' - Thorne.

(3) θ and ϕ are usual angular coordinates on the 2-sphere (which has area $4\pi r^2$).

(4) Singularities: The metric form has singularities at $r = 0$ and at $r = 2m$. The first is a physical singularity, the second is a coordinate singularity associated with the event horizon, and can be removed by changing to other coordinates. We explore this later.

(5) Acceleration: the preferred timelike orbits of 'static observers' – those at rest in our chosen coordinate system – are Killing vector orbits (i.e. symmetry paths) without rotation, expansion, or distortion, but are accelerating (i.e. are non-geodesic). Their normalised (unit) tangent vector is

$$u^a = \frac{1}{\sqrt{1 - 2m/r}} \delta^a_0$$

and has acceleration

$$\dot{u}^a{}_{;b}u^b = u^a_{,b}u^b + \Gamma^a_{bc}u^b u^c = \frac{1}{1-2m/r}\Gamma^a_{00} = \delta^a_1 \frac{m}{r^2} \tag{24}$$

pointing radially out and with magnitude diminishing with distance (this is the acceleration we feel that keeps us firmly on the earth, instead of falling freely through to its centre).

4 INTERIOR SOLUTIONS

4.1 Junction conditions

A complete solution comprises interior (fluid) and exterior (vacuum) solutions, correctly joined at the surface of the star. By Birkhoff's theorem, the exterior solution will be the Schwarzschild exterior solution (independent of the structure of the star).

At the surface of the star: we require that the 1st and second fundamental forms are continuous - i.e. continuity of the metric and its first radial derivative, as measured in geodesic normal coordinates.

Now $p = 0$ is equivalent to the latter, because of the momentum conservation equations. This is the more usual form of the second condition; then we require

> The surface of the star occurs where $p = 0$; we demand that the metric tensor be continuous there.

There will be a jump in density there; consequently, the metric and its first radial derivative will be continuous there, but its second derivative will be discontinuous.

4.2 General equations

The Oppenheimer-Volkov form of the EFE for a static, spherically symmetric star [see MTW: pp. 608-609; ST, pp.124-126] are as follows.

The metric can be written as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{25}$$

for $r < r_S$, the value of r at the surface of the star, where

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr, \quad m(r_S) = m_S \tag{26}$$

is a mass parameter but is *not* the integral of density through the star with respect to proper volume (it differs from this by the gravitational binding energy of the star); and m_S is the exterior Schwarzschild mass. Equations of state relate the energy density ρ and pressure p to the number density n of the fluid:

$$p = p(n), \quad \rho = \rho(n) \quad \Rightarrow \quad \rho = \rho(p) \tag{27}$$

where n is the number density of the fluid. The Oppenheimer-Volkov equation of hydrostatic equilibrium - basically the momentum equation - is

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} \tag{28}$$

(Nb: the GR terms that make gravitational collapse so much worse than in the Newtonian case are the terms in p and $2m$ on the right hand side. The Newtonian equation is identical except that these terms do not occur in that case).

The source equation for $\Phi(r)$ (the acceleration equation) is

$$\frac{d\Phi}{dr} = -\frac{dP}{dr}\frac{1}{\rho + p} = \frac{(m + 4\pi r^3 p)}{r(r - 2m)} \tag{29}$$

The boundary conditions are

$$p(r = 0) = p_c, \quad p(r = r_S) = 0, \quad \Phi(r = r_S) = \frac{1}{2} ln(1 - 2m_S/r_S), \tag{30}$$

where p_c is the central pressure.

Procedure: choose equation of state, Φ_0, central pressure p_c. Integrate the coupled equations outwards until $p = 0$. This defines the star's surface; the value of the radius there is r_S; the value of the mass m there is the star's mass ($m_S = m(r_S)$). One can continue to integrate the equations into the vacuum region, regaining the exterior Schwarzschild solution there (up to a constant multiple of $e^{\Phi}(r)$). The result is a relativistic stellar model whose structure equations Φ, m, ρ, p, and n satisfy the equations of structure. One can renormalise $\Phi(r)$ by adding to it a constant so that it gives the standard Schwarzschild solution in the exterior region (and thus obeys the boundary condition given above).

In more detail: (26) is valid through the surface of the star and into the vacuum around it, showing that $m(r)$ increases in the interior of the star and then is constant for $r > r_S$. The Oppenheimer-Volkov equation (28) will be trivially satisfied in the vacuum (as p andρ are zero there); and in the vacuum region, the equation (29) for $\Phi(r)$ becomes

$$\frac{d\Phi}{dr} = \frac{m}{r(r - 2m)} = \frac{1}{2}\left(\frac{1}{r - 2m} - \frac{1}{r}\right) \tag{31}$$

(where m is const), so

$$\Phi(r) = \frac{1}{2}\ln\left(\frac{r-2m}{r}\right) + C \quad \Leftrightarrow \quad e^{2\Phi(r)} = (1-\frac{2m}{r})e^{-C} \tag{32}$$

where appropriate choice of Φ_0 can be used to set $C = 0$. Thus we regain the Schwarzschild exterior solution (21) from these equations; so we can integrate the OV equations in the star and through the surface to outside it, obtaining the complete interior plus exterior solution.

