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BLACK HOLES AND GENERAL RELATIVITY

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3 BLACK HOLES

The Schwarzschild vacuum (exterior) solution is given in suitable coordinates by

$$ds^2 = -(1-\frac{2m}{r})dt^2 \ + \ (1-\frac{2m}{r})^{-1}dr^2 \ + \ r^2 d\Omega^2 \tag{90}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \,. \tag{91}$$

We have up to now assumed $r > r_S > 2m$. Now we drop that condition and consider the nature of the singularity at $r = 2m$.

We consider what happens if we assume there is a spherically symmetric vacuum solution everywhere, and extend it as far as we can, to obtain the Schwarzschild exterior solution maximal extension. That is, we try to extend the vacuum solution across $r = 2m$ to $r = 0$.

3.1 The nature of $r = 2m$

What characterises the 'singularity' at $r = 2m$?

(1) *Singular metric*: It is clear that the metric components are singular at $r = 2m$.

(2) *Infinite redshift*: From the redshift formula (86,87),

$$1 + z = \frac{\lambda_R}{\lambda_E} = \left(\frac{1 - 2m/R_0}{1 - 2m/r_E}\right)^{1/2} \tag{92}$$

shows that $z \to \infty$ as $r_E \to 2m$. Furthermore from (82), on radial null geodesics the coordinate time is

$$t = \pm\left[r + 2m \ln\left(\frac{r}{2m} - 1\right)\right] + const$$

where the constant labels the geodesic. Consequently $t \to \infty$ on a radially ingoing null geodesic, as $r \to 2m$.

(3) *Radial infall*: Considering radial infall of a test object (with non-zero restmass). The proper time taken is given by (65)

$$\tau = \frac{1}{\sqrt{2m}} \int_r^{r_0} \left(\frac{r r_0}{r_0 - r}\right)^{1/2} dr \tag{93}$$

giving

$$\tau = \frac{r_0^{3/2}}{\sqrt{2m}} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{r}{r_0}\right)^{1/2} + \left(\frac{r}{r_0}\right)^{1/2}\left(1 - \frac{r}{r_0}\right)^{1/2}\right]. \tag{94}$$

which is finite as $r \to 2m$. Thus the proper time taken to fall from $r = r_0 > 2m$ to $r = 2m$ is finite.

However the coordinate time is given by

$$dt/d\tau = k/(1 - 2m/r) = (1 - 2m/r_0)^{1/2}/(1 - 2m/r)$$

(by (36), evaluating k by $\dot{r_0} = 0$, see (63)), and so

$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = \frac{(1 - 2m/r_0)^{1/2}}{(r - 2m/r)}\frac{1}{\sqrt{2m}}\left(\frac{rr_0}{r_0 - r}\right)^{1/2} \tag{95}$$

that is

$$t = \left(\frac{r_0 - 2m}{2m}\right)^{1/2} \int_r^{r_0} \frac{r^{3/2}dr}{(r - 2m)(r_0 - r)^{1/2}} \tag{96}$$

which is infinite as $r \to 2m$. To see this note that for $r > 2m$ we have $r^{3/2} > (2m)^{3/2}$, $1/(r_0 - r)^{1/2} > 1/r_0^{1/2}$, so

$$t > \left(\frac{r_0 - 2m}{2m}\right)^{1/2} \frac{(2m)^{3/2}}{r_0^{1/2}} \int_r^{r_0} \frac{dr}{(r - 2m)^{1/2}} \tag{97}$$

The integral is

$$ln\left(\frac{r_0 - 2m}{r - 2m}\right)$$

which diverges as $r \to 2m$. Thus the particle traverses an infinite coordinate time in a finite proper time as it approaches $r = 2m$. This suggests that an observer at $r > 2m$ observing an object falling radially inward towards $r = 2m$ will find that the object takes an infinite coordinate time to reach $r = 2m$ – indeed that it never actually reaches that radial value – whereas the object measures a finite proper time for the trip. By the previous, the object becomes more and more redshifted as well.

(4) *Change of nature of coordinates*: If we ignore these problems and consider the solution for $0 < r < 2m$, we find there that $(1 - 2m < r) < 0$, so the role of the coordinates has changed – t has become a spatial coordinate (measuring spatial distance along the curves $\{t$ only varies $\}$, orthogonal to timelike surfaces $\{t = const\}$), while r has become a time coordinate (measuring proper time along the curves $\{r$ only varies $\}$, which are orthogonal to spacelike surfaces $\{r = const\}$).

Associated with this, the solution has changed from being static to being *spatially homogeneous* but *evolving with time*, for the essential metric dependence is with the coordinate r, which has changed from being a coordinate measuring spatial distances to one measuring time changes. Thus the nature of the space-time symmetry changes completely for $r < 2m$.

(5) *Asymptotic null cones*: If we look at the radial null rays in this solution, from (1)

$$0 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 \quad (98)$$

on radial null geodesics, where $ds^2 = 0 = d\theta^2 = d\phi^2$. Thus on these geodesics,

$$\frac{dt}{dr} = \pm(1 - \frac{2m}{r})^{-1} \quad (99)$$

which is the equation of the local null cones. This shows that $dt/dr \to \pm 1$ as $r \to \infty$, as in flat space-time in standard coordinates, but $dt/dr \to \pm\infty$ as $r \to 2m$. Accordingly light rays from the outside region ($r > 2m$) cannot reach the surface $r = 2m$: rather they become asymptotic to it as they approach it. The same will hold true for timelike geodesics.

(6) *Geodesic incompleteness*: however these timelike and null geodesics are *incomplete*, that is, they cannot be extended to arbitrarily large values of their affine parameter, in the region $2m < r < \infty$. This follows already for timelike geodesics from the discussion (3) above: for the proper time along the geodesics from r_0 to $2m$ is finite. This shows the geodesic, moving inward as t increases, cannot be extended to infinite values of its affine parameter (namely, proper time). The same is true for timelike geodesics moving inwards as t decreases; and the analogous result holds for null geodesics.

(7) *Surface gravity*: we established (equation (24)) that the acceleration of the static observers is of magnitude $\dot{u} = m/r^2$. At the limit surface $r = 2m$, with limiting surface area $A_* = 4\pi(2m)^2 = 16\pi m^2$, this acceleration takes the limiting value

$$\kappa \equiv \{\dot{u}|_{r\to 2m}\} = \frac{1}{4m} = \left(\frac{\pi}{A_*}\right)^{1/2} \quad (100)$$

which is finite. This is the *surface gravity* at $r = 2m$.

(8) *Physical singularity?* To try to see if the singularity at $r = 2m$ is a physical or coordinate singularity, we look at scalars (because they are coordinate-independent quantities) constructed from a mathematical objects that describe the curvature. Because the solution is a vacuum solution ($R_{ab} = 0$), both $R = R^a{}_a$ and $R^{ab}R_{ab}$ vanish. The simplest non-zero scalar is the Kretschmann scalar

$$R_{abcd}R^{abcd} = \frac{48m^2}{r^6} \quad (101)$$

This is finite at $r = 2m$, but diverges as $r \to 0$. This suggests – but does not prove – that $r = 2m$ is a coordinate singularity: that is, there is no problem with the space-time, rather the coordinates break down there, and so we can get rid of the singularity by choosing different coordinates.

We prove this supposition is correct by making different extensions of the solution across the surface $r = 2m$. One can do this by attaching coordinates to either timelike or null geodesics that cross this surface; we choose the latter. As a preliminary, we look at the nature of null coordinates in flat space-time.

3.1.1 Flat space-time in null coordinates

Flat space in standard spherical coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{102}$$

The radial null geodesics are given by $ds^2 = 0 = d\theta^2 = d\phi^2$, so on them

$$\frac{dt}{dr} = \pm 1 \Leftrightarrow t = \pm r + const \tag{103}$$

(the light rays are at $\pm 45^0$). If we define the constants to be v, w, then:

$$v = t + r, \quad w = t - r \tag{104}$$

and the outgoing null geodesics are $\{w = const\}$ while the ingoing null geodesics are $\{v = const\}$. The inverse of these relations are

$$t = \frac{1}{2}(v + w), \quad r = \frac{1}{2}(v - w), \tag{105}$$

Since

$$dv = dt + dr, \quad dw = dt - dr \tag{106}$$

we can see the form taken by the flat space metric on using these coordinates. We consider three forms that arise.

1] Using the null coordinate v instead of t, we have coordinates (v, r, θ, ϕ), and the metric (102) is

$$ds^2 = -(dv - dr)^2 + dr^2 + r^2 d\Omega^2,$$

giving the single-null form

$$ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2 \tag{107}$$

which is time asymmetric. The radial null geodesics are given by $dv = 0 \Leftrightarrow v = const$ (ingoing), $dv = 2dr \Leftrightarrow v = 2r + const$ (outgoing).

