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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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THE MODULI SPACE OF N=(2,2) SUPERCONFORMAL FIELD THEORIES

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Basic CFT for our purposes: "Gaußian model"

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \partial x \, \bar\partial x \, d^2z \qquad \Sigma = \text{world-sheet.}$$

$(\alpha' = \frac{1}{2})$.

x is a real variable

This is a conformal field theory with $c = 1$.

$$T(z) \simeq \, :\partial x \partial x:$$

$$T(z)T(w) = \frac{1}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \dots$$

So CFT has "modelled" $\mathbb{R}$.

$\mathbb{R}^n$ is modelled by n of these theories ... $c = n$.
"central charge ⇒ dim"

We may impose periodic conditions $x \simeq x + 2\pi R$.
... now have a circle
may rescale $x \to x/R$ so that $x \simeq x + 2\pi$

$$S = \frac{R^2}{4\pi\alpha'}\int_\Sigma \partial x \, \bar\partial x \, d^2z$$

If we add $\int_\Sigma \varphi \, d^2z$ onto action where $\varphi = \partial x \bar\partial x$

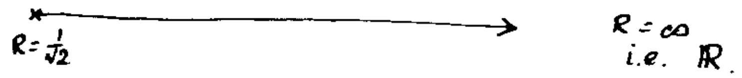
then we change R.

φ is a "truly marginal operator".

1/2. φ builds up a moduli space of theories from original.

At $R = \frac{1}{\sqrt{2}}$ an extra symmetry appears such that $\varphi \cong -\varphi$ for deformation.

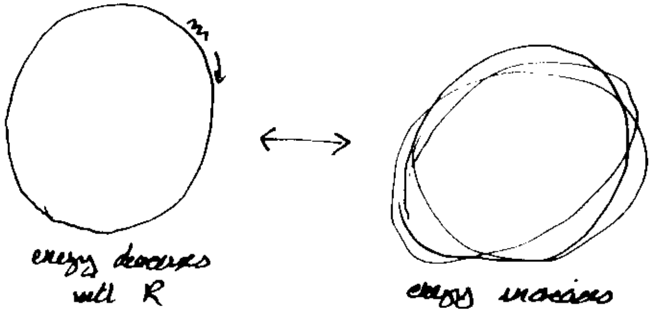
⇒



Actually moduli space of $c = 1$ theories is believed to be:

Actually, the CFT for a circle radius R
$\cong$ circle for radius $\frac{1}{2R}$.
(hence all "runs out" at $R = \frac{1}{\sqrt{2}}$)

This can be pictured in terms of momenta & winding modes



Sim. for n dimensions we have n^2 marginal operators 1/4

Narain has shown that a CFT on an n-torus is described by an even self-dual lattice of signature (n,n)

i.e. $\langle e_i \cdot e_j \rangle$ lattice vectors given by

$$2n\left\{\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}\right.$$

moduli space of such lattices is

trivial rotations of lattice

$$O(n,n,\mathbb{Z})\cancel{O(d}\backslash O(n,n)\Big/O(n)\times O(n)$$

generalized $R \leftrightarrow \frac{1}{2R}$ identifications
automorphisms of lattice.

indeed, $\dim = n^2$.

Note that this is <u>not</u> the moduli space classically for a torus: $GL(n)\big/O(n)$ $\dim = \frac{1}{2}n(n+1)$

Dimension is not even right! To get dimension right we add "B-field": - antisymmetric constant

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \left\{ h^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^j g_{ij} + \epsilon^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^j B_{ij} \right\}$$

(constant) metric.

One can show that this also gives even self-dual lattices.

In order to solve harder problems we need to introduce supersymmetry.

Conformal algebra → superconformal algebra
Add fermions into action to make it supersymmetric

A free fermion ≡ Ising model has $c = \frac{1}{2}$.

∴ For super-target-space we expect
$c = \frac{3}{2} \times$ dimension

supercircle has $c = \frac{3}{2}$.

Note, moduli space of $N=1$, $c = \frac{3}{2}$ theories is messy.

For $N=2$ theories, we pair bosons & fermions to form a complex structure $\mathbb{R}^{2d} \to \mathbb{C}^d$
$c = 3d$. d = COMPLEX dim of s.-t.

Moduli space of $c=3$, $N=2$ theories is very messy!

However we may restrict to theories which have a good geometrical interpretation.

$N=(2,2)$ theories have left- and right- $U(1)$ currents
- fields may be labelled by charge $(Q,\bar{Q})$

Let us restrict to $Q,\bar{Q} \in \mathbb{Z}$ for NS strings
(links to S.-T. SUSY).

Gepner has shown that any such CFT ~~is~~ corresponds to a torus:

for $c=3$,

$$J(z)J(w) = \frac{1}{(z-w)^2} + \dots$$

$\Rightarrow$ $J(z) = i\partial\varphi$ for some free boson $\varphi(z)$.

Since $G^\pm$ have charge ± 1

$$G_\pm = \hat{G}_\pm :e^{i\varphi}:$$

$$\Rightarrow \quad \hat{G}_+(z)\hat{G}_{-}(w) = \frac{2}{(z-w)^2} + \dots$$

$\Rightarrow$ $\hat{G}_+ = \sqrt{2}\,\partial H$, $G_- = \sqrt{2}\,\partial H^{\cancel{*}\dagger}$ for some complex boson H.

If all fields have charge $\in \mathbb{Z}$, φ lives on circle of radius 1
$\cong$ complex fermion ψ.
$J = :\psi^\dagger\psi:$

H periodic boundary conditions give torus.

Thus from Narain we have $\mathcal{M} = O(2,2,\mathbb{Z})\backslash O(2,2)/O(2)\times O(2)$

$$\frac{O(2,2)}{\cancel{O(2)}\,O(2)\times O(2)} = \left[\frac{Sl(2)}{U(1)}\right]^2 \quad \text{up to } \mathbb{Z}_2 \text{ factors}$$

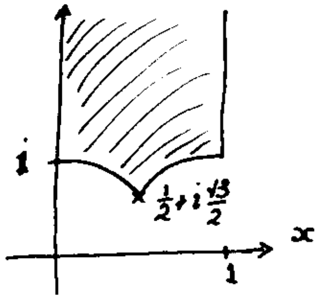
$\uparrow$ given by $x = \frac{ai+b}{ci+d}$ for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in Sl(2)$

$O(2,2,\mathbb{Z})$ contains $Sl(2,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ for $\mathrm{Im}(x) > 0$.