4.3 Schwarzschild interior solution

This is the simplest interior solution: See Stephani for details. In summary: we assume $\mu = const$ in the interior [rather than assuming a given equation of state]. The resulting *Interior Schwarzschild solution* can be solved analytically, and matched to the exterior solution to give a complete explicit solution. However it is not physically realistic because the pressure varies while the density stays constant. One can give an imbedding diagram for the complete solution [Stephani Figure 10.1].

Note 1: for a given total mass m the interior solution is regular only if the stellar radius r_S is large enough: a finite pressure everywhere demands that

$$r_S > \frac{9}{8}2m \tag{33}$$

(which agrees with what we expect: $r_S > r_G$).

Note 2: the spatial sections of the interior solution are spaces of constant curvature (as in the Robertson-Walker case).

5 PARTICLE ORBITS

Every freely moving particle (that is, every particle on which no forces except gravity acts) travels along a geodesic of space-time. Thus it is important to examine geodesics in space-time (both massive and massless, i.e. corresponding to particles moving at less than and equal to the speed of light) to understand the gravitational fields they represent. We shall work them out for a Schwarzschild exterior solution for values of $r > 2m$ (thus including the black hole case where r continues down to $r = 2m$). If the surface of a star or planet intervenes for some value r_* of r, the free particle orbit will stop (or start) there.

5.1 General equations

The Lagrangian for geodesics is

$$L = g_{ab}(x^c)\dot{x}^a\dot{x}^b$$

where $\dot{x}^a = dx^a/dv$, v an affine parameter along the geodesics. Choosing $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, in the Schwarzschild exterior case we have (from (19))

$$L = -(1 - \frac{2m}{r})\dot{t}^2 + (1 - \frac{2m}{r})^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2 \qquad (34)$$

where $\dot{f} = df/dv$. The geodesics are given by

$$\left(\frac{\partial L}{\partial x^a}\right) - \frac{d}{dv}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0 \qquad (35)$$

Set $a = 0$ to get

$$\frac{d}{dv}\left((1 - \frac{2m}{r})\dot{t}\right) = 0 \quad \Leftrightarrow \quad (1 - \frac{2m}{r})\,\dot{t} = k = const \qquad (36)$$

This is the particle energy equation. For a timelike or null geodesic, $k \neq 0$. Set $a = 1$ to get

$$\left(1 - \frac{2m}{r}\right)^{-1}\ddot{r} + \frac{m\dot{t}^2}{r^2} - \left(1 - \frac{2m}{r}\right)^{-2}\frac{m\dot{r}^2}{r^2} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0 \qquad (37)$$

This is the radial equation. Set $a = 2$ to get

$$\frac{d}{dv}(-r^2\dot{\theta}) + r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad (38)$$

Set $a = 3$ to get

$$\frac{d}{dv}(-r^2\sin^2\theta\dot{\phi}) = 0 \qquad (39)$$

The last two are the angular equations. There is also the fact that L is constant:

$$L = -\epsilon \qquad (40)$$

where $\epsilon = 1$ for timelike geodesics, $\epsilon = 0$ for null geodesics, that is

$$\epsilon = \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 \qquad (41)$$

This is a first integral of the other equations.

We can solve the angular equations by setting $\theta = \pi/2$ which implies $\dot{\theta} = 0$, and find

$$\theta = \pi/2, \quad r^2\dot{\phi} = h, \quad h = const. \qquad (42)$$

These equations are the equations of conservation of angular momentum (the first says that the motion takes place in a plane; the second shows conservation of angular momentum in that plane). The first integral (40) becomes

$$\epsilon = \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 \qquad (43)$$

5.1.1 Effective radial equation

Substituting the energy and angular momentum equations (36,42) into the integral (43) gives

$$\epsilon = \left(1 - \frac{2m}{r}\right)^{-1} k^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - \frac{h^2}{r^2} \tag{44}$$

which implies

$$(\epsilon + \frac{h^2}{r^2}) = \left(1 - \frac{2m}{r}\right)^{-1} (k^2 - \dot{r}^2) \tag{45}$$

i.e.

$$\left(1 - \frac{2m}{r}\right) (\epsilon + \frac{h^2}{r^2}) = k^2 - \dot{r}^2 \tag{46}$$

finally giving the effective radial equation of motion

$$\dot{r}^2 + V^2(h,r) = k^2 \tag{47}$$

where

$$V^2(h,r) \equiv \left(1 - \frac{2m}{r}\right) (\epsilon + \frac{h^2}{r^2}) \tag{48}$$