2] Similarly, using the null coordinate w instead of t, we have coordinates (w, r, θ, ϕ), and the metric takes the single-null form

$$ds^2 = -dw^2 - 2dwdr \ + \ r^2 d\Omega^2 \tag{108}$$

which is also time asymmetric. It is identical to (107) except for the sign of the cross-term (which has the opposite sign, because we have used the outgoing instead of incoming null coordinate). The radial null geodesics are given by $dw = 0 \ \Leftrightarrow \ w = const$ (outgoing), $dw = -2dr \ \Leftrightarrow \ w = -2r + const$ (ingoing).

3] If we use both null coordinates (v, w) instead of (t, r), because the metric (102) is

$$ds^2 = -(dt + dr)(dt - dr) \ + \ r^2 d\Omega^2$$

we get the double null form

$$ds^2 = -dvdw \ + \ r^2 d\Omega^2 \tag{109}$$

which is time symmetric (v and w appear on an equal footing). The radial null geodesics are given by $dv = 0 \ \Leftrightarrow \ v = const$ (ingoing), $dw = 0 \ \Leftrightarrow \ w = const$ (outgoing).

We see the characteristic feature of use of null coordinates is zeros down the diagonal of the metric. This is perfectly allowable, as the determinant remains non-zero.

3.1.2 Schwarzschild null coordinates

We now set up similar null coordinates in the Schwarzschild solution. From (99), the radial null geodesics are given by

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2m}{r}} \ \Leftrightarrow \ t = \pm \ r^* + const \tag{110}$$

where

$$dr^* = \frac{dr}{1 - \frac{2m}{r}} \ \Leftrightarrow \ r^* = r + 2m \ln\left(\frac{r}{2m} - 1\right) \tag{111}$$

Defining the constants in (110) to be v, w:

$$v = t + r^*, \quad w = t - r^* \tag{112}$$

then the (outgoing) null geodesics are $\{w = const\}$ and the (ingoing) null geodesics are $\{v = const\}$. Using

$$dv = dt + dr^* = dt + \frac{dr}{1 - \frac{2m}{r}}, \quad dw = dt - dr^* = dt - \frac{dr}{1 - \frac{2m}{r}} \tag{113}$$

we can obtain the three null forms for the Schwarzschild metric corresponding to those above for flat space-time.

3.2 The Eddington-Finkelstein extension

We change to coordinates (v, r, θ, ϕ). By (90), (113) the metric is

$$ds^2 = -(1 - \frac{2m}{r})\left(dv - \frac{dr}{1 - \frac{2m}{r}}\right)^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

which is

$$ds^2 = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{114}$$

corresponding to (107). This is the Eddington-Finkelstein form of the metric (A S Eddington: Nature 113, 192 (1924)). The transformation has succeeded in getting rid of the singularity at $r = 2m$; indeed at that radius, the metric takes the (double null) flat space-time form (109). The coordinate transformation (which is singular at $r = 2m$) extends the original space-time region **I**, defined by $2m < r < \infty$, to a new region **II**, defined by $0 < r < 2m$. In more detail: equation (114) for $r > 2m$ is just the same as (90), but given in different coordinates; also equation (114) for $0 < r < 2m$ is the same as (90), but given in different coordinates; and the solution (114) is analytic for $r > 0$, so it is an analytic extension across the surface $r = 2m$ of the outside region **I** of (1) to the inside region **II** of (1). Indeed if we were to seek vacuum solutions of the field equations of the form

$$ds^2 = -A(r)dv^2 + 2dvdr + r^2 d\Omega^2$$

we would obtain the solution (114) *ab initio*, without ever experiencing any problems at $r = 2m$.

From this metric form, we can determine the radial null geodesics as usual: setting $ds^2 = 0 = d\theta = d\phi$ in (114), we find

$$0 = dv\left(-(1 - \frac{2m}{r})dv + 2dr\right)$$

so the radial geodesics obey

$$dv = 0, \quad \frac{dv}{dr} = \frac{2}{1 - 2m/r} \tag{115}$$

that is, $v = const$, $v = 2r^* = const$ (on using (111)), and one can plot the null cones from this.

However we are unused to plotting metrics using null coordinates, so it is convenient to change to the time and radial coordinates (t^*, r) corresponding to (v, r) in the same way as they do in flat spacetime:

$$v = t^* + r \tag{116}$$

(cf (104)), with r^* given by (111). Note that here r is the same as in the original metric, but t^* is a new time coordinate. It is given by

$$t^* = v - r = t + r^* - r \quad \Leftrightarrow \quad t = t^* + 2m\ln\left(\frac{r}{2m} - 1\right) \tag{117}$$

From (116), $dv = dt^* + dr$ so (114) becomes

$$ds^2 = -(1 - \frac{2m}{r})(dt^* + dr)^2 + 2(dt^* + dr)dr \ + \ r^2 d\Omega^2$$

giving

$$ds^2 = -(1 - \frac{2m}{r})dt^{*2} - (1 - \frac{2m}{r})2dt^* dr - (1 - \frac{2m}{r})dr^2 + 2dt^* dr + 2dr^2 \ + \ r^2 d\Omega^2$$

that is: in terms of the coordinate (t^*, r, θ, ϕ), the Schwarzschild metric takes the form

$$ds^2 = -(1 - \frac{2m}{r})dt^{*2} + \frac{4m}{r}dt^* dr + (1 + \frac{2m}{r})dr^2 \ + \ r^2 d\Omega^2 \tag{118}$$

This is just the metric (114), but now transformed to 'standard' time and radial coordinates. It is again regular at $r = 2m$.

Now the radial null geodesics of (118) are given by

$$0 = -(1 - \frac{2m}{r})dt^{*2} + \frac{4m}{r}dt^* dr + (1 + \frac{2m}{r})dr^2$$

which is the same as

$$0 = (dt^* + dr)\left(-(1 - \frac{2m}{r})dt^* + (1 + \frac{2m}{r})dr\right)$$

The solutions are

$$0 = (dt^* + dr), \quad 0 = -(1 - \frac{2m}{r})dt^* + (1 + \frac{2m}{r})dr$$

so the local light cones are given by

$$\frac{dt^*}{dr} = -1, \quad \frac{dt^*}{dr} = \frac{(1 + \frac{2m}{r})}{(1 - \frac{2m}{r})} \tag{119}$$

when $r \neq 2m$; when $r = 2m$, they are given by $dt*/dr = -1$, $dr = 0$.

Thus the ingoing light rays are always at -45^0 (as in flat space time) relative to the coordinates (t^*, r);; but the 'outgoing' light rays have a slope that varies with distance r from the centre. As $r \to \infty$, the slope $\to +45^0$; however at

$r = 2m$ they are vertical, and as $r \to 0$ they have the limit of -45^0. The integral light rays are given by: ingoing,

$$t^* = -r + const \tag{120}$$

(the constant just being v), and, outgoing: when $r \neq 2m$,

$$t^* = \int \frac{1 + \frac{2m}{r}}{1 - \frac{2m}{r}} dr = \int \left(1 + \frac{4m}{r - 2m}\right) dr$$

that is

$$t^* = r + 4m \ln(r - 2m) + const. \tag{121}$$

When $\{r = 2m\}$ we have a particular ('outgoing') null geodesic. Thus while the ingoing rays cross the surface $r = 2m$ without any problem, the 'outgoing' rays become tangent to it and cannot cross this surface. The reason for this is that it is itself a null geodesic.

Plotting these local null cones and light rays in the space with coordinates (t^*, r), we get the Eddington-Finkelstein diagram. We see from this, the following features.

(1) The surfaces of constant r are vertical lines in this diagram. The surfaces of constant t are (from (117)) nearly flat at large distances but bend down and never cross the surface $r = 2m$. In fact t diverges at this surface; and it is this bad behaviour of the t-coordinate that is responsible for the coordinate singularity at $r = 2m$ in (90), and is why the coordinate time diverges for a freely falling particle that crosses this surface (see (97)).

(2) The outer region is asymptotically flat, as $r \to \infty$ (the cross-term in $dtdr$ becomes indefinitely small as $r \to \infty$).

