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PERTURBATIVE QCD AND CHIRAL LANGRANGIANS

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Please note: These are preliminary notes intended for internal distribution only.

CHIRAL THEORIES (REFS.)

NASON

S. WEINBERG PHYSICA 96 A (1979) 327

Goleman, Wess and Zumino Phys. Rev. 177 (1969) 2239
Galler, Coleman, Wess and Zumino " " " " 2247
M. Peskin, SLAC-PUB-3021 (1982), Les Houches 1982

H. Leutwyler, BUTP-91/26 lectures given at
the advanced theoretical Study Institute
in Elementary Particle Physics,
Boulder, Golorado (1991)

G. Ecker, CERN-TH. 6660/92 Cargese lectures (1992)

CHIRAL THEORIES

FERMION SECTOR OF THE QCD LAGRANGIAN

$$\mathcal{L}_F = \sum_f \bar{\psi}_i^{(f)} \left((i\not\partial - m_f)\delta_{ij} - g t^a_{ij} \not{A}_a \right) \psi_j^{(f)}$$

Defining right and left-handed field components:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi \qquad , \quad \psi = \psi_L + \psi_R$$

$$\psi_R = \frac{1}{2}(1+\gamma_5)\psi$$

we have: (suppressing colour indices)

$$\mathcal{L}_F = \sum_f \bar{\psi}_L^{(f)} \left(i\not\partial - g t^a \not{A}_a \right) \psi_L^{(f)}$$

$$+ \sum_f \bar{\psi}_R^{(f)} \left(i\not\partial - g t^a \not{A}_a \right) \psi_R^{(f)}$$

$$+ \sum_f \left(\bar{\psi}_R^{(f)} m_f \psi_L^{(f)} + \bar{\psi}_L^{(f)} m_f \psi_R^{(f)} \right)$$

because $\bar{\psi}_L = \psi_L^\dagger \gamma^0 = \psi^\dagger \frac{1}{2}(1-\gamma^5)\gamma^0$

$$= \bar{\psi} \ \frac{1}{2}(1+\gamma^5)$$

and $\left(\frac{1}{2}(1+\gamma^5)\right)^2 = \frac{1}{2}(1+\gamma^5)$

If we could neglect fermion masses, the Lagrangian would have a large symmetry:

$$SU_L(N) \otimes SU_R(N) \otimes U_L(1) \otimes U_R(1)$$

$\uparrow N = \#$ of flavours

Meaning, the transformation:

$$\tilde{\psi}_L^{f'} = \sum_f U_L^{f'f} \psi_L^f \qquad U_L, U_R \in SU(N)$$

$$\tilde{\psi}_R^{f'} = \sum_f U_R^{f'f} \psi_R^f$$

would leave the Lagrangian invariant, and so would the $U_L(1)$, $U_R(1)$ transformations:

$$\tilde{\psi}_L^f = e^{i\varphi_L} \psi_L^f$$

$$\tilde{\psi}_R^f = e^{i\varphi_R} \psi_R^f$$

with φ_L, φ_R two independent (real) phases

(In fact, the whole $SU_L(N) \otimes SU_R(N) \otimes U_L(1) \otimes U_R(1)$ does not survive quantization, because of the so called "Chiral anomaly").

According to Noether's theorem, we would have a set of conserved currents:

FLAVOURED CURRENTS

$$SU_L(N) \Rightarrow \sum_{f,f'} \bar{\psi}_L^f \gamma^\mu \psi_L^{f'} t^a_{ff'} \qquad \mathrm{tr}\, t^a t^b$$

$$SU_R(N) \Rightarrow \sum_{f,f'} \bar{\psi}_R^f \gamma^\mu \psi_R^{f'} t^a_{ff'} \qquad \mathrm{tr}\, t = 0$$

t herm...