This gives the qualitative discussion of MTW Chapter 25 [also ST Chapter 12], determining the allowed range of r for each set of values (h,k): First choose ϵ (what type of particle are we considering?), then choose the angular momentum h. Now plot the curves $V^2(h,r)$; the allowed range of r, for any given energy k (and the chosen value of h) is the set of r-values where

$$\dot{r}^2 = k^2 - V^2(h,r) \geq 0$$

Now $V^2 \to 0$ as $r \to 2m$ and $V^2 \to \epsilon$ as $r \to \infty$. To see what happens we need to locate the minima/maxima of $V^2(h,r)$; there may be two, one, or no extrema. Now

$$\frac{d}{dr} V^2(h,r) = \frac{2m}{r^2}(\epsilon + \frac{h^2}{r^2}) + \left(1 - \frac{2m}{r}\right)(-2\frac{h^2}{r^3}) \tag{49}$$

that is

$$\frac{d}{dr} V^2(h,r) = \frac{2}{r^4}\left(m\epsilon r^2 - h^2 r + 3mh^2\right) \tag{50}$$

We need now to find the zeros of the rhs, giving the extreme values r_* of r; the results are different if the orbits are timelike or null. We will examine these separately below. Also

$$\frac{d^2}{dr^2} V^2(h,r) = -\frac{4}{r}\frac{d}{dr} V^2(h,r) + \frac{2}{r^4}\left(2m\epsilon r - h^2\right) \tag{51}$$

so at an extremum $r = r_*$ we have

$$\frac{d^2}{dr^2}V^2(h, r_*) = \frac{2}{r_*^4}\left(2m\epsilon r_* - h^2\right) \tag{52}$$

determining the nature of the extremum.

5.1.2 Radial motion

In this special case: $h = 0$ (angular momentum vanishes) so $\dot{\theta} = 0 = \dot{\phi}$. Now the integral (43) becomes

$$\epsilon = \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 \tag{53}$$

that is

$$\dot{r}^2 + \epsilon\left(1 - \frac{2m}{r}\right) = k^2 \tag{54}$$

giving $dr/dv = \dot{r}$ on radial geodesics in the form

$$\frac{dr}{dv} = \left(k^2 - \epsilon(1 - \frac{2m}{r})\right)^{1/2} \tag{55}$$

In terms of coordinate time: by (36), $\dot{t} = dt/dv = k/(1 - \frac{2m}{r})$ so

$$\frac{dr}{dt} = \frac{dr}{dv}\frac{dv}{dt} = (1 - \frac{2m}{r})\left(1 - \frac{\epsilon}{k^2}(1 - \frac{2m}{r})\right)^{1/2} \tag{56}$$

relates the radial and time coordinates on radial geodesics.

In these cases the orbital shape is simply a radial line, movingaway from or toward the centre.

5.1.3 Orbital Shape Equations

When $h \neq 0$ (angular momentum is non-zero), on dividing by $\dot{\phi}^2 = h^2/r^4$, the first integral equation (43) becomes

$$\frac{\epsilon}{\dot{\phi}^2} = (1 - \frac{2m}{r})\frac{\dot{t}^2}{\dot{\phi}^2} - (1 - \frac{2m}{r})^{-1}\frac{\dot{r}^2}{\dot{\phi}^2} - r^2 \tag{57}$$

Now $\dot{r}/\dot{\phi} = dr/d\phi$, so on using equation (42) we get

$$\frac{\epsilon r^4}{h^2} - (1 - \frac{2m}{r})^{-1}\frac{k^2 r^4}{h^2} + (1 - \frac{2m}{r})^{-1}\left(\frac{dr}{d\phi}\right)^2 + r^2 = 0 \tag{58}$$

giving

$$\left(\frac{dr}{d\phi}\right)^2 - \frac{k^2r^4}{h^2} + r^2(1-\frac{2m}{r}) + (1-\frac{2m}{r})\frac{\epsilon r^4}{h^2} = 0 \tag{59}$$

which is

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = \frac{r^4}{h^2}(k^2-\epsilon) + \frac{2m\epsilon r^3}{h^2} + 2mr$$

Now put $r = 1/u$ which gives $dr/d\phi = (-1/u^2)(du/d\phi)$, so $(dr/d\phi)^2 = (1/u^4)(du/d\phi)^2$, to get

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{h^2}(k^2-\epsilon) + \frac{2m\epsilon}{h^2}u + 2mu^3 \tag{60}$$

Recalling $m = MG/c^2$ and defining $E = \frac{1}{h^2}(k^2-\epsilon)$, we find

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E + \frac{2MG\epsilon}{h^2}u + \frac{2MG}{c^2}u^3 \tag{61}$$

This is the first form of the GR orbital equation, like an energy equation, determining its shape $u(\phi) \Leftrightarrow r(\phi)$ (and valid for both timelike and null geodesics). The last term on the right contains the General relativity effects (the other terms are identical to the Newtonian case).

