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STRONG WEAK COUPLING DUALITY IN STRING THEORY

ASHOKE SEN
Tata Institute of Fundamental Research
Bombay
INDIA

Strong-Weak Coupling Duality in Four Dimensional String Theory

Ashoke Sen

Tata Institute of Fundamental Research

Homi Bhabha Road, Bombay 400005, INDIA.

sen@theory.tifr.res.in, sen@tifrvax.bitnet

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Abstract

We present several pieces of evidence for strong-weak coupling duality symmetry in the heterotic string theory, compactified on a six dimensional torus. These include symmetry of the 1) low energy effective action, 2) allowed spectrum of electric and magnetic charges in the theory, 3) allowed mass spectrum of particles saturating the Bogomol'nyi bound, and 4) Yukawa couplings between massless neutral particles and massive charged particles saturating the Bogomol'nyi bound.

This duality transformation exchanges the electrically charged elementary string excitations with the magnetically charged soliton states in the theory. It is shown that the existence of a strong-weak coupling duality symmetry in four dimensional string theory makes definite prediction about the existence of new stable monopole and dyon states in the theory with specific degeneracies, including certain supersymmetric bound states of monopoles and dyons. The relationship between strong-weak coupling duality transformation in string theory and target space duality transformation in the five-brane theory is also discussed.

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1 Introduction

String theory has many surprising symmetries which completely change our understanding of the geometry and topology of space-time. Among them are the familiar duality symmetries of string theory compactified on a torus and the mirror symmetries of string theories compactified on a Calabi-Yau manifold. From the world sheet point of view, these symmetries provide an equivalence relation between two dimensional quantum field theories and not between their classical limits. Thus this equivalence cannot be seen if we expand both the theories in the σ-model loop expansion parameter g_σ, and compare terms order by order in g_σ. In this sense, these symmetries are non-perturbative from the world-sheet point of view. However, all of these symmetries are valid order by order in string perturbation theory, i.e., the two σ models related by such a symmetry transformation give rise to equivalent quantum field theories on a two dimensional surface of any arbitrary genus.

In these notes we shall present evidence that string theory in four dimensions, resulting from the compactification of the heterotic string theory on a six-dimensional torus, possesses another kind of symmetry, which acts non-trivially on the string loop expansion parameter g_{st}, and hence is not a property of each term in the expansion in powers of g_{st}. In particular, at the level of states, this duality transformation, acting on the elementary excitations in string theory carrying electric charge, gives rise to magnetically charged solitons. For definiteness, we shall call this duality transformation S-duality, and the usual target space duality transformation in the four dimensional string theory T-duality. Since at present the only way we know of calculating anything in string theory is as a power series expansion in g_{st}, we have no way of actually proving the existence of S-duality symmetry in string theory. However, there are several quantities in string theory, where the tree level answers are believed to be the exact answers. It is possible to check if these quantities are invariant under the S-duality transformation mentioned above. We shall focus on four such sets of quantities.

1. Low Energy Effective Field Theory: It is well known that string theory at low energies is described by an effective field theory of massless fields. A priori there is no reason to expect that this field theory will not be modified by quantum corrections, and in fact, for a generic string compactification, the low energy effective field theory will be modified by quantum corrections. However, the theory that we shall consider, namely the toroidal compactification of the heterotic string theory, possesses a local $N = 4$ supersymmetry in four dimensions. There is strong evidence that for such theories, specifying the gauge symmetry group determines the low energy effective field theory completely[10]. Thus we expect that the low energy effective field theory at the tree level is not modified by string quantum corrections (up to possible redefinitions of various fields).[1] Thus if S-duality is a genuine symmetry of the theory, this low energy effective field theory must possess S-duality invariance.

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[1] We are implicitly assuming that the computation of the effective action does not suffer from any infra-red or collinear divergences, so that the effective action can be expressed as the integral of a local Lagrangian density. Since we shall be working at a generic point in the moduli space of compactification where the unbroken gauge symmetry group is abelian, and all the charged particles are massive, this is a plausible assumption.

2. Allowed Spectrum of Electric and Magnetic Charges: At a generic point in the moduli space of vacuum configurations, the theory under consideration has an unbroken gauge symmetry $U(1)^{28}$. The U(1) charges of different states in the theory are described by 28 dimensional vectors belonging to an even, self-dual, Lorentzian lattice. For N=4 supersymmetric string compactification, we expect these gauge charges not to be renormalized by quantum corrections[33]. Since the spectrum of magnetic charges in the theory is determined from the spectrum of electric charges by the Dirac-Schwinger-Zwanziger-Witten[11, 50] quantization rules, it follows that the spectrum of allowed magnetic charges in the theory is also not renormalized by quantum corrections. Hence the spectrum of electric and magnetic charges, calculated from the tree level theory, must be invariant under the S-duality transformation if it is to be a symmetry of the theory.

3. Allowed Mass Spectrum of Particles Saturating the Bogomol'nyi Bound: The mass of a generic string state is most certainly renormalized by quantum corrections. However, there is a special class of string states for which the tree level formulæ for the masses are expected to be exact[49, 37]. These states are characterized by the fact that 1) they belong to the 16-dimensional representation of the $N = 4$ supersymmetry algebra, and 2) their masses are determined completely in terms of their electric and magnetic charges by the so called Bogomol'nyi formula, which also gives a lower bound to the mass of any state in the theory carrying a given amount of electric and magnetic charges. In fact, the supersymmetry algebra itself constrains the mass of a state in the 16-component supermultiplet to saturate the Bogomol'nyi bound. Since the representation of a state is not expected to be modified by quantum corrections, the masses of these states are also expected to be unaffected by quantum corrections. As a result, if the S-duality transformation is to be a symmetry of the theory, the allowed mass spectrum of the states in the 16-component supermultiplet, calculated at the tree level, must be invariant under this transformation.

4. Yukawa Couplings Between Massless Scalars and Massive Charged States in the 16-component Supermultiplet: As in the case of the mass spectrum, the three point couplings between generic string states will most certainly be modified by quantum corrections. However, as we shall see, the Yukawa couplings of all the massless scalar fields of the theory to various string states can be determined in terms of the dependence of the masses of these states on various modular parameters. Since we have already argued that the masses of the string states belonging to the 16-component supermultiplet are not modified by quantum corrections, the Yukawa couplings of the massless scalars of the theory to these states also remain unmodified. Hence, these Yukawa couplings, calculated at the tree level, must also remain invariant under the S-duality transformation, if the latter is a symmetry of the theory.

We shall analyze the S-duality transformation properties of each of these quantities, and show that they are, indeed, invariant under this transformation.

S-duality transformation of elementary string states correspond to monopole and dyon states in the string theory. We shall show that whereas many of these states can be identified with known monopole and dyon states in the theory, there are many others which do not correspond

to any known state. Existence of these states can be taken to be a prediction of the S-duality symmetry.

Besides the conjecture of S-duality symmetry of the four dimensional string theory, there has been yet another independent conjecture in string theory which is even harder to test. This conjecture claims that string theory in ten dimensions is equivalent to the theory of five-branes (five dimensional extended objects) in ten dimensions. The reason that this conjecture is difficult to prove is that 1) the theory of five branes at lowest order is described by an interacting six dimensional field theory and has not been solved, and, 2) the relationship between the loop expansion parameter of the string theory and that of the five brane theory is somewhat non-trivial[14], so that the duality conjecture does not relate a given order term in the string loop expansion parameter to the same order term in the five-brane loop expansion parameter.

If we accept the equivalence of the string theory and five-brane theories in ten dimensions despite these difficulties, then it would also imply the equivalence between the corresponding theories compactified on a six-dimensional torus. It will then be natural to ask how the S-duality transformation in string theory acts on the states of the five-brane theory. It turns out that the S-duality transformation has a very natural action on the states of the five-brane theory, namely, it interchanges the Kaluza-Klein modes of the theory (states carrying non-zero momenta in the internal direction) with the five-brane winding modes on the torus. Thus this is an exact analog of the target space duality (T-duality) transformation in string theory, under which the Kaluza Klein modes of the theory get exchanged with the string winding modes on the torus. In this sense, the string five-brane duality interchanges the roles of the T-duality and S-duality. We call this 'duality of dualities'.

These notes will be divided into two main parts. In the first part (§2-§6) we shall discuss the evidence for the S-duality symmetry in four dimensional string theory. In the second part (§7) we shall show how the electric-magnetic duality transformation in string theory can be interpreted as the target space duality transformation in the five-brane theory compactified on a six dimensiional torus. Much of the material in these notes will be a review of Refs.[42, 43, 45, 39, 40]. For earlier discussions of the possibility of a strong-weak coupling duality in four dimensional field theory see Refs.[34, 37], and in four dimensional string theory, see Ref.[17].

2 Symmetry of the Effective Action

We shall begin this section by carrying out the dimensional reduction of the $N = 1$ supergravity theory coupled to $N = 1$ super Maxwell theory from ten dimensions to four dimensions. In §2.2 we discuss the O(6,22) and SL(2,R) symmetry of the resulting effective field theory. We shall see that O(6,22) and SL(2,R) symmetries appear on a somewhat different footing; the former is a symmetry of the effective action, while the latter is only a symmetry of the equations of motion. In §2.3 we shall show that it is possible to give an alternative formulation of the theory in which SL(2,R) becomes a symmetry of the action. Finally in §2.4 we shall show that the

manifestly SL(2,R) invariant formulation of the theory can be obtained from the dimensional reduction of the dual formulation of the $N = 1$ supergravity theory from ten to four dimensions. In later sections we shall see that the discrete SL(2,Z) subgroup of the SL(2,R) group can be identified as the S-duality group, just as the discrete O(6,22;Z) subgroup of the O(6,22) group can be identified as the T-duality group[23].

2.1 Dimensional Reduction of the Ten Dimensional Theory

We consider heterotic string theory compactified on a six dimensional torus. The simplest way to derive the low energy effective action for this theory is to start with the $N = 1$ supergravity theory coupled to $N = 1$ super Yang-Mills theory in ten dimensions, and dimensionally reduce the theory from ten to four dimensions[16, 27, 32]. Since at a generic point in the moduli space only the abelian gauge fields give rise to massless fields in four dimensions, it is enough to restrict to the $U(1)^{16}$ part of the ten dimensional gauge group. The ten dimensional action is given by,

$$\frac{1}{32\pi} \int d^{10}z \sqrt{-G^{(10)}}\, e^{-\Phi^{(10)}} \Big(R^{(10)} + G^{(10)MN}\partial_M \Phi^{(10)}\partial_N \Phi^{(10)}$$
$$-\frac{1}{12}H^{(10)}_{MNP}H^{(10)MNP} - \frac{1}{4}F^{(10)I}_{MN}F^{(10)IMN}\Big), \tag{1}$$

where $G^{(10)}_{MN}$, $B^{(10)}_{MN}$, $A^{(10)I}_M$, and $\Phi^{(10)}$ are ten dimensional metric, anti-symmetric tensor field, U(1) gauge fields and the scalar dilaton field respectively ($0 \le M, N \le 9$, $1 \le I \le 16$), and,

$$F^{(10)I}_{MN} = \partial_M A^{(10)I}_N - \partial_N A^{(10)I}_M$$
$$H^{(10)}_{MNP} = (\partial_M B^{(10)}_{NP} - \frac{1}{2}A^{(10)I}_M F^{(10)I}_{NP}) + \text{cyclic permutations in } M, N, P. \tag{2}$$

We have ignored the fermion fields in writing down the action (1); we shall discuss them in §2.5. Also note that we have included a factor of $(1/32\pi)$ multiplying the action for later convenience. This factor can be absorbed into $\Phi^{(10)}$ by shifting it by $\ln 32\pi$.

For dimensional reduction, it is convenient to introduce the 'four dimensional fields' $\hat{G}_{mn}$, $\hat{B}_{mn}$, $\hat{A}^I_m$, Φ, $A^{(a)}_\mu$, $G_{\mu\nu}$ and $B_{\mu\nu}$ ($1 \le m \le 6$, $0 \le \mu \le 3$, $1 \le a \le 28$) through the relations[32, 42, 38][2]

$$\hat{G}_{mn} = G^{(10)}_{m+3,n+3}, \quad \hat{B}_{mn} = B^{(10)}_{m+3,n+3}, \quad \hat{A}^I_m = A^{(10)I}_{m+3},$$

$$A^{(m)}_\mu = \frac{1}{2}\hat{G}^{mn}G^{(10)}_{n+3,\mu}, \quad A^{(I+12)}_\mu = -(\frac{1}{2}A^{(10)I}_\mu - \hat{A}^I_n A^{(n)}_\mu),$$

$$A^{(m+6)}_\mu = \frac{1}{2}B^{(10)}_{(m+3)\mu} - \hat{B}_{mn}A^{(n)}_\mu + \frac{1}{2}\hat{A}^I_m A^{(I+12)}_\mu,$$

$$G_{\mu\nu} = G^{(10)}_{\mu\nu} - G^{(10)}_{(m+3)\mu}G^{(10)}_{(n+3)\nu}\hat{G}^{mn},$$

$$B_{\mu\nu} = B^{(10)}_{\mu\nu} - 4\hat{B}_{mn}A^{(m)}_\mu A^{(n)}_\nu - 2(A^{(m)}_\mu A^{(m+6)}_\nu - A^{(m)}_\nu A^{(m+6)}_\mu),$$

$$\Phi = \Phi^{(10)} - \frac{1}{2}\ln \det \hat{G}, \qquad 1 \le m, n \le 6, \quad 0 \le \mu, \nu \le 3, \quad 1 \le I \le 16. \tag{3}$$

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[2]The normalization and sign conventions used here are slightly different from those used in Ref.[42]. Care has been taken to ensure that we use the same normalization convention throughout this paper.

Here $\hat{G}^{mn}$ denotes the inverse of the matrix $\hat{G}_{mn}$. We now combine the scalar fields $\hat{G}_{mn}$, $\hat{B}_{mn}$, and $\hat{A}^I_m$ into an $O(6,22)$ matrix valued scalar field M. For this we regard $\hat{G}_{mn}$, $\hat{B}_{mn}$ and $\hat{A}^I_m$ as 6×6, 6×6, and 6×16 matrices respectively, and $\hat{C}_{mn} = \frac{1}{2}\hat{A}^I_m\hat{A}^I_n$ as a 6×6 matrix, and define M to be the 28×28 dimensional matrix

$$M = \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1}(\hat{B}+\hat{C}) & \hat{G}^{-1}\hat{A} \\ (-\hat{B}+\hat{C})\hat{G}^{-1} & (\hat{G}-\hat{B}+\hat{C})\hat{G}^{-1}(\hat{G}+\hat{B}+\hat{C}) & (\hat{G}-\hat{B}+\hat{C})\hat{G}^{-1}\hat{A} \\ \hat{A}^T\hat{G}^{-1} & \hat{A}\hat{G}^{-1}(\hat{G}+\hat{B}+\hat{C}) & I_{16}+\hat{A}^T\hat{G}^{-1}\hat{A} \end{pmatrix}. \tag{4}$$

satisfying

$$MLM^T = L, \qquad M^T = M, \qquad L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}, \tag{5}$$

where I_n denotes the $n \times n$ identity matrix.

The effective action that governs the dynamics of the massless fields in the four dimensional theory is obtained by substituting the expressions for the ten dimensional fields in terms of the four dimensional fields in Eq.(1), and taking all field configurations to be independent of the internal coordinates. The result is

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-G}\, e^{-\Phi}[R_G + G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'}$$
$$-G^{\mu\mu'}G^{\nu\nu'}F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)}_{\mu'\nu'} + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)] \tag{6}$$

where

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$$
$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} + 2A^{(a)}_\mu L_{ab}F^{(b)}_{\nu\rho}) + \text{cyclic permutations of } \mu, \nu, \rho, \tag{7}$$

and R_G is the scalar curvature associated with the four dimensional metric $G_{\mu\nu}$. In deriving this result we have taken $\int d^6y = 1$, where y^m ($1 \le m \le 6$) denote the coordinates labeling the six dimensional torus.

2.2 O(6,22) and SL(2,R) Symmetries of the Effective Field Theory

This effective action can easily be seen to be invariant under an $O(6,22)$ transformation[23]

$$M \to \Omega M\Omega^T, \quad A^{(a)}_\mu \to \Omega_{ab}A^{(b)}_\mu, \quad G_{\mu\nu} \to G_{\mu\nu}, \quad B_{\mu\nu} \to B_{\mu\nu}, \quad \Phi \to \Phi \tag{8}$$

where Ω is an $O(6,22)$ matrix, satisfying

$$\Omega^T L\Omega = L. \tag{9}$$

An $O(6,22; Z)$ subgroup of this is known to be an exact symmetry of the full string theory and will be called the T-duality group in this paper. Part of this symmetry exchanges the Kaluza-Klein modes of the theory, i.e. the states carrying momenta in the internal directions, with the

string winding modes, – states corresponding to a string wrapped around one of the compact directions.