(3) The null cones tilt over, the inner ray always being at 45^0 inwards, but the outer one (pointing outward for $r > 2m$) becomes vertical at $r = 2m$ and points inwards for $r < 2m$. Thus the surface $r = 2m$ is a *null surface* (a light ray emitted outwards at $r = 2m$ stays forever at that distance from the centre). Because of this, it is a *trapping surface*: particles that have fallen in and crossed this surface from the outside region **I** to the inside **II**, can never get out again. Indeed because they must always move locally at less than the speed of light, and so are confined between the ingoing and outgoing light rays, once in region **II** their future is to inevitably fall into the singularity at $r = 0$, where they are crushed by infinite tidal forces (cf. the divergence of the scalar (101)).

(4) Conversely, $r = 2m$ is an *event horizon*, hiding its interior from the view of outside observers. If we consider an observer static at $r = r_1 > 2m$, her world

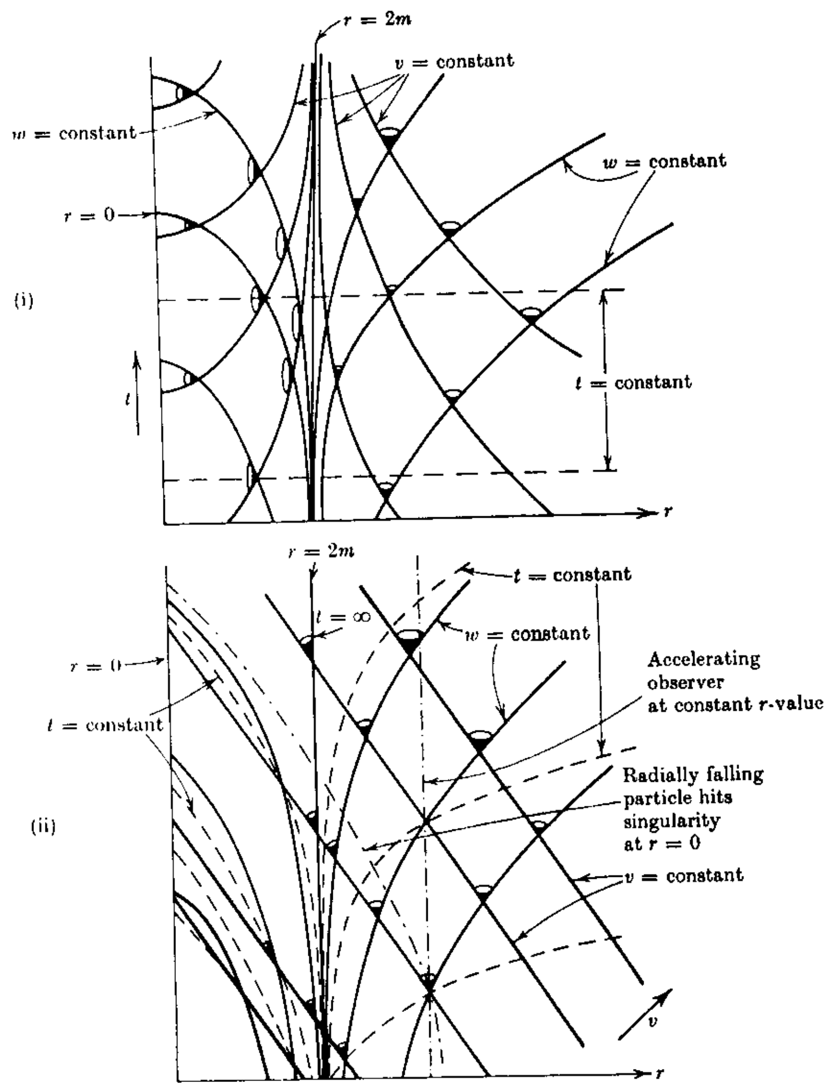

FIGURE 23. Section (θ, ϕ) constant of the Schwarzschild solution.
(i) Apparent singularity at $r = 2m$ when coordinates (t, r) are used.
(ii) Finkelstein diagram obtained by using coordinates (v, r) (lines at 45° are lines of constant v). Surface $r = 2m$ is a null surface on which $t = \infty$.

isometric to $(\mathscr{M}, \mathbf{g})$. A construction of this larger manifold has been given by Kruskal (1960). To obtain it, consider $(\mathscr{M}, \mathbf{g})$ in the coordinates (v, w, θ, ϕ); then the metric takes the form

$$ds^2 = -\left(1-\frac{2m}{r}\right) dv\, dw + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where r is determined by

$$\tfrac{1}{2}(v-w) = r + 2m \log(r - 2m).$$

This presents the two-space (θ, ϕ constant) in null conformally flat coordinates, as the space with metric $ds^2 = -dv\, dw$ is flat. The most general coordinate transformation which leaves this two-space expressed in such conformally flat double null coordinates is $v' = v'(v)$, $w' = w'(w)$ where v' and w' are arbitrary C^1 functions. The resulting metric is

$$ds^2 = -\left(1-\frac{2m}{r}\right) \frac{dv}{dv'}\frac{dw}{dw'} dv'\, dw' + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

To reduce this to a form corresponding to that obtained earlier for Minkowski space-time, define

$$x' = \tfrac{1}{2}(v' - w'), \quad t' = \tfrac{1}{2}(v' + w').$$

The metric takes the final form

$$ds^2 = F^2(t', x')(-dt'^2 + dx'^2) + r^2(t', x')(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (5.23)$$

The choice of the functions v', w' determines the precise form of the metric. Kruskal's choice was $v' = \exp(v/4m)$, $w' = -\exp(-w/4m)$. Then r is determined implicitly by the equation

$$(t')^2 - (x')^2 = -(r - 2m)\exp(r/2m) \quad (5.24)$$

and F is given by

$$F^2 = \exp(-r/2m) \,.\, 16m^2/r. \quad (5.25)$$

On the manifold $\mathscr{M}^*$ defined by the coordinates (t', x', θ, ϕ) for $(t')^2 - (x')^2 < 2m$, the functions r and F (defined by (5.24), (5.25)) are positive and analytic. Defining the metric $\mathbf{g}^*$ by (5.23), the region I of $(\mathscr{M}^*, \mathbf{g}^*)$ defined by $x' > |t'|$ is isometric to $(\mathscr{M}, \mathbf{g})$, the region of the Schwarzschild solution for which $r > 2m$. The region defined by $x' > -t'$ (regions I and II in figure 24) is isometric to the advanced Finkelstein extension $(\mathscr{M}', \mathbf{g}')$. Similarly the region defined by $x' > t'$ (regions I and II′ in figure 24) is isometric to the retarded Finkelstein extension $(\mathscr{M}'', \mathbf{g}'')$. There is also a region I′, defined by $x' < -|t'|$,

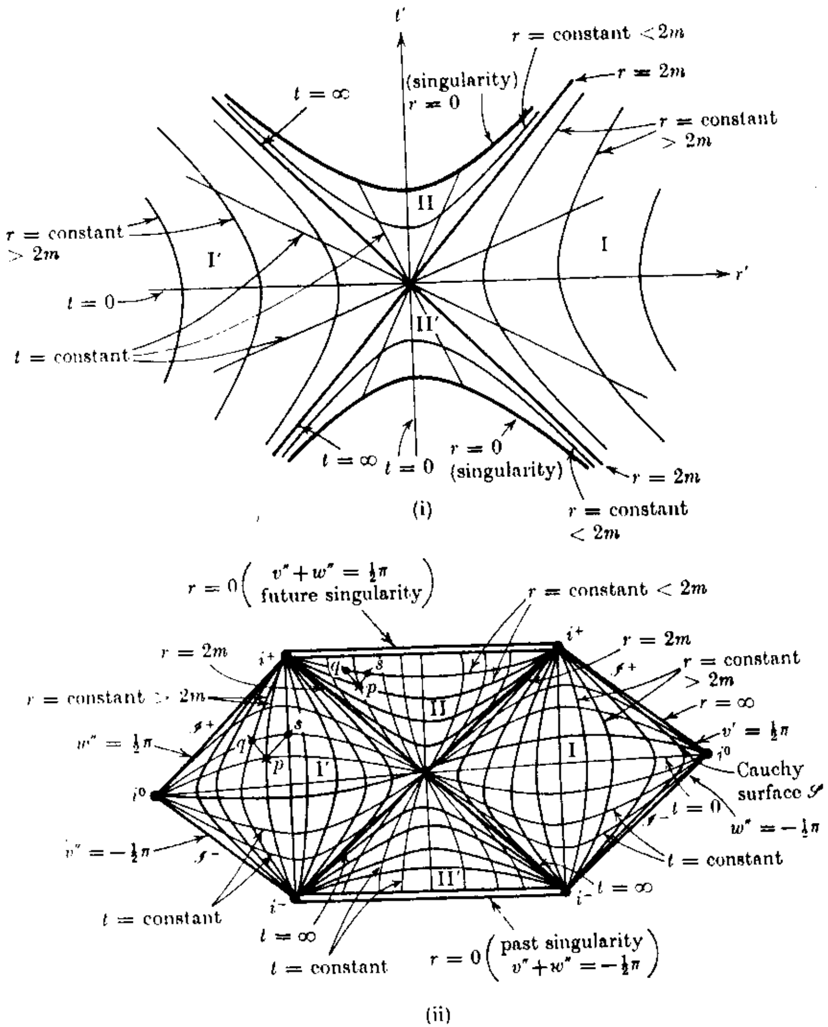

FIGURE 24. The maximal analytic Schwarzschild extension. The θ, ϕ coordinates are suppressed; null lines are at $\pm 45°$. Surfaces $\{r = \text{constant}\}$ are homogeneous.