$Sl(2,\mathbb{Z})$ acts on x by $x \to \frac{ax+b}{cx+d}$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in Sl(2,\mathbb{Z})$

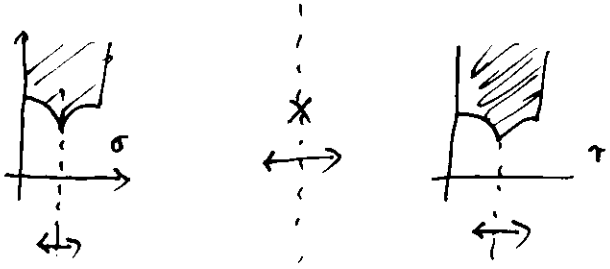
21/6.

$Sl(2,\mathbb{Z})\backslash Sl(2)/U(1)$ can be ~~w~~ drawn as

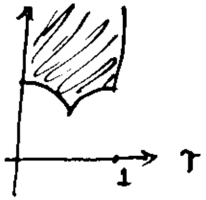


(often drawn )

Taking all the little $\mathbb{Z}_2$-factors into account we end up with



classically, the moduli space of a torus is of fixed area.



Thus moduli space $\mathcal{M}_{c=3}$ contains classical moduli space of complex structures

+ Minor map to σ co-ord
+ other $\mathbb{Z}_2$-flip

~~σ may be written as $B + iJ$~~

~~where $B_{ij} = \epsilon_{ij} B$~~

~~and $z\,g_{ij} =$~~ [illegible]

Once a complex structure has been put on ~~torus~~ target space we write

$$S = \frac{i}{4\pi\alpha'}\int \left\{ 2\, g_{i\bar{j}} (\partial x^i \bar{\partial} x^{\bar{j}} + \bar{\partial} x^i \partial x^{\bar{j}}) - i B_{i\bar{j}} (\partial x^i \bar{\partial} x^{\bar{j}} - \bar{\partial} x^i \partial x^{\bar{j}}) \right\} d^2 z$$

$$= \frac{1}{4\pi^2\alpha'} \left\{ \underset{\substack{\uparrow \\ \text{some} \\ \text{function}}}{K(g_{i\bar{j}})} + 2\pi i \int_\Sigma x^*(B) \right\}$$

B is a 2-form $= \frac{1}{2} B_{i\bar{j}}\, dx^i \wedge dx^{\bar{j}}$

viewing x as a map $x : \Sigma \to \underset{\substack{\uparrow \\ \text{target space.}}}{X}$

Let us fix $4\pi^2\alpha' = 1$ from now on

then if we have $B = e$, where $e \in H^2(X, \mathbb{Z})$

$$\Rightarrow \int_\Sigma x^*(B) = \int_{\mathrm{Im}(\Sigma)} e \in \mathbb{Z}$$

$\Rightarrow$ makes no contribution to field theory

$\Rightarrow B \to B + e$ is a symmetry of the action

In this case $\dim H^2 = 1$ and e is only generator of $H^2(X, \mathbb{Z})$

Actually $B + iJ = \sigma . e$

$\sigma \to \sigma + 1$ is just B-field symmetry

$\sigma \to -1/\sigma$ is "area $\leftrightarrow \frac{1}{\text{area}}$" (for $B = 0$)

for area $= \int_\Sigma J$.

To study more complicated target spaces we need to develope tools further.

Here, we could use metric - because it was constant & $\therefore$ flat. In general life will be harder.

The moduli space $\mathcal{M}$ contained the classical moduli space of complex structures. This may be got from algebraic geometry without reference to the metric.

— HEREIN LIES AN IMPORTANT POINT !!

One might try 3 ways of doing geometry.

1) Think about it directly analytical
2) Chop the space into ~~space~~ small bits and look closely - differential
3) Look at functions on it - algebraic

~~String the~~ 1) is too hard!

2) might <u>not</u> be reasonable in string theory

3) Needs a field to work over

$N = 2 \Rightarrow k = \mathbb{C}$.

$\therefore$ $N = (2,2)$ lies at the heart of all that follows.

1/9. Algebraic description of a torus:

$$W = x_0^3+\overset{x_1}{\cancel{y}}{}^3+\underset{x_2}{\cancel{z}}{}^3-3\psi xyz = 0.$$

in $\mathbb{P}^2$ $[x,y,z]$.

etc.

$y \to x_1$
$z \to x_2$

(complex version of "elliptic curve").

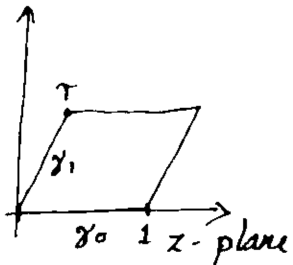
complex structure depends on ψ (not a metric in sight)

but how does τ depend on ψ?

A torus has 2 generators of H_1 ~~B~~ γ_0, γ_1.

We also have a holomorphic 1-form ~~dz~~ $\Omega = f.dz$ on the torus unique up to a constant

$$\tau = \int_{\gamma_1}\Omega \Big/ \int_{\gamma_0}\Omega$$



Fix area & orientation
$\mathrm{Im}(\tau) > 0$.

↑ fix constant in Ω by $\int_{\gamma_0}\Omega = 1$ and $f = 1$.

$\tau \to \tau + 1$ symmetry is $\gamma_1 \to \gamma_1 + \gamma_0$

$\tau \to -1/\tau$ is $\gamma_0 \leftrightarrow \gamma_1$

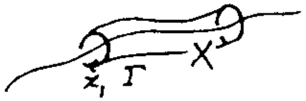
1/10 How do we build $\int_\gamma \Omega$? $= \varpi$

~~A~~ For a manifold $W(x_i) = 0$
first set one coordinate $= 1$ to get affine patch

then $\Omega = \dfrac{dx_2 \wedge dx_3 \wedge \ldots}{\partial W/\partial x_1}$

is a nowhere vanishing, finite n-form.