The effective four dimensional theory is invariant under another set of symmetry transformations, which correspond to a symmetry of the equations of motion, but not of the effective action given in Eq.(6). To exhibit this symmetry, we introduce the canonical metric,

$$g_{\mu\nu} = e^{-\Phi} G_{\mu\nu}, \tag{10}$$

and use the convention that all indices are raised or lowered with respect to this canonical metric. Also, we denote by D_μ the standard covariant derivative constructed from the metric $g_{\mu\nu}$. The $B_{\mu\nu}$ equations of motion, as derived from the action (6), are given by

$$D_\rho(e^{-2\Phi} H^{\mu\nu\rho}) = 0 \tag{11}$$

which allows us to introduce a scalar field Ψ through the relation

$$H^{\mu\nu\rho} = -(\sqrt{-g})^{-1} e^{2\Phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi. \tag{12}$$

Let us introduce a complex scalar field

$$\lambda = \Psi + i e^{-\Phi} \equiv \lambda_1 + i\lambda_2 \tag{13}$$

The equations of motion of the fields $G_{\mu\nu}$, $A_\mu^{(a)}$ and Φ, derived from the action given in (6), together with the Bianchi identity for the field strength $H_{\mu\nu\rho}$, may now be written as,

$$R_{\mu\nu} = \frac{\partial_\mu \bar\lambda \partial_\nu \lambda + \partial_\nu \bar\lambda \partial_\mu \lambda}{4(\lambda_2)^2} + 2\lambda_2 F_{\mu\rho}^{(a)}(LML)_{ab} F_\nu^{(b)\rho} - \frac{1}{2}\lambda_2 g_{\mu\nu} F_{\rho\sigma}^{(a)}(LML)_{ab} F^{(b)\rho\sigma},$$

$$D_\mu(-\lambda_2(ML)_{ab} F^{(b)\mu\nu} + \lambda_1 \tilde{F}^{(a)\mu\nu}) = 0,$$

$$\frac{D^\mu D_\mu \lambda}{(\lambda_2)^2} + i\frac{D_\mu \lambda D^\mu \lambda}{(\lambda_2)^3} - iF_{\mu\nu}^{(a)}(LML)_{ab} F^{(b)\mu\nu} + \tilde{F}_{\mu\nu}^{(a)} L_{ab} F^{(b)\mu\nu} = 0, \tag{14}$$

where $R_{\mu\nu}$ is the Ricci tensor calculated with the metric $g_{\mu\nu}$, and,

$$\tilde{F}^{(a)\mu\nu} = \frac{1}{2}(\sqrt{-g})^{-1} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(a)}. \tag{15}$$

Derivation of the equations of motion for the field M is a little bit more complicated, since M is a constrained matrix. The simplest way to derive these equations is to introduce a set of independent parameters $\{\phi^i\}$ that label the symmetric O(6,22) matrix M. (We can take ϕ^i to be the set $\{\hat{G}_{mn}, \hat{B}_{mn}, \hat{A}_m^I\}$, but any other parametrization will also do.) Varying the action with respect to these parameters ϕ^i, we get the following set of equations of motion,

$$\frac{1}{4} Tr\left(\frac{\delta M}{\delta \phi^i} L D_\mu D^\mu M L\right) + \lambda_2 F_{\mu\nu}^{(a)}\left(L \frac{\delta M}{\delta \phi^i} L\right)_{ab} F^{(b)\mu\nu} = 0. \tag{16}$$

Finally, the Bianchi identities satisfied by the gauge field strengths $F_{\mu\nu}^{(a)}$ are given by;

$$D_\mu \tilde{F}^{(a)\mu\nu} = 0. \tag{17}$$

It is now straightforward to check that the set of equations (14), (16) and (17) are invariant under the following set of SL(2,R) transformations[10, 47, 42, 38]:

$$\lambda \to \lambda' = \frac{a\lambda + b}{c\lambda + d}, \quad F_{\mu\nu}^{(a)} \to F_{\mu\nu}'^{(a)} = (c\lambda_1 + d)F_{\mu\nu}^{(a)} + c\lambda_2(ML)_{ab}\tilde{F}_{\mu\nu}^{(b)}, \quad g_{\mu\nu} \to g_{\mu\nu}, \quad M \to M. \tag{18}$$

where a, b, c and d are real numbers satisfying $ad - bc = 1$. In particular, if we consider the element $a = 0$, $b = 1$, $c = -1$ and $d = 0$, then the transformations take the form:

$$\lambda \to -\frac{1}{\lambda}, \quad F_{\mu\nu}^{(a)} \to -\lambda_1 F_{\mu\nu}^{(a)} - \lambda_2(ML)_{ab}\tilde{F}_{\mu\nu}^{(b)}. \tag{19}$$

For $\lambda_1 = 0$, this transformation takes electric fields to magnetic fields and vice versa. It also takes λ_2 to $1/\lambda_2$. Since $(\lambda_2)^{-1} = e^\Phi$ can be identified with the coupling constant of the string theory, we see that the duality transformation takes a strong coupling theory to a weak coupling theory and vice-versa. We shall refer to the transformations (19) as the strong-weak coupling duality transformation, or electric-magnetic duality transformation. Note that the full SL(2,R) group of transformations is generated as a combination of the transformation (19) and the trivial duality transformation

$$\lambda_1 \to \lambda_1 + b, \tag{20}$$

with all other fields remaining invariant.

It can be easily checked that the set of equations (14) and (16) can be derived from the action

$$S = \frac{1}{32\pi}\int d^4x \sqrt{-g}[R - \frac{1}{2(\lambda_2)^2}g^{\mu\nu}\partial_\mu\lambda\partial_\nu\bar\lambda - \lambda_2 F_{\mu\nu}^{(a)}(LML)_{ab} F^{(b)\mu\nu}$$
$$+\lambda_1 F_{\mu\nu}^{(a)} L_{ab}\tilde{F}^{(b)\mu\nu} + \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)] \tag{21}$$

This form of the action will be useful to us for later analysis.

We wish to know whether any subgroup of this SL(2,R) group can be an exact symmetry of string theory, in the same way that the O(6,22;Z) subgroup of O(6,22) is an exact symmetry of string theory. However, before we address this question, we notice that even at the level of effective action, there is an asymmetry between the O(6,22) and SL(2,R) symmetry transformtions. The former is a symmetry of the effective action, whereas the latter is only a symmetry of the equations of motion. We shall now show how to reformulate the theory so that SL(2,R) becomes a symmetry of the effective action[39, 41].

2.3 Manifestly SL(2,R) Invariant Action

We begin by defining the matrices,

$$\mathcal{M} = \frac{1}{\lambda_2}\begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{22}$$

We also introduce a set of auxiliary gauge fields[28, 39] $A_\mu^{(a,2)}$ ($1 \le a \le 28$), and define,

$$A_\mu^{(a,1)} = A_\mu^{(a)},\tag{23}$$

$$F_{\mu\nu}^{(a,\alpha)} = \partial_\mu A_\nu^{(a,\alpha)} - \partial_\nu A_\mu^{(a,\alpha)}, \qquad E_i^{(a,\alpha)} = F_{0i}^{(a,\alpha)}, \qquad B^{(a,\alpha)i} = \tilde{F}^{(a,\alpha)0i} = (\sqrt{-g})^{-1}\epsilon^{0ijk}\partial_j A_k^{(a,\alpha)},\tag{24}$$

for $1 \le \alpha \le 2$. It can be checked that the set of equations (14), (16) and (17) are identical to the equations of motion and Bianchi identities derived from the action[39]

$$\begin{aligned}
S = \frac{1}{32\pi}\int d^4x\Big[\sqrt{-g}\,\{R - \frac{1}{4}g^{\mu\nu}tr(\partial_\mu\mathcal{M}\mathcal{L}\partial_\nu\mathcal{M}\mathcal{L}) + \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)\}\\
-2\{B^{(a,\alpha)i}\mathcal{L}_{\alpha\beta}L_{ab}E_i^{(b,\beta)} + \epsilon^{ijk}\frac{g^{0k}}{g^{00}}B^{(a,\alpha)i}\mathcal{L}_{\alpha\beta}L_{ab}B^{(b,\beta)j}\\
-\frac{g_{ij}}{\sqrt{-g}g^{00}}B^{(a,\alpha)i}(\mathcal{L}^T\mathcal{M}\mathcal{L})_{\alpha\beta}(LML)_{ab}B^{(b,\beta)j}\}\Big].
\end{aligned}\tag{25}$$

where tr and Tr denote traces over the indices α,β and a,b respectively. The simplest way to check that this action gives rise to the same set of equations as (14), (16) and (17) is to note that the $A_i^{(a,2)}$ equations of motion give

$$\epsilon^{ijk}\partial_j\Big[L_{ab}E_k^{(b,1)} + \epsilon^{klm}\frac{g^{0m}}{g^{00}}L_{ab}B^{(b,1)l} + \frac{g_{kl}}{\sqrt{-g}g^{00}}(\mathcal{M}\mathcal{L})_{1\beta}(LML)_{ab}B^{(b,\beta)l}\Big] = 0\tag{26}$$

where $\epsilon^{ijk} = \epsilon^{0ijk}$ is the three dimensional totally anti-symmetric tensor density. Since these equations do not involve any time derivative of the fields $A_i^{(a,2)}$, we can treat $A_i^{(a,2)}$ as auxiliary fields, and eliminate them from the action (25) by using their equations of motion. The resulting action is identical to the action (21).

The action (25) is invariant under manifest SL(2,R) transformation

$$\mathcal{M} \to \omega\mathcal{M}\omega^T, \qquad A_\mu^{(a,\alpha)} \to \omega_{\alpha\beta}A_\mu^{(a,\beta)},\tag{27}$$

and O(6,22) transformations

$$M \to \Omega M\Omega^T, \qquad A_\mu^{(a,\alpha)} \to \Omega_{ab}A_\mu^{(b,\alpha)},\tag{28}$$

where

$$\omega = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \qquad ad - bc = 1,\tag{29}$$

is an SL(2,R) matrix, satisfying,

$$\omega^T\mathcal{L}\omega = \mathcal{L}.\tag{30}$$

The transformation laws of λ, induced by Eq.(27), can be seen to be identical to those given in Eq.(18). Also, after we eliminate the fields $A_i^{(a,2)}$ by their equations of motion, the O(6,22) and SL(2,R) transformation laws of the rest of the fields coincide with those given in Eqs.(8) and (18). The loss of manifest SL(2,R) invariance of the action after integrating out the gauge field

components $A_i^{(a,2)}$ can be traced to the fact that the set of fields $A_i^{(a,2)}$ is not an SL(2,R) invariant set, since they transform to linear combinations of $A_i^{(a,1)}$ and $A_i^{(a,2)}$ under SL(2,R) transformations. In contrast, this set is invariant under O(6,22) transformation, since the fields in this set transform to linear combinations of the fields in the same set under O(6,22) transformations.

The action (25) is also invariant under the gauge transformations

$$\delta A_\mu^{(a,\alpha)} = \partial_\mu\Lambda^{(a,\alpha)}, \qquad \delta A_0^{(a,\alpha)} = \Psi^{(a,\alpha)},\tag{31}$$

where $\Lambda^{(a,\alpha)}$ and $\Psi^{(a,\alpha)}$ are the gauge transformation parameters. Note that the action does not depend on $A_0^{(a,\alpha)}$. Finally, although (25) is not manifestly general coordinate invariant, it is invariant under a hidden 'general coordinate transformation'

$$\begin{aligned}
\delta A_i^{(a,\alpha)} &= \xi^j\partial_j A_i^{(a,\alpha)} + (\partial_i\xi^j)A_j^{(a,\alpha)}\\
&\quad -\xi^0\Big\{\frac{g_{ij}}{\sqrt{-g}g^{00}}(\mathcal{M}\mathcal{L})_{\alpha\beta}(ML)_{ab}B^{(b,\beta)j} + \frac{g^{0k}}{g^{00}}\epsilon^{ijk}B^{(a,\alpha)j}\Big\},\\
\delta M &= \xi^\mu\partial_\mu M, \qquad \delta\mathcal{M} = \xi^\mu\partial_\mu\mathcal{M}, \qquad \delta g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu\xi^\rho + g_{\mu\rho}\partial_\nu\xi^\rho.
\end{aligned}\tag{32}$$

This transformation does not look like the usual general coordinate transformation. However, if we use the equations of motion of $A_i^{(a,2)}$ given in (26), the transformation laws of all other fields reduce to the usual general coordinate transformation laws[39].

Thus we see that the low energy effective theory of the four dimensional heterotic string can be described by a manifestly SL(2,R)×O(6,22) invariant action. This action is not manifestly general coordinate invariant, but has a hidden general coordinate invariance. One can now ask if it is possible to find another action describing the same theory, which is manifestly SL(2,R) and general coordinate invariant. It turns out that this is possible for a restricted class of configurations where we set all fields originating from the ten dimensional gauge fields $A_M^{(10)I}$ to zero[39]. In terms of four dimensional fields this would correspond to replacing the 28 component gauge field $A_\mu^{(a)}$ by a 12 component gauge field $\check{A}_\mu^{(b)}$ ($1 \le b \le 12$), and M by a 12×12 matrix $\check{M}$, satisfying,

$$\check{M}^T = \check{M}, \qquad \check{M}\check{L}\check{M}^T = \check{L}, \qquad \check{L} = \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix}.\tag{33}$$

The action (21) is now replaced by,

$$\begin{aligned}
S = \frac{1}{32\pi}\int d^4x\sqrt{-g}\,[R - \frac{1}{2(\lambda_2)^2}g^{\mu\nu}\partial_\mu\lambda\partial_\nu\bar{\lambda} - \lambda_2\check{F}_{\mu\nu}^{(a)}(\check{L}\check{M}\check{L})_{ab}\check{F}^{(b)\mu\nu}\\
+\lambda_1\check{F}_{\mu\nu}^{(a)}\check{L}_{ab}\check{\tilde{F}}^{(b)\mu\nu} + \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu\check{M}\check{L}\partial_\nu\check{M}\check{L})]
\end{aligned}\tag{34}$$

where

$$\check{F}_{\mu\nu}^{(a)} = \partial_\mu\check{A}_\nu^{(a)} - \partial_\nu\check{A}_\mu^{(a)}.\tag{35}$$

The indices a, b run from 1 to 12. This action has manifest O(6,6) symmetry. As in the previous case, the equations of motion are invariant under SL(2,R) transformation, but the effective action

is not SL(2,R) invariant. As before, this theory may be shown to be equivalent to a manifestly SL(2,R) and O(6,6) invariant, but not manifestly general coordinate invariant, action

$$
S = \frac{1}{32\pi} \int d^4x \Big[\sqrt{-g} \, \{ R - \frac{1}{4} g^{\mu\nu} tr(\partial_\mu \mathcal{M} \mathcal{L} \partial_\nu \mathcal{M} \mathcal{L}) + \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu \check{M} \check{L} \partial_\nu \check{M} \check{L}) \}
$$
$$
-2\{ \check{B}^{(a,\alpha)i} \mathcal{L}_{\alpha\beta} \check{L}_{ab} \check{E}_i^{(b,\beta)} + \varepsilon^{ijk} \frac{g^{0k}}{g^{00}} \check{B}^{(a,\alpha)i} \mathcal{L}_{\alpha\beta} \check{L}_{ab} \check{B}^{(b,\beta)j}
$$
$$
- \frac{g_{ij}}{\sqrt{-g} g^{00}} \check{B}^{(a,\alpha)i} (\mathcal{L}^T \mathcal{M} \mathcal{L})_{\alpha\beta} (\check{L} \check{M} \check{L})_{ab} \check{B}^{(b,\beta)j} \} \Big]. \tag{36}
$$

The SL(2,R) and O(6,6) transformations act on the various fields as

$$
\mathcal{M} \to \omega \mathcal{M} \omega^T, \qquad \check{M} \to \check{\Omega} \check{M} \check{\Omega}^T, \qquad \check{A}_\mu^{(a,\alpha)} \to \omega_{\alpha\beta} \check{\Omega}_{ab} \check{A}^{(b,\beta)}, \tag{37}
$$

where $\check{\Omega}$ is an O(6,6) matrix satisfying $\check{\Omega} \check{L} \check{\Omega}^T = \check{L}$. If we eliminate the O(6,6) invariant set of fields $\check{A}_i^{(b,2)}$ for $1 \leq b \leq 12$ by their equations of motion, we recover the original action (34). Instead of doing that, we can also eliminate the SL(2,R) invariant set of fields $\check{A}_i^{(m+6,\beta)}$ for $1 \leq \beta \leq 2$, and $1 \leq m \leq 6$ by their equations of motion, since these equations do not contain any time derivative of these fields. The resulting action is[39]

$$
\frac{1}{32\pi} \int d^4x \sqrt{-g} \Big[R - \frac{1}{4} g^{\mu\nu} tr(\partial_\mu \mathcal{M} \mathcal{L} \partial_\nu \mathcal{M} \mathcal{L}) + \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu \check{M} \check{L} \partial_\nu \check{M} \check{L})
$$
$$
- \check{F}_{\mu\nu}^{(m,\alpha)} \hat{G}_{mn} (\mathcal{L}^T \mathcal{M} \mathcal{L})_{\alpha\beta} \check{F}_{\rho\sigma}^{(n,\beta)} g^{\mu\rho} g^{\nu\sigma} - \check{F}_{\mu\nu}^{(m,\alpha)} \hat{B}_{mn} \mathcal{L}_{\alpha\beta} \check{F}_{\rho\sigma}^{(n,\beta)} g^{\mu\rho} g^{\nu\sigma} \Big],
$$
$$
1 \leq m, n \leq 6, \tag{38}
$$

which is manifestly general coordinate and SL(2,R) invariant, but is not O(6,6) invariant. The equations of motion, however, are invariant under the O(6,6) transformations.[3]

Thus we see that at the level of the effective action, we have been able to put O(6,6)(O(6,22)) transformations and the SL(2,R) transformations on an equal footing. First, there is a formulation of the theory in which O(6,22) is a manifest symmetry of the action whereas SL(2,R) is only a symmetry of the effective action. Second, there is a different formulation of the theory where the action is manifestly O(6,22) and SL(2,R) invariant, but not manifestly general coordinate invariant. Finally, in the special case when we ignore the ten dimensional gauge fields, there is a third formulation of the theory where the action is manifestly SL(2,R) and general coordinate invariant, but O(6,6) is only a symmetry of the equations of motion.

Despite these three alternate formulations of the action, one of them, namely (6), appears to be more fundamental, since this is the action that comes from the dimensional reduction of the $N = 1$ supergravity action in ten dimensions. We shall now show that if we start with the dual formulation of the $N = 1$ supergravity theory in ten dimensions, then we recover a manifestly SL(2,R) invariant form of the action after dimensional reduction[39, 3].

---

[3] Note that this procedure cannot be carried out for the action (25), since in that case we cannot find an SL(2,R) invariant set of fields whose equations of motion do not contain time derivative of the fields being eliminated.

2.4 Manifestly SL(2,R) Invariant Effective Action from Dimensional Reduction of the Dual $N = 1$ Supergravity Theory in Ten Dimensions

The dual formulation of the $N = 1$ supergravity theory in ten dimensions is based on the metric $\tilde{G}_{MN}^{(10)}$, a six-form field $\tilde{B}_{M_1...M_6}^{(10)}$, and the dilaton field $\tilde{\Phi}^{(10)}$. (We are ignoring the ten dimensional gauge fields and the fermionic fields in the analysis of this section.) The action is given by[13],

$$
S = \frac{1}{32\pi} \int d^{10}z \sqrt{-\tilde{G}^{(10)}} \, e^{\tilde{\Phi}^{(10)}/3} \Big(\tilde{R}^{(10)}
$$
$$
- \frac{1}{2 \times 7!} \tilde{G}^{(10)M_1 N_1} \cdots \tilde{G}^{(10)M_7 N_7} \tilde{H}_{M_1...M_7}^{(10)} \tilde{H}_{N_1...N_7}^{(10)} \Big), \tag{39}
$$

where

$$
\tilde{H}_{M_1...M_7}^{(10)} = \partial_{[M_1} \tilde{B}_{M_2...M_7]}^{(10)}. \tag{40}
$$

The equations of motion and the Bianchi identities derived from this action can be shown to be identical to those derived from the action (1) provided we make the identifications

$$
\tilde{\Phi}^{(10)} = \Phi^{(10)}, \qquad \tilde{G}_{MN}^{(10)} = e^{-\Phi^{(10)}/3} G_{MN}^{(10)},
$$
$$
\sqrt{-\tilde{G}^{(10)}} \, e^{\tilde{\Phi}^{(10)}/3} \tilde{G}^{(10)M_1 N_1} \cdots \tilde{G}^{(10)M_7 N_7} \tilde{H}_{N_1...N_7}^{(10)} = -\frac{1}{3!} \epsilon^{M_1...M_{10}} H_{M_8 M_9 M_{10}}. \tag{41}
$$

Note that the Bianchi identity for the field strength $H_{MNP}^{(10)}$ in ten dimensions

$$
\epsilon^{M_1...M_{10}} \partial_{M_7} H_{M_8 M_9 M_{10}}^{(10)} = 0, \tag{42}
$$

now corresponds to the equation of motion for the six form field $\tilde{B}_{M_1...M_6}^{(10)}$. Similarly, the Bianchi identity for the field strength $\tilde{H}_{M_1...M_7}$ in ten dimensions

$$
\epsilon^{M_1...M_{10}} \partial_{M_3} \tilde{H}_{M_4...M_{10}}^{(10)} = 0, \tag{43}
$$

corresponds to the equation of motion of the anti-symmetric tensor field $B_{MN}^{(10)}$.