(SINGLET CURRENTS)

$U_L(1) \Rightarrow \sum_f \bar{\psi}_L^f \gamma^\mu \psi_L^f$

$U_R(1) \Rightarrow \sum_f \bar{\psi}_R^f \gamma^\mu \psi_R^f$

The conserved currents associated with the given simmetries are obtained with the usual method; for example, the infinitesimal transformation associated to $SU_L(N)$ are given by:

$$\delta \psi_L^f = \sum_{f'} i M_{ff'} \psi_L^{f'}$$

where M is an infinitesimal traceless Hermitean matrix

(Remember: an infinitesimal $SU(N)$ transformation can be written: $U = \mathbb{1} + iM$, with M infinitesimal. Then unitarity gives:

$$U U^\dagger = \mathbb{1} + iM - iM^\dagger + \mathcal{O}(M^2) = \mathbb{1}$$

from which we get: $M = M^\dagger$

Furthermore $SU(N)$ matrices have unit determinant:

$$\det U = \det e^{iM} = e^{i \,\mathrm{tr} M} = \mathbb{1}$$

from which we get $\mathrm{tr} M = 0$)

Using then Noether's theorem we can find the associated conserved current:

$$J^\mu = \frac{\partial \mathcal{L}_0}{\partial\, \partial^\mu \psi_L^f}\, \delta \psi_L^{(f)}$$

$$= \bar{\psi}_L^{(f)} \gamma^\mu \delta \psi_L^{(f)} = i\, \bar{\psi}_L^{(f)} \gamma^\mu \psi_L^{(f')} M_{ff'}$$

We can now chose a "basis" of conserved currents of the form:

$$J_a^{\mu\,(L)} = \sum_{ff'} \bar{\psi}_L^{(f)} \gamma^\mu t^a_{ff'} \psi_L^{(f')}$$

Sometimes it is convenient to use another basis of conserved currents:

ECTOR $$J_V^\mu = J_L^\mu + J_R^\mu$$

$$= \bar{\psi} \gamma^\mu \frac{(1-\gamma_5)}{2} \psi + \bar{\psi} \gamma^\mu \frac{1+\gamma_5}{2} \psi = \bar{\psi}\gamma^\mu \psi$$

XIAL $$J_A^\mu = J_L^\mu - J_R^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$$

Vector currents are associated to a full subgroup of the $SU_L(N) \otimes SU_R(N) \otimes U_L(1) \otimes U_R(1)$, which is the subgroup:

$$\psi'_L = U_L \psi_L \qquad \text{with } U_L = U_R$$
$$\psi'_R = U_R \psi_R$$

Axial transformations do not form a subgroup in general; if we define:

$$\psi'_L = U_L \psi_L \qquad \text{with } U_L = U_R^{-1}$$
$$\psi'_R = U_R \psi_R$$

we see that the group composition law does not work:

$$\psi'_L = U'_L U_L \psi_L \qquad \text{with } U_L = U_R^{-1}$$
$$\psi'_R = U'_R U_R \psi_R \qquad U'_L = U'^{-1}_R$$

is not the same as:

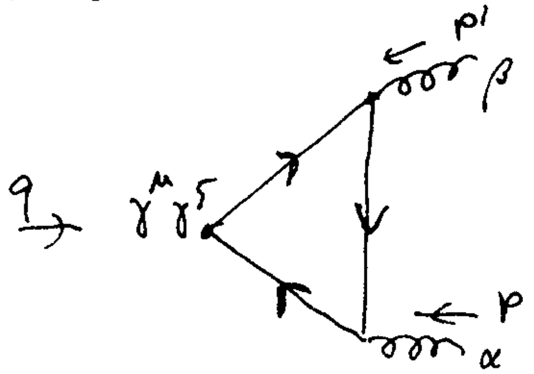
$$\psi'_L = (U'_L U_L) \psi_L$$
$$\psi'_R = (U'_L U_L)^{-1} \psi_R$$

if U' and U ~~[illegible]~~ do not commute.

On the other hand the $U_L(1)$, $U_R(1)$ are abelian groups, so in this case axial transformations do form a group; in general we can write the full invariance group as:

$$SU_L(N) \otimes SU_R(N) \otimes U_V(1) \otimes U_A(1)$$

The $U_A(1)$ symmetry that we found at the classical level does not survive quantization; This is because the triangle graph:



$$= \Gamma^{\mu\alpha\beta}(q, p, p')$$

violates the current conservation condition:

$$q_\mu \cdot \Gamma^{\mu\alpha\beta}(q, p, p') \neq 0$$

The problem manifests itself only for the singlet axial current, because for flavour currents we have a flavour factor:

$$\sum_i t^a_{ii} = 0$$

so that the diagram vanishes.