(i) The Kruskal diagram, showing asymptotically flat regions I and I' and regions II, II' for which $r < 2m$.

(ii) Penrose diagram, showing conformal infinity as well as the two singularities.

which turns out to be again isometric with the exterior Schwarzschild solution $(\mathcal{M}, \mathbf{g})$. This can be regarded as another asymptotically flat universe on the other side of the Schwarzschild 'throat'. (Consider the section $t = 0$. The two-spheres $\{r = \text{constant}\}$ behave as in Euclidean

space, for large r; however for small r, they have an area which decreases to the minimum value $16m^2$ and then increases again, as the two spheres expand into the other asymptotically flat three-space.) The regions I' and II are isometric with the advanced Finkelstein extension of region I', and similarly I' and II' are isometric with the retarded Finkelstein extension of I', as can be seen from figure 24. There are no timelike or null curves which go from region I to region I'. All future-directed timelike or null curves which cross the part of the surface $r = 2m$ represented here by $t' = |x'|$ approach the singularity at $t' = (2m + (x')^2)^{\frac{1}{2}}$, where $r = 0$. Similarly past-directed timelike or null curves which cross $t' = -|x'|$ approach another singularity at $t' = -(2m + (x')^2)^{\frac{1}{2}}$, where again $r = 0$.

The Kruskal extension $(\mathcal{M}^*, \mathbf{g}^*)$ is the unique analytic and locally inextendible extension of the Schwarzschild solution. One can construct the Penrose diagram of the Kruskal extension by defining new advanced and retarded null coordinates

$$v'' = \arctan(v'(2m)^{-\frac{1}{2}}), \quad w'' = \arctan(w'(2m)^{-\frac{1}{2}})$$

for $\quad -\pi < v'' + w'' < \pi \quad$ and $\quad -\frac{1}{2}\pi < v'' < \frac{1}{2}\pi, \; -\frac{1}{2}\pi < w'' < \frac{1}{2}\pi$

(see figure 24 (ii)). This may be compared with the Penrose diagram for Minkowski space (figure 15 (ii)). One now has future, past and null infinities for each of the asymptotically flat regions I and I'. Unlike Minkowski space, the conformal metric is continuous but not differentiable at the points i^0.

If we consider the future light cone of any point outside $r = 2m$, the radial outwards geodesic reaches infinity but the inwards one reaches the future singularity; if the point lies inside $r = 2m$, both these geodesics hit the singularity, and the entire future of the point is ended by the singularity. Thus the singularity may be avoided by any particle outside $r = 2m$ (so it is not 'universal' as it is in the Robertson–Walker spaces), but once a particle has fallen inside $r = 2m$ (in region II) it cannot evade the singularity. This fact will turn out to be closely related to the following property: each point inside region II represents a two-sphere that is a closed trapped surface. This means the following: consider any two-sphere p (represented by a point in figure 24) and two two-spheres q, s formed by photons emitted radially outwards, inwards at one instant from p. The area of q (which is given by $4\pi r^2$) will be greater than the area of p, but the area of s will be less than the area of p, if all three lie in a region $r > 2m$. However if they all lie in the region II where $r < 2m$, then the areas of *both* q and s will be less line is a vertical line in this diagram. Her past light cone never reaches inside $r = 2m$, so no signal from that region can reach her. Thus this space-time may reasonably be called a *black hole*: for no light or other radiation emitted by the inside region **II**, can reach the outside region **I**.

(5) If the outside observer drops a probe into the centre, then it crosses the event horizon $r = 2m$ in a finite proper time (cf. (94)) but takes an infinite coordinate time t to get there (cf. (97)) - because t diverges there (cf. (117)). If it emits pulses at regular intervals (say every second), these will be received by the outside observer at longer and loner time intervals. Indeed if the probe crosses the event horizon at 12 : 00 according to its internal clock, and sends out a special radio signal then, this signal will never reach the outside observer - it forever stays at $r = 2m$. Every signal sent before then will (eventually) reach the outside observer, and every signal sent afterwards will fall into the central singularity. The infinite slowing down of the received signals as the probe approaches the event horizon will result in the redshift in received signals diverging as $r_e \to 2m$ (see (92)).

After crossing the event horizon, the infalling probe reaches the central singularity in a finite time, where it is torn apart by unbounded tidal forces. None of this is visible to the outside world.

This analysis shows convincingly that $r = 2m$ is a null surface (the event horizon) where the original coordinates go wrong; the space-time can be extended across this null surface by a change to new coordinates. However there is a problem with what we have so far: namely the original solution (90) is time symmetric. The extension (114), or equivalently (118), is not; as is obvious from the Eddington-Finkelstein diagram.

3.2.1 The time-reversed extension

Reflection on the relation between flat space forms (107) and (108) will show that we can similarly make another Eddington-Finkelstein extension, in which in effect we choose the other direction of time for the extension, by using the other null coordinate.

In more detail: changing from (t, r, θ, ϕ) to coordinates (w, r, θ, ϕ), the metric (90) now takes the form

$$ds^2 = -(1 - \frac{2m}{r})dw^2 - 2dwdr \; + \; r^2 d\Omega^2 \tag{122}$$

which is the time-reverse of (114) (cf. the relation of (107) and (108)). This is also an Eddington-Finkelstein form of the metric, now extending the original space-time region **I**, defined by $2m < r < \infty$, to a further new region **II**$'$, defined by $0 < r < 2m$.

Following the same steps as before, we obtain a a metric equivalent to (118) but with a minus sign for the cross-term, leading to the associated time-reversed Eddington-Finkelstein picture of the local light cones and possible particle paths. This time it is the ingoing null geodesics that are badly behaved at $r = 2m$ (the outgoing ones cross this surface with no trouble). The surface $r = 2m$ is again a null cone, but this time it represents the surrounding horizon of a *white hole*: signals can come out from it, but not go into it. The exterior observer can receive messages from region **II**$'$, but never send signals there. Investigation will show that the surface $r = 2m$ is now the same as $t = -\infty$ (rather than $+\infty$, as in the previous case). Again, this divergence is the reason the original metric went wrong at this surface.

How do we know region **II** is different from region **II**$'$? (implying the associated horizon $r = 2m$ is a different surface in each of the two cases). The essential point is that in the original metric, both the past and the future inward-pointing radial null geodesics through each event q in region **I** were incomplete. The first extension completed the future-ingoing null geodesics, but not the past-ingoing ones; the second extension completed the past- ingoing null geodesics, but not the future-ingoing ones. The question now is whether we can make both extensions simultaneously.

3.3 Kruskal-Szekeres

The time asymmetry of each extension is because we used only one null coordinate in each case. To obtain a time-symmetric extension, we must use both null coordinates (as in (109)). Indeed if we start with (90) and change to coordinates (v, w, θ, ϕ), with v, w defined by (112), we obtain the double null form of the metric

$$ds^2 = -(1 - \frac{2m}{r})dv dw \;+\; r^2 d\Omega^2 \tag{123}$$

where from (111,112)

$$\frac{1}{2}(v - w) = r + 2m \ln\left(\frac{r}{2m} - 1\right) \tag{124}$$

defines $r(v, w)$ (both quantities being equal to r^*), and

$$t = \frac{1}{2}(v + w).$$

This is a time-symmetric double null form (analogous to (109)) – but is singular at $r = 2m$. So it is no good for our purposes. However if, following Kruskal, we rescale the null coordinates, we can attain what we want. Defining

$$V = e^{v/4m}, \quad W = -e^{-w/4m}, \tag{125}$$

we obtain a new set of double null coordinates for the Schwarzschild solution. The metric becomes

$$ds^2 = -\frac{32m^3}{r}e^{-r/2m}dVdW \ + \ r^2\,d\Omega^2 \tag{126}$$

with $r(V, W)$ determined by (124,125). This metric form is regular at $r = 2m$, and gives us the extension we want. The light cones are given by $V = const$, $W = const$.

