INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.762 - 40

IV

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

13 June - 29 July 1994

AN INTRODUCTION TO QUANTUM GEOMETRY

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Please note: These are preliminary notes intended for internal distribution only.

LECTURE 4

SUMMARY OF LECTURES 1,2,3:

I) INTRODUCTION/MOTIVATION

II) THE $N=2$ SUPERCONFORMAL ALGEBRA

- GENERAL PROPERTIES (REP THEORY; (C,C), (a,c) RINGS, etc.)
- EXAMPLES (FREE, CY σ-MODEL, L-G, MIN MODELS)

III) FAMILIES OF $N=2$ THEORIES

- TRULY MARGINAL OPERATORS (BASED ON (C,C), (a,c) FIELDS
- MODULI SPACES I: CFT/NLσM

IV) INTERRELATIONS BETWEEN REALIZATIONS OF $N=2$ SCA

- LG/MIN MODELS $\left(W_{LG}=\Psi^{P+2} \longleftrightarrow \text{LEVEL } P \text{ (A-INV) MM}\right.$
- LG/CALABI-YAU $\left(W_{LG}(\Psi_1,\ldots,\Psi_5)\right) \underset{\text{RELATED}}{\overset{\text{"PHASE"}}{\longleftrightarrow}} CY: W_{LG}(z_1,\ldots,z_5)=0$
- MODULI SPACE II

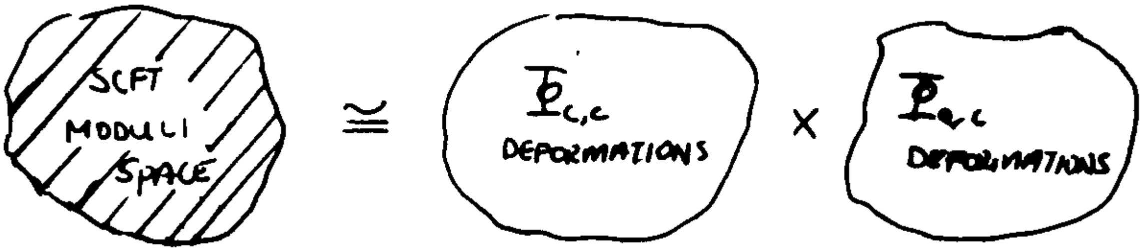
V) MIRROR MANIFOLDS

- MOTIVATION, • CONSTRUCTION
- MODULI SPACE III
- IMPLICATIONS

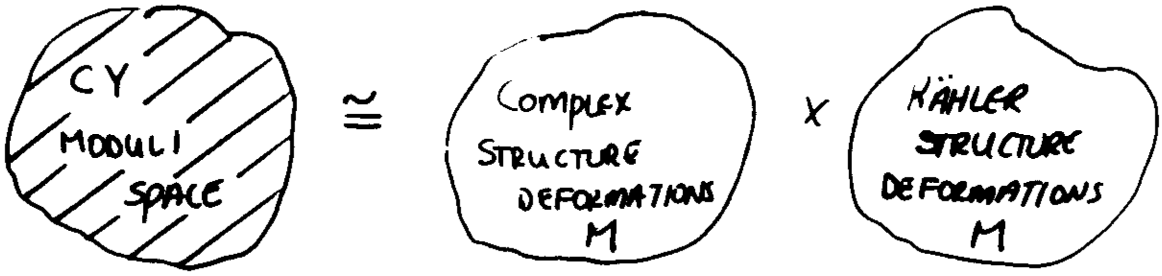
VI) SPACETIME TOPOLOGY CHANGE

Moduli Spaces I :

SCFT :



GEOMETRY :



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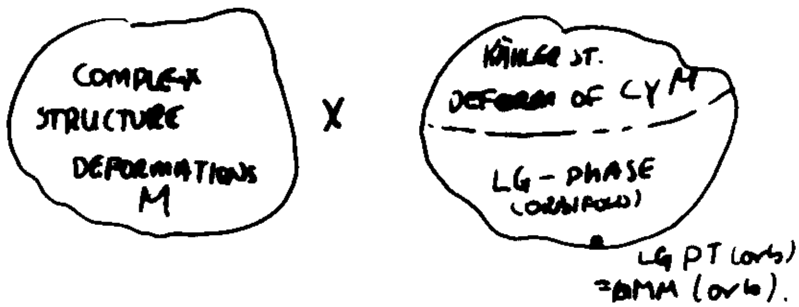
Moduli Spaces II :