The full quantum symmetry is then

$$SU_L(N) \otimes SU_R(N) \otimes U_V(1)$$

Let us focus on the case of 2 flavours.

Do we see the $SU_L(2) \otimes SU_R(2) \otimes U_V(1)$ symmetry?

It is easy to convince ourself that the $U_V(1)$ symmetry is associated to barion number conservation. It is also apparent that the $SU_V(2)$ symmetry is the well known Isospin symmetry, which is a very good symmetry of strong interactions.

There is instead no trace of the axial symmetries.

One way out to the problem would be to assume that the quark masses are not small, and we have m_u nearly equal to m_d. Then the vector currents will only survive. However, this is very unnatural since the other quarks (s, c, b, t) have very different masses. Furthermore, to have sizeable axial symmetry breaking, we should have m_u and m_d of the order of Λ_{QCD} at least. Bound states of up's and downs should then have a mass which is roughly proportional to the number of quarks: $m_p \approx \frac{3}{2} m_\pi$

Instead: $m_\pi \simeq 140$ MeV, $m_p \simeq 940$ MeV.

And why is the ρ so much heavier than the pion?

There is a second possibility:

Axial symmetry is spontaneously broken;

In $SU(2) \times SU(2)$ there are 6 generators,
3 for the axial symmetry; there will be
therefore 3 ~~of~~ Goldstone bosons, and they
will be pseudoscalars, since the symmetry is axial.

The quark masses will slightly break the
axial symmetry, thereby giving a mass to
the pion.

This picture is very promising; it explains
why the pion are so light with respect to the ρ.
It also allows for some quantitative predictions.

(8 bis)

Broken Symmetry

Given a dynamical system invariant under some group G, there are two ways in which the symmetry can be realized

"Wigner" type: the vacuum is a singlet under the action of the group

In this case the vacuum is annihilated by the charges associated to the generators of the symmetry groups; invariance implies:

$$e^{i\varepsilon Q}|0\rangle = |0\rangle \Rightarrow Q|0\rangle = 0$$

In this case, particles are grouped into degenerate multiplets of the group. This is certainly the "simple" way to realize a symmetry.

A second possibility is the "Goldstone" mode: the vacuum is not invariant under the action of the group. This is sometime expressed by the fact that the charge densities do not annihilate the vacuum:

$$J^0|0\rangle \neq 0$$

From translational invariance, it follows that $Q|0\rangle$ is a state of infinite norm. It can be written as:

$$\|Q|0\rangle\|^2 = \langle 0|Q\,Q|0\rangle = \int d^3x \, \langle 0|J^0(0)\,Q|0\rangle$$

so, if it is non-zero, is infinite

In the "Goldstone" mode, one can prove that there are massless particles associated to each group generator that does not annihilate the vacuum.

The "Goldstone" phenomenon is present also in classical (non-quantum) field theory

Examples:

massless scalar field:

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$$

invariant under $\phi \to \phi + c$

the vacuum (ground state) has ϕ = constant which breaks the symmetry; there is 1 massless excitation: the field ϕ itself

Examples from real world:

phonons in solids (broken translational invariance)

spin waves in ferromagnets,

etc.

Linear σ model

$$\vec{\Phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$$

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \vec{\phi}\, \partial^\mu \vec{\phi} - \frac{\lambda}{4}\left(\vec{\phi}^{\,2} - v_0^2\right)$$

The vacuum has $|\vec{\phi}| = v_0$ (Broken symmetry)

We expect goldstone particles

- $\mathcal{L}_0$ is invariant under $O(4)$;

The generators are 4×4 antisimmetric real matrices : 6 generators

The unbroken subgroup is the one that keeps one vector fixed: $O(3)$, 3 generators;

we expect:

$$6 - 3 = 3 \quad \text{goldstone bosons}$$

Originally we had 4 degrees of freedom; expect 3 massless + 1 massive particles

Rewrite the lagrangian introducing the matrix :

$$M = \begin{vmatrix} \chi_1 & -\chi_2^* \\ \chi_2 & \chi_1^* \end{vmatrix} \qquad \begin{aligned} \chi_1 &= \phi_1 + i\phi_2 \\ \chi_2 &= \phi_3 + i\phi_4 \end{aligned}$$

$$\mathcal{L}_0 = \frac{1}{4} \operatorname{tr} \partial_\mu M\, \partial_\mu M^\dagger - \frac{\lambda}{4}\left(\det M - v_0^2\right)$$