As before, it is convenient to define the associated time and radial coordinates, by equations analogous to (105): let (T, X) be defined by

$$X = \frac{1}{2}(V - W), \quad T = \frac{1}{2}(V + W), \tag{127}$$

then the metric becomes

$$ds^2 = \frac{32m^3}{r}e^{-r/2m}(-dT^2 + dX^2) \ + \ r^2 d\Omega^2 \tag{128}$$

where now

$$X^2 - T^2 = e^{r/2m}\left(\frac{r}{2m} - 1\right) \tag{129}$$

implicitly defines $r(X, T)$, and the original Schwarzschild time coordinate t is given by

$$\frac{t}{2m} = \ln\left(\frac{X+T}{X-T}\right) = 2\,\tanh^{-1}\left(\frac{T}{X}\right) \tag{130}$$

These are the Kruskal-Szekeres coordinates. The radial light rays are now given by $dX = dT \ \Leftrightarrow \ X = T + const$ (ingoing), $dX = -dT \ \Leftrightarrow \ X = -T + const$ (outgoing). We can obtain from this the Kruskal Diagram, giving the complete time-symmetric extension of the Schwarzschild solution with light rays at $\pm 45^0$.

It is important to remember this is a cross-section of the full space-time; in fact each point represents a 2-sphere of area $4\pi r^2$ of the full space-time (we have suppressed the coordinates (θ, ϕ) in order to draw this diagram). The null cones in this cross-section are at $\pm 45^0$ everywhere,in these coordinates. The whole solution is time symmetric, as desired. The most important new feature is that there is a new region **I′** in addition to the three regions **I**, **II**, and **II′** already identified. Let us see why this is so.

The region **I** in this diagram corresponds to the *same* region **I** in *both* the Eddington-Finkelstein diagrams. The region $T > X$ (bounded on the left by the line at -45^0 through the origin) corresponds to the first extension to region **II** completing the future-ingoing null geodesics. This part of the Kruskal diagram corresponds point by point with the first Eddington-Finkelstein diagram;

in particular the vertical null geodesics at $r = 2m$ in Eddington-Finkelstein correspond to the null geodesic at $+45^0$ through the origin in Kruskal. The region $T < X$ (bounded on the left by the line at $+45^0$ through the origin) corresponds to the second extension to region **II′**, completing the past-ingoing null geodesics (moving in the opposite direction to the future-ingoing ones). This part of the Kruskal diagram corresponds point by point with the second (time-reversed) Eddington-Finkelstein diagram.

Consider now a point in region **II**. The past-outgoing (i.e. moving to the right) null geodesics cross $r = 2m$ to the asymptotically-flat region **I**. The past-ingoing (i.e. moving to the left) null geodesics are completely symmetric with them: they must cross $r = 2m$ to an asymptotically-flat region **I′** which is identical to region **I**. Similarly for points in **II′**: the outgoing future-directed null geodesics crosses $r = 2m$ to region **I**, and the ingoing future-directed null geodesics must cross $r = 2m$ to an identical region **I″**. What is perhaps not immediately obvious is that this is the *same* region as **I′**. This follows if we assume the cross-over where the two null surfaces $r = 2m$ meet, is regular (we discuss this intersection further below).

The following features then arise, corresponding to the Eddington–Finkelstein features discussed above.

(1) The surfaces of constant r are given by $X^2 - T^2 = const$ (by (129)), and so correspond to the hyperbola at constant distance from the origin in flat space-time. They are spacelike for $0 < r < 2m$ and timelike for $r > 2m$ (with *two* surfaces occurring for each value of r). There are *two* singularities at $r = 0$: one in the past ($t^* < 0$) and one in the future ($t^* > 0$). The surfaces $r = 2m$ are the two intersecting null surfaces through the origin. The surfaces of constant t are (from (130)) the straight lines through the origin. This coordinate diverges at both surfaces $r = 2m$.

(2) The null surface segments $\{r = 2m,\ t^* > 0\}$ – obviously representing motion at the speed of light – are *trapping surfaces*: particles that have fallen in and crossed this surface from either outside region to the inside **II**, can never get out again; they inevitably fall into the future singularity at $\{r = 0,\ t^* > 0\}$, where they are crushed by infinite tidal forces. Each of these segments is also an *event horizon*, hiding its interior from the view of outside observers. They bound the *black hole* region **II** of the space-time, from which no light or other radiation emitted can reach the outside. The details of dropping in a probe from the outside can be followed in this diagram, revealing again the same affects as discussed above. Thus we have (vacuum) black holes, bounded by event horizons: from which light cannot escape

(3) There are two curvature singularities, each of which has a spacelike lim-

iting nature- one in the past and one in the future. The complete solution has both a white hole and a black hole singularity (the former emitting particles and radiation into the space-time, the latter receiving them).

It is particularly important to note here that we do *not* find a singular timelike world-line at the centre of the Schwarzschild solution, representing a particle generating the solution (as in the electromagnetic case). Here the strong gravitational field for $r < 2m$ profoundly alters the nature of the singularity from what we first expected.

(4) There is also an unexpected global topology, with two asymptotically flat spaces back to back, joined by a wormhole (to see this consider any particular surface $\{t = const\}$; the area of the 4-spheres which diverges at infinity decreases to a value A_*, which is $4\pi(2m^2)$ if it passes though the central 2-sphere where the two event horizons meet, and then increases again to infinity). hence an imbedding diagram which gives the correct picture of circumferences to radial distances will show two asymptotically flat 3-spaces joined by a wormhole. However one cannot communicate between the two asymptotically flat regions through the wormhole, because only spacelike curves can pass through.

Wheeler has emphasized one can take a dynamic view of the wormhole, by considering successive slices $\{T = const\}$ of the full solution. This will show initially two disjoint regions; the wormhole then opens up, its throat grows to the maximum area A_*, and then decreases again to zero when the wormhole pinches off and the two spaces are again separate. This dynamic evolution is a consequence of examining the universe by means of this particular time coordinate T (it does not happen if we use the coordinate t).

(5) There is change of nature of the space-time symmetry at the surfaces $r = 2m$: here there is a transition from a static to an evolving (Kantowski-Sachs) universe. Thus the Event horizon is also a *Killing horizon*, generated by null Killing vectors which are spacelike on one side and timelike on the other.

It inevitably follows when there is such a symmetry transition, that there is a bifurcation of the Killing horizon, with the bifurcation set being fixed points of the Killing vector field [see R H Boyer: *Proc Roy Soc* **A 311**, 245-252 (1969); this happens also for example in the boosts around a point in flat space-time]. This is the reason there are two null Killing surfaces $r = 2m$, intersecting in the bifurcation 2-sphere with area A_*. This bifurcation plays a central role in the Hawking radiation mechanism in the Schwarzschild solution.

(6) This solution is indeed the maximal extension of the initial Schwarzschild space-time: all geodesics in it *either* are complete (that is, they go to infinity) *or* run into one of the singularities where $r = 0$. Hence no further extensions

are possible (all curves end up at a singularity or infinity).

(7) Closed trapped surfaces exist. Consider any point p in region **II**; this represents a complete 2-sphere. Now the outgoing and ingoing null geodesics are at 45^0, and both lead to 2-spheres of smaller surface area (as the are coordinate r necessarily decreases along both geodesics as one moves to the future). Thus the surface are of light expanding out from p to the future decreases on both the 'outgoing' and the 'ingoing' null directions. This can only happen in very strong gravitational fields, and indeed guarantees that there will be a singularity in the future (Penrose, 1965).

(8) Cauchy surfaces in the space: finally, despite the singularities and horizons, there exist spacelike surfaces in this maximally extended space which intersect every timelike and null curve (e.g. the surface $T = 0$). These are *Cauchy Surfaces*: that is, surfaces on which one can put initial data for the fields in the space-time (and for the space-time itself), leading to a unique evolution of the fields (and the space-time) to the past and future of that surface, through the whole maximal space-time, provided their equations of motion are well-defined (e.g. the source-free Maxwell equations and the vacuum Einstein equations).