In order to carry out the dimensional reduction of this theory from ten to four dimensions, it is convenient to introduce the 'four dimensional fields' λ, $\mathcal{C}_\mu^m$, $\mathcal{D}_\mu^m$, $\hat{G}_{mn}$, $\mathcal{B}_{\mu\nu}^{mn}$, $\mathcal{E}_{\mu\nu\rho}^{mnp}$ and $g_{\mu\nu}$ through the relations[39]:

$$
\hat{G}_{mn} = e^{\tilde{\Phi}^{(10)}/3} \tilde{G}_{m+3,n+3}^{(10)}, \qquad \lambda_1 = \frac{1}{6!} \tilde{B}_{m_1+3,...m_6+3}^{(10)} \epsilon^{m_1...m_6}, \qquad \lambda_2 = \sqrt{\det \hat{G}} \, e^{-\tilde{\Phi}^{(10)}},
$$

$$
\mathcal{C}_\mu^m = e^{\tilde{\Phi}^{(10)}/3} \hat{G}^{mn} \tilde{G}_{(n+3)\mu}^{(10)}, \qquad \mathcal{D}_\mu^{m_1} = \frac{1}{5!} \epsilon^{m_1...m_6} \tilde{B}_{\mu(m_2+3)...(m_6+3)}^{(10)} - \lambda_1 \mathcal{C}_\mu^{m_1}
$$

$$
\mathcal{B}_{\mu\nu}^{m_1 m_2} = \frac{1}{4!} \epsilon^{m_1...m_6} \tilde{B}_{\mu\nu(m_3+3)...(m_6+3)}^{(10)}
$$
$$
- [(\lambda_1 \mathcal{C}_\mu^{m_1} \mathcal{C}_\nu^{m_2} + \frac{1}{2} \mathcal{D}_\mu^{m_1} \mathcal{C}_\nu^{m_2} - \frac{1}{2} \mathcal{D}_\nu^{m_1} \mathcal{C}_\mu^{m_2}) - (m_1 \leftrightarrow m_2)]
$$

$$
\mathcal{E}_{\mu\nu\rho}^{m_1 m_2 m_3} = \frac{1}{3!} \epsilon^{m_1...m_6} \tilde{B}_{\mu\nu\rho(m_4+3)...(m_6+3)}^{(10)},
$$
$$
g_{\mu\nu} = (\lambda_2)^{2/3} (\det \hat{G})^{\frac{1}{2}} (\tilde{G}_{\mu\nu}^{(10)} - \tilde{G}_{(m+3)(n+3)}^{(10)} \mathcal{C}_\mu^m \mathcal{C}_\nu^n), \tag{44}
$$

and the corresponding field strengths,

$$F_{\mu\nu}^{(C)m} = \partial_\mu \mathcal{C}_\nu^m - \partial_\nu \mathcal{C}_\mu^m, \qquad F_{\mu\nu}^{(D)m} = \partial_\mu \mathcal{D}_\nu^m - \partial_\nu \mathcal{D}_\mu^m$$

$$K_{\mu\nu\rho}^{mn} = \left([\partial_\mu \mathcal{B}_{\nu\rho}^{mn} - \frac{1}{2}\{(\mathcal{C}_\rho^n F_{\mu\nu}^{(D)m} + \mathcal{D}_\rho^n F_{\mu\nu}^{(C)m}) - (m \leftrightarrow n)\}] \right.$$
$$\left. + \text{ cyclic permutations of } \mu,\, \nu,\, \rho \right)$$

$$\mathcal{K}_{\mu\nu\rho\sigma}^{mnp} = [\partial_\mu \mathcal{E}_{\nu\rho\sigma}^{mnp} + (-1)^P \cdot \text{ cyclic permutations of } \mu,\, \nu,\, \rho,\, \sigma]$$
$$- [(\mathcal{C}_\sigma^p K_{\mu\nu\rho}^{mn} + \text{ cyclic permutations of } m,\, n,\, p)$$
$$+ (-1)^P \cdot \text{ cyclic permutations of } \mu,\, \nu,\, \rho,\, \sigma]$$
$$- [\{\mathcal{C}_\sigma^p \mathcal{C}_\rho^n (F_{\mu\nu}^{(D)m} + \lambda_1 F_{\mu\nu}^{(C)m}) + (-1)^P \cdot \text{ all permutations of } m,\, n,\, p\}$$
$$+ (-1)^P \cdot \text{ inequivalent permutations of } \mu,\, \nu,\, \rho,\, \sigma]$$
$$- [(\mathcal{C}_\sigma^p \mathcal{C}_\rho^n \mathcal{C}_\nu^m \partial_\mu \lambda_1 + (-1)^P \cdot \text{ all permutations of } m,\, n,\, p)$$
$$+ (-1)^P \cdot \text{ cyclic permutations of } \mu,\, \nu,\, \rho,\, \sigma]. \tag{45}$$

Using the relationship between the fields in the two formulations of the ten dimensional $N = 1$ supergravity theory given in Eq.(41), and the definition of the fields λ_1, λ_2, $\widehat{G}_{mn}$ and $g_{\mu\nu}$ in the two formulations, one can easily verify that the two sets of definitions lead to identical λ, $\widehat{G}_{mn}$ and $g_{\mu\nu}$.

The action (39), expressed in terms of these 'four dimensional fields', is given by,

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-g} \Big[R - \frac{1}{2(\lambda_2)^2} g^{\mu\nu} \partial_\mu \bar{\lambda} \partial_\nu \lambda + \frac{1}{4} g^{\mu\nu} \text{Tr}(\partial_\mu \widehat{G} \partial_\nu \widehat{G}^{-1})$$
$$- \frac{1}{4} \widehat{G}_{mn} g^{\mu\rho} g^{\nu\sigma} \left(F_{\mu\nu}^{(C)m} \quad -F_{\mu\nu}^{(D)m} \right) \mathcal{L}^T \mathcal{M} \mathcal{L} \begin{pmatrix} F_{\rho\sigma}^{(C)n} \\ -F_{\rho\sigma}^{(D)n} \end{pmatrix}$$
$$- \frac{1}{2 \times 2! \times 3!} \widehat{G}_{m_1 n_1} \widehat{G}_{m_2 n_2} g^{\mu_1 \nu_1} \cdots g^{\mu_3 \nu_3} K_{\mu_1 \mu_2 \mu_3}^{m_1 m_2} K_{\nu_1 \nu_2 \nu_3}^{n_1 n_2}$$
$$- \frac{\lambda_2}{2 \times 3! \times 4!} \widehat{G}_{m_1 n_1} \cdots \widehat{G}_{m_3 n_3} g^{\mu_1 \nu_1} \cdots g^{\mu_4 \nu_4} \mathcal{K}_{\mu_1 \cdots \mu_4}^{m_1 \cdots m_3} \mathcal{K}_{\nu_1 \cdots \nu_4}^{n_1 \cdots n_3} \Big], \tag{46}$$

where $\mathcal{M}$ has been defined in Eq.(22), and Tr denotes trace over the indices m, n $(1 \le m, n \le 6)$. The equation of motion for $\mathcal{E}_{\mu_1 \mu_2 \mu_3}^{m_1 m_2 m_3}$ gives

$$\partial_{\nu_1} [\lambda_2 \sqrt{-g} \, \widehat{G}_{m_1 n_1} \cdots \widehat{G}_{m_3 n_3} g^{\mu_1 \nu_1} \cdots g^{\mu_4 \nu_4} \mathcal{K}_{\nu_1 \cdots \nu_4}^{n_1 \cdots n_3}] = 0. \tag{47}$$

Since $\mathcal{K}_{\nu_1 \cdots \nu_4}^{n_1 \cdots n_3}$ is antisymmetric in $\nu_1, \ldots \nu_4$, we may write

$$\lambda_2 \sqrt{-g} \, \widehat{G}_{m_1 n_1} \cdots \widehat{G}_{m_3 n_3} g^{\mu_1 \nu_1} \cdots g^{\mu_4 \nu_4} \mathcal{K}_{\nu_1 \cdots \nu_4}^{n_1 \cdots n_3} = \epsilon^{\mu_1 \cdots \mu_4} H_{m_1 m_2 m_3} \tag{48}$$

for some H_{mnp}. The equation (47) then takes the form:

$$\partial_\nu H_{m_1 m_2 m_3} = 0, \tag{49}$$

showing that H_{mnp} is a constant. Comparison with the original formulation of the theory shows that H_{mnp} are proportional to the internal components of the three form field strength $H_{MNP}^{(10)}$.

During the dimensional reduction of the original ten dimensional $N = 1$ supergravity theory, we had set these constants to zero. Hence, if we want to recover the same theory, we must set them to zero here too. This gives

$$\mathcal{K}_{\mu_1 \cdots \mu_4}^{m_1 \cdots m_3} = 0. \tag{50}$$

The action (46) now reduces to

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-g} \Big[R - \frac{1}{4} g^{\mu\nu} tr(\partial_\mu \mathcal{M} \mathcal{L} \partial_\nu \mathcal{M} \mathcal{L}) + \frac{1}{4} g^{\mu\nu} \text{Tr}(\partial_\mu \widehat{G} \partial_\nu \widehat{G}^{-1})$$
$$- \frac{1}{4} \widehat{G}_{mn} g^{\mu\rho} g^{\nu\sigma} \left(F_{\mu\nu}^{(C)m} \quad -F_{\mu\nu}^{(D)m} \right) \mathcal{L}^T \mathcal{M} \mathcal{L} \begin{pmatrix} F_{\rho\sigma}^{(C)n} \\ -F_{\rho\sigma}^{(D)n} \end{pmatrix}$$
$$- \frac{1}{2 \times 2! \times 3!} \widehat{G}_{m_1 n_1} \widehat{G}_{m_2 n_2} g^{\mu_1 \nu_1} \cdots g^{\mu_3 \nu_3} K_{\mu_1 \mu_2 \mu_3}^{m_1 m_2} K_{\nu_1 \nu_2 \nu_3}^{n_1 n_2} \Big], \tag{51}$$

and has manifest SL(2,R) invariance

$$\mathcal{M} \to \omega \mathcal{M} \omega^T, \qquad \begin{pmatrix} \mathcal{C}_\mu^m \\ -\mathcal{D}_\mu^m \end{pmatrix} \to \omega \begin{pmatrix} \mathcal{C}_\mu^m \\ -\mathcal{D}_\mu^m \end{pmatrix}, \tag{52}$$

with all other fields remaining invariant under the SL(2,R) transformation. Although this action is not identical to the manifestly SL(2,R) invariant action (38), the equations of motion derived from these two actions can be seen to be identical, provided we make the identification

$$\sqrt{-g} \, \widehat{G}_{m_1 n_1} \widehat{G}_{m_2 n_2} g^{\mu_1 \nu_1} \cdots g^{\mu_3 \nu_3} K_{\mu_1 \mu_2 \mu_3}^{m_1 m_2} = -\epsilon^{\nu_1 \nu_2 \nu_3 \sigma} \partial_\sigma \widehat{B}_{n_1 n_2},$$
$$\mathcal{C}_\mu^m = 2\check{A}_\mu^{(m,1)}, \qquad \mathcal{D}_\mu^m = -2\check{A}_\mu^{(m,2)}. \tag{53}$$

Under this identification, the equations of motion of the scalar field $\widehat{B}_{mn}$ becomes identical to the Bianchi identity of the field strength $K_{\mu\nu\rho}^{mn}$, and the bianchi identity of $\partial_\mu B_{mn}$ becomes identical to the equations of motion of the field $\mathcal{B}_{\mu\nu}^{mn}$.

This shows that the SL(2,R) symmetry arises naturally in the four dimensional theory obtained from the dimensional reduction of the dual formulation of the $N = 1$ supergravity theory in ten dimensions, just as the O(6,6) or O(6,22) symmetry arises naturally in the dimensional reduction of the usual $N = 1$ supergravity theory from ten to four dimensions. Yet, the O(6,22) symmetry is more fundamental from the point of view of string theory, since the fields $G_{MN}^{(10)}$, $B_{MN}^{(10)}$, which arise in the *usual* formulation of the N=1 supergravity theory, couple naturally to the string. On the other hand, it is known[14] that the fields $\widetilde{G}_{MN}^{(10)}$ and $\widetilde{B}_{M_1 \cdots M_6}^{(10)}$ couple naturally to the five-brane, which has been conjectured to be equivalent to the theory of strings[12, 48, 13, 14]. Hence one would expect that the SL(2,R) symmetry will play a more fundamental role in the theory of five-branes. In §7 we shall show that there is a natural interpretation of the SL(2,Z) subgroup of SL(2,R) as the group of target space duality transformations in the five-brane theory.

2.5   Inclusion of the Fermions

So far we have concentrated on the bosonic part of the action. However, in order to establish the SL(2,Z) invariance of the full string theory, it is necessary to show that the low energy

effective field theory is SL(2,Z) invariant even after inclusion of the massless fermionic fields in the theory. For this we need to carry out the dimensional reduction of the full action of the ten dimensional N=1 supergravity theory, and show the SL(2,Z) invariance of the equations of motion derived from this action. We shall not do this here. However, we shall give an indirect argument showing that the equations of motion do remain SL(2,R) invariant after inclusion of the fermionic fields. This is done by comparing the dimensionally reduced theory to the N=4 Poincare supergravity theory coupled to abelian gauge field multiplets[10]. It can be seen that the bosonic part of the two theories are identical if we make the identification[44, 16]

$$M = UOO^T U^{-1}, \qquad \frac{i}{\lambda} = \frac{\phi_1 - \phi_2}{\phi_1 + \phi_2}, \qquad (54)$$

and a redefinition of the gauge fields $F^{(a)} \to U_{ab} F^{(b)}$. Here U is the matrix that diagonalizes L to $(I_6, -I_{22})$, and O, ϕ_1, ϕ_2 are fields defined in Ref.[10]. Since the bosonic parts of the two theories are identical, and furthermore, both the theories have local $N = 4$ supersymmetry, we have a strong evidence that the two theories are indeed the same. We shall proceed with the assumption that this is the case.

It was shown in Ref.[10] that the gauge field equations of motion in the Poincare supergravity theory are invariant under SL(2,R) transformation, even after including the fermionic fields. There is also a general argument due to Gaillard and Zumino[18], that if the gauge field equations in a theory have an SL(2,R) symmetry, then all other equations of motion also have this symmetry. From this we can conclude that the full set of equations of motion in the N=4 Poincare supergravity theory, and hence also in the dimensionally reduced low energy heterotic string theory, are invariant under SL(2,R) transformation.

3 Symmetry of the Charge Spectrum

In this section we shall analyze the possibility that part of the SL(2,R) symmetry can be realised as an exact symmetry of the theory. Thus the first question that we need to answer is, which part of SL(2,R) has a chance of being a symmetry of the full quantum theory. We shall see in §3.1 that the SL(2,R) symmetry group is necessarily broken down to SL(2,Z) due to the instanton corrections. Hence the question is whether this SL(2,Z) group of transformations can be a symmetry group of the full quantum string theory. As pointed out in the introduction, we shall refer to this group of SL(2,Z) transformations as the S-duality transformation, and the target space duality group O(6,22;Z) as the T-duality transformation.

We have already stated that since the S-duality transformation acts non-trivially on the coupling constant, it is not a symmetry of the theory order by order in the string perturbation theory, but could only be a symmetry of the full string theory. Thus, in order to test this symmetry we must look for quantities which can be calculated in the full string theory and see if those quantities are invariant under this symmetry transformation. We have pointed out in the introduction that there are four sets of such quantities. Of these, the low energy effective action has already been

shown to possess the SL(2,Z) invariance. In §3.2, we shall study the SL(2,Z) transformation properties of the allowed spectrum of electric and magnetic charges in the theory and show that this spectrum is invariant under the S-duality transformation.

3.1 Breaking of SL(2,R) to SL(2,Z)

In §2 we wrote down the effective action of the four dimensional theory in various different forms. From Eq.(21) we see that the field λ_1 couples to the topological density $F^{(a)}_{\mu\nu} L_{ab} \tilde{F}^{(b)\mu\nu}$, and hence the part of the SL(2,R) group that corresponds to a translation symmetry of λ_1 must be broken down to a discrete group of translations by instanton effects. Actually we have to be somewhat careful, since so far we have introduced only abelian gauge fields in the theory which do not have any instantons. However, we should keep in mind that the full string theory contains non-abelian gauge fields as well. The non-abelian group is spontaneously broken at a generic point of the moduli space, but nevertheless gives rise to instanton corrections to the theory. (At special points in the moduli space, e.g., where some of the $\hat{A}^I_m$ vanish, part of the non-abelian symmetry group is restored.) Thus to find how the instanton effects modify the translation symmetry of λ_1, we must first study the embedding of the abelian gauge group in the non-abelian group, and then compute the (quantized) topological charge that couples to the zero mode of the field λ_1.

To take a concrete case, note that the gauge field $A^{(28)}_\mu$ can be regarded as the gauge field associated with one of the three generators of an SU(2) group, such that the unbroken phase of this SU(2) group is restored when $\hat{A}^{16}_m$ vanishes for all m. Let $\mathcal{A}^i_\mu$ ($1 \le i \le 3$) denote these SU(2) gauge fields. Using the scaling freedom $\lambda \to c\lambda$, $F^{(a)}_{\mu\nu} \to \frac{1}{\sqrt{c}} F^{(a)}_{\mu\nu}$, under which the action remains invariant, we can always ensure that the field $A^{(28)}_\mu$ is equal to $\sqrt{2}\mathcal{A}^3_\mu$. Let us assume that this has been done. In that case, the $-\frac{1}{32\pi} \int d^4x \sqrt{-g}\, \lambda_1\, \tilde{F}^{(28)}_{\mu\nu} F^{(28)\mu\nu}$ term in the action can be regarded as a part of the term

$$-\frac{1}{16\pi} \int d^4x \sqrt{-g}\, \lambda_1 \sum_{i=1}^{3} \tilde{\mathcal{F}}^i_{\mu\nu} \mathcal{F}^{i\mu\nu} \qquad (55)$$

where $\mathcal{F}^i_{\mu\nu}$ are the components of the SU(2) field strength,

$$\mathcal{F}^i_{\mu\nu} = \partial_\mu \mathcal{A}^i_\nu - \partial_\nu \mathcal{A}^i_\mu + \varepsilon^{ijk} \mathcal{A}^j_\mu \mathcal{A}^k_\nu. \qquad (56)$$

Now, it is well known that for a single SU(2) instanton,

$$\frac{1}{16\pi} \int d^4x \sqrt{-g} \sum_{i=1}^{3} \tilde{F}^i_{\mu\nu} F^{i\mu\nu} = 2\pi. \qquad (57)$$

As a result, e^{iS} remains invariant under $\lambda_1 \to \lambda_1 + $ integer. Thus the presence of this instanton in the theory breaks the continuous translation symmetry of λ to $\lambda \to \lambda + 1$[47, 43].