In fact: M can be written as:

$$M = U\, v \quad , \quad v \text{ positive real}$$

where U is an $SU(2)$ matrix.
It is easy to show that

$$\frac{M}{\sqrt{\vec{\Phi}^2}} \in SU(2)$$ (Just show that $MM^\dagger = \vec{\Phi}^2$ and $\det M = \vec{\Phi}^2$

Next one should prove that for any $SU(2)$ matrix U and $v > 0$, Uv can be written in the form of M

Proof:
Given $U = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $U^{-1} = \frac{1}{\det U} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

if $U \in SU(2)$ then $\det U = 1$ and

$$U^{-1} = \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = U^\dagger = \begin{vmatrix} a^* & c^* \\ b^* & d^* \end{vmatrix}$$

therefore: $a^* = d$, $d^* = a$ $c^* = -b$, $b^* = -c$

$$\Rightarrow \quad U = \begin{vmatrix} a & -c^* \\ c & a^* \end{vmatrix} \qquad \text{with } aa^* + cc^* = 1$$

multiplying by an arbitrary $v > 0$ we get the form of M.

It is known that $O(4)$ is equivalent to $SU(2) \otimes SU(2)$

An $O(4)$ transformation on $\vec{\phi}$ corresponds to a transformation on M of the form:

$$M \to M' = U_L M U_R^\dagger$$

where U_L and U_R are $SU(2)$ Matrices:

1) $M = U v$ maintains the same form: in fact: $U_L M U_R^\dagger = U_L U U_R^\dagger v = U' v$

U' is still an $SU(2)$ matrix, an v $(= |\phi|)$ stays the same: i.e. the corresponding linear transformation upon ϕ belongs to $O(4)$

2) Counting parameters: $O(4)$ has 6 parameters, $SU_L(2) \otimes SU_R(2)$ has 6 parameters also

3) We have not proven that any $O(4)$ transformation is an $SU_L(2) \otimes SU_R(2)$ Transformation

Now write:

$$M = U\, v = U\,(v_0 + h)$$

We get:

$$\mathcal{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu U\, \partial^\mu U^\dagger\right)(v_0+h)^2 +$$

$$\partial_\mu U\, U^\dagger + U\,\partial_\mu U^\dagger = 0$$

$$+ \frac{1}{4}\overbrace{\left[\mathrm{tr}\left((\partial_\mu U)\, U^\dagger\, (v_0+h)\right)\partial^\mu h + \mathrm{tr}\left(U\left(\partial_\mu U^\dagger\right)(v_0+h)\right)\partial^\mu h\right]}^{=0,}$$

$$+ \frac{1}{2}\,\partial_\mu h\, \partial^\mu h - \frac{\lambda}{4}\left((v_0+h)^2 - v_0^2\right)^2$$

$$= \frac{v_0^2}{4}\,\partial_\mu U\, \partial^\mu U^\dagger + \frac{1}{4}\left(2v_0 h + h^2\right)\partial_\mu U\, \partial^\mu U^\dagger$$

$$+ \frac{1}{2}\,\partial_\mu h\, \partial^\mu h - \frac{\lambda}{4}\left(2v_0 h + h^2\right)^2$$

$$\downarrow$$

$\lambda\, v_0^2 h^2 \leftarrow$ mass term for h

Vacuum: $h = 0$, $U = \mathbb{1}$ (arbitrary)

The symmetry is explicit: $\mathcal{L}$ is still invariant under $SU_L(2) \otimes SU_R(2)$

From the example of the linear σ model we will accept the following fact: the goldstone bosons of the broken axial symmetry are represented by an $SU(N)$ matrix

$$U = e^{i\frac{\pi^a \cdot t^a}{v_0}}$$

which transforms as:

$$U_L\, U\, U_R^\dagger$$

under an $SU_L(N) \otimes SU_R(N)$ transformation.

(For a complete proof, see Coleman, Wess, Zumino, Callan; for a satisfactory (but less complete) argument see pag. 13 bis, 13 bisbis of the notes)

Having established what is the goldstone boson's field, we can try to write down a Lagrangian, invariant under $SU_L(N) \otimes SU_R(N)$.