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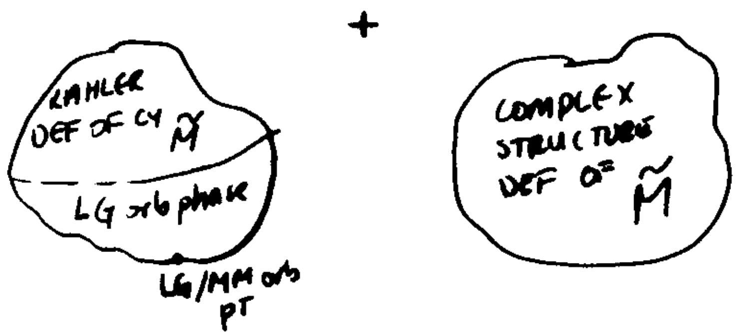
Moduli Spaces III :



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QUICK MIRROR REVIEW

- WHY MIGHT ONE EXPECT EXISTENCE OF [illegible] CFT

$$\left\{ \begin{matrix} (Q=1, \bar{Q}=-1) \\ (Q=1, \bar{Q}=1) \end{matrix} \right\} \longleftrightarrow \left\{ [illegible] \right\}$$

TRIVIAL DIFFERENCE : SIGN OF U(1) CHARGE $\longleftarrow$ STRANGE ASYMMETRY $\longrightarrow$ MAJOR DIFFERENCE : DISTINCT [illegible] OBJECT

- FURTHERMORE : HOW DETERMINE RHS [illegible] LHS ?

- **POSSIBLE RESOLUTION** : (DIXON; [illegible])

CFT

$A_{(\omega)}$ $(Q=1, \bar{Q}=1)$

$B_{(\varphi)}$ $(Q=1, \bar{Q}=-1)$

"QUANTUM" [illegible]
("STRINGY" [illegible])

M

$C_{(\kappa)}$ HARMONIC (2,1) FORMS

$K_{(\varphi)}$ HARMONIC (1,1) FORMS

EXISTENCE OF MIRRORS

- AT LEAST FOR ONE CLASS OF CY MANIFOLDS, MIRROR PAIRS $(M, \tilde{M})$ CAN BE EXPLICITLY CONSTRUCTED (BPG, PLESSER) (MODULI SPACE CONTAINS A "MINIMAL MODEL" POINT).

- NB: $(M, \tilde{M})$ MIRROR PAIR $\Rightarrow$ $h^{1,1}_M = h^{2,1}_{\tilde{M}}$; $h^{2,1}_M = h^{1,1}_{\tilde{M}}$

(a,c) (c,c)

- BUT: GIVEN $M, \tilde{M}$ S.T. $h^{1,1}_M = h^{2,1}_{\tilde{M}}$; $h^{2,1}_M = h^{1,1}_{\tilde{M}}$ DOES NOT IMPLY $(M, \tilde{M})$ FORM A MIRROR PAIR.
  - MUST SHOW THAT NLσM ON M & $\tilde{M}$ YIELD ISOMORPHIC SCFT'S.

- We therefore have explicit examples of

Physically Identical in String Theory

Any Observable

Formulation #1 $\qquad$ Formulation #2

$M_4 \times M_{CY}$ $\qquad\qquad$ $M_4 \times \tilde{M}_{CY}$

Mathematically Distinct Spacetimes

- Lets study this

- Both sides depend on location in moduli space.

Observables

- $$\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle \equiv \int [\mathcal{D}X]\, \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n}\, e^{i S_{CFT}}$$

- FOR CFT BUILT AS $N=2$ NLσM ON CY M:

$$= \int ([\mathcal{D}X] \cdots)\, \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n}\, e^{i S_{NL\sigma M}(M)}$$

- EXPANSION PARAMETER(S): "α' / R_{CY}^2" ← FROM KÄHLER FORM J ON M

- IF J CHOSEN TO BE IN REGION OF MODULI SPACE SUCH THAT PERTURBATION THEORY CONVERGES (i.e. "CY PHASE" AS DISCUSSED)

$$\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle \overset{\text{(TYPICALLY)}}{=} \int_M \text{INTEGRAND}(\mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n}) + \sum_{\substack{\text{INSTANTONS} \\ \text{i.e. RATIONAL CURVES} \\ \text{ON } M}} e^{-\frac{1}{\alpha'} \int X^*(J)} \ \text{I}_n \ (\text{INTEGRALS})$$

(AT ZERO MOMENTUM)

- WHY? STATIONARY PHASE: "POINT PARTICLE CONTRIBUTION" + "STRINGY CONTRIBUTIONS"

= "CLASSICAL GEOMETRY" + "QUANTUM" (i.e. STRINGY) CORRECTIONS

• ONE EXAMPLE (BRG, PLESSER)

CFT

$$\langle \mathcal{O}_{(\alpha)} \mathcal{O}_{(\beta)} \mathcal{O}_{(\gamma)} \rangle$$