Differentiating (61) with respect to ϕ and dividing by $\frac{du}{d\phi}$ (non-zero as $h \neq 0$), we find

$$\frac{d^2u}{d\phi^2} + u = \frac{MG\epsilon}{h^2} + \frac{MG}{c^2}3u^2 \tag{62}$$

This is the second form of the orbital equation, now like a force equation; again the last term on the right contains the GR effects.

These equations apply to both massive and massless particles. We look at them in turn.

5.2 Massive particles

For massive particles, set $\epsilon = 1$; then the affine parameter is proper time τ.

5.2.1 Radial motion

$h = 0$. Determine the constant k in (54): Starting at rest at r_0, then $\{r = r_0, \dot{r} = 0\} \Rightarrow k^2 = \epsilon(1 - \frac{2m}{r_0})$ Thus

$$\frac{1}{2}\dot{r}^2 = \epsilon\frac{MG}{c^2}(\frac{1}{r} - \frac{1}{r_0}) \tag{63}$$

which leads to

$$(\frac{d\tau}{dr})^2 = \frac{1}{2m}\left(\frac{rr_0}{r_0 - r}\right) \tag{64}$$

Thus the proper time taken to get from r_0 to r is

$$\tau = \frac{1}{\sqrt{2m}}\int_r^{r_0}\left(\frac{rr_0}{r_0 - r}\right)^{1/2} dr \tag{65}$$

This is finite even if $r \to 2m$ [see ST, p.343-344].

5.2.2 Effective radial equations

When $h \neq 0$, so the particles are moving around the centre, the effective potential (48) is

$$V^2(h, r) \equiv \left(1 - \frac{2m}{r}\right)(1 + \frac{h^2}{r^2}) \tag{66}$$

so $V \to 1$ as $r \to \infty$. To see the shape of the potential: from (50),

$$\frac{d}{dr}V^2(h, r) = \frac{2m}{r^4}\left(r^2 - \frac{h^2}{m}r + 3h^2\right) \tag{67}$$

Given h, this is zero for

$$r = r_* = \frac{h^2}{2m} \pm h\sqrt{\frac{h^2}{4m^2} - 3} \tag{68}$$

At such points, from (51)

$$\frac{d^2}{dr^2}V^2(h, r_*) = \frac{2}{r_*^4}(2mr_* - h^2) \tag{69}$$

Thus existence of an extremum demands $h^2 \geq 12m^2$; in the limiting case $h^2 = 12m^2$, and $r_* = -\frac{h^2}{2m} = 6m$, which is a minimum. We can solve $dV^2(h, r)/dr = 0$ for h_*^2 (the angular momentum required to get a circular orbit at that radius), to get (from (66))

$$\frac{dV^2(h_*, r)}{dr} = 0 \quad \Leftrightarrow \quad h_*^2 = \frac{mr^2}{r - 3m} \tag{70}$$

showing circular orbits can exist down to $r = 3m$. Putting this together, for $h^2 > 12m^2$ there is a minimum at r_- where $r_- > 6m$, and a maximum at r_+ where $3m < r_+ < 6m$. Thus stable circular orbits are possible for all $r \geq 6m$, while unstable circular orbits are possible for $3m < r < 6m$.

> Hence the minimum radius for a stable circular orbit is $r = 6m$. The limiting minimum radius for an unstable circular orbit is $r = 3m$.

[MTW, pp. 655-672; ST, pp. 344-348] Furthermore when $h^2 < 12m^2$ there is no minimum or maximum: the orbit goes from $R = 2m$ to infinity (or vice versa). The shape of the effective potential curves is shown in MTW, pp. 639 and 662.

from:
GRAVITATION:
Misner Thorne & Wheeler

Figure 25.2.
Effective potential for motion of a test particle in the Schwarzschild geometry of a concentrated mass M. Energy, in units of the rest mass μ of the particle, is denoted $\tilde{E} = E/\mu$; angular momentum, $\tilde{L} = L/\mu$. The quantity r denotes the Schwarzschild r coordinate. The effective potential (also in units of μ) is defined by equation (25.16) or, equivalently, by the equation

$$\left(\frac{dr}{d\tau}\right)^2 + \tilde{V}^2(r) = \tilde{E}^2$$

(see also §25.5) and has the value

$$\tilde{V} = [(1 - 2M/r)(1 + \tilde{L}^2/r^2)]^{1/2}.$$

It represents that value of $\tilde{E}$ at which the radial kinetic energy of the particle, at r, reduces to zero ($\tilde{E}$-value that makes r into a "turning point": $\tilde{V}(r) = \tilde{E}$. Note that one could equally well regard $\tilde{V}^2(r)$ as the effective potential, and define a turning point by the condition $\tilde{V}^2 = \tilde{E}^2$. Which definition one chooses depends on convenience, on the intended application, on the tie to the archetypal differential equation $\frac{1}{2}\dot{x}^2 + V(x) = E$, and on the stress one wishes to put on correspondence with the effective potential of Newtonian theory). Stable circular orbits are possible (representative point sitting at minimum of effective potential) only for $\tilde{L}$ values in excess of $2\sqrt{3}\,M$. For any such fixed $\tilde{L}$ value, the motion departs slightly from circularity as the energy is raised above the potential minimum (see the two heavy horizontal lines for $\tilde{L} = 3.75\,M$). In classical physics, the motion is limited to the region of positive kinetic energy. In quantum physics, the particle can tunnel through the region where the kinetic energy, as calculated classically, is negative (dashed prolongations of heavy horizontal lines) and head for the "pit in the potential" (capture by black hole). Such tunneling is absolutely negligible when the center of attraction has any macroscopic dimension, but in principle becomes important for a black hole of mass 10^{17} g (or 10^{-11} cm) if such an object can in principle exist.