One can verify that the $\lambda \to \lambda+1$ symmetry survives the effect of all other non-abelian instantons in the theory. Furthermore, one can show that the subgroup of SL(2,R), generated by the transformation $\lambda \to \lambda+1$, and the strong-weak coupling duality transformation $\lambda \to -1/\lambda$, is SL(2,Z). This corresponds to the subgroup of the SL(2,R) group of transformations generated by matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d integers and satisfying $ad - bc = 1$. The effect of these transformations on the various fields in the theory is the same as that given in Eq.(18).

In the rest of this section we shall find out whether SL(2,Z) can be an exact symmetry of the charge spectrum of the full string theory.

3.2 SL(2,Z) Invariance of the Electric and Magnetic Charge Spectrum

So far in our analysis we have only analyzed the effective action involving the neutral massless fields in the theory. The full string theory, of course, also contains charged fields (of which the non-abelian gauge fields discussed in the previous subsection are examples). Although at a generic point in the moduli space of compactification these fields are all massive, and hence decouple from the low energy effective field theory, we must show that the spectrum and the interaction of these charged fields remain invariant under the SL(2,Z) transformation, in order to establish the SL(2,Z) invariance of the full string theory.

We start by analyzing the charge spectrum of the states in string theory[43, 28]. In the presence of charged fields, the fields $A_\mu^{(a)}$ acquire new coupling in the action of the form

$$-\frac{1}{2}\int d^4x \sqrt{-g} A_\mu^{(a)}(x) J^{(a)\mu}(x) \tag{58}$$

where $J_\mu^{(a)}$ is the electric current associated with the charged fields. (The normalization factor of $-\frac{1}{2}$ is purely a matter of convention.) Let $e^{(a)}$ be the conserved charge associated with this current,

$$e^{(a)} = \int \sqrt{-g} J^{(a)0} d^3x. \tag{59}$$

We also define the quantity $Q_{el}^{(a)}$ through the relation

$$F_{0r}^{(a)} \simeq \frac{Q_{el}^{(a)}}{r^2} \qquad \text{for large } r. \tag{60}$$

Using the equations of motion derived from the sum of the actions (21) and (58), we see that

$$Q_{el}^{(a)} = \frac{1}{\lambda_2^{(0)}} M_{ab}^{(0)} e^{(b)}, \tag{61}$$

where the superscript (0) denotes the asymptotic values of the various fields.

From the analysis of Narain[30], we know that the allowed set of electric charge vectors $\{e^{(a)}\}$ are proportional to vectors $\{\alpha^a\}$ belonging to an even, self dual, Lorenzian lattice Λ with metric L defined in Eq.(5).[4] The constant of proportionality is fixed as follows. On the one hand, from the analysis of Ref.[30] we know that the states associated with the quanta of SU(2) gauge fields $\mathcal{A}_\mu^\pm$ have electric charge vectors of (length)$^2 = -2$. On the other hand, knowing that the Lagrangian of the SU(2) Yang-Mills theory is proportional to $\mathcal{F}_{\mu\nu}^i \mathcal{F}^{i\mu\nu}$, and using the relation $A_\mu^{(28)} = \sqrt{2}\mathcal{A}_\mu^3$, and the definition of $e^{(a)}$ given in Eqs.(58) and (59), we can calculate $e^{(a)}$ for the quanta of states created by the $\mathcal{A}_\mu^\pm$ fields out of the vacuum. The answer is $e^{(a)} = \pm\sqrt{2}\delta_{a,28}$. This shows that the constant of proportionality between $e^{(a)}$ and α^a is unity, i.e.

$$e^{(a)} = \alpha^a. \tag{62}$$

String theory also contains magnetically charged soliton states. The magnetic charge of such a state is characterized by a vector $Q_{mag}^{(a)}$ defined through the equation

$$\tilde{F}_{0r}^{(a)} \simeq \frac{Q_{mag}^{(a)}}{r^2} \qquad \text{for large } r. \tag{63}$$

The electric and magnetic charges of a generic state are characterized by the pair of 28 dimensional vectors $(Q_{el}^{(a)}, Q_{mag}^{(a)})$. Since elementary string states do not carry any magnetic charge, we see that they are characterized as

$$(Q_{el}^{(a)}, Q_{mag}^{(a)}) = (\frac{1}{\lambda_2^{(0)}} M_{ab}^{(0)}\alpha^b, 0). \tag{64}$$

Let us now consider a generic state carrying both electric and magnetic charges. By analyzing the system containing a pair of particles, one corresponding to an elementary string excitation carrying charges given in Eq.(64), and the other, a generic solitonic state carrying charges $(Q_{el}^{(a)}, Q_{mag}^{(a)})$, and taking into account the non-standard form of the gauge field kinetic term given in (21), we get the following form of the Dirac-Schwinger-Zwanziger[11] quantization rule,

$$\lambda_2^{(0)} Q_{mag}^{(a)}(LM^{(0)}L)_{ab}\frac{1}{\lambda_2^{(0)}}M_{bc}^{(0)}\alpha^c = \text{integer}. \tag{65}$$

The most general solution of this equation is

$$Q_{mag}^{(a)} = L_{ab}\beta^b, \qquad \vec{\beta} \in \Lambda, \tag{66}$$

where Λ is the self-dual Lorenzian lattice introduced before.

We now ask the question, 'what are the allowed values of $Q_{el}^{(a)}$ for a given $Q_{mag}^{(a)}$?' Naively one might think that $Q_{el}^{(a)}$ is given by Eq.(64) irrespective of the value of $Q_{mag}^{(a)}$, but this is not the

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[4] We can, for definiteness, take Λ to be the direct product of the root lattice of $E_8 \times E_8$ and the 12 dimensional lattice of integers.

case. From the analysis of Ref.[50] we know that the quantization laws for electric charge get modified in the presence of a magnetic charge. For standard normalization of the gauge field kinetic term, the shift is proportional to the magnetic charge, and also the θ angle, which, in this case, is equal to $2\pi\lambda_1^{(0)}$. Taking into account the non-standard normalization of the kinetic term, and calculating the overall normalization factor using the method of Ref.[50] (see also Ref.[6]), we get the following spectrum of electric and magnetic charges,

$$(Q_{el}^{(a)}, Q_{mag}^{(a)}) = (\frac{1}{\lambda_2^{(0)}}M_{ab}^{(0)}(\alpha^b + \lambda_1^{(0)}\beta^b), L_{ab}\beta^b). \tag{67}$$

We now want to test if this spectrum is invariant under the SL(2,Z) transformation given in Eq.(18) with a, b, c, d integers. To test the SL(2,Z) invariance of this spectrum, we need to calculate the transformation laws of $Q_{el}^{(a)}$ and $Q_{mag}^{(a)}$. This is straightforward, since both $Q_{el}^{(a)}$ and $Q_{mag}^{(a)}$ are given in terms of the asymptotic values of the field strength $F_{\mu\nu}^{(a)}$, whose transformation laws are already given in Eq.(18). We get,

$$\begin{aligned} Q_{el}^{(a)} \to Q_{el}^{(a)\prime} &= (c\lambda_1^{(0)} + d)Q_{el}^{(a)} + c\lambda_2^{(0)}(M^{(0)}L)_{ab}Q_{mag}^{(b)} \\ &= \frac{1}{\lambda_2^{\prime(0)}}M_{ab}^{(0)}(\alpha^{\prime b} + \lambda_1^{\prime(0)}\beta^{\prime b}), \\ Q_{mag}^{(a)} \to Q_{mag}^{(a)\prime} &= (c\lambda_1^{(0)} + d)Q_{mag}^{(a)} - c\lambda_2^{(0)}(M^{(0)}L)_{ab}Q_{el}^{(b)} \\ &= \frac{1}{\lambda_2^{\prime(0)}}L_{ab}\beta^{\prime b}, \end{aligned} \tag{68}$$

where,

$$\begin{pmatrix} \vec{\alpha}^\prime \\ \vec{\beta}^\prime \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} = \mathcal{L}\omega\mathcal{L}^T \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}, \tag{69}$$

and ω and $\mathcal{L}$ have been defined in Eqs.(29) and (22) respectively. Since a, b, c, d are all integers, both $\vec{\alpha}^\prime$ and $\vec{\beta}^\prime$ belong to the lattice Λ. This, in turn, shows that the $(Q_{el}^{(a)\prime}, Q_{mag}^{(a)\prime})$, when expressed in terms of the transformed variables, have exactly the same form as $(Q_{el}^{(a)}, Q_{mag}^{(a)})$ before the transformation. Hence the allowed spectrum of electric and magnetic charges in the theory is indeed invariant under the SL(2,Z) transformation. The crucial ingredient in this proof is that $\vec{\alpha}$ and $\vec{\beta}$ belong to the same lattice Λ, which, in turn, follows from the fact that the lattice Λ is self-dual.

Note that the charge spectrum that we have found refers to the charge spectrum of all states in the theory, and not just the single particle states. Whereas invariance of this charge spectrum under SL(2,Z) transformation is a necessary condition for the SL(2,Z) invariance of the theory, it is, by no means, sufficient. In order to establish the SL(2,Z) invariance of the spectrum, we need to calculate the degeneracy $N(\vec{\alpha}, \vec{\beta}, m)$ of single particle states of mass m, characterized by charge vectors $(\vec{\alpha}, \vec{\beta})$, and show that it is invariant under the SL(2,Z) transformation. In particular, given any elementary string excitation, we must be able to identify its SL(2,Z) transforms with specific monopole and dyon states in the theory, carrying the same mass as the elementary string state. This will be the subject of our analysis in §4 and §6.

Before we conclude this subsection, we note that under the O(6,22) transformation given in Eq.(8),

$$Q_{el}^{(a)} \to \Omega_{ab}Q_{el}^{(b)}, \qquad Q_{mag}^{(a)} \to \Omega_{ab}Q_{mag}^{(b)}, \qquad M^{(0)} \to \Omega M^{(0)}\Omega^T \tag{70}$$

which gives

$$\begin{pmatrix} \alpha^a \\ \beta^a \end{pmatrix} \to \begin{pmatrix} (L\Omega L)_{ab}\alpha^b \\ (L\Omega L)_{ab}\beta^b \end{pmatrix}. \tag{71}$$

Thus the charge spectrum is invariant under the O(6,22) transformation Ω if $L\Omega L$ preserves the lattice Λ. It can be shown that the group of such matrices form an O(6,22;Z) subgroup of O(6,22)[23]. This establishes O(6,22;Z) invariance of the charge spectrum.

4 Symmetry of the Mass Spectrum

If the string theory under consideration really has an SL(2,Z) symmetry, then not only the allowed spectrum of electric and magnetic charges, but also the full mass spectrum of the theory must be invariant under the SL(2,Z) transformation. However, unlike the spectrum of electric and magnetic charges, the mass spectrum of the theory does receive non-trivial quantum corrections, and hence we cannot test the SL(2,Z) invariance of the full mass spectrum with the help of the perturbative techniques available to us today. However, there is a special class of states in the theory whose masses do not receive any quantum corrections. These are the states that belong to the 16 dimensional representation of the $N = 4$ super-algebra, are annihilated by half of the sixteen supersymmetry generators of the theory, and satisfy a definite relation between mass and charge, known as the Bogomol'nyi bound[49]. In fact, 16-component supermultiplets exist only for states with this special relation between mass and charge. Since quantum corrections cannot change the representation to which a given supermultiplet belongs, it cannot change the mass-charge relation of the corresponding states either. As a result, the masses of these states do not receive any quantum corrections[49].

Thus a consistency test of the postulate of SL(2,Z) invariance of the theory would be to check whether the mass spectrum of the states saturating the Bogomol'nyi bound remains invariant under the SL(2,Z) transformation. The relationship between mass and charge for such states can be calculated using standard techniques[21]. It turns out that in this case, the relevant charges that determine the mass are the ones that also determine the asymptotic value of the field[25]

$$T_{(m)\mu\nu} \equiv \partial_\mu G_{(m+3)\nu}^{(10)} - \partial_\nu G_{(m+3)\mu}^{(10)} - H_{(m+3)\mu\nu}^{(10)}. \tag{72}$$

Let $\tilde{T}_{(m)}^{\mu\nu} \equiv \frac{1}{2}(\sqrt{-g})^{-1}\epsilon^{\mu\nu\rho\sigma}T_{(m)\rho\sigma}$, and let us stick to the convention that all indices are raised and lowered with the canonical metric $g_{\mu\nu}$. We now define charges Q_m and P_m through the asymptotic values of the fields $T_{(m)0r}$ and $\tilde{T}_{(m)0r}$:

$$T_{(m)0r} \simeq \frac{Q_m}{r^2}, \qquad \tilde{T}_{(m)0r} \simeq \frac{P_m}{r^2}. \tag{73}$$

In the normalization convention that we have been using, the mass m of a particle saturating the Bogomol'nyi bound is determined by the following formula[25]

$$m^2 = \frac{1}{64}\lambda_2^{(0)}(\widehat{G}^{(0)mn}Q_m Q_n + \widehat{G}^{(0)mn}P_m P_n), \tag{74}$$

where the matrix $\widehat{G}_{mn}$ and its inverse $\widehat{G}^{mn}$ have been defined in Eq.(3), and the superscript (0) denotes the asymptotic value as usual. Using Eqs.(3), (60) and (63) we can express Q_m and P_m, and hence m^2, in terms of $Q_{el}^{(a)}$ and $Q_{mag}^{(a)}$. The final answer is[45]

$$m^2 = \frac{\lambda_2^{(0)}}{16}\left(Q_{el}^{(a)}(LM^{(0)}L + L)_{ab}Q_{el}^{(b)} + Q_{mag}^{(a)}(LM^{(0)}L + L)_{ab}Q_{mag}^{(b)}\right), \tag{75}$$

which, with the help of Eq.(67) may be written as[40]

$$m^2 = \frac{1}{16}(\,\alpha^a \quad \beta^a\,)\mathcal{M}^{(0)}(M^{(0)} + L)_{ab}\begin{pmatrix} \alpha^b \\ \beta^b \end{pmatrix}. \tag{76}$$

The right hand side of this expression is manifestly invariant under the O(6,22;Z) transformation given in (8) and (71), and the SL(2,Z) transformations given in (27) and (69)[45, 40, 36].

This shows that two states saturating the Bogomol'nyi bound have the same mass if their electric and magnetic charge quantum numbers, and the asymptotic values of moduli fields M and λ, are related by an SL(2,Z) transformation. This does not completely establish the SL(2,Z) invariance of the mass spectrum for such states, but shows that if the degeneracy $N_{16}(\vec{\alpha},\vec{\beta})$ of 16-component supermultiplets, saturating the Bogomol'nyi bound and carrying charge vectors $\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}$, is SL(2,Z) invariant, then the mass spectrum of such states will also automatically be SL(2,Z) invariant. We shall analyze this question in §6. In particular, we shall identify the spectrum of elementary string excitations saturating the Bogomol'nyi bound, and show that for at least a subclass of these states, the dual magnetically charged states are in one to one correspondence to the elementary string excitations.

The result of this and the previous section indicates that it is more natural to combine the two vectors $\vec{\alpha}$ and $\vec{\beta}$ into a single 56 component vector $\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}$. This vector belongs to a 56 dimensional lattice $\Gamma = \Lambda \otimes \Lambda$.

5 Symmetry of the Yukawa Couplings

If SL(2,Z) is a symmetry of the theory, then all correlation functions of the theory must be invariant under the SL(2,Z) transformation. In particular, various Yukawa couplings, which represent the three point coupling between a zero momentum scalar and two fermions (and are related to various other couplings in the theory due to the $N = 4$ supersymmetry) must also

be invariant under the SL(2,Z) transformation. However, as in the case of mass spectrum, this symmetry can be checked only for those sets of Yukawa couplings which do not receive any quantum corrections, i.e., for which the tree level answer is the exact answer. Fortunately, such Yukawa couplings do exist in the theory under consideration, and, as we shall see, they are indeed invariant under the SL(2,Z) transformation. Analysis of these Yukawa couplings will be the subject of discussion of this section.

The Yukawa couplings under consideration are those between the massless scalars in the theory, corresponding to the fields M and λ (or, equivalently, $\mathcal{M}$), and massive charged fermions saturating the Bogomol'nyi bound. The reason that these Yukawa couplings are given by their tree level answer is that they can be related to the mass spectrum of the fermions, which is given by the tree level answer. This also indicates that these Yukawa coupling must be invariant under the SL(2,Z) (and O(6,22;Z)) transformation, since the fermion mass spectrum has this invariance. We shall now see in some detail how this happens.

Let $M^{(0)}$ and $\mathcal{M}^{(0)}$ be the vacuum expectation values of the fields M and $\mathcal{M}$ respectively. We now introduce fluctuations Φ and ϕ of these fields through the relations

$$M = M^{(0)} + \Phi, \qquad \mathcal{M} = \mathcal{M}^{(0)} + \phi, \tag{77}$$

where Φ and ϕ are 28×28 and 2×2 matrices respectively, satisfying,

$$\begin{aligned} \Phi^T &= \Phi, & \Phi LM^{(0)} + M^{(0)}L\Phi + \Phi L\Phi &= 0 \\ \phi^T &= \phi, & \phi\mathcal{L}\mathcal{M}^{(0)} + \mathcal{M}^{(0)}\mathcal{L}\phi + \phi\mathcal{L}\phi &= 0. \end{aligned} \tag{78}$$

The O(6,22;Z) and SL(2,Z) transformation properties of the fields Φ and ϕ are given by

$$\Phi \to \Omega\Phi\Omega^T \qquad \phi \to \omega\phi\omega^T, \tag{79}$$

respectively. The quanta of the fields Φ and ϕ are characterized by 'polarization tensors' E_{ab} and $e_{\alpha\beta}$, which are symmetric 28×28 and 2×2 matrices respectively, satisfying,

$$ELM^{(0)} + M^{(0)}LE = 0, \qquad e\mathcal{L}\mathcal{M}^{(0)} + \mathcal{M}^{(0)}\mathcal{L}e = 0. \tag{80}$$

The Yukawa couplings between the Φ or ϕ quanta, and the fermion fields saturating the Bogomol'nyi bound, may now be calculated by operating $E_{ab}\frac{\delta}{\delta M_{ab}^{(0)}}$ and $e_{\alpha\beta}\frac{\delta}{\delta\mathcal{M}_{\alpha\beta}^{(0)}}$ on the fermion mass matrix. This gives the following Yukawa coupling C and $\tilde{C}$ between the fermions characterized by the electric and magnetic charge vectors $\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}$ and $\begin{pmatrix} \vec{\gamma} \\ \vec{\delta} \end{pmatrix}$, and the scalar fields Φ and ϕ characterized by polarization vectors E and e respectively:

$$\begin{aligned} C\left(\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}, \begin{pmatrix} \vec{\gamma} \\ \vec{\delta} \end{pmatrix}, E\right) &= \frac{1}{16}(\,\gamma^a \quad \delta^a\,)\mathcal{M}^{(0)}E_{ab}\begin{pmatrix} \alpha^b \\ \beta^b \end{pmatrix} \times \frac{1}{2m(\vec{\alpha},\vec{\beta})}\delta_{\vec{\alpha},\vec{\gamma}}\delta_{\vec{\beta},\vec{\delta}} \\ \tilde{C}\left(\begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix}, \begin{pmatrix} \vec{\gamma} \\ \vec{\delta} \end{pmatrix}, e\right) &= \frac{1}{16}(\,\gamma^a \quad \delta^a\,)e(M^{(0)} + L)_{ab}\begin{pmatrix} \alpha^b \\ \beta^b \end{pmatrix} \times \frac{1}{2m(\vec{\alpha},\vec{\beta})}\delta_{\vec{\alpha},\vec{\gamma}}\delta_{\vec{\beta},\vec{\delta}}. \end{aligned} \tag{81}$$

These couplings are clearly invariant under the SL(2,Z)×O(6,22;Z) transformations:

$$\begin{pmatrix} \alpha^a \\ \beta^a \end{pmatrix} \to \mathcal{L}\omega\mathcal{L}^T \begin{pmatrix} (L\Omega L)_{ab}\alpha^b \\ (L\Omega L)_{ab}\beta^b \end{pmatrix}, \qquad \begin{pmatrix} \gamma^a \\ \delta^a \end{pmatrix} \to \mathcal{L}\omega\mathcal{L}^T \begin{pmatrix} (L\Omega L)_{ab}\gamma^b \\ (L\Omega L)_{ab}\delta^b \end{pmatrix},$$

$$E \to \Omega E \Omega^T, \qquad e \to \omega e \omega^T,$$ (82)

together with the transformations of the background

$$M^{(0)} \to \Omega M^{(0)} \Omega^T, \qquad \mathcal{M}^{(0)} \to \omega \mathcal{M}^{(0)} \omega^T.$$ (83)

This shows that the Yukawa couplings in a given background are equal to the Yukawa couplings around a new background, related to the original background by an SL(2,Z) (or O(6,22;Z)) transformation, after appropriate transformations on the quantum numbers of the external states.