$(Q = \bar{Q} = 1)$

$M_4 \times M_{CY}$

)GER, WITTEN;
BERG, WGN, WITTEN;
n et al

$$\langle K_{(\alpha)} K_{(\beta)} K_{(\gamma)} \rangle = \int_{M_{CY}} b_{(\alpha)} \wedge b_{(\beta)} \wedge b_{(\gamma)} + \sum_{\substack{\text{All rational} \\ \text{curves in } M_{CY}}} e^{-\frac{1}{\alpha'} \int_{I_n} X^*(J)} \int_{I_n} b_{(\alpha)} \int_{I_n} b_{(\beta)} \int_{I_n} b_{(\gamma)}$$

ll, MORRISON

$M_4 \times \tilde{M}_{CY}$

ER
, BRG

$$\langle C_{(\alpha)} C_{(\beta)} C_{(\gamma)} \rangle = \int_{\tilde{M}_{CY}} \Omega \wedge h^m_{(\alpha)} \wedge h^n_{(\beta)} \wedge h^p_{(\gamma)} \Omega_{mnp} + 0$$

• EXPLICITLY:

$$M = \sum z_i^5 = 0 \text{ IN } \mathbb{CP}^4 \; ; \; \tilde{M} = M / (\mathbb{Z}_5)^3$$

$$h^{1,1}_M = 1 \qquad\qquad h^{2,1}_{\tilde{M}} = 1$$

$$\text{LHS} = 5 + \sum_{d=1}^{\infty} n_d \; e^{2\pi i (B+iR) d} / 1 - e^{2\pi i (B+iR) d}$$

n_d ← # RATIONAL CURVES IN M OF DEGREE d

$$\text{RHS} = \int_{\tilde{M}} \Omega \wedge h^m \wedge h^n \wedge h^p \Omega_{mnp} \quad (\text{CALCULABLE})$$

• MATHEMATICS: USE RHS TO CALCULATE n_d
MIRROR MAP

¡ CANDELAS, de la OSSA, GREEN, PARKES → EVALUATE "n_d" $\forall d$!

VI) Spacetime Topology Change

(Aspinwall, BRG, Morrison ; Witten)

- Basic Ideas: Geometry of universe is dynamical. Might topology be as well?

- Only possible in a theory differing from classical general relativity on small scales (cf: singularity results)

  - "Quantum" gravity:

Curvature fluctuations: $(\hbar G / c^3)^{1/2} / L^3 = \frac{L_{pl}}{L^3}$

$$L \longrightarrow L_{pl} \Rightarrow \text{HUGE}$$

  - String Theory: Just need classical string theory for topology change

VI. A) A Puzzle for Mirror Symmetry

M		$\tilde{M}$
COMPLEX STRUCTURE DEFORMATIONS	$\longleftrightarrow$	KÄHLER STRUCTURE DEFORMATIONS
KÄHLER STRUCTURE DEFORMATIONS	$\longleftrightarrow$	COMPLEX STRUCTURE DEFORMATIONS

PROBLEM: THE PARAMETER SPACES OF COMPLEX STRUCTURE AND KÄHLER STRUCTURE DEFORMATIONS ARE QUITE DIFFERENT: DO NOT APPEAR TO BE ISOMORPHIC.

KÄHLER STRUCTURE

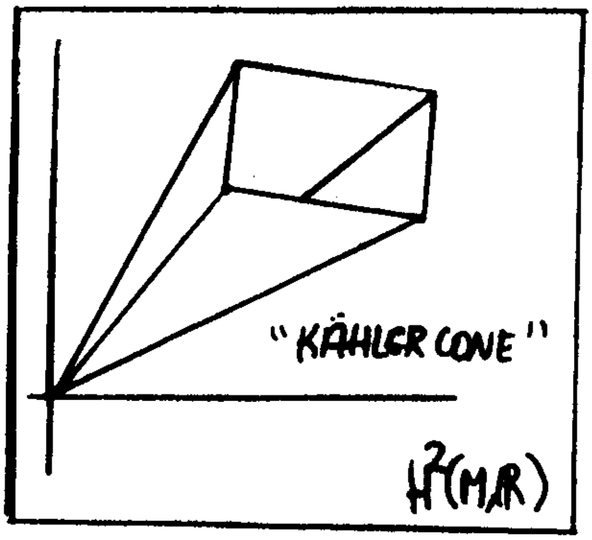
- J = KÄHLER FORM = $i g_{i\bar{j}}\, dx^i \wedge dx^{\bar{j}}$

- POSITIVITY CONDITIONS:

$$\int_M J \wedge J \wedge J > 0$$

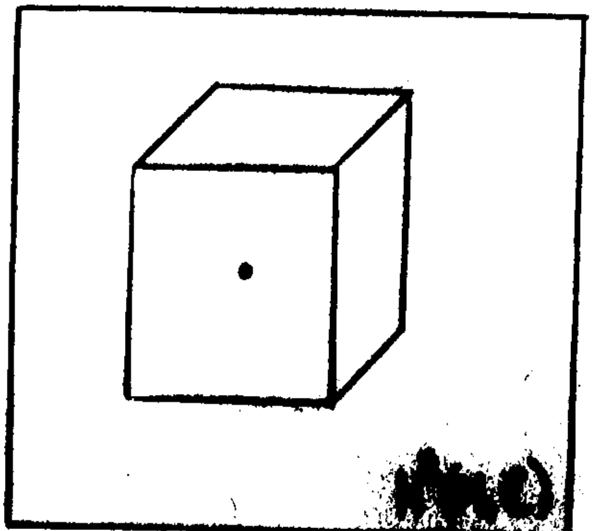
$$\int_{S\subset M} J \wedge J > 0$$

$$\int_{C\subset M} J > 0$$



J SATISFIES ⇒ $sJ \quad s>0 \in \mathbb{R}$.