Consider a tubular neighbourhood of γ ~~X~~ in ambient space



Since the residue of $\frac{1}{W}$ is $1\Big/\frac{\partial W}{\partial x_1}$

We have $\displaystyle\int_\gamma \Omega = \frac{1}{2\pi i}\int_\Gamma \frac{dx_1\wedge dx_2 \wedge \ldots}{W} = \frac{1}{2\pi i}\int_\Gamma x_0 \frac{dx_1 \wedge \ldots}{W}$

~~rcd~~ $\displaystyle\sim \int_{\Gamma\times \text{circle round } x_0} \frac{dx_0\wedge dx_1 \wedge}{W}.$

e.g. $\displaystyle\sim \int_{\substack{|x|=\epsilon \\ \cancel{|y|=\epsilon} \\ |z|=\epsilon}} \frac{\psi\, dx_0 \cancel{dy\, dz}\, dx_1\, dx_2}{x_0^3+y^3+z^3-3\psi xyz\; x}$ $|x_i| = \epsilon$.

$$\sim \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}(3\psi)^{3n}$$

$$\sim {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \cancel{\psi^3} z\right)$$

'11. which solves

$$\left(z\frac{d}{dz}\right)^{2\cancel{3}}\varpi - z\left(z\frac{d\cancel{\varpi}}{dz}+\frac{1}{3}\right)\left(z\frac{d}{dz}+\frac{2}{3}\right)\varpi = 0$$

$$z = \psi^{-3}$$

i.e. the periods solve a hypergeometric equation

– the "Picard - Fuchs" equation

Actually, all periods satisfy this equation

eneral solution to eqn is

$$f(z) = A\left(1+\frac{2}{9}z+\frac{10}{81}z^2+\frac{560}{6561}z^3+\dots\right)$$
$$+ B\left\{\left(1+\frac{2}{9}z+\frac{10}{81}z^2+\dots\right)\log z + \frac{5}{9}z+\frac{19}{54}z^2+\dots\right\}$$

~~log z - Note that ψ = ∞ pt has monodromy due to τ → τ+1~~

Singularity occurs at $z = 0$, $\psi = \infty$ – must correspond to $\tau = i\infty$

log z has branch pt at $z = 0$

– gives $\tau = \tau + 1$ symmetry

$$\Rightarrow \tau\cancel{\theta} = \frac{1}{2\pi i}\left(\frac{B\text{- part}}{A\text{ - part}} + \text{const}\right)$$

$$= \frac{1}{2\pi i}\left\{\log z + \frac{5}{9}z + \frac{37}{162}z^2 + \frac{2669}{19683}z^3 + \dots \;\; \cancel{\text{const}}\; K\right\} \;\; \cancel{+\text{const}}$$

We know that $\psi = 0$, torus has $\mathbb{Z}_3$ symmetry – must be $\tau = e^{\frac{2\pi i}{3}}$

above series may be analytically continued (rather messy)

via $\psi=0$, to show that $K = -\log 27$.

$$\tau = \frac{1}{2\pi i}\left\{\log\frac{z}{27} + \frac{5}{9}z + \frac{37}{162}z^2 + \frac{2669}{19683}z^3 + \dots\right\}.$$

''12. This provides the link between algebraic method & τ.

Note that for strings σ & τ are very similar

⇒ algebraic method for σ too.

Consider ways of doing geometry again

~~A torus~~ Classically a torus with a complex structure has 3 real moduli – it's area and τ - the complex structure. Algebraic methods may be used to describe τ but not A.

In string theory, the area combines with B-field to form σ - behaves like τ. ∴ Algebraic methods work for it too.

– Stringy geometry is MORE algebraic that even algebraic geometers hoped!

The target space is given <u>2</u> complex structures, - one is the classical one, the other one gives B+iJ.

Mirror symmetry now appears naturally. It happens when these complex structures are exchanged.

Mirror symmetry is <u>NOT</u> a symmetry of classical geometry.

Intrinsically string theory is probably most happy with 2 complex structures. To make contact with B+iJ we need to solve P.F. eqns.

//1. So much for flat space.

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \left\{ h^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^j g_{ij}(x) + \epsilon^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^j B_{ij}(x) \right\}$$

+ fermions
(+ dilaton).

let g_{ij} and B_{ij} vary over X.
σ-model now is interacting
coupling constant $\sim \alpha'/R^2$ — R some typical size of X.

∴ If X is large (and thus fairly flat) we can do perturbative theory.

To see if F.T. is conformally invariant, we look at β-functions ~~of fields, parameters~~ of theory. This yields a complicated set of equations which to 1-loop may be satisfied by

$$dB = 0 \qquad (\text{"Zero torsion"})$$
$$R_{ij} = 0 \qquad - \text{ Ricci-flat.}$$

$N=(2,2) \Rightarrow$ complex structure Kähler i.e. X is CY

In $1_{\mathbb{C}}$-dim, only CY is a torus – we solved that exactly.

In $2_{\mathbb{C}}$-dim, CY's are 4-torus and K3.

Let us study K3.

First look at $N=(2,2)$ algebra for $c=6$.

For L sector:

2.//2. Spectral flow of identity gives a field with
$Q=2,\ h=1. \qquad \Omega^+$
also gives $\qquad Q=-2,\ h=1. \qquad \Omega^-$

We also have $J = i\sqrt{2}\,\partial\varphi$ for some boson φ
$\Rightarrow \quad \Omega^\pm = e^{\pm i\sqrt{2}\varphi}$

$J, \Omega^\pm$ form an $SU(2)$ affine algebra (Frenkel-Kac)
i.e. $U(1)$ elevated to $SU(2)$
$G^\pm(z)$ may be split into 4 fields transforming as $2+\tilde{2}$ rep of $SU(2)$

This is an $N=4$ S.C.A.

I.e. $N=(2,2)$ raised to $N=(4,4)$ in $2_{\mathbb{C}}$-dim.

X now has quaternionic structure & hyper-Kähler metric

$dB=0,\ R_{ij}=0$ is EXACT. just like it was for torus.

So, what is a K3 surface?
Kummer's construction:
Take a "square" torus $(z_1,z_2) \in \mathbb{C}^2/(\mathbb{Z}[i])^2$.

Divide by $\mathbb{Z}_2$-action $(z_1,z_2) \cong (-z_1,-z_2)$

This has fixed points – 16 of them $(0,0),(\tfrac{1}{2},0),(\tfrac{i}{2},0),\ldots$
$\Rightarrow$ space is singular "orbifold"
let us "blow-up" singularities.

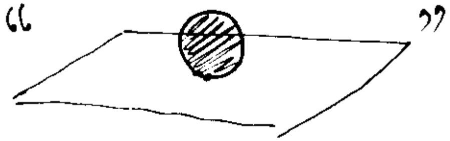
4/3½

First think of space

$$\mathcal{O}(-1) = \{[a,b], (x,y) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid ay = bx\}$$

If $(x,y) \neq (0,0)$ we determine $\frac{a}{b}$ from $(x,y) \Rightarrow$ pt on $\mathbb{P}^1$
$\Rightarrow$ isomorphic to $\mathbb{C}^2 \setminus \{0\}$.