6 Where are the SL(2,Z) Transform of the Elementary String Excitations?

In this section we shall first identify the elementary excitations in string theory that saturate the Bogomol'nyi bound, and then try to identify the magnetically charged soliton states in the theory, related to the elementary string states via SL(2,Z) transformations[45]. We start with a discussion of the spectrum of known elementary string excitations.

6.1 Where Do the Known Elementary String Excitations Fit in?

The mass formula (76) for $\vec{\beta} = 0$ takes the form:

$$m^2 = \frac{1}{16\lambda_2^{(0)}} \alpha^a (M^{(0)} + L)_{ab} \alpha^b.$$ (84)

In order to compare the above formula with the mass formula for the elementary excitations in string theory, we use the observation of Refs.[30, 31] that the physics remains unchanged under a simultaneous rotation of the background $M^{(0)}$ and the lattice Λ of the form:

$$M^{(0)} \to \Omega M^{(0)} \Omega^T, \qquad \Lambda \to L\Omega L\Lambda.$$ (85)

where Ω is an O(6,22) matrix. Certainly the mass formula is invariant under this transformation. If we choose Ω in such a way that $\Omega M^{(0)} \Omega^T \equiv \widehat{M}^{(0)} = I_{28}$, and if $\widehat{\vec{\alpha}} \equiv L\Omega L\vec{\alpha}$ denotes the vector in the new lattice $\widehat{\Lambda} \equiv L\Omega L\Lambda$, then Eq.(84) takes the simple form:

$$m^2 = \frac{1}{16\lambda_2^{(0)}} \widehat{\alpha}^a (I + L)_{ab} \widehat{\alpha}^b = \frac{1}{8\lambda_2^{(0)}} (\widehat{\alpha}_R)^2,$$ (86)

where,

$$\widehat{\alpha}_R^a \equiv \frac{1}{2}(I + L)_{ab}\widehat{\alpha}^b, \qquad \widehat{\alpha}_L^a \equiv \frac{1}{2}(I - L)_{ab}\widehat{\alpha}^b.$$ (87)

We now write down the mass formula for elementary string excitations[24]. Since the Ramond sector states are degenerate with the Neveu-Schwarz (NS) sector states due to space-time supersymmetry, it is enough to study the mass formula in the NS sector. With the normalization that we have chosen, it is given by

$$M^2 = \frac{1}{8\lambda_2^{(0)}} \{ (\widehat{\alpha}_R)^2 + 2N_R - 1 \} = \frac{1}{8\lambda_2^{(0)}} \{ (\widehat{\alpha}_L)^2 + 2N_L - 2 \}.$$ (88)

In this expression $(\widehat{\alpha}_R)^2$ and $(\widehat{\alpha}_L)^2$ denote the internal momenta contributions, N_L and N_R denote the oscillator contributions, and -1 and -2 denote the ghost contributions to L_0 and $\bar{L}_0$ in the world-sheet theory respectively. Note that in our convention the world-sheet supersymmetry appears in the right moving sector of the theory. The appearance of $1/\lambda_2^{(0)}$ factor in these expressions can be traced to the fact that we are using the canonical metric $g_{\mu\nu}$ to measure distances instead of the string metric $G_{\mu\nu}$. GSO projection requires N_R to be at least $1/2$, since we need a factor of $\psi^M_{-1/2}$ to create the lowest mass state in the NS sector. This clearly shows that $M^2 \geq m^2$ with m^2 given by Eq.(86). Furthermore the elementary string states that saturate the Bogomol'nyi bound all have

$$N_R = \frac{1}{2}$$ (89)

so that

$$M^2 = \frac{1}{8\lambda_2^{(0)}} \{ (\widehat{\alpha}_R)^2 \} = \frac{1}{8\lambda_2^{(0)}} \{ (\widehat{\alpha}_L)^2 + 2N_L - 2 \}.$$ (90)

For these states, $M^2 = m^2$. We also see that

$$N_L - 1 = \frac{1}{2} ((\widehat{\alpha}_R)^2 - (\widehat{\alpha}_L)^2) = \frac{1}{2} \widehat{\alpha}^a L_{ab} \widehat{\alpha}^b \equiv \frac{1}{2} (\widehat{\alpha})^2.$$ (91)

Since space-time supersymmetry generators act only on the right-moving fermions ψ^M, it is also easy to analyze the supersymmetry transformation properties of these states. In particular, for a fixed oscillator state in the left-moving sector, states created by $\psi^M_{-1/2}$ for eight transverse M, together with their Ramond sector counterparts, give rise to a 16 dimensional super-multiplet of the N=4 supersymmetry algebra. The transformation laws of these states depend on the left moving oscillator content also. In N=4 super-Poincare algebra, however, depend on the left-moving oscillator content also. In particular, if the left moving oscillators involved in the construction of a state transform as a scalar, then the resulting supermultiplet will contain states with maximum spin 1, we shall call this the vector supermultiplet. On the other hand, if the left-moving oscillators transform as a vector, then the resulting supermultiplet contains states with maximum spin 2. We shall refer to this representation of the super-Poincare algebra as the spin 2 supermultiplet. It should be clear from this discussion that super-multiplets of arbitrarily high spin can be constructed this way. However, each of these super-multiplets decompose into several copies of the 16 dimensional

super-multiplet if we look at their transformation laws under the supersymmetry subalgebra of the full super-Poincare algebra.

Before we conclude this subsection, let us analyse the stability of the various elementary string excitations discussed above.[5] Since we are concentrating on states saturating the Bogomol'nyi bound, we are guaranteed that in the rest frame these states are the lowest energy states in the given charge sector, and hence there is no multiparticle state in theory that carries the same amount of charge and has *less* energy than the single particle state. It is, however, possible that there exists a multiparticle state, with all the particles at rest, which has the *same* energy as the particular elementary particle state under consideration. From the mass relation $m^2 \propto (\widehat{\alpha}_R)^2$ it is clear that such a situation can arise if the right hand component of the charge vector $(\widehat{\alpha}_R)$ of the original state, and those of the states constituting the multi-particle state, are parallel to each other. If such a situation has to hold for a generic choice of the lattice, then it would imply that the full charge vectors $\widehat{\alpha}$ of the original particle, and of the decay products, must also be parallel to each other. For, if the right hand components of the charge vectors are parallel to each other but the left hand components are not, then a slight O(6,22) rotation of the lattice, which mixes the right and the left hand components of the charge vectors, will destroy the alignment of the right hand components. This implies that in order for a particle carrying charge vector $\widehat{\alpha}$ to decay into two or more particles at rest, $\widehat{\alpha}$ must be an integral (n) multiple of some other lattice vector $\widehat{\alpha}_0$. In this case the original particle can decay into n other particles, each carrying charge vector $\widehat{\alpha}_0$. From this we can conclude that for a generic choice of the lattice, an elementary string state, saturating the Bogomol'nyi bound, and characterized by the charge vector $\widehat{\alpha}$, is absolutely stable as long as $\widehat{\alpha}$ is not an integral multiple of another vector in the lattice $\widehat{\Lambda}$.

In §6.3 we shall try to identify soliton states of the theory which are related to these elementary string excitations via SL(2,Z) transformations. But first we need to know how the soliton solutions in the theory fit into the mass formula given in Eq.(76).

6.2 Where Do the Known Solitons Fit in?

We now turn our attention to the spectrum of known magnetically charged soliton solutions in string theory. Many such solutions are known[2, 25, 29, 19]. We shall focus our attention only on the non-singular solutions with asymptotically flat space-time geometry, since it is only these solutions which have a clear interpretation as new particle like states in the theory.

BPS Gauge Monopole Solutions: These solutions were constructed in Ref.[25] (see also Ref.[2]) and were further explored in Ref.[19]. We work in a gauge where asymptotically the Higgs field is directed along a fixed direction in the gauge space (and is identified with the field $A_4^{(10)16}$) except along a Dirac string singularity. In this gauge, after appropriate rescaling of the ten dimensional coordinates z^0 and z^4, the asymptotic values of various ten dimensional fields

associated with this solution are given by

$$B_{\mu\nu}^{(10)} \simeq 0, \qquad G_{\mu\nu}^{(10)} \simeq e^{2\phi_0}\eta_{\mu\nu}, \qquad \Phi^{(10)} \simeq 2\phi_0,$$

$$G_{(m+3)\mu}^{(10)} = 0, \qquad H_{(m+3)0i}^{(10)} \simeq O(\frac{1}{r^3}), \qquad H_{(m+3)ij}^{(10)} \simeq 8Ce^{-\phi_0}\delta_{m,1}\varepsilon_{ijk}\frac{x^k}{r^3},$$

$$F_{0i}^{(10)I} \simeq O(\frac{1}{r^3}), \qquad F_{ij}^{(10)I} \simeq -2\sqrt{2}\,\delta_{I,16}\varepsilon_{ijk}\frac{x^k}{r^3},$$

$$B_{(m+3)(n+3)}^{(10)} \simeq 0, \qquad A_{m+3}^{(10)I} \simeq 2\sqrt{2}Ce^{-\phi_0}\delta_{I,16}\delta_{m,1}, \qquad G_{(m+3)(n+3)}^{(10)} \simeq \delta_{mn},$$

$$1 \le i,j \le 3, \quad 1 \le m,n \le 6, \quad 0 \le \mu,\nu \le 3, \tag{92}$$

where C and ϕ_0 are two arbitrary constants. Using Eqs.(3) we see that the asymptotic values of various four dimensional fields are given by,

$$\widehat{G}_{mn} \simeq \delta_{mn}, \qquad \widehat{B}_{mn} \simeq 0, \qquad \widehat{A}_m^I \simeq 2\sqrt{2}\,Ce^{-\phi_0}\delta_{I,16}\delta_{m,1}, \qquad \Phi = 2\phi_0,$$

$$F_{0r}^{(a)} \simeq O(\frac{1}{r^3}), \qquad \widetilde{F}_{0r}^{(a)} \simeq -\sqrt{2}\,\delta_{a,28}\frac{1}{r^2}, \qquad g_{\mu\nu} \simeq \eta_{\mu\nu}, \qquad B_{\mu\nu} \simeq 0, \tag{93}$$

Note that even though $H_{(m+3)ij}^{(10)}$ is asymptotically non-trivial, $F_{ij}^{(m+6)}$ is trivial. This happens due to the cancellation between various terms appearing in the expression for $A_\mu^{(m+6)}$ given in Eq.(3).

This solution can be generalized in several ways. In particular, we can generate a multi-parameter family of solutions, if, *keeping the lattice* Λ *fixed*, we make the following transformations on the original solution:

$$G_{(m+3)(n+3)}^{(10)} \to S_m^{\ p}S_n^{\ q}G_{(p+3)(q+3)}^{(10)}, \qquad A_{(m+3)}^{(10)I} \to S_m^{\ p}A_{p+3}^{(10)I} + T_m^I,$$

$$B_{(m+3)(n+3)}^{(10)} \to S_m^{\ p}S_n^{\ q}B_{(p+3)(q+3)}^{(10)} + R_{mn} + \frac{1}{2}(S_m^{\ p}A_{p+3}^{(10)I}T_n^I - S_n^{\ p}A_{p+3}^{(10)I}T_m^I),$$

$$G_{(m+3)\mu}^{(10)} \to S_m^{\ p}G_{(p+3)\mu}^{(10)}, \qquad B_{(m+3)\mu}^{(10)} \to S_m^{\ p}B_{(p+3)\mu}^{(10)} - \frac{1}{2}A_\mu^{(10)I}T_m^I, \tag{94}$$

where $S_m^{\ p}$ is an arbitrary constant 6×6 matrix, R_{mn} is a constant anti-symmetric 6×6 matrix, and T_m^I is a constant 6×16 matrix, satisfying,

$$T_m^{16} = 0. \tag{95}$$

All other 10 dimensional fields remain invariant under these transformations. The freedom of shifting $B_{(m+3)(n+3)}^{(10)}$ and $A_{(m+3)}^{(10)I}$ by constant matrices R_{mn} and T_m^I stem from the fact that the equations of motion involve only the field strengths $H_{MNP}^{(10)}$ and $F_{MN}^{(10)I}$. These field strengths are invariant under these transformations, as can be seen from Eqs.(2). The reason that T_m^{16} need to vanish is that the solution contains SU(2) gauge fields $A_M^{(10)i}$ ($1 \le i \le 3$) at its core, with $A_M^{(10)16}$ identified to $2\sqrt{2}\,A_M^{(10)3}$. Thus a constant shift in $A_M^{(10)16}$ will change the SU(2) field strengths, and the resulting configuration will not remain a solution of the equations of motion.

Performing the transformations (94) on the solution (92), and using Eqs.(3) again, we get the following asymptotic form of various four dimensional fields,

$$\widehat{G}_{mn} \simeq S_m{}^p S_n{}^p, \qquad \widehat{B}_{mn} \simeq R_{mn}, \qquad \widehat{A}_m^I \simeq 2\sqrt{2}\,C e^{-\phi_0}\delta_{I,16}S_m{}^1 + T_m^I, \qquad \Phi = 2\phi_0 - \det S,$$

$$F_{0r}^{(a)} \simeq O(\frac{1}{r^3}), \qquad \widetilde{F}_{0r}^{(a)} \simeq -\sqrt{2}\,\delta_{a,28}\frac{1}{r^2}, \qquad g_{\mu\nu} \simeq \eta_{\mu\nu}, \qquad B_{\mu\nu} \simeq 0. \tag{96}$$

It can be checked that by appropriately adjusting the matrices S, T and R, and the constant C, we can choose $\widehat{G}_{mn}^{(0)}$, $\widehat{B}_{mn}^{(0)}$ and $\widehat{A}_m^{(0)I}$ to be completely arbitrary, consistent with their symmetry properties. Thus the monopole solution given in Eq.(96) is characterized by an arbitrary value of $M^{(0)}$.

Using Eqs.(60), (63), and (67) we see that this monopole carries quantum numbers

$$\begin{pmatrix} \alpha^a = 0 \\ \beta^a = \sqrt{2}\delta_{a,28} \end{pmatrix}, \tag{97}$$

with

$$\lambda_1^{(0)} = 0. \tag{98}$$

The BPS dyon solutions, saturating the Bogomol'nyi bound were also constructed in Ref.[25] following the method of Ref.[9]. For these solutions,

$$F_{0r}^{(a)} \simeq \sqrt{2}\,\frac{Q_e}{\lambda_2^{(0)}}M_{a,28}^{(0)}\frac{1}{r^2}, \tag{99}$$

instead of being zero. Here Q_e is an arbitrary constant. Using Eqs.(60) and (67), it is easy to see that these solutions correspond to non-zero values of $\lambda_1^{(0)}$ and carry quantum numbers,

$$\begin{pmatrix} \alpha^a = p\sqrt{2}\delta_{a,28} \\ \beta^a = \sqrt{2}\delta_{a,28} \end{pmatrix}, \tag{100}$$

with $\lambda_1^{(0)}$ and the integer p determined (up to the SL(2,Z) transformation $\lambda_1^{(0)} \to \lambda_1^{(0)} - n$, $p \to p+n$ for some integer n) in terms of the parameter Q_e by the relation $Q_e = p+\lambda_1^{(0)}$. Following the arguments of Ref.[37] one can show that these states belong to the vector supermultiplet of the super-Poincare algebra.

In the next subsection we shall compare these states with the SL(2,Z) transform of the elementary string excitations discussed in the last subsection. Note, however, that the analysis of the last subsection was carried out in a representation where the matrix $M^{(0)}$ was transformed to the identity matrix via an O(6,22) rotation, and all the modular parameters were encoded in the lattice $\widehat{\Lambda}$. In order to facilitate comparison, it is convenient to bring $M^{(0)}$ to identity in this case also, with a simultaneous rotation of the lattice Λ to $\widehat{\Lambda}$. Under this rotation, the vector $\sqrt{2}\,\delta_{a,28}$ is transformed to some vector $\vec{l}$ with $\vec{l}^2 \equiv l^a L_{ab}l^b = -2$. Thus the resulting dyon solution has charge quantum numbers

$$\begin{pmatrix} \widehat{\alpha} = p\vec{l} \\ \widehat{\beta} = \vec{l} \end{pmatrix}. \tag{101}$$

Applying this argument in reverse, we can construct dyon solutions at any point in the moduli space, characterized by some self-dual Lorentzian lattice $\widehat{\Lambda}$, with $\widehat{M}^{(0)} = I_{28}$. For, given such a configuration, we can always find an O(6,22) transformation Ω such that $\widehat{\Lambda} = L\Omega L\Lambda$. This transformation rotates $\widehat{M}^{(0)} = I_{28}$ to $M^{(0)} = (\Omega^T\Omega)^{-1}$. This rotation also brings some vector $\vec{l} \in \widehat{\Lambda}$ with $\vec{l}^2 = -2$ to the vector $\sqrt{2}\,\delta_{a,28} \in \Lambda$. Since for the compactification lattice Λ we know how to construct a dyon solution with charge vector (100) for any value of $M^{(0)}$, the O(6,22) rotation of this solution by Ω will give us a dyon solution carrying charge quantum numbers (101) in the vacuum characterized by the lattice $\widehat{\Lambda}$ and $\widehat{M}^{(0)} = I_{28}$. Also, note that the transformation Ω that gives $\widehat{\Lambda} = L\Omega L\Lambda$ is not unique, since $L\Omega L$ can always be multiplied from the right by any element of the O(6,22;Z) subgroup of O(6,22) that constitutes the group of automorphisms of the lattice Λ. Using this freedom, different vectors $\vec{l} \in \widehat{\Lambda}$ can be mapped to the vector $\sqrt{2}\,\delta_{a,28} \in \Lambda$. This gives us a way of constructing dyon solutions carrying charge quantum numbers (101) for different vectors $\vec{l} \in \widehat{\Lambda}$ with $\vec{l}^2 = -2$.