3.4 Penrose/Carter diagrams

The final step that is helpful is, following Roger Penrose, to make infinity visible in the space-time diagrams. This is achieved by defining

$$v'' = arc\tan(V(2m)^{-1/2}), \quad w'' = arc\tan(W(2m)^{-1/2}) \tag{131}$$

which 'makes infinity finite' while preserving the light cones at $\pm 45^0$ (for null coordinates are being mapped to null coordinates, e.g. $V = const$ is a null surface, and therefore so is $v'' = const$, and the same holds for w). The whole space time, including its boundaries at infinity, is now contained in the region

$$-\pi < v'' + w'' < \pi, \quad -\frac{1}{2}\pi < v'' < \frac{1}{2}\pi, \quad -\frac{1}{2}\pi < w'' < \frac{1}{2}\pi$$

Thus we can see in the resulting diagram the whole space time and boundary at infinity (as well as singularities) This makes the causal structure of the singularities clear. One can again see all the properties mentioned above in this diagram (which shows accurately all causal relationships, at the cost of wildly distorting spatial and timelike distances).

3.5 Orbits

Particle orbits - stable circular orbits down to $r = 6m$ (last stable circular orbit) Unstable circular orbits in range $3m < r < 6m$

Light rays – Highly bent if close. Indeed circular orbits at $r = 3m$.

4 SPHERICAL GRAVITATIONAL COLLAPSE

The black holes discussed in the previous section have lived for ever - they have not been formed from collapse of a star or other massive object, for they are the maximal vacuum solutions consistent with the Schwarzschild geometry. The issue now is how does this picture get modified if we consider collapse of a massive object to form a black hole.

4.1 Astrophysical non-rotating Black holes

Spherically symmetric analysis: based on the Eddington-Finkelstein diagram. Schwarzschild solution represents non-rotating black holes: final state of collapse of non-rotating star after falls within event horizon

depends on idea that anisotropies will be radiated away, through decay of perturbations, so the spherical situation is a good description of realistic collapse This has to be investigated (see later)

4.1.1 Mass and radius limits

Black hole: can't have static solution with $r < 2m$

Schwarzschild interior solution mass limits: apply also to general stars (and limit the redshift possible from a static star's surface)

Broad physical basis for these limits: see 'Black holes as the final state of evolution of massive bodies', B Carter *Journal de Physique* C7-39 to C7-46 (1973)

4.1.2 Major features

If an object gets very near the critical circumference, then gravity necessarily overwhelms all other forces inside it, and squeezes it into a catastrophic implosion which forms a black hole

Oppenheimer and Schneider Phys Rev 56, 455-459 (1939)

K S Thorne Gravitational Collapse Sci Am 217, 88 (1965)

Tolman-Bondi solutions of this type can be explicitly constructed with Coordinates fixed to infalling matter [Stephani pp224-225]

* an imploding star really does shrink through the critical circumference without hesitation. That it appears to freeze as seen from far away is an illusion.

Eddington-Finkelstein Picture of gravitational collapse

* diagram of collapse, observer, matter falling in, event horizon

* contrast of internal and external views; Exterior view of gravitational collapse: can't see the collapse. Appears to slow down, take an infinite time with infinite redshift

* Throat blocked off by collapsing star

* closed trapped surfaces, singularity theorems: inevitability of singularity [stable property]

* Cauchy surfaces exist

4.1.3 Appearance

Appearance of collapsing star:

WL Ames and K S Thorne ApJ 151 659-670 (1968)

K S Thorne, Sci Am ? 98-88

K Lake and R C Roeder Ap J 232:277-281 (1979)

The endpoint will be hidden by the horizon; it will be invisible

however:

1: bending of light will cause lensing

2: will leave a gravitational field that can be detected in a binary system (a star and a bh in orbit around each other) giving a telltale shift of spectra from red to blue

3: travelling through gas will cause a shock front that will heat gas which will then emit x-rays so

ideal bh candidate is a binary with an optically bright but x-ray dark star, plus an optically dark but x-ray bright object (the bh) - cygnus x-1

4: x-ray emitting gas should form a accretion disk around the hole

falling in matter: forms disk, heats up by friction, accretion disk [Lynden Bell] emits x-rays eg 3c273

Bardeen: rotating bh acts as gyroscope, spin direction fixed and unchanging, swirl of space created by the spin remains firmly oriented in the same direction This swirl grabs the accretion disk and holds it in the equatorial plane The infalling gas spins up the bh to nearly maximal speed

this could lead to hot-spots orbiting the hole and so producing pulsar-like pulses of radiation through a swinging beam BUT the gas moves turbulently, so lumps do not remain coherent for long; there is chaotic x-rays from the rest of the gas; most of the x-rays should come from about 10 times the horizon radius, where gravity is relatively weak and lacking specific black hole signatures

see 'Astrophysical processes near black holes'. Eardley and Press *Ann Rev Ast Ast* **13**: 381-422 (975) either pass within its Schwarzschild radius, or manage to eject sufficient matter that its mass is reduced to less than M_L.

Ejection of matter has been observed in supernovae and planetary nebulae, but the theory is not yet very well understood. What calculations there have been suggest that stars up to $20M_L$ may possibly be able to throw off most of their mass and leave a white dwarf or neutron star of mass less than M_L (see Weymann (1963), Colgate and White (1966), Arnett (1966), Le Blanc and Wilson (1970), and Zel'dovich and Novikov (1971)). However it is not really credible that a star of more than $20M_L$ could manage to lose more than 95 % of its matter, and so one would expect that the inner part of the star at any rate would collapse within its Schwarzschild radius. (Present calculations in fact indicate that stars of mass $M > 5M_L$ would not be able to eject sufficient mass to avoid a relativistic collapse.)

Going to larger masses, consider a body of about $10^8 M_L$. If this collapsed to its Schwarzschild radius, the density would only be of the order of 10^{-4} gm cm^{-3} (less than the density of air). If the matter were fairly cold initially, the temperature would not have risen sufficiently either to support the body or to ignite the nuclear fuel; thus there would be no possibility of mass loss, or uncertainty about the equation of state. This example also shows that the conditions when a body passes through its Schwarzschild radius need not be in any way extreme.

To summarize, it seems that certainly some, and probably most, bodies of mass $> M_L$ will eventually collapse within their Schwarzschild radius, and so give rise to a closed trapped surface. There are at least 10^9 stars more massive than M_L in our galaxy. Thus there are a large number of situations in which theorem 2 predicts the existence of singularities. We discuss the observable consequences of stellar collapse in the next sections.

9.2 Black holes

What would a collapsing body look like to an observer O who remained at a large distance from it? One can answer this if the collapse is exactly spherically symmetric, since then the solution outside the body will be the Schwarzschild solution. In this case, an observer O' on the surface of the star would pass within $r = 2m$ at some time, say 1 o'clock, as measured by his watch. He would not notice anything special at that time. However after he passes $r = 2m$ he will not be

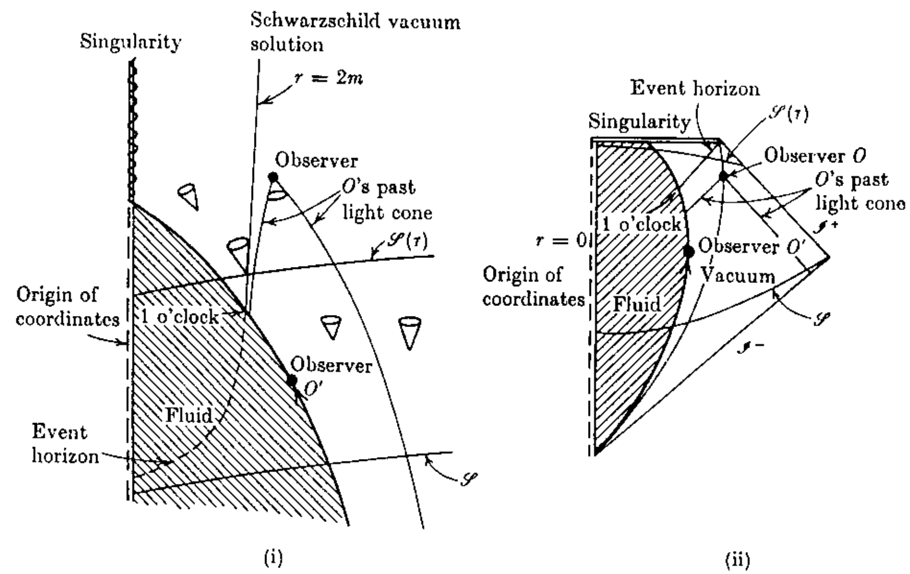

FIGURE 57. An observer O who never falls inside the collapsing fluid sphere never sees beyond a certain time (say, 1 o'clock) in the history of an observer O' on the surface of the collapsing fluid sphere.
(i) Finkelstein diagram; (ii) Penrose diagram.

visible to the observer O who remains outside $r = 2m$ (figure 57). However long the observer O waits, he will never see O' at a time later than 1 o'clock as measured by O''s watch. Instead he will see O''s watch apparently slow down and asymptotically approach 1 o'clock. This means that the light he receives from O' will have a greater and greater shift of frequency to the red and as a consequence a greater and greater decrease of intensity. Thus although the surface of the star never actually disappears from O's sight, it soon becomes so faint as to be invisible in practice. In fact O would first see the centre of the disc of the star become faint, and then this faint region would spread outwards to the limb (Ames and Thorne (1968)). The time scale for this diminution of intensity is of the order for light to travel a distance $2m$.