If $(x,y) = (0,0)$, a,b are undetermined so we have whole $\mathbb{P}^1$.



$\mathbb{P}^1$ replaces origin

"$\mathbb{C}^2$ has been blown-up at 0"
actually smooth!

so consider

$$\mathcal{O}(-2) = \{[a,b], (x,y) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid a^2 y = b^2 x\}$$

One may show that away from $(x,y) = (0,0)$
isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.
$(\xi, \eta) \simeq (-\xi, -\eta)$

But this is smooth.

Thus, this is a space that is smooth but looks like the singular space $\mathbb{C}^2/\mathbb{Z}_2$. The space $\mathbb{C}^2/\mathbb{Z}_2$ is singular at the origin. $\mathcal{O}(-2)$ has a $\mathbb{P}^1$ at the "origin". Singularity has been "blown-up" to give a $\mathbb{P}^1$.
$\mathcal{O}(-2)$ is a Kähler manifold with $\cancel{\mathbb{R}^2}$ dim $H^2 = \cancel{0}1$.

Case over quadric:

$\mathbb{C}^2/\mathbb{Z}_2$ (s,t) form $\mathbb{C}^2$
$(s,t) \cong (-s,-t)$

Let $x = s^2$
$y = t^2$
$z = st$ then $xy = z^2 \subset \mathbb{C}^3 \simeq \mathbb{C}^2/\mathbb{Z}_2$.

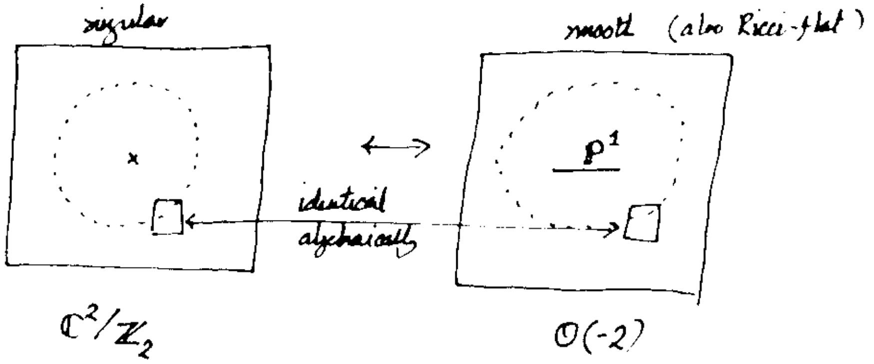
Suppose $[x,y,z]$ are now considered hom. of $\mathbb{P}^2$.
Then $xy - z^2 = 0$ is locus of a smooth $\mathbb{P}^1 \subset \mathbb{P}^2$
$[a,b] \rightsquigarrow \mathbb{P}^1$ $x = a^2, y = b^2, z = ab$. (1-1 fn. hom)

Consider $\{[a,b], (\overset{x,y,z}{\cancel{u,v,w}}) \mid xy = z^2, a^2 z = abx, a^2 y = b^2 x,$
$aby = b^2 z\}$

~~Actually~~ Then away from $x = y = z = 0$ we have $\mathbb{C}^2/\mathbb{Z}_2$
at $x = y = z = 0$ we have $\mathbb{P}^1$.

Actually z is uniquely determined from eqns - we may drop it

$$\mathcal{O}(-2) \simeq \{[a,b], (x,y) \mid a^2 y = b^2 x\}$$



Very algebraic
Very un-differential } geometry

1/4. The Kähler form $J = Ae$ e generates H^2.

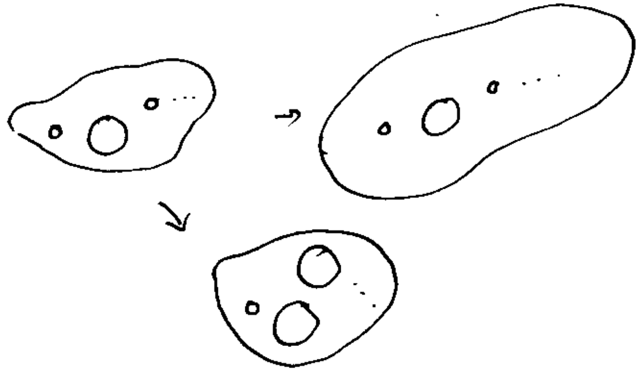
$\cdot$ Area of $\mathbb{P}^1 \tilde{=} \int_{\mathbb{P}^1} e\ \cancel{2}A \sim A$.

$\therefore$ Size of $\mathbb{P}^1$ is varied by Kähler form J.
In the limit $J \to 0$, $\mathbb{P}^1 \to pt$ and we recover singular space. Orbifold lies at the boundary of the space of Kähler forms for a manifold

K3 has 16 such $\mathbb{P}^1$'s

$$\begin{array}{ccccc} &&1&& \\ &2&&2& \\ 1&&4&&1 \\ &2&&2& \\ &&1&& \end{array} \xrightarrow{/\mathbb{Z}_2} \begin{array}{ccccc} &&1&& \\ &0&&0& \\ 1&&4&&1 \\ &0&&0& \\ &&1&& \end{array} \xrightarrow[\text{up}]{\text{blow}} \begin{array}{ccccc} &&1&& \\ &0&&0& \\ 1&&20&&1 \\ &0&&0& \\ &&1&& \end{array}$$

$$J = \sum_{i=1}^{20} J_i e_i$$



Any smooth CY build by any method ~~b~~ in $2_{\mathbb{C}}$ in

2/5. Actually, K3 is a little awkward for $\mathcal{M}$.

J is a Kähler form $\Rightarrow$ J is a $(1,1)$-form.

The space of 2-forms is 22-dimensional, "Hodge decomp"
$$H_2^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

This decomp. varies with complex structure –
just as $\int_\gamma \Omega$ varied for the torus

Now $\int_\gamma \Omega \leftarrow \in H^{2,0}(X)$ varies
$\gamma \nwarrow \in H_2(X, \mathbb{Z})$

– the ~~lattice $H_2(X)$~~ direction of $H^{2,0}$ varies within the $H_2(X,\mathbb{Z})$ lattice

~~but this~~ but this time J is affected by c.s.

This means that deformations of ~~f~~ K.f. + C.S. "mix"
Note also that $B \in H^2$ has 22-degrees of freedom but $J \in H^{1,1}$ has only 20.
$\hookrightarrow$ $B + iJ$ cannot be naturally built.