We should note, however, that the solutions of Ref.[25] were constructed by ignoring the higher derivative terms in the string effective action, and hence are valid for small C, which in this case translates to small $(\vec{l}_R)^2$.[6] Nevertheless we expect that the general features of the solution, e.g., partially broken supersymmetry, will continue to hold for all C, and consequently, it will continue to represent a state in the vector representation of the super-Poincare algebra, saturating the Bogomol'nyi bound.

H-**Monopole Solutions**: We now turn to the next class of solutions in string theory, which carry magnetic charge associated with the ten dimentional field $H_{MNP}^{(10)}$ but not the ten dimensional gauge fields[29, 19]. A non-singular, asymptotically flat solution of this kind was constructed in Ref.[19] by wrapping *a finite sized gauge five-brane solution* around the torus. After appropriate rescaling of the ten dimensional coordinates z^0 and z^4, the only non-trivial asymptotic fields for this solution are given by,

$$\Phi^{(10)} \simeq 2\phi_0, \qquad G_{\mu\nu}^{(10)} = e^{2\phi_0}\eta_{\mu\nu}, \qquad H_{(m+3)ij}^{(10)} \simeq 2Q\delta_{m,1}\epsilon_{ijk}\frac{x^k}{r^3}, \tag{102}$$

where Q is a constant. From this we can determine the asymptotic values of various four dimensional fields. They are,

$$\widehat{G}_{mn} \simeq \delta_{mn}, \qquad \widehat{B}_{mn} \simeq 0, \qquad \widehat{A}_m^I \simeq 0, \qquad \Phi = 2\phi_0,$$

$$F_{0r}^{(a)} \simeq O(\frac{1}{r^3}), \qquad \widetilde{F}_{0r}^{(a)} \simeq Q\delta_{a,7}\frac{1}{r^2}, \qquad g_{\mu\nu} \simeq \eta_{\mu\nu}, \qquad B_{\mu\nu} \simeq 0. \tag{103}$$

Using Eqs.(4), (60), (63), and (67) we see that this monopole carries quantum numbers

$$\begin{pmatrix} \alpha^a = 0 \\ \beta^a = Q\delta_{a,1} \end{pmatrix}, \tag{104}$$

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[6]To see this, note that small C with the standard choice of the lattice Λ implies small mass for the particles carrying charge quantum numbers $\pm\sqrt{2}\delta_{a,28}$, — these particles can be interpreted as the SU(2) gauge bosons that have acquired mass due to spontaneous breakdown of the SU(2) symmetry by the Higgs vacuum expectation value $\propto C$. On the other hand, in the picture where $M^{(0)}$ has been set to identity by an O(6,22) rotation, Eq.(88) tells us that for $N_R = 1/2$, particles carrying electric charge vector $\vec{l}$ has mass^2 proportional to $(\vec{l}_R)^2$. This shows that small C in one picture implies small $(\vec{l}_R)^2$ in the other picture.

with,

$$\lambda_1^{(0)} = 0. \tag{105}$$

Since $\vec{\beta}$ lies on the lattice Λ, we see that the parameter Q must be quantized. Since this solution does not carry any electric charge, the corresponding value of $\lambda_1^{(0)}$ is 0. Although the corresponding dyon solutions have not been constructed, there is, in principle, no reason to expect that they do not exist. These dyon solutions will correspond to non-zero values of $\vec{\alpha}$ and $\lambda_1^{(0)}$ as before.

For the solution given in Eq.(103), $M^{(0)} = I_{28}$, but as in the previous case, we can get more general class of solutions using the transformations (94). Since this monopole solution contains SU(2) gauge fields at its core[19], the transformation parameter T_m^I must satisfy an equation similar to Eq.(95). In fact if we take $A_M^{(10)16}$ to be the third component $\mathcal{A}_M^{(10)3}$ of the SU(2) gauge field, then the condition on T_m^I is precisely the one given in (95). As a result, even after the transformation, we have $\widehat{A}_m^{(28)} = 0$ asymptotically. This shows that by this method, monopole solutions carrying charge quantum numbers (104) cannot be constructed for arbitrary choice of $M^{(0)}$, but only for a specific class of $M^{(0)}$.

As in the previous case, we can bring $M^{(0)}$ to I_{28} by an O(6,22) rotation, simultaneously rotating the lattice Λ to a new lattice $\widehat{\Lambda}$. The vector $Q\delta_{a,1}$ gets rotated into some new vector $\vec{m}$ satisfying $\vec{m}^2 = 0$. Thus the charge quantum numbers of the monopole are now given by,

$$\begin{pmatrix} \widehat{\vec{\alpha}} = 0 \\ \widehat{\vec{\beta}} = \vec{m} \end{pmatrix}. \tag{106}$$

The fact that the H-monopole solutions can be constructed only for a special class of $M^{(0)}$ now translates into the statement that such solutions exist only for a special class of lattice $\widehat{\Lambda}$, — those which correspond to the existence of an unbroken SU(2) gauge group.

6.3 SL(2,Z) Transform of the Elementary String States

In this subsection we shall try to identify soliton solutions related to the elementary string excitations via SL(2,Z) transformation. We begin by reminding the reader that the SL(2,Z) transformation acts non-trivially on the vacuum, and hence relates elementary string excitations in one vacuum to the monopole and dyon solutions constructed around different vacua. Throughout this discussion we shall be implicitly assuming that the theory is in a single phase in the entire upper half $\lambda^{(0)}$ plane, unlike the cases discussed in Refs.[5, 46], so that the dyon spectrum computed at weak coupling can be continued to the strong coupling regime.[7]

We shall concentrate on the states belonging to the 16 dimensional representation of the supersymmetry algebra. The mass spectrum of such states has been given in Eq.(90). We shall discuss the three cases, $(\widehat{\vec{\alpha}})^2 = -2$, $(\widehat{\vec{\alpha}})^2 = 0$, and $(\widehat{\vec{\alpha}})^2 > 0$ separately.

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[7]This is analogous to the fact that the theory is in the same phase for all values of $M^{(0)}$, except possibly on surfaces of high codimension in the moduli space, where part of the non-abelian gauge symmetry is unbroken.

$(\widehat{\vec{\alpha}})^2 = -2$: In this case Eq.(91) gives $N_L = 0$. Since there are no left moving oscillators, by our previous argument, these states, together with their Ramond sector counterparts, constitute a vector supermultiplet of the super-Poincare algebra. Note also that each of these particles are absolutely stable, since the lattice Λ, being even and self dual, cannot contain $\widehat{\alpha}/n$ as a lattice vector for any integer n. Under the SL(2,Z) transformation

$$\mathcal{L}\omega\mathcal{L}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{107}$$

an elementary string state carrying charge quantum numbers $\begin{pmatrix} \widehat{\vec{\alpha}} = \vec{l} \\ \widehat{\vec{\beta}} = 0 \end{pmatrix}$ is mapped onto a soliton state carrying charge quantum numbers given in Eq.(101) with $p = 0$. Furthermore, as we have seen, these magnetically charged states can be constructed for any choice of the vacuum characterized by the lattice $\widehat{\Lambda}$. This agrees with the fact that the elementary string states of the form discussed above also exist for any choice of the lattice $\widehat{\Lambda}$. Finally, as has already been mentioned before, these soliton states belong to the vector supermultiplet of the $N = 4$ super-Poincare algebra[37]. This shows that for elementary string states saturating the Bogomol'nyi bound and having $N_L = 0$, we do have soliton states in the theory related to these elementary string states via the SL(2,Z) transformation (107), and belonging to the same representation of the super-Poincare algebra.

Let us now analyze the effect of a general SL(2,Z) transformation on an elementary string state labeled by the quantum numbers $\begin{pmatrix} \widehat{\alpha}^a = l^a \\ \widehat{\beta}^a = 0 \end{pmatrix}$, with $\vec{l}^2 = -2$. Acting on such a state, an SL(2,Z) transformation

$$\mathcal{L}\omega\mathcal{L}^T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad ps - qr = 1, \tag{108}$$

produces a state with quantum numbers

$$\begin{pmatrix} \widehat{\alpha}^a = pl^a \\ \widehat{\beta}^a = rl^a \end{pmatrix}. \tag{109}$$

Note that the quantum numbers of the final state depend only on p and r. Given p and r which are relatively prime, it is always possible to find q and s satisfying $ps - qr = 1$. Furthermore, the choice of q and s is unique up to a translation $s \to s + nr$, $q \to q + np$ for some integer n. This freedom can be understood by noting that

$$\begin{pmatrix} p & q+np \\ r & s+nr \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \tag{110}$$

The SL(2,Z) transformation $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, acting on an elementary string state carrying only electric charge, leaves its quantum numbers unchanged. Acting on the field λ, it produces the trivial transformation $\lambda \to \lambda - n$. Thus we see that up to this trivial transformation, different SL(2,Z) group elements, labeled by the integers p and r, produce different charge quantum numbers.

From this analysis we conclude that in order to establish SL(2,Z) invariance of the spectrum in this sector, one needs to show the existence of non-singular, asymptotically flat, dyon solutions

carrying charge quantum numbers given in Eq.(109) for all relatively prime integers p and r. Furthermore, these dyon states must saturate the Bogomol'nyi bound and belong to the vector representation of the super-Poincare algebra. The soliton states carrying charge quantum numbers given in (101) are special cases of these with $r = 1$.

Existence of these new dyon states in the theory can be taken to be a prediction of the SL(2,Z) invariance of the theory. Let us now give a plausibility argument for the existence of these states. We begin with the observation that the charge quantum numbers with $r > 1$ correspond to states with multiple units of magnetic charge. Multi-dyon solutions in ordinary Yang-Mills-Higgs system have been constructed in the BPS limit[7, 35], and there is good reason to believe that they also exist in the full string theory[25]. It is quite plausible that when we quantize the bosonic and the fermionic zero modes of these solutions, then in each charge sector, the ground state will have partially broken supersymmetry, and will belong to the vector supermultiplet of the super-Poincare algebra, as in the case of singly charged monopoles. What is not so obvious is what is special about the cases when p and r are relatively prime. We shall now show that dyons carrying quantum numbers given in Eq.(109) represent absolutely stable single particle states if and only if p and r are relatively prime. These dyons could then be regarded as stable, supersymmetric, bound states of monopoles and dyons, each carrying one unit of magnetic charge.

Suppose p and r are not relatively prime, so that there exist integers p_0, r_0 and n such that $p = np_0$ and $r = nr_0$. It is easy to verify that a dyon with quantum number

$$\begin{pmatrix} \widehat{\alpha}^a = np_0 l^a \\ \widehat{\beta}^a = nr_0 l^a \end{pmatrix} \tag{111}$$

and saturating the Bogomol'nyi bound, has mass and charge identical to that of n dyons with quantum numbers

$$\begin{pmatrix} \widehat{\alpha}^a = p_0 l^a \\ \widehat{\beta}^a = r_0 l^a \end{pmatrix} \tag{112}$$

and hence is indistinguishable from such a state. Thus these dyons should not be regarded as new states in the spectrum. On the other hand, if p and r are relatively prime, then the dyon with charge quantum numbers given in Eq.(109) cannot be regarded as a state containing multiple dyons, since the mass of this dyon is strictly less than the sum of the masses of the dyons whose charge quantum numbers add up to those given in Eq.(109). To see this, let us compare the mass of the dyon with charge quantum numbers given in (109) to the sum of the masses of the dyons carrying charge quantum numbers

$$\begin{pmatrix} \widehat{\alpha}^a = p_1 l^a \\ \widehat{\beta}^a = r_1 l^a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widehat{\alpha}^a = p_2 l^a \\ \widehat{\beta}^a = r_2 l^a \end{pmatrix}, \quad \text{with} \quad p = p_1 + p_2, \; r = r_1 + r_2. \tag{113}$$

One can easily verify that the mass of the dyon carrying charge quantum numbers given in Eq.(109) is smaller than the sum of the masses of the dyons carrying charge quantum numbers given in Eq.(113), by using the triangle inequality

$$\left[(p \quad r) \mathcal{M}^{(0)} \begin{pmatrix} p \\ r \end{pmatrix} \right]^{\frac{1}{2}} \le \left[(p_1 \quad r_1) \mathcal{M}^{(0)} \begin{pmatrix} p_1 \\ r_1 \end{pmatrix} \right]^{\frac{1}{2}} + \left[(p_2 \quad r_2) \mathcal{M}^{(0)} \begin{pmatrix} p_2 \\ r_2 \end{pmatrix} \right]^{\frac{1}{2}}, \tag{114}$$

and noting that the equality holds if and only if $p_1/r_1 = p_2/r_2 = p/r$, which cannot happen if p and r are relatively prime. Thus for p and r relatively prime, the dyons carrying quantum numbers given in Eq.(109) are absolutely stable, and should be regarded as new states in the theory.

$(\widehat{\alpha})^2 = 0$: In this case Eq.(91) gives $N_L = 1$. The contribution to N_L here can come from the oscillators associated with any of the 22 internal directions, or the four space-time directions. The oscillators associated with the 22 internal directions transform as scalars under the four dimensional Lorentz transformation, and hence give rise to vector super-multiplets of the super-Poincare algebra. The requirement that the corresponding vertex operator is a primary operator gives one constraint, which reduces the number of independent choices of the left moving oscillator to 21. Thus there are 21 distinct vector supermultiplets of the super-Poincare algebra at this level. On the other hand, the left moving oscillators associated with the space-time coordinates transform as vectors under the four dimensional Lorentz transformation. By our previous argument, this gives rise to a spin two supermultiplet of the super-Poincare algebra.

Note that given any light-like vector $\widehat{\alpha} \in \widehat{\Lambda}$, $n\widehat{\alpha}$ is also a like-like vector in the lattice Λ. However, the later state can decay into n particles at rest, each carrying charge vector $\widehat{\alpha}$.

The SL(2,Z) transformation (107) maps elementary string states carrying charge quantum numbers $\begin{pmatrix} \widehat{\alpha} = \vec{m} \\ \widehat{\beta} = 0 \end{pmatrix}$ with $\vec{m}^2 = 0$ to monopole states carrying charge quantum numbers $\begin{pmatrix} \widehat{\alpha} = 0 \\ \widehat{\beta} = \vec{m} \end{pmatrix}$. This coincides with the quantum numbers of the H-monopole solution given in (106). However, note that these H-monopole solutions have been constructed only for a subclass of vacuum configurations, whereas the elementary string states carrying the quantum number $\widehat{\alpha} = \vec{m}$ exist for all choices of the vacuum.

If SL(2,Z) is a genuine symmetry of the theory, then there should be a one to one correspondence between the elementary string states and monopole solutions of this kind, and hence one must be able to construct the H-monopole solutions for a generic choice of the lattice $\widehat{\Lambda}$. Also, there should be 21 distinct H-monopole states in the vector representation and 1 H-monopole state in the spin 2 representation of the super-Poincare algebra, carrying the same magnetic charge, since the elementary string state carrying a given electric charge has this degeneracy. Finally there should be H-dyon states carrying p units of electric charge and r units of magnetic charge for p and r relatively prime. Existence of these states can be taken to be a prediction of the SL(2,Z) invariance of the theory. One already sees evidence of large degeneracies in the construction of the H-monopole solution in Ref.[19], since an SU(2) gauge group is necessary to construct the solution, and different choices of this SU(2) group will lead to different H-monopole solutions carrying the same charge quantum numbers.[8] However, a proper understanding of this degeneracy will be possible only after we are able to construct the H-monopole solution in a generic background where the non-abelian gauge group of the theory is completely broken, and then quantize the bosonic and fermionic zero modes of the solution.

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[8]The charge quantum numbers of the H-monopole are not affected by the choice of the SU(2) group.

$(\widehat{\alpha})^2 > 0$: In this case, from Eq.(91) we get $N_L \geq 2$. These states carry charge quantum numbers of the form

$$\begin{pmatrix} \widehat{\vec{q}} = \vec{n} \\ \widehat{\vec{\beta}} = 0 \end{pmatrix}, \qquad (115)$$

with $\widehat{\vec{n}}^2 = 2(N_L - 1) > 0$. The monopoles, related to these states by the SL(2,Z) transformation (107), have quantum numbers

$$\begin{pmatrix} \widehat{\vec{q}} = 0 \\ \widehat{\vec{\beta}} = \vec{n} \end{pmatrix}. \qquad (116)$$

There are no known monopole solutions carrying these quantum numbers. This, however, is not surprising, since, as we shall argue now, there is no *a priori* reason why such monopole solutions can be constructed in terms of the massless fields of the low energy effective field theory. Note that in the previous two cases, there is a limit $((\widehat{\alpha}_R)^2 \to 0)$ in which the monopole mass vanishes, and hence, at least in this limit, the monopole solution must be constructed purely in terms of the massless fields of the theory. In the present case, however, there is no such limit since $(\widehat{\alpha}_R)^2 \geq 2$, and these monopoles always have mass of order M_{Pl}. Thus there is no reason to expect that these monopoles can be constructed in terms of the massless fields in the low energy effective field theory. Construction of monopole solutions carrying these quantum numbers remains another open problem in this field.

7 SL(2,Z) Duality in String Theory as Target Space Duality of the Five Brane Theory

In the previous sections we have presented several pieces of evidence that the SL(2,Z) symmetry, which exchanges the strong and weak coupling limits of the string theory, is a genuine symmetry of the theory. The purpose of this section is somewhat different; instead of producing more evidence for the SL(2,Z) symmetry, we shall try to find a geometrical understanding of this symmetry.

We begin with the observation that the O(6,22;Z) symmetry already has a nice geometrical interpretation. It generalizes the symmetry that sends the size of the compact manifold, measured in appropriate units, to its inverse, and, at the same time, exchanges the usual Kaluza-Klein modes of the string theory carrying momentum in the internal directions, with the string winding modes, — states corresponding to a string wrapped around one of the compact directions. One way to see this is to note that the six dimensional vector α^m ($1 \leq m \leq 6$) has the interpretation as the components of momentum of a state in the internal directions, and α^{m+6} ($1 \leq m \leq 6$) has the interpretation as the winding number of a state along the compact directions. Thus the O(6,22;Z) transformation $\begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & I_{16} \end{pmatrix}$ gives $\alpha^m \leftrightarrow \alpha^{m+6}$ for $1 \leq m \leq 6$, thereby interchanging the quantum numbers associated with internal momenta and winding numbers.