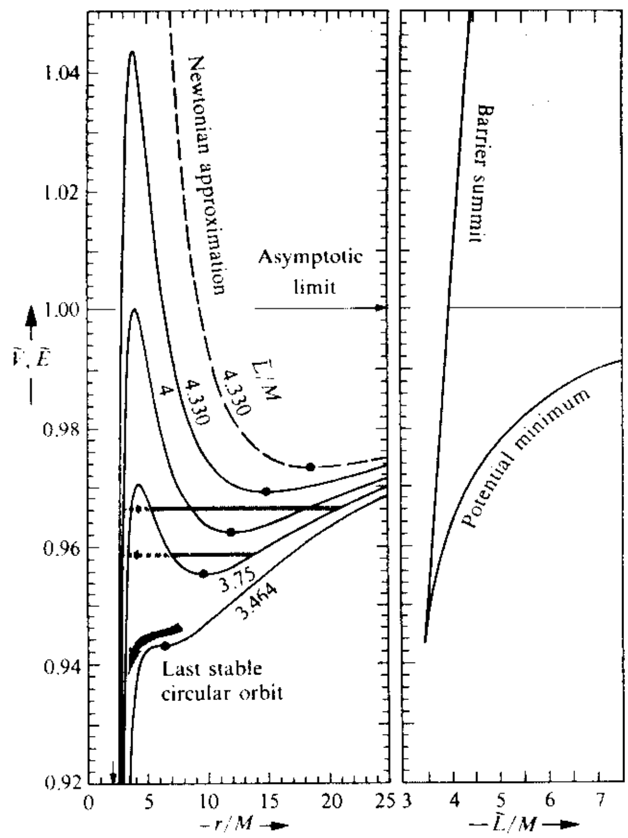
The diagram at the right gives values of the minimum and maximum of the potential as they depend on the angular momentum of the test particle. The roots of $\partial\tilde{V}/\partial r$ are given in terms of the "reduced angular momentum parameter" $L^\dagger = \tilde{L}/M = L/M\mu$ by

$$r = \frac{6M}{1 \mp (1 - 12/L^{\dagger 2})^{1/2}},$$

$$\tilde{E}^2 = \frac{(L^{\dagger 2} + 36) \pm (L^{\dagger 2} - 12)(1 - 12/L^{\dagger 2})^{1/2}}{54}$$

$[= 8/9$ for $L^\dagger = (12)^{1/2}$; 1 for $L^\dagger = 4$; $(L^{\dagger 2}/27) + (1/3) + (1/L^{\dagger 2}) + \cdots$ for $L^\dagger \longrightarrow \infty]$

(plus root for maximum of the effective potential; minus root for minimum; see exercise 25.18).



Box 25.6 (*continued*)

1. Orbits with periastrons at $r \gg M$ are Keplerian in form, except for the periastron shift (exercise 25.16; §40.5) familiar for Mercury.
2. Orbits with periastrons at $r \lesssim 10M$ differ markedly from Keplerian orbits.
3. For $\tilde{L}/M \leq 2\sqrt{3}$ there is no periastron; any incoming particle is necessarily pulled into $r = 2M$.
4. For $2\sqrt{3} < \tilde{L}/M < 4$ there are bound orbits in which the particle moves in and out between periastron and apastron; but any particle coming in from $r = \infty$ (unbound; $\tilde{E}^2 \geq 1$) necessarily gets pulled into $r = 2M$.
5. For $L^\dagger = \tilde{L}/M > 4$, there are bound orbits; particles coming in from $r = \infty$ with

$$\tilde{E}^2 < \tilde{V}_{\max}{}^2 = (1 - 2u_m)(1 + L^{\dagger 2}u_m{}^2),$$

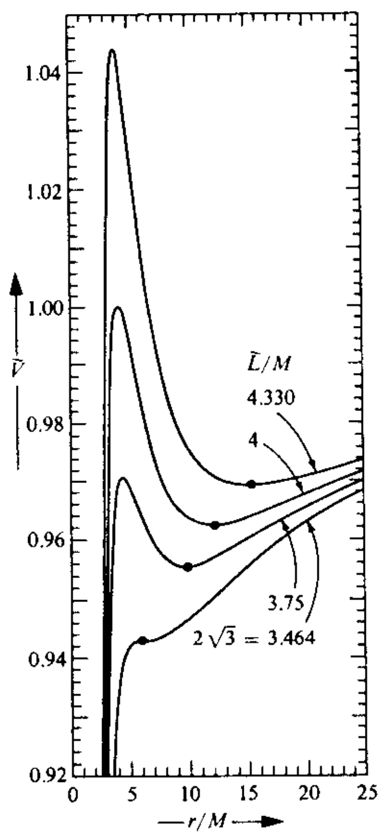
$$u_m \equiv \frac{1 + \sqrt{1 - 12/L^{\dagger 2}}}{6}$$

reach periastrons and then return to $r = \infty$; but particles from $r = \infty$ with $\tilde{E}^2 > \tilde{V}_{\max}{}^2$ get pulled into $r = 2M$.