Actually these problems may be overcome to build

$$\mathcal{M}_{K3} = O(4,20,\mathbb{Z}) \backslash O(4,20) / O(4) \times O(20)$$

– Just like a heterotic torus!
(Noone quite knows why)

Now let us go to complex dim 3.
There are many (infinite?) CY 3-folds.
Some are just obtained from earlier ones – torus, torus × K3 / G
Ignore these by putting ~~h^2~~ $b^1 = 0 \Rightarrow h^{2,0} = 0$.

$\pi_1(X)$ finite (or 0)

3-fold is not flat & we do not have $N=4$.
This makes σ-model worse.

w $\underset{\text{of metric}}{\beta_{i\bar{j}}} = -\frac{1}{2\pi} R_{i\bar{j}} - \frac{4\zeta(3)}{3(4\pi)^4} \partial_i \partial_{\bar{j}} \Big[R_{k\bar{l}m\bar{n}} R^{p\bar{l}q\bar{n}} R_{p\ q}^{\ k\ m} - R_{k\bar{l}m\bar{n}} R^{m\bar{n}p\bar{q}} R_{p\bar{q}}^{\ \ k\bar{l}} \Big] + \ldots$

to 4-loop

$= 0$

X is no longer Ricci-flat. It becomes so at L.R.L.
X is a CY with a deformed metric. (4-loop term not obstructed).

However, even perturbation theory breaks down.

Action S has local minimum if x is a holomorphic map
Tree level
$x : \Sigma \to \Gamma \subset X$ Γ is a "rational curve"

$S = \cancel{2\int_{\Delta} B+iJ} \quad -2\pi i \int_{\Gamma} (B+iJ)$

rrelation function ~~f~~ depends on this quantity as $e^{-S} = e^{2\pi i \int_{\Gamma} (B+iJ)}$

this effect, introduce e_k – a basis for $H^2(X, \mathbb{Z})$, $k = 1, \ldots, h^{1,1}$

Put $B + iJ = \sum_k (B+iJ)_k e_k$
and $q_k = e^{2\pi i (B+iJ)_k}$

3/12 Instanton effects will be polynomial in q_k's.

Since $h^{2,0} = 0$, Kf & c.s. do not interfere with each other (except at special pts in $\mathcal{M}$)

and we may factorize $\mathcal{M} = \mathcal{M}_{Kf} \times \mathcal{M}_{cs}$.

~~Also~~ The $N = (2,2)$ theory may be twisted into a T.A.F.T.
$T(z) \to T(z) \pm \frac{1}{2} \partial J(z) \qquad \bar{T}(\bar{z}) \to \bar{T}(\bar{z}) + \frac{1}{2} \bar{\partial} \bar{J}(\bar{z})$.

The 2 sign choices give A-model or B-model.
The moduli space of these TAFT's (preserving CFT) are $\mathcal{M}_{Kf}$ & $\mathcal{M}_{cs}$ respectively.

B-model knows about c.s. of X. It has no instantons.
A-model knows about K.f. of X. It has instanton dep (i.e. q_k dep.)
– only converges for small $|q_k|$
i.e. large J_k, i.e. large radius

∴ $\mathcal{M}_{c.s.}$ is found classically – just like the torus.

Finding $\mathcal{M}_{Kf}$ is the subject ~~of~~ to be studied.

Take particular example "Quintic 3-fold"

defined by $W = x_0^5 + x_1^5 + .. + x_4^5 = 0$ in $\mathbb{P}^4$.

has 101 deformations of c.s. (= defs of W). $h^{2,1} = 101$
has 1 def of K.f. – size of $\mathbb{P}^4$. $h^{1,1} = 1$

Let us first use Witten's LINEAR σ-model.

$N = (2,2)$ field theory
6 chiral superfields $x_0, \dots, x_4$, p $= \Phi_i$
$U(1)$ guage symmetry $+1, \dots, +1, -5$

[we will confuse superfields and their cap]

$$S = \int d^2z\, d^4\theta \sum_i \bar{\Phi}_i \Phi_i - \frac{1}{4e^2} \int d^2z\, d^4\theta\, \bar{\Sigma}\Sigma$$

$$- \int d^2z\, \underset{\substack{\uparrow \\ \text{notation ??}}}{d^2\theta^+}\, \underline{W}(\Phi_i) + \frac{i}{2\sqrt{2}} (\beta + ir) \int d^2z\, d\theta^+ d\bar{\theta}^-\, \Sigma$$

$+ h.c.$

$\Sigma = \frac{1}{2\sqrt{2}} \{ \bar{\mathcal{D}}_+, \mathcal{D}_- \}$ super-(field-strength) of $U(1)$ guage symmetry

$\underline{W} = p(x_0^5 + x_1^5 + .. + x_4^5)$ – has 101 defs as above
$\beta + ir$ is "$B + iJ$" degree of freedom.

Look at vacuum – minimize potential.

$$U = \mathcal{D}^2 + \sum \left| \frac{\partial \underline{W}}{\partial \Phi_i} \right|^2 + \dots$$

$$= \left[|x_0|^2 + \dots + |x_4|^2 - 5|p|^2 - r \right]^2 + |W|^2 + |p|^2 \sum_{i=0}^4 \left| \frac{\partial W}{\partial x_i} \right|^2 + \dots$$

For $r >> 0$ one of $x_i \neq 0$ $\Rightarrow p = 0$

$|x_0|^2 + .. + |x_4|^2 = r$ is S^9
$/U(1)$ $S^9/S^1 \simeq \mathbb{P}^4$

Also $W = 0$ $\Rightarrow$ Quintic target space – as required

For $r << 0$ $p \neq 0$ $\Rightarrow$ $x_i = 0$ $\forall i$
$|p| = r$

$U(1)$ degree of freedom allows us to fix p.

Target space is a point
Potential $|W|^2$ gives Landau-Ginzburg type modes around pt.
– excitations in x_i
Note that p had charge 5 $\Rightarrow$ fixing p by $U(1)$ leaving $\mathbb{Z}_5$ acting on x_i's.

$\Rightarrow$ Theory is a Landau-Ginzburg theory of $x_0, .., x_4$ with potential W, divided by $\mathbb{Z}_5$ action $(x_0, .., x_4) \mapsto (\alpha x_0, .., \alpha x_4)$
$\alpha = e^{2\pi i/5}$.

Thus $r >> 0$ appears to have a different geom interpret to $r << 0$.