No such simple geometric interpretation exists for SL(2,Z) transformation in string theory. In fact, as we have seen, the non-trivial part of the SL(2,Z) transformation exchanges the Kaluza-Klein states, carrying momenta in the internal directions, with the magnetically charged soliton states in the theory. Such a symmetry is necessarily non-perturbative, and cannot be understandood from the point of view of the string world-sheet theory, which is designed to produce the perturbation expansion in string theory.

This distinction between the roles played by the SL(2,Z) and O(6,22;Z) symmetries in string theory was already manifest in §2.2, where we saw that in the low energy effective field theory describing the four dimensional string theory, the two symmetries appear on a somewhat different footing. O(6,22;Z) is a symmetry of the effective action, whereas SL(2,Z) is only a symmetry of the equations of motion. However, in §2.4 we saw that with the restriction to field configurations without any ten dimensional gauge fields, and by going to a dual formulation of the theory, the roles of the SL(2,Z) and O(6,22;Z) symmetries can be reversed. In this new formulation SL(2,Z) becomes a symmetry of the action, whereas an O(6,6;Z) subgroup of the O(6,22; Z) group becomes a symmetry only of the equations of motion.

This leads us to believe that if there is an alternate formulation of the heterotic string theory, where the dual formulation of the $N = 1$ supergravity theory in ten dimensions (or its dimensional reduction) appears naturally as the low energy effective field theory in ten (or four) dimensions, then SL(2,Z) transformations will have a more natural action on the states in this new formulation. Fortunately, it has already been conjectured that such a dual formulation of the heterotic string theory exists. It has been argued in Ref.[12] that heterotic string theory is equivalent to a theory of 5 dimensional extended objects, also known as 5-branes. The fields $\widetilde{G}^{(10)}_{MN}$, and $\widetilde{B}^{(10)}_{M_1...M_6}$, that appear in the dual formulation of the $N = 1$ supergravity theory, have natural couplings to the five-brane. (Unfortunately, at present there is no satisfactory way of coupling the ten dimensional gauge fields to the five-brane, so we shall leave them out of the analysis of this section. This difficulty may be related to the difficulty that we encountered in §2.3 in writing down a manifestly SL(2,R) and general coordinate invariant effective action in the presence of ten dimensional gauge fields.) Thus one might hope that the SL(2,Z) transformation has a natural action on the five-brane world volume theory.

We shall now see that this is indeed the case[40]. In particular, we shall show that the quantum numbers α^m and β^m ($1 \leq m \leq 6$) have interpretation as the internal momenta and the five-brane winding numbers[1] along the internal direction respectively. Thus the SL(2,Z) matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which corresponds to the transformation $\alpha^m \to \beta^m$, $\beta^m \to -\alpha^m$, exchanges the Kaluza-Klein modes carrying internal momenta with the five-brane winding modes on the torus. On the other hand, the quantum numbers α^{m+6}, β^{m+6} ($1 \leq m \leq 6$) correspond to magnetic type charges in the five-brane theory, and only the soliton solutions in the five-brane theory carry these charges. As a result, part of the O(6,22;Z) symmetry, $\alpha^m \leftrightarrow \alpha^{m+6}$, now interchanges elementary excitations of the five-brane theory with the solitons in this theory.

The world-volume swept out by the five-brane is six dimensional. If ξ^r denote the coordinates of this world volume ($0 \leq r \leq 5$) and Z^M denote the coordinates of the ten dimensional embedding

space ($0 \leq Z \leq 9$), then in the presence of the background $\tilde{G}_{MN}^{(10)}$ and $\tilde{B}_{M_1 \ldots M_6}^{(10)}$, the five-brane world-volume theory is described by the action[14]

$$\int d^6 \xi [\frac{1}{2}\sqrt{-\gamma}\gamma^{rs}\tilde{G}_{MN}^{(10)}\partial_r Z^M \partial_s Z^N - 2\sqrt{-\gamma} + \frac{1}{6!}\tilde{B}_{M_1 \ldots M_6}^{(10)}\epsilon^{r_1 \ldots r_6}\partial_{r_1}Z^{M_1}\cdots\partial_{r_6}Z^{M_6}]. \tag{117}$$

Here γ_{rs} is the metric on the five-brane world volume. Upon compactification, the coordinates Z^M split into the space time coordinates $X^\mu = Z^\mu$ ($0 \leq \mu \leq 3$) and internal coordinates $Y^m = Z^{m+3}$ ($1 \leq m \leq 6$). Let us first consider a background where all fields are independent of the internal coordinates Y^m, and the only non-vanishing components of the fields are

$$\tilde{G}_{mn}^{(10)}, \qquad \tilde{G}_{\mu\nu}^{(10)}, \qquad \text{and} \qquad \tilde{B}_{m_1 \ldots m_6}^{(10)} = \lambda_1 \epsilon_{m_1 \ldots m_6}. \tag{118}$$

Furthermore, $\tilde{G}_{\mu\nu}^{(10)}$ is adjusted so that $g_{\mu\nu} = \eta_{\mu\nu}$ asymptotically. The corresponding world-volume theory has two conserved current densities, given by,

$$j_m^r = (\sqrt{-\gamma}\gamma^{rs}\tilde{G}_{mn}^{(10)}\partial_s Y^n + \frac{\lambda_1}{5!}\epsilon^{rr_2 \ldots r_6}\epsilon_{mm_2 \ldots m_6}\partial_{r_2}Y^{m_2}\cdots\partial_{r_6}Y^{m_6})$$

$$\tilde{j}_m^r = \frac{1}{5!}\epsilon^{rr_2 \ldots r_6}\epsilon_{mm_2 \ldots m_6}\partial_{r_2}Y^{m_2}\cdots\partial_{r_6}Y^{m_6}, \tag{119}$$

which can be interpreted as the current densities associated with the five-brane internal momenta and winding numbers respectively. The total internal momenta p_m and winding numbers w_m of the five-brane are given by,

$$p_m = \int d^5\xi j_m^0, \qquad w_m = \int d^5\xi \tilde{j}_m^0. \tag{120}$$

In order to find the relationship between these conserved charges, and the quantum numbers α^m and β^m, we shall proceed in three stages. In the first stage we shall determine the coupling of the background gauge fields C_μ^m and $\mathcal{D}_\mu^m$, defined through Eq.(44), to the current densities j_m^r and $\tilde{j}_m^r$. In the second stage, we shall calculate the asymptotic values of the fields $F_{\mu\nu}^{(C)m}$ and $F_{\mu\nu}^{(D)m}$ in the presence of a five-brane carrying a fixed amount of p_m and w_m charges. In the third stage, we shall relate the asymptotic values of $F_{\mu\nu}^{(C)m}$ and $F_{\mu\nu}^{(D)m}$ to the asymptotic values of $F_{\mu\nu}^{(a)}$, and hence to α^a and β^a.

In order to carry out the first step, we switch on the background fields $\tilde{G}_{m\mu}^{(10)}$ and $\tilde{B}_{\mu m_2 \ldots m_6}^{(10)}$, and calculate the resulting contribution to the five-brane world volume action to linear order in these fields. Using Eqs.(117) and (44) we find that the extra contribution to the action to linear order in C_μ^m and $\mathcal{D}_\mu^m$ is given by

$$\int d^6\xi(C_\mu^m j_m^r \partial_r X^\mu + \mathcal{D}_\mu^m \tilde{j}_m^r \partial_r X^\mu). \tag{121}$$

Using the identification (53), we can rewrite this coupling as

$$2\int d^6\xi(\check{A}_\mu^{(m,1)}j_m^r \partial_r X^\mu - \check{A}_\mu^{(m,2)}\tilde{j}_m^r \partial_r X^\mu). \tag{122}$$

If we work in the static gauge $\xi^0 = X^0$, then the coupling of $\check{A}_0^{(m,\alpha)}$ is given by,

$$2\int d^5\xi dX^0(\check{A}_0^{(m,1)}j_m^0 - \check{A}_0^{(m,2)}\tilde{j}_m^0). \tag{123}$$

We now add (123) to the action (38) (or, equivalently, (51)), derive the equations of motion for the gauge fields $\check{A}_\mu^{(m,\alpha)}$, and compute the fields $\check{F}_{\mu\nu}^{(m,\alpha)}$ induced by the 5-brane source. The resulting asymptotic values of these fields are given by the equations

$$\hat{G}_{mn}^{(0)}(\mathcal{L}^T\mathcal{M}^{(0)}\mathcal{L})\begin{pmatrix}\check{F}_{0r}^{(n,1)}\\\check{F}_{0r}^{(n,2)}\end{pmatrix} + \hat{B}_{mn}^{(0)}\mathcal{L}\begin{pmatrix}\check{F}_{0r}^{(n,1)}\\\check{F}_{0r}^{(n,2)}\end{pmatrix} \simeq \frac{4}{r^2}\begin{pmatrix}-\int d^5\xi j_m^0\\\int d^5\xi \tilde{j}_m^0\end{pmatrix} = \frac{4}{r^2}\begin{pmatrix}-p_m\\w_m\end{pmatrix}, \tag{124}$$

and

$$\begin{pmatrix}\check{F}_{0r}^{(m,1)}\\\check{F}_{0r}^{(m,2)}\end{pmatrix} \simeq 0. \tag{125}$$

This determines the asymptotic values of the fields $\check{F}_{\mu\nu}^{(m,\alpha)}$ for $1 \leq m \leq 6$ and $1 \leq \alpha \leq 6$. On the other hand, the quantum numbers α^a and β^a are related to the asymptotic values of the fields $\check{F}_{\mu\nu}^{(a)}$ for $1 \leq a \leq 12$, as can be seen from Eqs.(60), (63) and (67). In the source free region, the relationship between the two sets of fields $\check{F}_{\mu\nu}^{(m,\alpha)}$ and $\check{F}_{\mu\nu}^{(a)}$ can be found by starting with the action (36), writing down the gauge field equations of motion in this theory, and noting that $\check{F}_{\mu\nu}^{(a)} \equiv \check{F}_{\mu\nu}^{(a,1)}$ for $1 \leq a \leq 12$. These equations let us express $\check{F}_{\mu\nu}^{(a)}$ in terms of the fields $\check{F}_{\mu\nu}^{(m,\alpha)}$, from which we can calculate the asymptotic values of the fields $\check{F}_{\mu\nu}^{(a)}$ in terms of p_m and w_m. Comparing these asymptotic values with Eqs.(60), (63) we get,

$$Q_{el}^{(m)} = \frac{4}{\lambda_2^{(0)}}\hat{G}^{(0)mn}(-p_n + \lambda_1^{(0)}w_n), \qquad Q_{mag}^{(m)} = 0$$

$$Q_{el}^{(m+6)} = -\frac{4}{\lambda_2^{(0)}}\hat{B}_{mq}^{(0)}\hat{G}^{(0)qn}(-p_n + \lambda_1^{(0)}w_n), \qquad Q_{mag}^{(m+6)} = 4w_m. \tag{126}$$

(Note that when $A_M^{(10)I} = 0$, then $\check{F}_{\mu\nu}^{(a)} = F_{\mu\nu}^{(a)}$ for $1 \leq a \leq 12$.) Finally, comparison with Eq.(67) yields

$$\alpha^m = -4p_m, \qquad \beta^m = 4w_m, \qquad \alpha^{m+6} = \beta^{m+6} = 0, \qquad \text{for} \quad 1 \leq m \leq 6. \tag{127}$$

(Note that here $\vec{\alpha}$ and $\vec{\beta}$ are 12 dimensional vectors, since we have ignored the charges associated with the ten dimensional gauge fields.) This establishes the desired relation, i.e. the quantum numbers α^m and β^m are related to the five-brane momenta and winding numbers in the internal direction respectively. Thus we see that the SL(2,Z) transformations do interchange the Kaluza-Klein modes with the five-brane winding modes.[9] Note also that the quantum numbers α^{m+6} and β^{m+6} for $1 \leq m \leq 6$ now have to be interpreted as topological charges in the five-brane theory.

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[9]This conclusion is also consistent with the fact that the H-monopole solutions can be regarded as five branes wrapped around the torus[19].

There are in fact further analogies between the target space duality transformations in string theory and the SL(2,Z) transformations in the five-brane theory. Let us define

$$\mathcal{G}_{mn} = \tilde{G}^{(10)}_{m+3,n+3} \tag{128}$$

as the internal components of the five-brane metric. From Eqs.(44) we see that the complex field λ has a natural expression in terms of the variables in the five-brane theory:

$$\lambda = \tilde{B}^{(10)}_{4\ldots9} + i\sqrt{\det\mathcal{G}}. \tag{129}$$

This is very similar to the expression for the complex structure moduli field τ for string theory compactified on a two dimensional torus:

$$\tau = B^{(10)}_{89} + i\sqrt{\det\bar{G}}, \tag{130}$$

where 8 and 9 denote the compact directions and $\bar{G}$ denote the components of $G^{(10)}$ in the two internal directions. Here $B^{(10)}$ and $G^{(10)}$ are the variables that couple naturally to the string. Under the target space duality transformation, the variable τ transforms to $(a\tau + b)/(c\tau + d)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an SL(2,Z) matrix, exactly as λ transforms under the S-duality transformation.

The existence of target space duality symmetry in string theory implies the existence of a minimum compactification radius, since the T-duality transformation relates tori of small radius to tori of large radius, with distances measured in the string metric $G^{(10)}_{MN}$. In the same spirit, the S-duality symmetry in string theory implies the existence of a maximum value of the string coupling constant. The discussion in the previous paragraph shows that this result may also be interpreted as the existence of a minimum size of the compact manifold, but now measured in the five-brane metric $\tilde{G}^{(10)}_{MN}$.

We end this section by summarising the roles of SL(2,Z) and O(6,6;Z) transformations in the string theory and the five-brane theory. This is best illustrated in the following table:

String Theory	Five Brane Theory
O(6,6;Z) is the symmetry of the low energy effective action	SL(2,Z) is the symmetry of the low energy effective action
SL(2,Z) is the symmetry of the low energy equations of motion	O(6,6;Z) is the symmetry of the low energy equations of motion
O(6,6;Z) exchanges Kaluza-Klein modes with string winding modes	SL(2,Z) exchanges Kaluza-Klein modes with 5-brane winding modes
SL(2,Z) exchanges elementary string excitation with solitons in string theory	O(6,6;Z) exchanges elementary 5-brane excitations with solitons in 5-brane theory
O(6,6;Z) implies a minimum size of the compact manifold measured in the string metric	SL(2,Z) implies a minimum size of the compact manifold measured in the 5-brane metric

8   Discussion and Open Problems

We conclude these notes with a discussion of some specific features of the SL(2,Z) symmetry, and some open problems in this area.

8.1   SL(2,Z) as a Discrete Gauge Symmetry

We have already argued that S-duality transformation has the possibility of being a symmetry of the four dimensional heterotic string theory. We shall now show that if SL(2,Z) is a symmetry of the theory, then it must act as a discrete gauge symmetry, *i.e.* we must *identify* field configurations that are related by any SL(2,Z) transformation. To start with, we note that the full SL(2,Z) group is generated by two elements,

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{131}$$

$\mathcal{T}$ generates the transformation $\lambda \to \lambda + 1$. It is well known[51, 8] that λ changes by 1 as we go around an elementary string. As a result, the very existence of elementary string states forces us to identify field configurations related by the transformation $\mathcal{T}$. Now, if $\mathcal{S}$ is a symmetry of the theory, then, acting on an elementary string state it must produce a valid state in the theory. But when we go around this new state, the field configuration changes by the SL(2,Z) transformation $\mathcal{S}\mathcal{T}\mathcal{S}^{-1}$. Thus we must also identify field configurations that are related by the SL(2,Z) transformation $\mathcal{S}\mathcal{T}\mathcal{S}^{-1}$. Now, Eq.(131) gives

$$\mathcal{S} = \mathcal{T} \cdot \mathcal{S}\mathcal{T}\mathcal{S}^{-1} \cdot \mathcal{T}, \tag{132}$$

showing that the full SL(2,Z) group is generated by $\mathcal{T}$ and $\mathcal{S}\mathcal{T}\mathcal{S}^{-1}$. This shows that we must identify field configurations which are related by any SL(2,Z) transformation, *i.e.* SL(2,Z) must be treated as a discrete gauge symmetry of the theory.

8.2   Relation to Other Proposals

Electric-Magnetic duality in four dimensional string theory has been discussed from a different point of view in Refs.[15]. This duality transformation can be identified to the string - five-brane duality transformation, when both the string theory and the five-brane theory are compactified on a six dimensional torus. This differs from the duality symmetry discussed here in an essential way, namely the string - five-brane duality transformation relates two different theories, and in that sense, is not a *symmetry* of any theory, whereas the SL(2,Z) transformation discussed here relates two different vacua of the same theory. This can also be seen from the point of view of the low energy effective field theory, — SL(2,Z) acts as a transformation on the variables of the low energy effective field theory, and is a symmetry of the equations of motion in the theory, whereas the string - five-brane duality transformation relates variables of two different actions (34) and (51).

8.3 Open Problems

In this paper we have produced several pieces of evidence for the existence of SL(2,Z) symmetry in string theory compactified on a six dimensional torus. However, much work remains to be done. First of all, we need to explicitly construct the new monopole and dyon states in the theory which must exist in order for SL(2,Z) to be a genuine symmetry. These have been discussed in §6, but we shall list them again here.

1) SL(2,Z) symmetry predicts the existence of BPS dyon solutions (with space-like electric and magnetic charge vectors) carrying multiple units of magnetic and electric charge in the vector representation of the N=4 super-Poincare algebra. Furthermore, if p and r denote the number of units of electric and magnetic charges carried by the dyon, then p and r must be relatively prime. For $r > 1$, these dyons could be regarded as supersymmetric bound states of monopoles and dyons, each carrying single unit of magnetic charge. A careful quantization of the zero modes of the BPS multi-monopole solutions[7] should exhibit these features if SL(2,Z) is a genuine symmetry of the theory. Recent results of Ref.[4], as well as earlier results of Refs.[22, 26, 20] may be particularly useful for this purpose. Triangle inequality guarantees that the energy of a supersymmetric state carrying these charges is strictly less than the lowest energy state in the continuum, hence it is quite plausible that such bound states do exist in the theory.

2) SL(2,Z) symmetry also predicts the existence of H-monopole and dyon solutions (with light-like electric and magnetic charge vectors) carrying multiple units of electric and magnetic charge. As before, if p and r denote the number of units of electric and magnetic charge carried by the dyon, then p and r must be relatively prime. For each such pair (p, r) there should be 21 distinct dyon states in the vector supermultiplet of the N=4 super-Poincare algebra, and one dyon state in the spin 2 representation of the N=4 super-Poincare algebra, saturating the Bogomol'nyi bound. Finally these solutions must exist at any generic point in the compactification moduli space. At present the existence of such solutions has been shown only at special points in the moduli space, where there is one or more unbroken SU(2) gauge group.

3) Finally, SL(2,Z) symmetry predicts the existence of monopole and dyon solutions with time-like electric and magnetic charge vectors. However, there is no limit in which these states become massless. As a result we do not expect these states to be represented as solutions in the effective field theory involving (nearly) massless fields. Perhaps one might be able to construct them as exact conformal field theories.