6. There are stable circular orbits at the minimum of the effective potential; the minimum moves inward from $r = \infty$ for $\tilde{L} = \infty$ to $r = 6M$ for $L^\dagger = \tilde{L}/M = 2\sqrt{3}$. The most tightly bound, stable circular orbit ($\tilde{L}/M = 2\sqrt{3}$, $r = 6M$) has a fractional binding energy of

$$\frac{\mu - E}{\mu} = 1 - \tilde{E} = 1 - \sqrt{8/9} = 0.0572.$$

7. There are unstable circular orbits at the maximum of the effective potential; the maximum moves outward from $r = 3M$ for $\tilde{L} = \infty$ to $r = 6M$ for $\tilde{L}/M = 2\sqrt{3}$. A particle in such a circular orbit, if perturbed inward, will spiral into $r = 2M$. If perturbed outward, and if it has $\tilde{E}^2 > 1$, it will escape to $r = \infty$. If perturbed outward, and if it has $\tilde{E}^2 < 1$, it will either reach an apastron and then enter a spiraling orbit that eventually falls into the star (e.g., if $\delta\tilde{E} > 0$. with unchanged angular momentum); or it will move out and in between apastron and periastron, in a stable bound orbit (e.g., if $\delta\tilde{E} < 0$, again with unchanged angular momentum).



5.2.3 Orbital equations

The equation (62) for timelike orbits is

$$\frac{d^2u}{d\phi^2} + u = \frac{MG}{h^2} + \frac{MG}{c^2}3u^2 \tag{71}$$

This successfully gives the Newtonian limit (predicting the elliptical orbits of galaxies) and is indeed more accurate than Newtonian theory in that it also predicts the previously unexplained part of the perihelion precession of Mercury and other planets.

NB: this confirms the remarkable picture of highly curved spatial orbits arising from geodesics in space-time – because the space-time is curved. Thus for example the almost circular orbit of the earth around the sun represents an undeviating path in space-time.

5.2.4 Circular motion

This occur for values of r where $dV^2/dr = 0$ provided the energy k satisfies

$$\dot{r}^2 = 0 \quad \Leftrightarrow \quad V^2 = k^2 \tag{72}$$

(from (47)). From the radial equation (37) with $\theta = \pi/2$,

$$\frac{m^2\dot{t}^2}{r^2} = r\dot{\phi}^2 \tag{73}$$

so

$$m^2\dot{t}^2 = r^3\dot{\phi}^2$$

where r is constant. Thus

$$(\frac{d\phi}{dt})^2 = \frac{m}{r^3} \tag{74}$$

so for a whole circular orbit, where $\Delta\phi = 2\pi$, and

$$\Delta t = 2\pi\left(\frac{r^3}{m}\right)^{1/2} \tag{75}$$

which is Kepler's law for the relativistic orbit. The spatial distance travelled in one orbit is $2\pi r$ (as this is the circumference of the circle).

These equations tell us how freely falling particles will move in a weak field, for example the solar system, and also in a strong gravitational field: for example, when being accreted by a black hole. One can envisage here a cloud of particles falling in with all values of h and k; then some particles will hang around at radii near $r = 6m$ for a while, surrounding the centre by a cloud of particles, before falling in or escaping.

5.3 Massless particles

In this case $\epsilon = 0$ and the affine parameter is undetermined up to a constant multiple. We can therefore conveniently renormalize this parameter to set k or h to a chosen value.

5.3.1 Radial motion

$d\theta = d\phi = 0$ and from (53)

$$0 = -(1 - \frac{2m}{r})\dot{t}^2 + (1 - \frac{2m}{r})^{-1}\dot{r}^2 \tag{76}$$

i.e.

$$\frac{dt}{dr} = \pm\frac{1}{(1 - \frac{2m}{r})} \tag{77}$$

This shows how the gravitational potential controls the radial speed of photons in terms of these coordinates (they are of course all moving locally at the speed of light; but these coordinates do not directly measure proper distance or time). In terms of these coordinates, the particle slows down indefinitely as $r \to 2m$.