Instantons: For $r >> 0$ one may show are given by
$\bar{\partial} x_i = 0$ i.e. holomorphic maps as in σ-model

However, there is a subtle difference, the ~~[illegible]~~ x_i's may all simultaneously vanish for an instanton, but not (by defn of $\mathbb{P}^4$) for a rational curve. ∴ These linear σ-model instantons have a

! little extra bit to them. However, these extra bits are "massive" (cp. original dim of vacuum) — vary $\sim\frac{1}{\sqrt{r}}$ in size.

$\therefore$ Instantons = A-model type + "massive" ones.

At LG orb we also have instantons

$$\bar\partial p = 0$$

& v_{12} = world-sheet field strength of $U(1) = e^2(-5|p|^2 - r)$

"Nielsen - Olsen abelian vortex line"

Also appear to be "massive" scale $\sim 1/\sqrt{-r}$

At $r = +\infty$ we have exactly CY theory

At $r = -\infty$ we have exactly LG theory

For finite r, we have instanton corrections to this state of affairs.

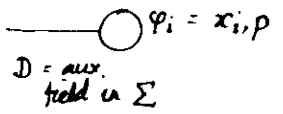
Does anything funny happen for finite r?

Yes — there is a field σ in the Σ supermultiplet we have ignored. This has a mass $\sim \sum_i Q_i^2|\Phi_i|^2 = 25|p|^2 + |x_0|^2 + \dots + |x_4|^2$

At $r \neq 0$, this field suddenly becomes massless — vacuum is suddenly so!

Actually there are a couple of subtleties to be noted here

Firstly there are "tadpole" diagrams



3/15 Those result in a shift in r by $\frac{1}{2\pi}\sum_i Q_i \log Q_i$

$$= \frac{5}{2\pi}\log 5$$

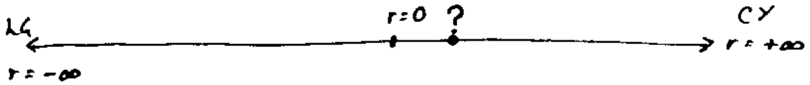
When the theory is analyzed properly for large $|\sigma|$ min. energy density of a state at large $|\sigma|$ is

$$\sim r^2 + \beta^2$$

Thus $\beta = 0$ for "singularity" too.

Thus, the funny thing actually happens at

$$\beta + ir = \frac{5i}{2\pi}\log 5$$



~~Rememb~~ Remember that this is NOT the $\mathcal{M}$ of CFT's known — the linear σ-model is not conformally inv.

To find $\mathcal{M}_{Kf}$ we may apply mirror symmetry.

$\exists\, Y$ — "the mirror of X"

such that $\mathcal{M}_{Kf}(X) = \mathcal{M}_{c.s.}(Y)$ & v.v.

The mirror of quintic is the quintic $/(\mathbb{Z}_5)^3$

gen. by $(x_0, x_1, x_2, x_3, x_4) \mapsto (\alpha x_0, \alpha^4 x_1, \dots)$

$(\alpha x_0, x_1, \alpha^4 x_2, \dots)$

$(\alpha x_0, x_1, x_2, \alpha^4 x_3, \dots)$

∴ General W inv. under this is

$$W = x_0^5 + x_1^5 + \dots + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4$$

ψ gives $\mathcal{M}_{c.s.}(Y)$

how does $\beta + ir$ depend on ψ?

Firstly let's make a single valued function for moduli space.

β ~~θ~~ appears in action as $i\beta \int V_{01}\, d^2z$

$$\int V_{01}\, d^2z = \int \text{curvature of field strength } d^2z$$

$$= 2\pi \int c_1 .\, d^2z \in 2\pi\mathbb{Z}$$ - a topological invariant

∴ Perturbatively does not contribute $-2\pi i.\beta.\mathbb{Z}$

Non-p only contributes as e

$$\Rightarrow \beta \cong \beta + 1$$

∴ Let us introduce $z = e^{2\pi i(\beta + ir)}$ ~~single~~

From W, we have symmetry $(x_0, x_1, x_2, x_3, x_4) \mapsto (\alpha x_0, \dots)$

taking $\psi \mapsto \alpha^{-1}\psi$.

∴ $\psi \cong \alpha\psi$

∴ ψ^5 is a good co-ordinate

From G.P. construction, we know that $\psi = 0$ is L.G. minor $r = -\infty$

At $\psi = 1$, W forms a singular space. CFT is bad $r = \frac{5}{2\pi}\log 5$

$\psi = \infty$ is the only other interesting pt $\Rightarrow$ CY-theory at $r = \infty$

(also see it is ∞ distance in Zam metric)

3/7. ∴ $\underline{\underline{z = \frac{1}{(5\psi)^5}}}$ is obvious choice

anything else will cause extra identifications

So, we have a map between $\mathcal{M}$ of linear σ-model $\beta + ir$ space

& $\mathcal{M}_{c.s.}(Y)$.

It is very simple! <u>Algebraic</u>.

In a way it would be nice to stop here. Formulate physics in terms of $z = \exp 2\pi i(\beta + ir)$.

(Unfortunately) we use $B + iJ$.

We thus need Picard-Fuch's eqn.

This may be determined as in torus

$$\left(z\frac{d}{dz}\right)^4 - 5^5 z\left(z\frac{d}{dz} + \frac{1}{5}\right) \dots \left(z\frac{d}{dz} + \frac{4}{5}\right) = 0$$

$$\hookrightarrow \varpi_0 = {}_4F_3\left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1, 1, 1; 5^5 z\right)$$

Other solutions go as $\underset{\varpi_1}{\underset{\parallel}{\log z}}, \underset{\varpi_2}{\underset{\parallel}{\log^2 z}}, \underset{\varpi_3}{\underset{\parallel}{\log^3 z}}$.

If we assume that the funny "massive" instantons in the linear σ-model are unimportant at large radius, we must have

$$q = z + O(z^2)$$

This fixes which periods to use completely

$$B + iJ = \frac{1}{2\pi i} \cdot \frac{\varpi_1}{\varpi_0}$$

$$\varpi_1 = \log z . \varpi_0 + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} z^n \left\{ \Psi(5n+1) - \Psi(n+1) \right\}$$

This gives

$$q = z + 770z^2 + 1014275z^3 + \ldots$$

converges for $|z| < 5^{-5}$

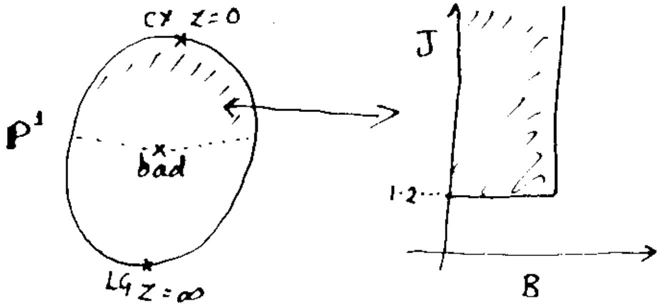
The boundary of the region of convergence is marked by the bad point at $\psi = 1$.