We should also remember that the parity transformation changes the L with the R (in QCD). A realization of parity is immediately given by:

$$P[U] = U^\dagger$$

So that:

$$U_L\, U\, U_R^\dagger = U'$$

$$U_R\, P[U]\, U_L^\dagger = P[U']$$

Also: $P[\pi] = -\pi$

so $U \to U^\dagger$

General argument for the Goldstone Boson field to be represented by an $SU(N)$ matrix, transforming under $SU_L(N) \otimes SU_R(N)$ via left and right multiplication.

The vacuum is represented by some point $\mathcal{O}$ in a manifold $\mathcal{M}$; oscillations of the vacuum (the Goldstone field) can then be represented by:

$$P = e^{i \pi^a \cdot A^a} (\mathcal{O})$$

meaning: the group element $e^{i\pi^a A^a}$ acting on $\mathcal{O}$. A^a are the generators of the axial symmetry. The parameters π^a represent the Goldstone fields.

To get the transformation properties of π^a, apply a generic $SU(N) \otimes SU(N)$ transformation

$$P' = \left(e^{i(\xi_L^a L^a + \xi_R^a R^a)} e^{i\pi^a \cdot A^a} \right)(\mathcal{O})$$

We can write $A^a = L^a - R^a$ (L, R are the left and right generators.) Then use the group composition law, use the fact that L and R commute, so that:

$$e^{i\pi^a \cdot A^a} = e^{i\pi^a L^a} e^{-i\pi^a R^a}$$

$$= e^{2i\pi^a L^a} e^{-i\pi^a V^a}$$

with $V^a = L^a + R^a$. But $e^{-i\pi^a V^a}$ leaves the vacuum invariant. So:

$$P' = \left(e^{i(\xi_L^a L^a + \xi_R^a R^a)} e^{2i\pi^a L^a} \right)(\mathcal{O})$$

Which can again be written:

$$p' = e^{i\xi_L^a L^a} e^{2i\pi^a L^a} e^{i\xi_R^a R^a}(\sigma)$$

$$= e^{i\xi_L^a L^a} e^{2i\pi^a L^a} e^{-i\xi_R^a L^a}(\sigma)$$

when I have again used the invariance of the vacuum under vector transformations

Now defining:

$$e^{2i\pi^{a\prime} t^a} = e^{i\xi_L^a t^a} e^{2i\pi^a t^a} e^{-i\xi_R^a t^a}$$

where now t^a are $SU(N)$ generators, we find:

$$p' = e^{2i\pi^{\prime a} t^a}(\sigma) = e^{i\pi^{\prime a} A^a}(\sigma)$$

Therefore the vacuum oscillations are parameterized by a field π^a which transforms under $SU_L(2) \otimes SU_R(2)$ in the following way:

$$e^{2i\pi^{a\prime} t^a} = e^{i\xi_L^a t^a} e^{2i\pi^a t^a} e^{-i\xi_R^a t^a}$$

$$U' = U_L \, U \, U_R^\dagger$$

C. V. D.

Invariant with lowest # of derivatives:

$$\mathrm{tr}\left(\partial_\mu U \, \partial^\mu U^\dagger\right)$$

which is clearly invariant under $SU_L(N) \otimes SU_R(N)$, and also under parity. We have:

$$\mathcal{L}_0 = \frac{f^2}{4} \, \mathrm{tr}\left(\partial_\mu U \, \partial^\mu U^\dagger\right) \quad + \text{higher derivatives terms}$$

The "effective" lagrangian written here can be used to make prediction regarding the dynamics of Goldstone particles at small momenta; in this case, in fact, higher derivative terms are small, and can be neglected.