5.3.2 Effective Radial equations

When $h \neq 0$ the effective radial potential (48) is

$$V^2(h,r) \equiv \left(1 - \frac{2m}{r}\right)\frac{h^2}{r^2} \tag{78}$$

so $V \to 0$ as $r \to \infty$. We can normalise h as desired, e.g. to $27m^2$, so the effective potential is essentially the same for all photons (unlike the massive case, whereit depends essentially on h). To see its shape, note that from (49), (50)

$$\frac{d}{dr}V^2(h,r) = \frac{2h^2}{r^4}(-r+3m), \quad \frac{d^2}{dr^2}V^2(h,r_*) = -\frac{2h^2}{r_*^4} \tag{79}$$

Thus the only extremum is a maximum that occurs when

$$r = r_+ = 3m, \quad V^2 = V_*^2 = \frac{1}{3}\frac{h^2}{9m^2} \tag{80}$$

and unstable circular orbits are possible for light rays travelling at $r = 3m$ (and only at that radius) – they are held at this distance by gravitational attraction as they orbit at the speed of light [See ST, pp. 350-353]. For $k^2 < V_*^2$ we can get light rays that are highly bent by the central mass, reaching a minimum radius and the returning to infinity. For $k^2 > V_*^2$, they plunge in to $r = 2m$.

This value should be compared with the geometrical capture cross section of a particle by a sphere of radius R in Newtonian theory:

$$\sigma_{\text{Newt}} = \pi R^2\left(1 + \frac{2M}{v_\infty^2 R}\right). \tag{12.4.39}$$

A black hole thus captures nonrelativistic particles like a Newtonian sphere of radius $R = 8M$.

from: SHAPIRO & TEUKOLSKY

12.5 Massless Particle Orbits in the Schwarzschild Geometry

For $m = 0$ (e.g., a photon), Eqs. (12.4.6)–(12.4.8) become

$$\frac{dt}{d\lambda} = \frac{E}{1 - 2M/r}, \tag{12.5.1}$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}, \tag{12.5.2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.3}$$

Now by the Equivalence Principle, we know that the particle's worldline should be independent of its energy. We can see this by introducing a new parameter

$$\lambda_{\text{new}} = l\lambda. \tag{12.5.4}$$

Writing

$$b \equiv \frac{l}{E} \tag{12.5.5}$$

and dropping the subscript "new," we find

$$\frac{dt}{d\lambda} = \frac{1}{b(1 - 2M/r)}, \tag{12.5.6}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}, \tag{12.5.7}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.8}$$

The worldline depends only on the parameter b, which is the particle's *impact parameter*, and not on l or E separately. Taking the limit $m \to 0$ of Eq. (12.4.35), we see that b of Eq. (12.5.5) is the same quantity defined in the previous section for massive particles.

We can understand photon orbits by means of an effective potential

$$V_{\text{phot}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right), \tag{12.5.9}$$

so that Eq. (12.5.8) becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V_{\text{phot}}(r). \tag{12.5.10}$$

Clearly the distance from a horizontal line of height $1/b^2$ to V_{phot} gives $(dr/d\lambda)^2$. The quantity V_{phot} has a maximum of $1/(27M^2)$ at $r = 3M$; it is displayed in Figure 12.5. We see that the critical impact parameter separating capture from scattering orbits is given by $1/b^2 = 1/(27M^2)$, or

$$b_c = 3\sqrt{3}\,M. \tag{12.5.11}$$

The capture cross section for photons from infinity is thus

$$\sigma_{\text{phot}} = \pi b_c^2 = 27\pi M^2. \tag{12.5.12}$$

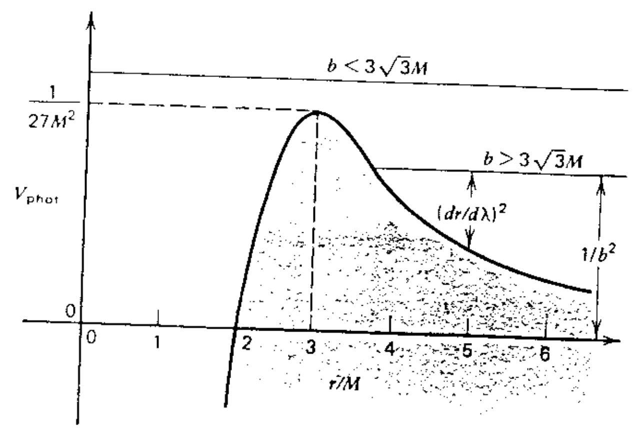


Figure 12.5 Sketch of the effective potential profile for a particle with *zero* rest mass orbiting a Schwarzschild black hole of mass M. If the particle falls from $r = \infty$ with impact parameter $b > 3\sqrt{3}\,M$ it is scattered back out to $r = \infty$. If, however, $b < 3\sqrt{3}\,M$ the particle is captured by the black hole.

5.3.3 Orbital equations

The equation (62) for the null geodesic orbit when $h \neq 0$ is

$$\frac{d^2u}{d\phi^2} + u = \frac{MG}{c^2} 3u^2 \tag{81}$$

This implies the bending of light when $M \neq 0$, which in turn predicts the gravitational lensing effect that has become such an interesting part of modern cosmology (see *Gravitational Lenses* by P Schneider, J Ehlers,and E Falco (Springer, 1992).