Here one may determine that

$$B + iJ \simeq i.(1 \cdot 2056)$$

Thus, for $J > 1 \cdot 2 . 4\pi^2\alpha'$ we may identify points in $\mathcal{M}$ corresponding to such $(B+iJ)^{'s}$ $0 \le B < 1$.

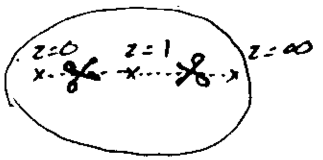
We have labelled all points on upper hemisphere



So what about lower hemisphere?

We may analytically continue the hypergeometric functions by Barne's contour integrals

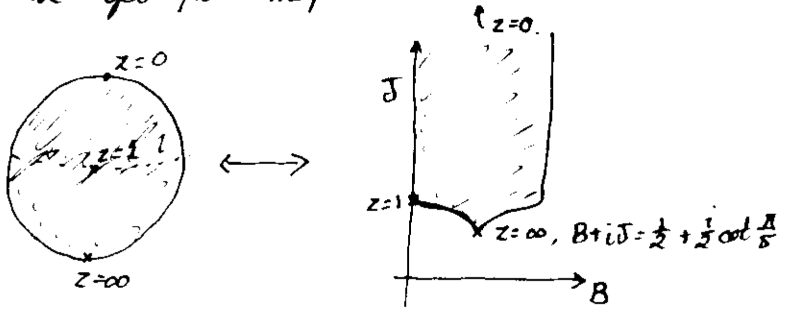
3./9. We need branch cuts to be able to do this. We have already cut from $z=0$ to $z=1$. Put other cut in by



Do this by imposing $-\frac{2\pi}{5} < \arg \psi < 0$

$$B + iJ = \frac{1}{2} + \frac{i}{2} \left\{ \cot\frac{\pi}{5} + \frac{\Gamma^4(\frac{4}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^4(\frac{3}{5})} \left(\cot\frac{\pi}{5} - \cot\frac{2\pi}{5} \right) e^{\frac{\pi i}{5}} \psi + o(\psi^2) \right\}$$

Then we get full map



Thus we get full moduli space of $B+iJ$.

Note that $J \geqslant \frac{1}{2}\cot \pi/5$ – i.e. we have a minimum size but that something special happened at $J \sim 1 \cdot 2$ anyway.

This region may look a bit like that for torus but there is a difference.

The torus could be written as $H / SL(2, \mathbb{Z})$.

The region for quintic cannot directly be written in this form of space / modular group.

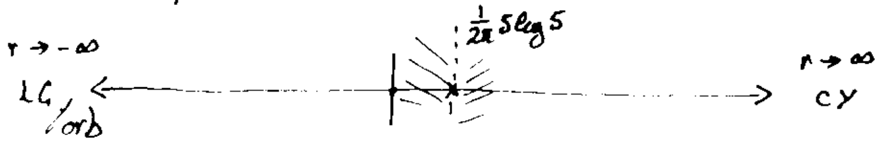
Instantons appear to muck up such a possibility.

As a curiosity, one may write this region [illegible] ...

4/2

$1/1$. **Phases**

Last time we had $h^{1,1}=1$. r diagram was simple:



If $h^{1,1} > 1$ we have more "r" parameters
i.e. the gauge group in linear σ-model is $U(1)^{h^{1,1}}$ and we can add lots of twisted F terms

e.g. $X = z_0^4 + z_1^4 + z_2^{4}, z_3^{8}, z_4^{8} \subset \mathbb{P}^4_{\{2,2,1,1,1\}}$

$k = 2,2,6,6,6$

$\frac{2}{4}+\frac{2}{4}+\frac{3}{8}+\frac{3}{4}+\frac{3}{4}$
$\frac{2}{4}+\frac{2}{4}+\frac{2}{4}+\frac{3}{4}+\frac{3}{4}$
$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{3}{2}=1$

$$\begin{array}{cccccccc} & p\,(& z_0^4 + & z_1^4 + & z_2^4 + & s^4(& z_3^4 + & z_4^2\,)) \\ 1 & -8 & 0 & 0 & 0 & -2 & 1 & 1 \\ , & -4 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

x_i^{5} !!! NOT z_i

Mirror of $X \simeq z_0^4 + z_1^4 + z_2^4 + z_3^8 + z_4^8 - 8\psi z_0 z_1 z_2 z_3 z_4 - 2\varphi z_3^4 z_4^4$
$/ \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$

$(\beta + ir)_1 = \frac{1}{2\pi i} \log \frac{1}{4\varphi^2}$ $\qquad z_1 = 1/4\varphi^2$

$(\beta + ir)_2 = \frac{1}{2\pi i} \log\left(-\frac{\varphi}{2^{11}\psi^4}\right)$ $\qquad z_2 = -\varphi/2^{11}\psi^4$

by hypothesized
mon-div map.

$\mathcal{D}_1 = |\varphi_3|^2 + |\varphi_4|^2 - 2|s|^2 - r_1 = 0$

$\mathcal{D}_2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 + |s|^2 - 4|p|^2 - r_2 = 0$

$r_1 << 0 \Rightarrow$ ~~$|s| \to$~~ 0 $\quad s \neq 0$

$r_1 + 2r_2 = |\varphi_3|^2 + |\varphi_4|^2 + 2(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2) - 8|p|^2$

$r_1 + 2r_2 << 0 \Rightarrow p \neq 0$.

$p \frac{\partial W}{\partial z_i} = 0 \Rightarrow z_0 = z_1 = z_2 = z_3 = z_4 = 0$.

L.G. phase.