- COMPLEXIFICATION:



STRING THEORY:

$J \to K = B + iJ$ (B: ANTISYMMETRIC TENSOR FIELD)

INVARIANCE: $B \to B + \Lambda$, $\Lambda \in H^2(M,\mathbb{Z}) \Rightarrow$

$$W_\ell = e^{2\pi i (B_\ell + i J_\ell)}$$

//

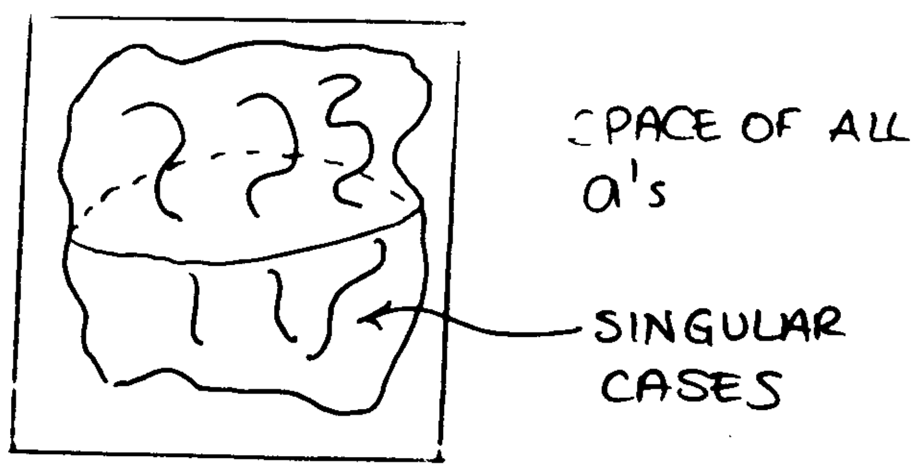
Complex Structure

- Typical CY's:

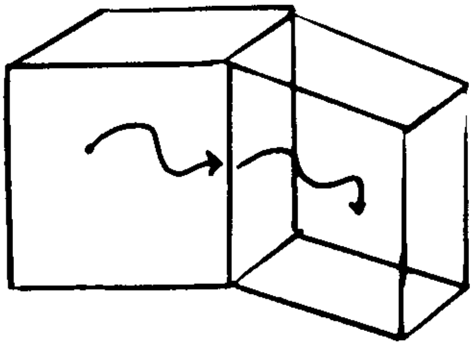
$$M =: z_1^{p_1} + z_2^{p_2} + z_3^{p_3} + z_4^{p_4} + z_5^{p_5} + a_1 z_1 z_2 z_3 z_4 z_5 + a_2 z_1^{q_1} z_2^{q_2} + \cdots + a_{\text{dim of C.S.}} z_1^{r_1} \cdots z_5^{r_5} = 0$$

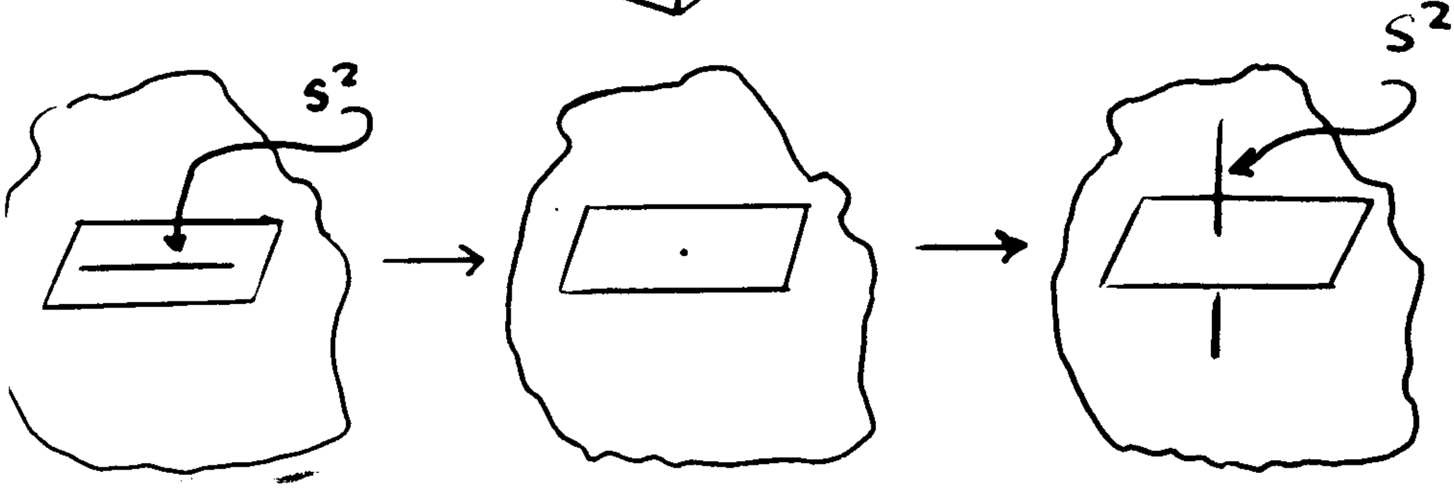
- Changing the values of the a's changes the complex structure