If we define: $\quad U = e^{i2\frac{\pi^a \cdot t^a}{f}}$

expanding in powers of the π field we find:

$$\mathcal{L}_0 = \frac{f^2}{4} \, \mathrm{tr}\left(\left[i\,2\,\partial_\mu \frac{\pi^a \cdot t^a}{f} - \frac{1}{2}\partial_\mu\left(2\frac{\pi^a \cdot t^a}{f}\right)^2 - \frac{i}{3!}\partial_\mu\left(2\frac{\pi^a \cdot t^a}{f}\right)^3 + \ldots \right] \right.$$

$$\left. \left[-i\,2\,\partial_\mu \frac{\pi^{a'} \cdot t^{a'}}{f} - \frac{1}{2}\partial_\mu\left(2\frac{\pi^{a'} \cdot t^{a'}}{f}\right)^2 + \frac{i}{3!}\partial_\mu\left(2\frac{\pi^{a'} \cdot t^{a'}}{f}\right)^3 + \ldots \right] \right)$$

$$= \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{f^2}\mathrm{tr}\Big[\partial_\mu\left(\pi^a \cdot t^a\right)^2 \partial^\mu\left(\pi^a \cdot t^a\right)^2$$

$$- \frac{4}{3}\partial_\mu\left(\pi^a \cdot t^a\right) \quad \partial_\mu\left(\pi^{a'} \cdot t^{a'}\right)^3\Big] \quad + \cdots$$

The axial current can be found by using Noethers theorem. An infinitesimal axial transformation acting on U has the form:

$$e^{i\xi^a \cdot t^a} \, U \, e^{i\xi^a \cdot t^a}$$

$$= U + i\xi^a t^a U + U i \xi^a t^a$$

So : $\delta U = i\xi^a t^a U + U i \xi^a t^a$

Noether's theorem:

$$\xi^a J_a^{\mu\,(A)} = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu U}\,\delta U + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu U^\dagger}\,\delta U^\dagger$$

$$= \frac{f^2}{4}\left\{ \mathrm{tr}\left(\delta U \, \partial^\mu U^\dagger \right) + \mathrm{tr}\left(\partial^\mu U \, \delta U^\dagger \right) \right\}$$

$$= i\xi^a \frac{f^2}{4}\left\{ \mathrm{tr}\left(t^a U \partial^\mu U^\dagger + U t^a \partial^\mu U^\dagger - \partial^\mu U \, t^a U^\dagger - \partial^\mu U \, U^\dagger t^a \right) \right\}$$

$$= i\xi^a \frac{f^2}{4}\left\{ \mathrm{tr}\left(t^a \left[U \partial^\mu U^\dagger - \partial^\mu U \, U^\dagger + \partial^\mu U^\dagger U - U^\dagger \partial^\mu U \right] \right) \right\}$$

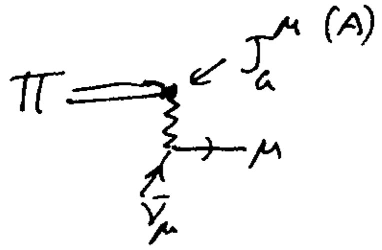
Setting: $U = e^{i\frac{2\pi^a t^a}{f}}$

$$J_a^{\mu\,(A)} = f\,\partial^\mu \pi^a + \text{terms with more } \pi \text{ fields}$$

It follows that:

$$\langle 0 | J_a^{\mu\,(A)}(0) | \pi \rangle = f \; i p^\mu$$

where p^μ is the pion momentum. The constant f is the pion decay constant; in fact, the pion decays via weak interaction with the mechanism:



and the hadronic part of this decay is precisely the matrix element of the axial current between a 1 pion state and the vacuum.

Masses

The mass term in the original lagrangian has the form:

$$\sum_f \bar{\psi}_L^{(f)} m_f \psi_R^{(f)} + \bar{\psi}_R^{(f)} m_f \psi_L^{(f)}$$

If we write it in the general form:

$$\sum_{f,f'} \bar{\psi}_L^{(f)} M_{ff'} \psi_R^{(f')} + \bar{\psi}_R^{(f)} M^{\dagger}_{ff'} \psi_L^{(f')}$$

then the lagrangian is invariant under a simultaneous change of:

$$\psi_L \to U_L \psi_L$$

$$\psi_R \to U_R \psi_R$$

together with:
$$M \to U_L M U_R^{\dagger}$$

So in the effective action we should write the term with lowest possible powers of M and derivatives invariant under the transformations:

$$U \to U_L U U_R^{\dagger}$$

$$M \to U_L M U_R^{\dagger}$$

We have:

$$\mathcal{L}_{eff} = \frac{f^2}{4} \operatorname{tr}\left(\partial_\mu U \partial^\mu U + 2B\left(M U^{\dagger} + U M^{\dagger}\right)\right)$$