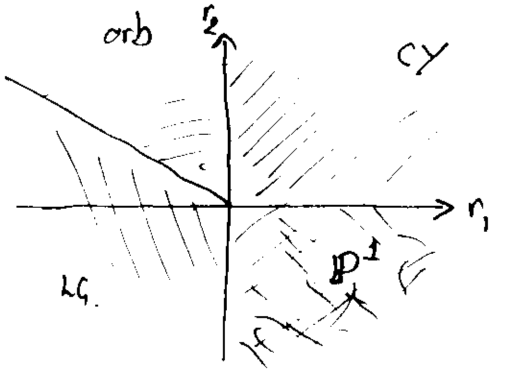
$r_1 + 2r_2 >> 0 \Rightarrow \Sigma Q_i |\varphi_i|^2 = 0$
i.e. not all φ_i^2 are zero.

$r_1 << 0 \Rightarrow s \neq 0$.
$\Rightarrow p = 0$
$\Rightarrow W = 0$ for some fixed s.
~ hypersurface in $\mathbb{P}^4_{\{2,2,2,1,1\}}$
- has orbifold $(\mathbb{Z}_2)$ along
$z_3 = z_4 = 0$ ~~$z_0 = z_1 = z_2 = 0$~~
Orbifold phase.

$r_1 >> 0 \Rightarrow$ ~~$s=0$~~ $|\varphi_3|^2 + |\varphi_4|^2 > 0$ ie φ_3, φ_4 not both zero.
$r_2 << 0 \Rightarrow p \neq 0 \Rightarrow \varphi_0 = \varphi_1 = \varphi_2 = 0$. $\quad s = 0$.
~ $\mathbb{P}^1$ with $[\varphi_3, \varphi_4]$ as coords
LG potential $\varphi_0^4 + \varphi_1^4 + \varphi_2^4 + s^4 / \mathbb{Z}_4$

Lat $r_1 >> 0 \Rightarrow$ ~~$|\varphi_3|^2$~~ φ_3, φ_4 not zero.
$r_2 >> 0 \Rightarrow \varphi_1, \varphi_2,$ ~~φ_3, φ_4~~ s not all zero. $\Big\}\ p = 0$.
away from ~~$\varphi_3 = \varphi_4 = \varphi_1 = \varphi_2 = \varphi_3 = 0$~~ $\varphi_3 = \varphi_4 = 0$ we have $s = 0$ ~ CY

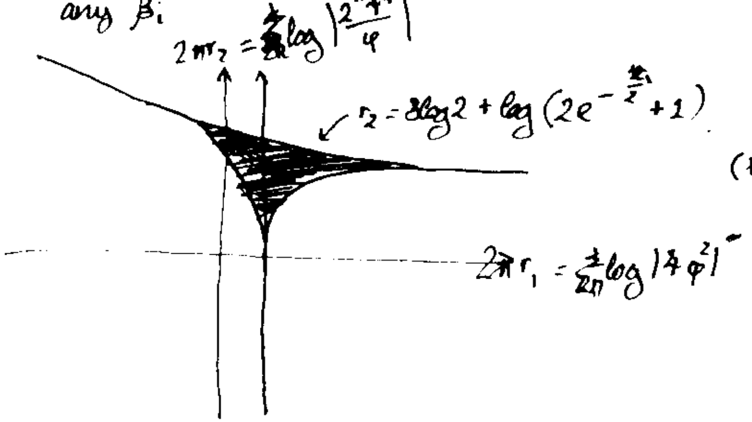
So roughly we have a "phase diagram"



"Secondary Fan"

with 4 phases – only one of which is a CY manifold.

Again we ~~put~~ have corrections that move singularities.
I.e. for ~~$\beta_i = 0$~~ any β_i. where are bad CFT's?



(Plesser, Morrison?

The above plots the discrim. of $W(x_i)$
Checked against by M.R.P.

4/14. Let us describe each phase

1. CY phase – rational curve instantons as quin
2. LG. phase – twist field instantons as before.
3. Orbifold – both present – rational curves in target space + twist fields around $\mathbb{Z}_2$-ring.
4. $\mathbb{P}^1$ – both – rational curve ITSELF! + twist field in LG fibre.

Note region in the middle where we have "no" interpretation as model + finite instantons.

All the above have the same right to be called the mirror of Y.

P.F. may be solved:

$$q_1 = z_1 + 2z_1^2 + 48z_1z_2 + 5z_1^3 + 7560z_1z_2^2 + \dots$$

$$q_2 = z_2 - z_1z_2 + 104z_2^2 - z_1^2z_2 - 56z_1z_2^2 + 15188z_2^2 + \dots$$

Let us now do the same thing in 5 dim'l $\mathcal{M}$.

eg. X is $W = x_0^3 + x_1^3 + x_2^6 + x_3^9 + x_4^{18} \subset \mathbb{P}^4_{\{6,6,3,2,1\}}$

Witten linear σ-model analysis too difficult (use toric methods?)
$U(1)^5$

We have 100 cones in secondary fan.

5 correspond to CY manifolds
related by flops.

A flop is similar to $\mathcal{O}(-2)$ blow-up of $\mathbb{C}^2/\mathbb{Z}_2$.

Rather than $xy - z^2 = 0$ in $\mathbb{C}^3$ $(\cong \mathbb{C}^2/\mathbb{Z}_2)$

put $xy - wz = 0$ in $\mathbb{C}^4$

- singularity in 3 dims

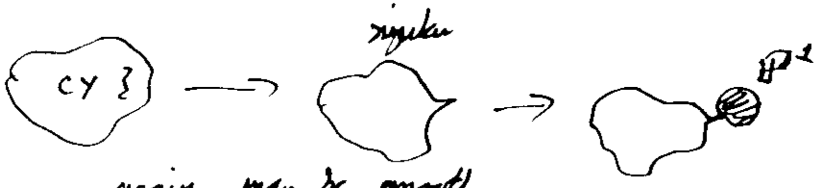
- may blow-up to $\mathcal{O}(-1,-1)$

I.e. $\mathcal{O}(-1,-1) \cong \{ [a,b], (x,y), (s,t) \in \mathbb{P}^1 \times \mathbb{C}^2 \times \mathbb{C}^2 \mid ay = bx, at = bs \}$

This puts $\mathbb{P}^1$ where sing was
However, this can be done in two different ways leading to a "flop"

Thus any 5 can be the mirror-manifold of Y.
All fit into the same moduli space.
- all have instanton sums
only at most one is convergent.
- "95%" of time none are.

We have other bizarre possibilities, LG, orb & hybrid as before. Plus EXOFLOP.



again may be smooth
by giving β's.

Conclusions

- The moduli space of $N=(2,2)$ is best described algebraically – No metric on X.

- Naturally divides into phases only some (if any) of which are CY manifolds