- Restriction on a's: $P=0$ & $\partial P/\partial z_i = 0\ \forall i$ must not have a common sol'n
  This is one eqn in a's.



VI.B) A Possible Resolution





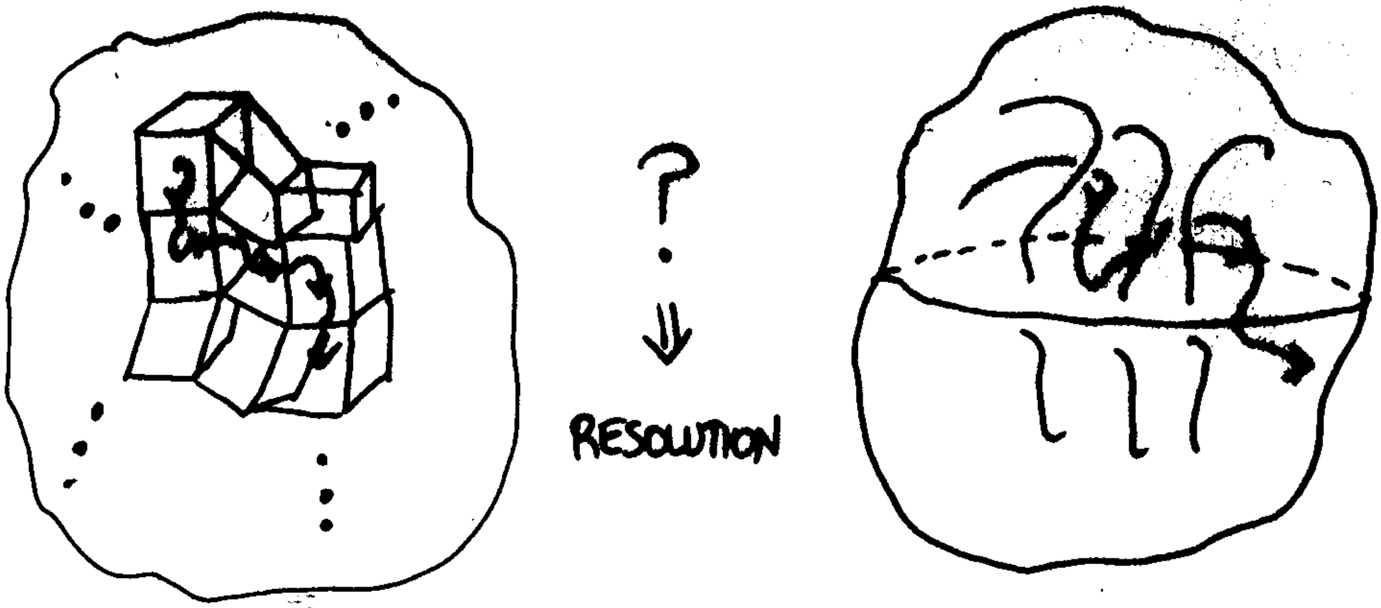
M_0 — SMOOTH

M_1 — SINGULAR

M_2 — SMOOTH

- Mathematical operation: "Flop"
- Topology of $M_2 \neq$ Topology of M_1
- Can repeat → Adjoin numerous regions each being Kähler parameter space for "flopped" manifolds

• SO, THE MATHEMATICS LEADS US TO A [illegible] NEW FORM FOR THE KÄHLER PARAMETER SPACE:



KÄHLER $_M$

Complex Structure $_{\tilde{M}}$

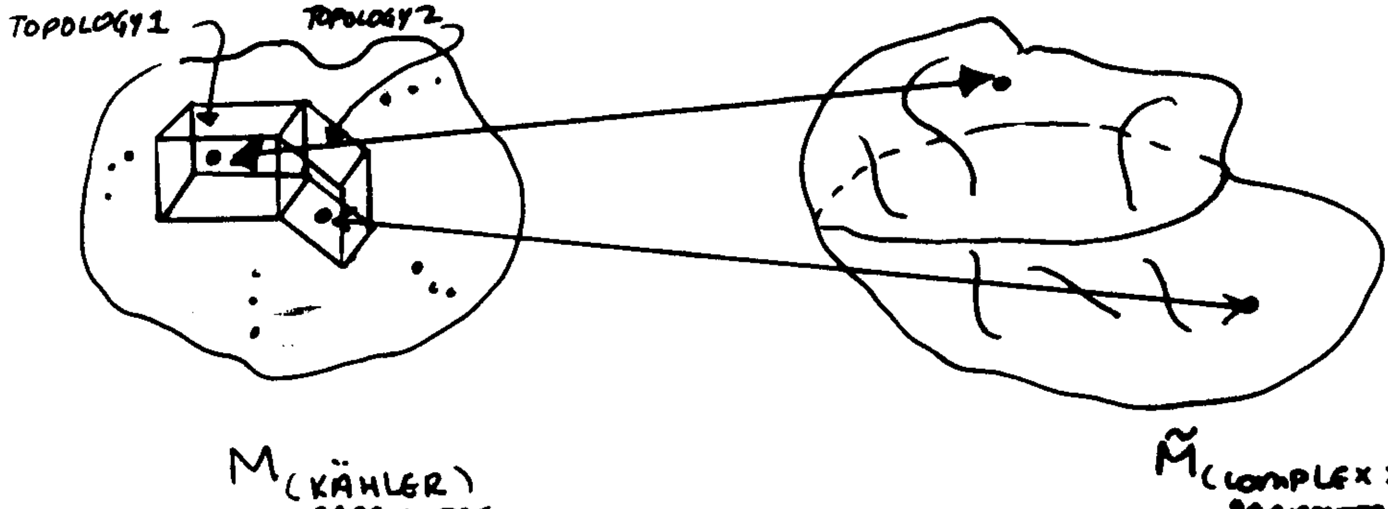
• HOW ABOUT PHYSICS?

i) IS THIS MATHEMATICAL OPERATION PHYSICALLY REALIZED?

ii) DOES CFT/STRING THEORY REQUIRE US TO INCLUDE THESE OTHER REGIONS?

SENSITIVE TEST:

- IF CFT REQUIRES ADDITIONAL REGIONS, i.e., IF FLOP TRANSITION IS PHYSICAL, IT MUST BE THAT **ANY** POINT ON KÄHLER SIDE HAS A CORRESPONDING POINT ON COMPLEX STRUCTURE SIDE YIELDING IDENTICAL PHYSICS.



$$\langle A_{(\alpha)} A_{(\beta)} A_{(\gamma)} \rangle$$

$$\int_{M_r} b_{(\alpha)} \wedge b_{(\beta)} \wedge b_{(\gamma)} + \mathcal{O}(e^{-r R}) \overset{?}{=} \int_{\tilde{M}_s} \Omega(s) \wedge h_{(\alpha)} \wedge h_{(\beta)} \wedge h_{(\gamma)} \Omega + 0$$

TOPOLOGICAL INVARIANT

EXAMPLE (CHOSEN FOR CALCULATIONAL EASE)

- $M =: z_0^3 + z_1^3 + z_2^6 + z_3^9 + z_4^{18} = 0$ IN $W\mathbb{P}^4_{(6,6,3,2,1)}$
- MIRROR : $\tilde{M} = M/G \qquad G = (\mathbb{Z}_3)^3$

$$(z_0, \ldots, z_4) \mapsto (\alpha\beta z_0, \alpha^{-1} z_1, \beta^{-1} z_2, \gamma^{-1} z_3, z_4)$$

$$\alpha^3 = \beta^3 = \gamma^3 = 1$$

- KÄHLER MODULI SPACE OF M : 5-DIMENSIONAL WITH 5 CELLS (CORRESPONDING TO 5 TOPOLOGICALLY DISTINCT SPACES).

- COMPLEX STRUCTURE OF $\tilde{M}$: 5-DIMENSIONAL :

$$z_0^3 + z_1^3 + z_2^6 + z_3^9 + z_4^{18} + a_1 z_0 \cdots z_4 + a_2 z_2^3 z_4^9 + a_3 z_3^6 z_4^6 + a_4 z_3^3 z_4^{12} + a_5 z_2^3 z_3^3 z_4^3 = 0$$

- WRITE : $a_i = s^{l_i}$

- SEEK 5 CHOICES OF l SO THAT AS $s \to \infty$ GET EQUALITY.
- TORIC GEOMETRY

FIVE DISTINCT TOPOLOGIES :