Since $M = \begin{vmatrix} m_u & & \\ & m_d & \\ & & m_s \end{vmatrix}$ we get:

$$\mathcal{L}_{eff} = \frac{f^2}{4} \, \mathrm{tr} \left(\partial_\mu U \, \partial^\mu U + 2B\, M \left(U^\dagger + U \right) \right)$$

We have:

$$U^\dagger + U = 2 - \left(\frac{2\pi^a \cdot t^a}{f} \right)^2 + \cdots$$

So our mass term is given by:

$$2B \; \mathrm{tr}\, M \left(\pi^a \cdot t^a \right)^2$$

Write:

$$\pi^a \cdot t^a = \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{ccc} \bar{u} & \bar{d} & \bar{s} \end{array} \\ \begin{array}{c} u \\ d \\ s \end{array} & \begin{vmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta^0 \end{vmatrix} \end{array}$$

The mass term is then:

$$B \times \Big[(m_u + m_d) \left(\pi^+ \pi^- + \frac{{\pi^0}^2}{2} \right) + \frac{1}{\sqrt{3}} \pi^0 \eta^0 (m_u - m_d)$$

$$+ \left(K^+ K^- + \bar{K}^0 K^0 \right) \left(m_s + \frac{m_u + m_d}{2} \right)$$

$$+ \left(K^+ K^- - \bar{K}^0 K^0 \right) \frac{m_u - m_d}{2} + \frac{2}{3} (\eta^0)^2 \left(m_s + \frac{m_u + m_d}{4} \right) \Big]$$

It follows that:

$$m^2_{\pi^+} = B\ (m_u + m_d)$$

$$m^2_{K^+} = B\ (m_s + m_u)$$

$$m^2_{K^0} = B\ (m_s + m_d)$$

$$m^2_{\eta^0} = B\ \frac{4}{3}\left(m_s + \frac{m_u+m_d}{4}\right)$$

π^0 and η^0 mix $\left(\approx \frac{m_u - m_d}{m_s - \hat{m}} \right)$

$\hookrightarrow = \frac{m_u + m_d}{2}$

this causes a mass difference at second order in the perturbation:

$$M^2_{\pi^0} = M^2_{\pi^+} - \frac{1}{4}\left(\frac{m_u - m_d}{m_s - \hat{m}}\right)^2 \left(M^2_K - M^2_\pi\right)$$

a numerically insignificant effect.
Most of the $\pi^0 - \pi^\pm$ mass difference comes from electromagnetic effects.

→ Gell-Mann - Okubo mass relation:

$$3\ M^2_\eta = 2\ M^2_{K^+} + 2\ M^2_{\bar{K}} - M^2_\pi$$

Mass formulae:

$$\frac{m_u + m_d}{2 m_s} = \frac{m_\pi^2}{2 m_K^2 - m_\pi^2} = \frac{1}{25}$$

$$\frac{m_d - m_u}{m_s + \hat{m}} = \frac{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2}{m_K^2} = \frac{1}{46}$$

or $\quad \dfrac{m_u}{m_s} = \dfrac{1}{35} \; ; \quad \dfrac{m_d}{m_s} = \dfrac{1}{19} \; ; \quad \dfrac{m_u}{m_d} = \dfrac{1}{1.8}$

Observe that $m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2$ is used to take away from the Kaon mass difference the electromagnetic part, which should be ~~[illegible]~~ the same as for the $\pi^0 \pi^+$ mass difference (Dashen theorem)

Other applications

Low energy pion scattering theorems:

$$T_{\pi^+\pi^0\to\pi^+\pi^0} = \frac{t - M_\pi^2}{f^2} + \mathcal{O}(p^4, p^2\mathcal{M}, \mathcal{M}^2)$$

Notice the power counting: $\mathcal{M}$ counts as p^2
(since $B\mathcal{M} \approx m_\pi^2$)

Also:

$$T_{\eta\to\pi^+\pi^-\pi^0} = \frac{\sqrt{3}}{4}\,\frac{m_u - m_d}{m_s - \hat{m}}\,\frac{s - \frac{4}{3}M_\pi^2}{f^2}$$

This results can be obtained as a simple exercise from the expansion of the chiral lagrangian up to terms with 4 pion fields.