One would be left with an object which, for all practical purposes, is invisible. However it would still have the same Schwarzschild mass, and would still produce the same gravitational field, as it did before it collapsed. One might be able to detect its presence by its gravitational effects, for instance its effects on the orbits of nearby objects, or by the deflection of light passing near it. It is also possible that gas

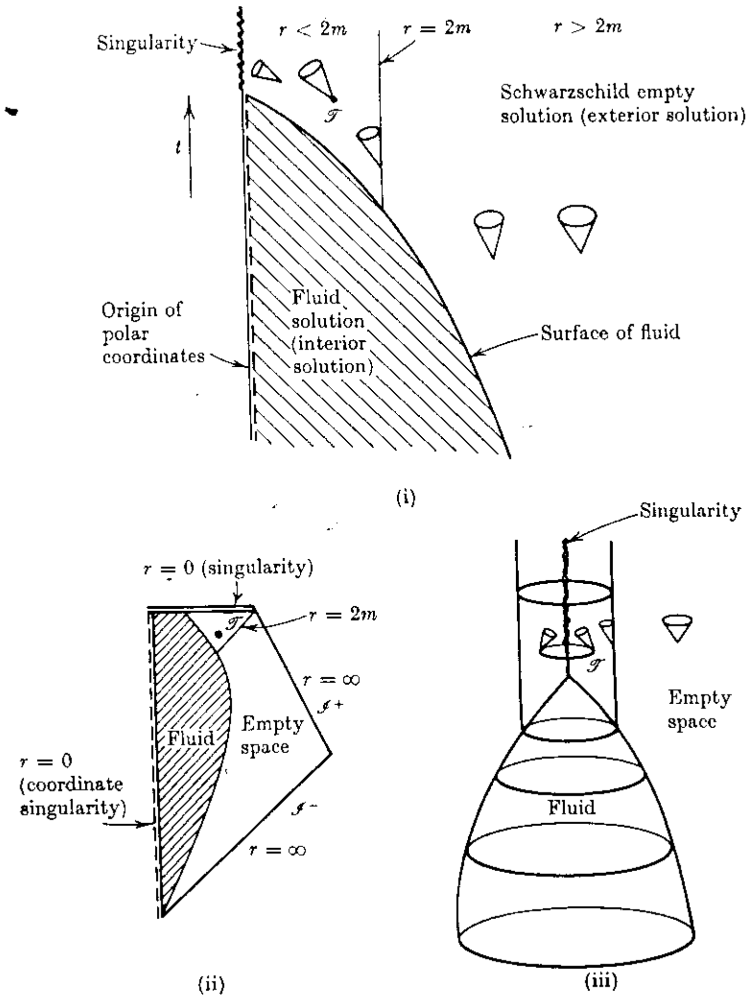

FIGURE 54. Collapse of a spherical star.
(i) Finkelstein diagram ((r, t) plane) of a collapsing spherically symmetric fluid ball. Each point represents a two-sphere.
(ii) Kruskal diagram of the collapsing fluid ball.
(iii) Diagram of the collapse with only one spatial dimension suppressed.

greater than $2m$ (the 'Schwarzschild radius'). This follows because the surface of a static star must correspond to the orbit of a timelike Killing vector, and in the Schwarzschild solution there is a timelike Killing vector only where $r > 2m$. If r_0 were less than $2m$, the surface of the star would be expanding or contracting. To get an idea of the magnitude of the Schwarzschild radius, we note that the Schwarzschild radius of the earth is 1.0 cm and that of the sun is 3.0 Km;

the ratios of the Schwarzschild radius to the radius of the earth and the sun are 7×10^{-10} and 2×10^{-6} respectively. Thus normal stars are a long way from their Schwarzschild radii.

The life of a typical star will consist of a long ($\sim 10^9$ years) quasi-static phase in which it is burning nuclear fuel and supporting itself against gravity by thermal and radiation pressure. However when the nuclear fuel is exhausted, the star will cool, the pressure will be reduced, and so it will contract. Now suppose that this contraction cannot be halted by the pressure before the radius becomes less than the Schwarzschild radius (we shall see below that this seems likely for stars of greater than a certain mass). Then since the solution outside the star is the Schwarzschild solution, there will be a closed trapped surface $\mathscr{T}$ around the star (see figure 54), and so, by theorem 2, a singularity will occur provided that causality is not violated and the appropriate energy condition holds. Of course in this case, because the exterior solution is the Schwarzschild solution, it is obvious (see figure 54) that there must be a singularity. However the point is that even if the star is not exactly spherically symmetric, a closed trapped surface will still occur providing the departures from spherical symmetry are not too great. This follows from the stability of the Cauchy development proved in §7.5; for one can regard the solution as developing from a partial Cauchy surface $\mathscr{H}$ (figure 55). Now if one changes the initial data by a sufficiently small amount on the compact region $J^-(\mathscr{T}) \cap \mathscr{H}$, the new development of $\mathscr{H}$ will still be sufficiently near the old in the compact region $J^+(\mathscr{H}) \cap J^-(\mathscr{T})$ that there will still be a closed trapped surface around the star in the perturbed solution. Thus we have shown that there is a non-zero measure set of initial conditions which lead to a closed trapped surface and hence to a singularity by theorem 2.

The two principal reasons why a star may depart from spherical symmetry are that it may be rotating or may have a magnetic field. One may get some idea of how large the rotation may be without preventing the occurrence of a trapped surface by considering the Kerr solution. This solution can be thought of as representing the exterior solution for a body with mass m and angular momentum $L = am$. If a is less than m there are closed trapped surfaces, but if a is greater than m they do not occur. Thus one might expect that if the angular momentum of the star were greater than the square of its mass, it would be able to halt the contraction of the star before a closed trapped surface developed. Another way of seeing this is that if $L = m^2$ and angular momentum is conserved during the collapse, then the velocity

5 THE REISSNER-NORDSTROM SOLUTION

The charged version: radial electric field, solution of maxwell's equations
Reissner-Nordstrom:

$$ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dt^2 \ + \ (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1}dr^2 \ + \ r^2 d\Omega^2 \qquad (132)$$

Two values of r givehorizons: at $r_\pm = m \pm (m^2 - e^2)^{1/2}$ when $e^2 < m^2$ Again define advanced and retarded null coordinates and find maximal extension

* Many asymptotically flat regions, many horizons, infinities
* Singularity is a timelike, so timelike and null curves can avoid hitting the singularities
* No Cauchy surfaces in the space
* Wormhole but unstable

6 NON-SPHERICAL GEOMETRIES

see 'Black Holes', W Israel: *Sc Prog Oxf* 68:333-363 (1983)

Settling down: rotating bh will go axisymmetric
there is No Birkhoff theorem for general or axisymmetric solutions But there are theorems showing uniqueness of the final state

Rotating black holes will settle down to the Kerr solution
Non-rotating will settle down to the Schwarzschild solution

6.1 Kerr solution

Rotating black holes and the ergosphere

Kerr solution, represents a spinning black hole - rather every spinning black hole: final state of collapse of rotating star that falls within event horizon

* Maximal extensions, horizon structure, Complex topologies
* causal violations, ring singularity,
* separation of event horizon and killing horizon - spinning bh stores energy in the swirl of space around itself; this energy can be extracted and used to power things: Penrose Black Holes Sci Am 226, 38-46 (1972)
* Teukolsky: equations of pulsation of spinning black holes – these give natural frequencies of bh pulsations, and show its stability to perturbations [no matter how fast it is spinning]

Chandrasekhar: completed into a complete system of perturbation equations

6.2 Binary systems

A Black hole ('bh') binary will emit gravitational waves with characteristic signature as they spiral towards each other and coalesce to produce a Kerr solution Can in principle be detected by gravitational wave detectors [bars, laser interferometers - LIGO] - the best characteristic signature of black holes [as opposed to other compact objects].