These equations tell us how light will move in a weak field, for example the solar system, and also in a strong gravitational field: for example, when being accreted by a black hole.

5.4 Tests

General relativity passes all the tests based on these equations successfully: the bending of light, perihelion precession, radar ranging in the solar system. [To carry out these tests in detail one must calculate all the perturbations caused by the other planets, the interplanetary medium, the rotation of the sun, and so on; this has been done in great detail.] Thus the space-time geometry of the solar system is accurately described by the Schwarzschild solution.

Additionally GR explains (a) the binary pulsar orbital data, taken as evidence for the emission of gravitational radiation that causes orbital decay, and (b) is to some extent tested by its successful use in cosmological models (in particular, through the successful nucleosynthesis calculations).

There is no experimental evidence at the classical level that disagrees with the predictions of general relativity [see the book on experimental tests of GR by Clifford Will].

5.5 Spectral shift

Consider a radial null ray (light ray) from a source that is stationary at $r = r_E$ to a receiver that is stationary at $r = r_R$.

Integrating the radial equation (77):

$$t_R - t_E = \int_{t_E}^{t_R} dt = \int_{r_E}^{r_R} \frac{dr}{(1 - \frac{2m}{r})} \tag{82}$$

also

$$(t_R + \Delta t_R) - (t_E + \Delta t_E) = \int_{t_E+\Delta t_E}^{t_R+\Delta t_R} dt = \int_{r_E}^{r_R} \frac{dr}{(1 - \frac{2m}{r})} = t_R - t_E \tag{83}$$

therefore

$$\Delta t_R = \Delta t_E \tag{84}$$

But these are coordinate times! what about proper times?

$$d\tau_E = (1 - \frac{2m}{r_E})^{1/2} dt_E$$

and also

$$d\tau_R = (1 - \frac{2m}{r_R})^{1/2} dt_R$$

so

$$\frac{d\tau_E}{d\tau_R} = \left(\frac{1 - \frac{2m}{r_E}}{1 - \frac{2m}{r_R}}\right)^{1/2} \tag{85}$$

which holds for arbitrary fixed emitters and receivers in a Schwarzschild solution .

If emitter is a pulsating atom and emits n pulses in time $\Delta\tau_E$ then the proper frequency is $\nu_E = n/\Delta\tau_E$. The receiver sees them in time $\Delta\tau_R$, and measures an apparent frequency $\nu_R = n/\Delta\tau_R$. Then we have

$$\frac{\nu_R}{\nu_E} = \frac{\Delta\tau_E}{\Delta\tau_R} = \left(\frac{1 - 2m/r_E}{1 - 2m/r_R}\right)^{1/2} \tag{86}$$

This immediately gives the (gravitational) redshift z, defined in terms of emitted and received wavelengths by

$$1 + z = \frac{\lambda_R}{\lambda_E} = \frac{\nu_E}{\nu_R} \tag{87}$$

So if $r_R > r_E$ then $\nu_R < \nu_E \Leftrightarrow \lambda_R > \lambda_E$: redshift
and if $r_R < r_E$ then $\nu_R < \nu_E \Leftrightarrow \lambda_R < \lambda_E$: blueshift.

This has been tested on Earth, by using an accurate radiation source at the bottom of the Harvard tower and an accurate receiver at the top. It is in principle also measurable for massive stars.

Thus: time warpage leads to gravitational redshift (Thorne). Furthermore we see that light emitted from the critical circumference is shifted in wavelength an infinite amount. A star as small as the critical radius must therefore appear completely dark, when viewed from far away (since intensity decreases as the redshift to the fourth power); it must be a black hole.

5.6 The size of the sky

A photon escapes to infinity if either (i) $\dot{r} > 0$ or (ii) (i) $\dot{r} < 0$ and $k^2 < V * 2$ (cf. equation (78)). Translating this into angles in the sky relative to a local orthonormal frame [ST, pp.353-354] we see that inward moving photons in a black hole situation (the vacuum extends to critical radius $r = 2m$) will escape to infinity iff

$$\sin\psi > \frac{3\sqrt{3}M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}; \tag{88}$$

otherwise it falls into the black hole at $r = 2m$. Thus at $r = 6m$, escape requires $\psi < 135^0$; at $r = 3m$, $\psi < 90^0$, so that all inward going photons are captured by the black holes. Thus considering the time-reversed photons, at that radius, half the sky appears dark. Similarly an outward going photon emitted between $r = 3m$ and $r = 2m$ escapes iff

$$\sin\psi < \frac{3\sqrt{3}M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}; \tag{89}$$

Thus as one approaches $r = 2m$, the sky closes up; only outward directed radial photons escape as the source approaches $r = 2m$, and in that limit, an observer will see the entire sky covered by the black hole.