Resolution	1	2	3	4	5
$\frac{(E_1^3)(E_4^3)}{(E_1^2 E_4)(E_1 E_4^2)}$	-7	$0/0$	$0/0$	∞	9
$\frac{(E_2^2 E_4)(E_3^2 E_4)}{(E_2 E_3 E_4)(E_2 E_3 E_4)}$	2	4	0	$0/0$	$0/0$
$\frac{(E_2 E_3 E_4)(H E_2^2)}{(E_2^2 E_4)(H E_2 E_3)}$	1	1	1	0	$0/0$
$\frac{(E_2 E_3 E_4)(H E_1^2)}{(E_1^2 E_4)(H E_2 E_3)}$	2	1	∞	$0/0$	0

Table 1: Ratios of intersection numbers
(3-POINT FUNCTIONS)

FIVE DIRECTIONS IN COMPLEX STRUCTURE MODULI SPACE OF $\tilde{M}$:

Resolution	1	2	3	4	5
Exponents	$(\frac{11}{9}, 1, \frac{4}{3}, \frac{5}{3}, \frac{5}{3})$	$(\frac{7}{6}, \frac{1}{2}, 1, 1, \frac{5}{2})$	$(\frac{3}{2}, \frac{1}{2}, 2, 3, \frac{3}{2})$	$(\frac{13}{9}, 2, \frac{5}{3}, \frac{4}{3}, \frac{4}{3})$	$(\frac{11}{6}, \frac{7}{2}, 1, 1, \frac{3}{2})$
$\frac{\langle\varphi_1^3\rangle\langle\varphi_4^3\rangle}{\langle\varphi_1^2\varphi_4\rangle\langle\varphi_1\varphi_4^2\rangle}$	$-7 - 181s^{-1} + \ldots$			$-\frac{2}{5}s^2 - \frac{129}{250}s + \ldots$	$9 + 289s^{-1} + \ldots$
$\frac{\langle\varphi_2^2\varphi_4\rangle\langle\varphi_3^2\varphi_4\rangle}{\langle\varphi_2\varphi_3\varphi_4\rangle\langle\varphi_2\varphi_3\varphi_4\rangle}$	$2 - 5s^{-1} + \ldots$	$4 - 22s^{-1} + \ldots$	$0 + 2s^{-1} + \ldots$		
$\frac{\langle\varphi_2\varphi_3\varphi_4\rangle\langle\varphi_0\varphi_2^2\rangle}{\langle\varphi_2^2\varphi_4\rangle\langle\varphi_0\varphi_2\varphi_3\rangle}$	$1 + \frac{1}{2}s^{-1} + \ldots$	$1 + \frac{3}{2}s^{-2} + \ldots$	$1 + 4s^{-1} + \ldots$	$0 - 2s^{-1} + \ldots$	
$\frac{\langle\varphi_2\varphi_3\varphi_4\rangle\langle\varphi_0\varphi_1^2\rangle}{\langle\varphi_1^2\varphi_4\rangle\langle\varphi_0\varphi_2\varphi_3\rangle}$	$2 + 27s^{-1} + \ldots$	$1 - \frac{1}{2}s^{-1} + \ldots$	$-2s - 33 + \ldots$		$0 + 4s^{-2} + \ldots$

Table 4: Asymptotic ratios of 3-point functions

REMARKS

1) GENERIC:

- ANY CY / N=2 SCFT
- GENERIC PATH IN MODULI SPACE RESULTS IN TOPOLOGY CHANGE

2) MICROSCOPIC ANALYSIS:

- $\langle \theta_i \theta_j \theta_k \rangle = \int_M b_i \wedge b_j \wedge b_k +$ STRING CORRECTIONS
- THROUGH WALL: DISCONTINUITY $\left(\int_M b_i \wedge b_j \wedge b_k \right) =$

  $-$ DISCONTINUITY (STRING CORRECTIONS)

3) OTHER REGIONS:

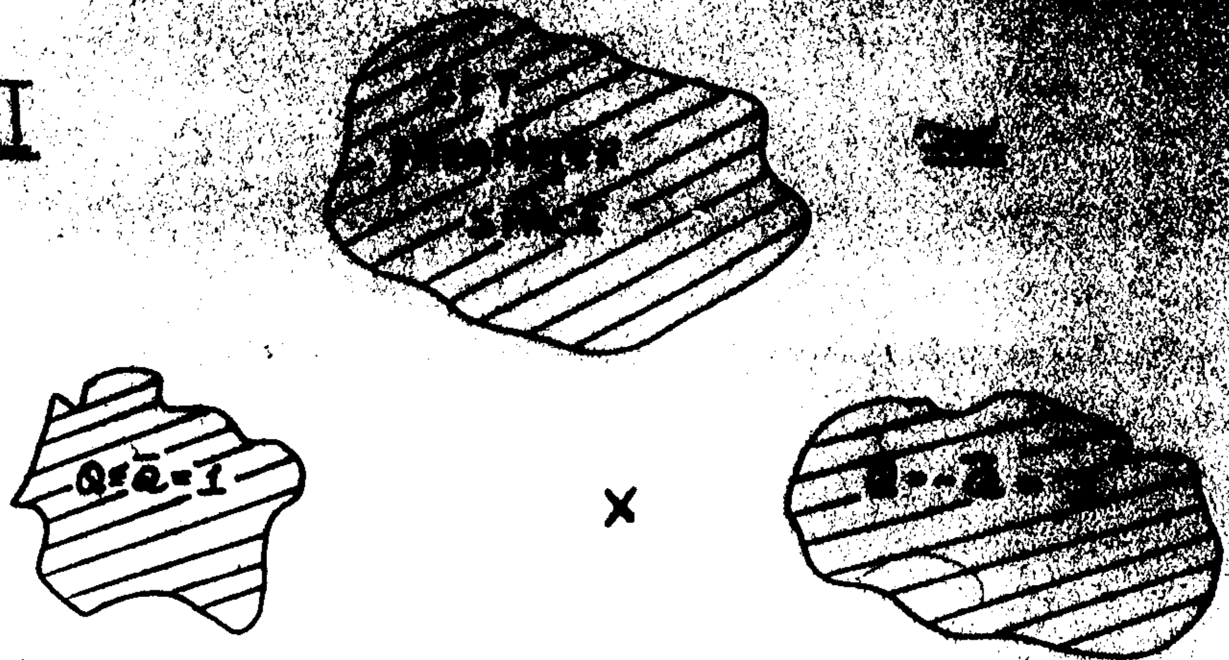
- IN THE EXAMPLE DISUSSED:

5 TOPOLOGICALLY DISTINCT CY REGIONS
27 ORBIFOLD REGIONS *
67 "HYBRID" REGIONS
1 LANAU-GINSBURG REGION
100 TOTAL

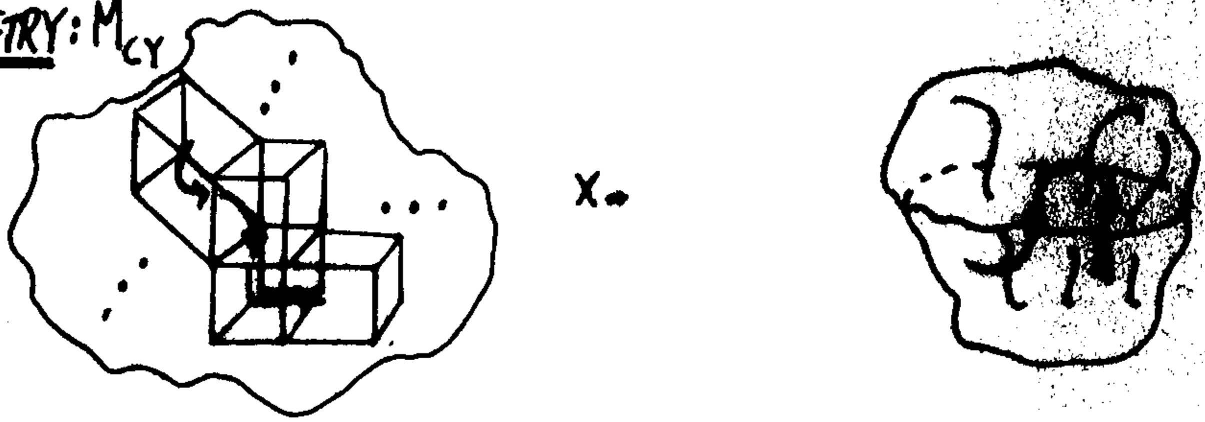
4) SUPERSYMMETRY BREAKING

MODULI SPACE OF N=2

CFT



EOMETRY: M_{CY}



OMETRY: $\tilde{M}_{CY}$

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CONCLUSIONS

1. CLASSICAL GEOMETRY IS A "POINT-PARTICLE" APPROXIMATION TO STRING PHYSICS. "QUANTUM GEOMETRY" DESCRIBES THE LATTER.

2. QUANTUM GEOMETRY CAN DIFFER SIGNIFICANTLY FROM CLASSICAL GEOMETRY

3. MIRROR MANIFOLDS ILLUSTRATE THIS: TWO SPACETIMES - DISTINCT AS CLASSICAL GEOMETRIC SPACES - YET PHYSICALLY IDENTICAL (POWERFUL CALCULATIONAL TOOL.)

4. SINGULAR CLASSICAL GEOMETRY CAN BE SMOOTH IN QUANTUM GEOMETRY & HENCE PHYSICALLY SMOOTH.

   IN PARTICULAR - USING MIRRORS WE HAVE SHOWN THAT A TOPOLOGY CHANGING TRANSITION THROUGH A SINGULAR SPACE IS PERFECTLY PHYSICALLY SMOOTH

   - FURTHERMORE - IT IS NOT EXOTIC!