6.3 general Non-spherical

See 'Black Holes in GR' S W Hawking *Comm Math Phys* **25**:152-166 (1972)

Definition? by event horizon

6.3.1 Event and apparent horizons, and singularities

general definitions: event horizon defines black hole

* closed trapped surface: apparent horizon, implies singularities must exist [Penrose]

* absolute horizon, the boundary in spacetime between those events outside the horizon that can send signals to the distant universe, and those inside the horizon that cannot

* its area must always increase, implying limits on energy radiation in black hole mergers [Hawking]

6.3.2 The uniqueness, censorship, and no-hair theorems

Teukolsky et al: perturbations die away (radiation), hence stable

* Black hole has no hair: radiated away

A black hole should provide no evidence of what it was that collapsed to create it

Israel theorem, Robinson (non-lineaR): 3 parameters for collapsed object all the properties of a black hole are predictable from just 3 numbers: its mass, rate of spin, and electric charge

A highly non-spherical (uncharged, non-spinning) implosion can have only two outcomes: either it produces no black hole at all, or else it produces a black hole that is precisely spherical

What makes the hole become spherical? gravitational radiation radiates away whatever can be radiated away (Price) influence of the holes mass, charge and spin are all that can remain behind when the radiation has cleared away If it is non-spinning it becomes asymptotically Schwarzschild; if it is spinning, it becomes asymptotically a Kerr solution

The underlying assumption is the *cosmic censorship conjecture*: that as a massive object collapses to form a singularity, this singularity will never be seen from the future: it will always be clothed by an event horizon. That is, naked singularities (visible at future infinity) will not occur. The status of this conjecture is not clear, as there are counter examples of high symmetry, and thus presumably unrealistic; we do not have a good statement of the theorem that captures what we want to show physically and is also provable.

Thus its status is uncertain.

see Black holes, Naked Singularities and cosmic censorship Shapiro and Teukolsky: *Am Scientist* **79**, 330-343 (1991)

Assuming this is true, then the battery of physical arguments and uniqueness theorems show that the space-time around a collapsing object will eventually settle down to a steady state, after radiation off inhomogeneities and anisotropies, and be well-described by the Kerr or Schwarzschild solutions. Then the properties of those solutions outlined above determine the astrophysical processes that will occur (for example formation of accretion disks).

7 EXISTENCE: EVIDENCE

see M J Rees Observational status of black holes. *Proc Roy Soc Lond* **A368**:27-32 (1979)

7.1 Stellar collapse

Various candidates: best is Cygnus x-1 (6000 ly from earth): x-rays inary with one component the x-ray bright bh of mass definitely greater than 3 sun masses, probably greater than 7 sun masses, and most likely about 16; the optically bright but x-ray dark companion has a mass probably greater than 20 sun masses and is about 20 times larger than the sun

7.2 Galactic Centres

Believed to occur in Radio galaxies, qso's, agn's, e.g. 3c273 which has high redshift 16% speed of light (and so great distance and radiates immense amounts of power - 100 times more luminous than most luminous galaxy ever seen), looks like a star, has fluctuating brightness within period of a month implying light comes from region smaller than a light-month in size - 10^{18} times smaller than volume in which galaxies produce their light

Light comes from massive, compact gaseous object heated by an enormously powerful small engine. Outgoing electrons are beamed around magnetic field lines spiralling around and producing synchrotron radiation in radio lobes

Energy source? chemical power is inadequate, so is nuclear power and conversion of matter to anti-matter. Gravity seems best possibility Radio galaxies emit radio waves not only from giant double lobes but also from the core of the central galaxy itself Thus a single engine can be responsible for all the galaxies radio waves through gas jet emerging from the central engine and creating the radio lobes [Rees and Blandford] Because the jest are straight the central engine has to fire the jets in the same direction for a very long time so the nozzles that collimate the jets must be attached to a superbly steady object - a long-lived gyroscope of some sort - a gigantic spinning black hole

Thus bhs are the best bet for powering quasars and radio sources and agns. Four mechanisms are known for making jets p348 Thorne In the Blandford-Znajek process the energy comes from the holes rotational energy The light emitting region is typically about a light-year in size but can be much smaller but gives of radiation hundreds of times brighter than all the stars in the surrounding galaxy together Brightness in various parts of spectra depends on feeding rate and magnetic fields

Alternative: massive spinning magnetic star

How common are bhs? probably at core of most radio galaxies and qsos but also quite likely at the cores of ordinary galaxies such as our own

– evidence from orbital motion of gas clouds near thy centre of the galaxy apparently orbiting around an object weighing about 3 million times mass of sun which is a radio object

velocities implying very concentrated mass, e.g. M82

In such cases, the bh results from cumulative interactions of stars in galaxy core with friction driving interstellar gas down into the core

see e.g. Galactic Nuclei and Quasars: Supermassive Black Holes M J Rees *New Sci* **80**, 188-191 (1978)

7.3 Primordial

There is a possibility of massive perturbations formed in big bang in early universe rapidly forming very small blackholes; these would be emitting gamma-rays as they decay by the Hawking process.

Could exist - but no evidence that they in fact do [rather, an upper limit on how many there could be].

7.4 Options

Collapse: to final state.

Massive object burns all its nuclear fuel: collapses, throws off some matter.

'If there is no other possible stellar graveyard but white dwarfs, neutron stars, and black holes, then when a star dies it must either eject enough mass to bring it below the maximum for neutron stars, or continue shrinking towards a black hole'

Possible final states

NON-SINGULAR	SINGULAR	
	Hidden	Visible
Jupiter White Dwarf Neutron Star Quark star (?)		
	Black Hole	
		Naked Singularity
Predictable	Predictable	Not Predictable

8 FURTHER ISSUES

8.1 Horizons and Radiation emission

The geometry of the horizon

8.1.1 Boyers bifurcation

Killing horizon properties: infinite rescaling of affine parameter relative to Killing vector (symmetry) parameter as fixed points are approached

8.2 The Rindler universe

The flat space analogue: a uniformly accelerating particle in Minkowski space has event horizons associated with Killing horizon, just as in the Schwarzschild case.

NB here r = 0 is not singular; every point is equivalent (there is a 3-d set of isometries in the plane) should only have rhs of the diagram or there are two points for each sphere

8.2.1 Black hole thermodynamics

Area of final horizon must always be greater than that of original horizons, even if holes merge

This resembles the 2nd law of thermodynamics
Bekenstein: black hole area IS its entropy multiplied by a constant
This leads to the 4 laws of black hole thermodynamics, see J M Bardeen B Carter S W Hawking: *Comm Math Phys* **31**:161-170 (1973)

replace horizon area by entropy and horizon surface gravity by temperature [strength of gravity's pull felt by someone just above the horizon] thus a bh must have a finite temperature and must emit radiation
J D Bekenstein: Phys Today 33, 24-31 (1980)

Issue of entropy: loss of information of what has been put into hole, thus entropy is a measure of all the ways the hole could have been made

8.3 Hawking radiation

The 'classical' view: virtual pairs exist near the horizon; if close enough, can become real and one falls in, one emitted to infinity

Quantum field calculations in a curved space-time show thermal radiation emission by a black hole (based on definition of incoming/outgoing particle states, which are related by the parameters associated with the Killing horizon bifurcation which in turn depend on the surface gravity. for Calculations: see Wald and Kay).

The radiation emitted slows the spin of a rotating black hole and then emits energy losing mass Spectrum is black body Horizon has temperature proportional to the surface gravity Stellar mass: $T = 3 \times 10^{-8}$ K so evaporation is slow: requires 10^{67} years
However this speeds up as it shrinks. Very small holes evaporate very rapidly by this process emitting gamma-rays.

8.4 Lower dimensional versions: issues and problems

Minimum dimension for full gravity is 4 for this is the first dimension in which there is a part of the curvature tensor the Weyl tensor that is not algebraically determined by the field equations. Thus the 'action at a distance' of gravity is not possible in lower dimensions - if the field equations are like the EFE.

8.5 Black Holes in the Expanding Universe: local definition

Problem of asymptotic definition: depends on idea of asymptotic flatness, which does not relate to the real universe but is very global in its nature.

How to obtain a usable definition for the real universe, allowing us to define a bh in a finite time at a finite distance from the collapsing object, without making irrelevant and probably wrong assumptions about what happens at infinity??

e.g. for the case when $k = +1$ so there is no infinity
see: Black holes in closed universes. F J Tipler *Nature* **270** :500-501 (1977)

8.6 The issue of the final state

Loss of coherence and quantum mechanics ??

Laws of Gr must fail at hole's centre and be replaced by new laws of quantum gravity
see: The lesson of the black hole, J A Wheeler *Proc Am Phil Soc* **125**: 25-37 (1981)

Conjecture; could this lead to re-expansion into new universe?? Gives basis of Phoenix universe idea, and possibility of Darwinian evolution of universe [Smolin]