Another useful direction of investigation may be the study of five-branes. We have argued that the SL(2,Z) transformations act naturally on the five-branes, and hence it might be possible to establish that the five brane theory has an exact SL(2,Z) symmetry, even if we cannot solve the five-brane theory. This would at least establish that the SL(2,Z) symmetry of the four dimensional string theory is an immediate consequence of the string–five-brane duality in arbitrary dimensions.

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DYON - MONOPOLE BOUND STATES, SELF-DUAL HARMONIC FORMS ON THE MULTI-MONOPOLE MODULI SPACE, AND SL(2,Z) INVARIANCE IN STRING THEORY

ASHOKE SEN[*]

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

ABSTRACT

Existence of SL(2,Z) duality in toroidally compactified heterotic string theory (or in the N=4 supersymmetric gauge theories), that includes the strong weak coupling duality transformation, implies the existence of certain supersymmetric bound states of monopoles and dyons. We show that the existence of these bound states, in turn, requires the existence of certain normalizable, (anti-)self-dual, harmonic forms on the moduli space of BPS multi-monopole configurations, with specific symmetry properties. We give an explicit construction of this harmonic form on the two monopole moduli space, thereby proving the existence of all the required bound states in the two monopole sector.

[*] e-mail addresses: sen@theory.tifr.res.in, sen@tifrvax.bitnet

Heterotic string theory, compactified on a six dimensional torus, has been conjectured to possess an SL(2,Z) symmetry, part of which interchanges the strong and weak coupling limits of the string theory [1]. This generalizes the strong-weak coupling duality conjecture of Montonen and Olive [2] for the N=4 supersymmetric gauge theories [3], which is the non-gravitational sector of the low energy effective field theory of toroidally compactified heterotic string theory. In Ref.[1] we presented several pieces of evidence in support of this conjecture, and also worked out several consequences of this duality symmetry. One of the consequences is the existence of certain bound states of the known BPS monopole and dyon states in the theory. In this paper we shall study the criteria for the existence of these bound states. By working in a region of the moduli space where the masses of the monopoles and dyons are small compared to the Planck mass, we can ensure that gravity does not play an important role in the analysis, so that our results are equally valid for string theory, as well as for the N=4 supersymmetric gauge theories. We shall show that the existence of these bound states requires the existence of certain normalizable, (anti-)self-dual, harmonic forms on the multi-monopole moduli space with specific symmetry properties, and explicitly construct such a form on the two monopole moduli space.

The relevant parameters characterizing the theory under consideration are the loop expansion parameter g^2 and the theta angle θ. In string theory these can be related to the asymptotic values of the dilaton and the axion fields respectively, but in N=4 supersymmetric theory they must be treated as external parameters. It is convenient to combine the two parameters into a single complex parameter $\lambda = (\theta/2\pi) + ig^{-2} \equiv \lambda_1 + i\lambda_2$. The electric and magnetic charges of a state are characterized by two integers m and n.[†] Taking into account the Witten effect [4]

[†] For toroidally compactified heterotic string theory, the unbroken gauge group at a generic point in the compactification moduli space is $U(1)^{28}$, and we must specify two vectors $\vec{\alpha}$ and $\vec{\beta}$ in an appropriate 28 dimensional lattice to specify the electric and magnetic charges of a state. Since we shall be interested in the case where the vectors $\vec{\alpha}$ and $\vec{\beta}$ are parallel to each other [1], we may write $\vec{\alpha} = m\vec{\alpha}_0$, $\vec{\beta} = n\vec{\alpha}_0$, where $\vec{\alpha}_0$ is the smallest lattice vector along the direction of $\vec{\alpha}$ and $\vec{\beta}$. Thus, for fixed $\vec{\alpha}_0$, the electric and magnetic charge vectors

the physical electric charge can be shown to be proportional to $(m + n\theta/2\pi)$ (see Ref.[1] for complete expression for the electric and magnetic charges). The SL(2,Z) transformations are generated by matrices of the form $A \equiv \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with p, q, r, s integers, satisfying $ps - qr = 1$, and act on the quantum numbers m and n, and the coupling constant λ [5], in the following way,

$$\begin{pmatrix} m \\ n \end{pmatrix} \to \begin{pmatrix} m' \\ n' \end{pmatrix} = A \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} pm + qn \\ rm + sn \end{pmatrix}, \qquad \lambda \to \frac{p\lambda - q}{-r\lambda + s}. \qquad (1)$$

The mass of a state in a sector characterized by the quantum numbers m and n is bounded from below by a function $f(m,n)$, known as the Bogomol'nyi bound [6], given by [7] [8],

$$\left(f(m,n) \right)^2 = C \times (m \quad n) \mathcal{M} \begin{pmatrix} m \\ n \end{pmatrix}, \qquad (2)$$

where C is a normalization constant,[‡] and [9]

$$\mathcal{M} = \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix}. \qquad (3)$$

It can be easily checked that $f(m,n)$ is invariant under the SL(2,Z) transformation (1). Particles saturating the Bogomol'nyi bound are annihilated by half of the 16 supersymmetry generators, and belong to a 16 component representation of the supersymmetry algebra. A class of elementary string excitations (e.g. the massive gauge supermultiplet of the N=4 supersymmetric theories with spontaneously broken non-abelian gauge symmetries) are of this kind, carrying quantum numbers $(m = \pm 1, n = 0)$.

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are specified by the two integers m and n. The BPS monopoles that we shall discuss here correspond to states for which $(\vec{\alpha}_0)^2 = -2$ in the notation of Ref.[1].

[‡] In general C depends on other modular parameters of the theory like the Higgs vacuum expectation value, and also the vector $\vec{\alpha}_0$ specifying the direction of the electric and the magnetic charge vectors. But since these parameters do not transform under the SL(2,Z) transformation, we can treat them as constants for the purpose of this paper.

Given the existence of a state with quantum number $(m = 1, n = 0)$, SL(2,Z) symmetry of the theory requires the existence of other states, whose quantum numbers are related to these by the SL(2,Z) transformation given in Eq.(1). The charge quantum numbers carried by these states are given by,

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix}. \qquad (4)$$

Note that SL(2,Z) transformation relates the charge spectrum for one value of λ to charge spectrum for a different value of λ, since it acts non-trivially on λ. We shall assume that the elementary excitations carrying charge quantum numbers $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exist for all values of λ in the upper half plane.[§] In that case states carrying charge quantum numbers given in Eq.(4) must also exist for all values of λ. States with $p = 0$, $r = 1$ correspond to the usual BPS monopole solution in the theory, and belong to a 16 component supermultiplet [3]. States with $p \neq 0$ and $r = 1$ can be identified with the BPS dyon states in the theory, and are also known to belong to the 16-component supermultiplet. The states with $r \geq 2$ will be the subject of discussion of the present paper.

Note that p and r cannot be arbitrary integers, but must be such that for some integers s and q, $ps - qr = 1$. We can find integers s and q satisfying this requirement if and only if p and r are relatively prime. Thus we need to prove the existence of a *single* 16 component supermultiplet, saturating the Bogomol'nyi bound, for each pair of values of p and r which are relatively prime.

The significance of the requirement of p and r being relatively prime was discussed in Ref.[1]. For completeness we shall review the main points. If p and r are not relatively prime, then there exist integers p_0, r_0 and n such that $p = np_0$, $r = nr_0$. In this case a state carrying quantum numbers $\begin{pmatrix} p \\ r \end{pmatrix}$ and saturating the

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[§] This follows from the general argument given in Ref.[6] if we assume that the spectrum varies continuously with λ.

Bogomol'nyi bound has the same mass as the sum of masses of n states carrying quantum numbers $\begin{pmatrix} p_0 \\ r_0 \end{pmatrix}$ and saturating the Bogomol'nyi bound. Thus even if there exists a state saturating the Bogomol'nyi bound and having quantum numbers $\begin{pmatrix} p \\ r \end{pmatrix}$, it is energetically indistinguishable from the lowest energy state in the continuum containing n particles with quantum numbers $\begin{pmatrix} p_0 \\ r_0 \end{pmatrix}$. On the other hand, if p and r are relatively prime, then the mass of any state with these quantum numbers and saturating the Bogomol'nyi bound will be necessarily lower than the sum of the masses of any two or more states whose charge quantum numbers add up to $\begin{pmatrix} p \\ r \end{pmatrix}$. This is seen by using the triangle inequality,

$$ f(p,r) \le f(p_1, r_1) + f(p_2, r_2), \tag{5} $$

for any p_i, r_i satisfying $p = p_1 + p_2$, $r = r_1 + r_2$. The equality holds if and only if $p_1/r_1 = p_2/r_2 = p/r$, which is impossible if p and r are relatively prime. Thus for p and r relatively prime, a state carrying quantum numbers $\begin{pmatrix} p \\ r \end{pmatrix}$ and saturating the Bogomol'nyi bound, has an energy strictly less than the lowest energy state in the continuum. Such states, if they exist, may be regarded as bound states of r monopoles and/or dyons, each carrying a single unit of magnetic charge.

In order to look for these bound states, we must study the quantization of the collective modes of the r monopole solution. We shall carry out our analysis for the case where $\theta = 0$, but the extension to non-zero values of θ should be straightforward following the procedure of Ref.[4]. Fortunately this problem has already been solved in the recent work of Refs.[10 − 12] . The bosonic part of the configuration space M_r of the r monopole solution is known to have the structure [13]

$$ M_r = R^3 \times \frac{S^1 \times M_r^0}{Z_r}, \tag{6} $$

where R^3 denotes the configuration space of the center of mass coordinate of the monopole, S^1 is labeled by the coordinate χ conjugate to the total charge of the monopole, and M_r^0 is a non-trivial $4(r - 1)$ dimensional space. The group of transformations Z_r is generated by an element g that acts on the coordinate of S^1 by a shift $\chi \to \chi + (2\pi/r)$, and also has a non-trivial action on the manifold M_r^0. In order to carry out the quantization on M_r, we can first carry out the quantization on $R^3 \times S^1 \times M_r^0$ and then restrict ourselves to Z_r symmetric states. If we are looking at a sector with total electric charge quantum number p, then the wave-function on $R^3 \times S^1$ is proportional to $e^{ip\chi}$ and picks up a multiplicative factor of $e^{2\pi ip/r}$ under the action of g. Thus the component of the wave-function on M_r^0 must pick up a factor of $e^{-2\pi ip/r}$ under the action of g.

For the monopoles in N=4 supersymmetric theories, it was shown in Refs.[11][12] that for each bosonic collective coordinate, there are fermionic collective coordinates represented by a two component Majorana fermion. Let X^α $(1 \le \alpha \le 4(r - 1))$ and Y^a $(1 \le a \le 4)$ denote the coordinates on M_r^0 and $R^3 \times S^1$ respectively, and let λ^α and η^a be the corresponding two component fermionic collective coordinates. The metric on $R^3 \times S^1$ is known to be flat. Let us normalize the coordinates Y^a such that this metric is equal to δ_{ab}. Also, let us denote by $g_{\alpha\beta}$ the metric on M_r^0, by $\Gamma^\alpha_{\beta\gamma}$ the corresponding Christoffel symbol, and by $R_{\alpha\beta\gamma\delta}$ the Riemann curvature tensor. The dynamics of the collective coordinates is described the lagrangian [11] [12]

$$ L = L_0 + L_{int}, \tag{7} $$

where,

$$ L_0 = \frac{1}{2} \partial_0 Y^a \partial_0 Y^a + \frac{i}{2} \bar\eta^a \gamma^0 \partial_0 \eta^a, \tag{8} $$

$$ L_{int} = \frac{1}{2} g_{\alpha\beta} \partial_0 X^\alpha \partial_0 X^\beta + \frac{i}{2} g_{\alpha\beta} \bar\lambda^\alpha \gamma^0 D_0 \lambda^\beta + \frac{1}{12} R_{\alpha\beta\gamma\delta} \bar\lambda^\alpha \lambda^\gamma \bar\lambda^\beta \lambda^\delta, \tag{9} $$

and,

$$ D_0 \lambda^\beta = \partial_0 \lambda^\beta + \Gamma^\beta_{\alpha\gamma} \partial_0 X^\alpha \lambda^\gamma. \tag{10} $$

The quantization of L_0 is straightforward and has been discussed before [11], so

let us just mention the main features. If P_a denote the momenta conjugate to the coordinates Y^a, then the Hamiltonian is given by $\frac{1}{2}P_aP_a$. The vector (P_1, P_2, P_3) can be interpreted as the spatial momenta, and P^4 may be interpreted as the electric charge, up to certain normalization constants. The corresponding eigenvalue of the Hamiltonian is known to saturate the Bogomol'nyi bound [11] (in the semi-classical approximation of large g that we are using). Also, since the Hamiltonian is independent of the fermionic coordinates, there is a large degeneracy. Since each η^a is two component, there are eight fermionic coordinates altogether; we can quantize them by treating half of them as creation operators and half of them as annihilation operators. This gives a $2^4 = 16$ fold degeneracy of each state.

Before we go on to quantize L_{int}, let us try to determine what we should expect in order for SL(2,Z) to be a valid symmetry of the theory. Since the contribution from H_0 already saturates the Bogomol'nyi bound, we need an eigenstate of H_{int} of zero eigenvalue, in order that the combined state saturates the Bogomol'nyi bound. Furthermore, since we want the final state to have a 16-fold degeneracy, and since the quantization of H_0 already gives rise to this degeneracy, we need to have a unique eigenstate of H_{int} with zero eigenvalue for each value of the electric charge quantum number p, for which p and r are relatively prime. (Note that although p does not appear explicitly in the expression for H_{int}, it enters through the requirement that the wave-functions on M_r^0 must pick up a phase $e^{-2\pi ip/r}$ under the action of the generator g of Z_r transformations.)

Let us now discuss quantization of L_{int}. Fortunately, this has already been carried out by Witten [14]. It was shown that the states in the Hilbert space of this system are in one to one correspondence to the differential forms on M_r^0, and a zero energy eigenstate corresponds to a harmonic form on M_r^0. Thus, in order to establish the SL(2,Z) invariance of the spectrum, we need to establish the existence of a *unique* harmonic form on M_r^0, which picks up a phase of $e^{-2\pi ip/r}$ under the action of the generator g of the Z_r group. Since by our previous argument it is guaranteed to represent a true bound state, such an harmonic form will automatically be normalizable.

Now, given a harmonic p form ω on M_r^0 satisfying the required properties, we can always construct a harmonic $4(r-1)-p$ form on M_r^0 by taking the Hodge dual of ω. This would violate the condition that ω must be the unique harmonic form on M_r^0. The only exception is the case where ω is an (anti-)self-dual $2(r-1)$ form on M_r^0. Thus we see that SL(2,Z) invariance of the spectrum not only demands the existence of a harmonic form with specific Z_r transformation properties, but it also requires this harmonic form to be an (anti-)self-dual $2(r-1)$ form.

To summarize, SL(2,Z) invariance of the spectrum requires that

For every integer p for which p and r are relatively prime, the space M_r^0 must contain a normalizable, (anti-)self-dual, harmonic $2(r-1)$ form, which picks up a phase $e^{-2\pi ip/r}$ under the action of the generator g of the Z_r transformations.

In the rest of the paper we shall discuss the construction of such a harmonic form in the special case $r = 2$. The metric on the space M_2^0 is known explicitly [13] [15]. This space is labeled by the coordinates $(\rho, \theta, \phi, \psi)$ ($0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$) with the identification [15],

$$(\rho, \theta, \phi, \psi) \equiv (\rho, \pi - \theta, \pi + \phi, -\psi). \tag{11}$$

ρ is a radial coordinate. The metric is given by [13] [15]

$$ds^2 = f^2 d\rho^2 + a^2(\sigma_1)^2 + b^2(\sigma_2)^2 + c^2(\sigma_3)^2, \tag{12}$$

where f, a, b and c are known functions of ρ, and,

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \\ \sigma_3 &= d\psi + \cos\theta d\phi. \end{aligned} \tag{13}$$

The σ_i's satisfy the relation,

$$d\sigma_i = \frac{1}{2}\epsilon^{ijk}\sigma_j \wedge \sigma_k. \tag{14}$$

The generator g of Z_2 transformation acts on the coordinates $(\rho, \theta, \phi, \psi)$ as,

$$(\rho, \theta, \phi, \psi) \rightarrow (\rho, \theta, \phi, \psi + \pi). \tag{15}$$

Since $r = 2$, the relevant values of p are the set of all odd integers. For each of these values of p, the action of g is required to send the (anti-)self-dual form to its negative.

Thus what we are looking for is an (anti-)self-dual harmonic two form ω, which is invariant under the transformation (11), and goes to its negative under the transformation (15). We take the following trial solution

$$\omega = F(\rho)\left(d\sigma_1 - \frac{fa}{bc}d\rho \wedge \sigma_1\right). \tag{16}$$

By construction, this form is anti-self-dual, and satisfies all the symmetry requirements. Thus we only need to make sure that $d\omega = 0$; since ω is anti-self-dual, this will automatically give $d^\dagger\omega = 0$. This gives,

$$\frac{dF}{d\rho} = -\frac{fa}{bc}F, \tag{17}$$

and hence,

$$F(\rho) = F_0 \exp(-\int_\pi^\rho \frac{fa}{bc}d\rho'), \tag{18}$$

where $F_0 \equiv F(\pi)$. With the parametrization of Ref.[15], we have the following asymptotic form for the functions f, a, b and c as $\rho \rightarrow \infty$:

$$f \simeq -1, \qquad a \simeq \rho, \qquad b \simeq \rho, \qquad c \simeq -2. \tag{19}$$

Thus, as $\rho \rightarrow \infty$, $\frac{fa}{bc} \rightarrow \frac{1}{2}$. This gives $F \simeq F_0 \exp(-\rho/2)$ asymptotically, showing that the harmonic form is normalizable. This is expected, since, according to our previous argument, the energy of this state is strictly less than that of the continuum.

Finally, we need to ensure that the harmonic form ω is non-singular near the 'Bolt', $\rho \simeq \pi$. The correct choice of coordinates in this region is $\widetilde{\rho} \equiv \rho - \pi$, and a new set of Euler angles $\widetilde{\theta}$, $\widetilde{\phi}$ and $\widetilde{\psi}$ [15]. The one forms σ_i are given in this coordinate system by,

$$\sigma_1 = d\widetilde{\psi} + \cos\widetilde{\theta}d\widetilde{\phi},$$
$$\sigma_2 = -\sin\widetilde{\psi}d\widetilde{\theta} + \cos\widetilde{\psi}\sin\widetilde{\theta}d\widetilde{\phi}, \tag{20}$$
$$\sigma_3 = \cos\widetilde{\psi}d\widetilde{\theta} + \sin\widetilde{\psi}\sin\widetilde{\theta}d\widetilde{\phi}.$$

The functions f, a, b and c are approximated by,

$$f \simeq -1, \qquad a \simeq 2\widetilde{\rho}, \qquad b \simeq \pi, \qquad c \simeq -\pi. \tag{21}$$

Thus from Eq.(12) we see that, for $\rho \simeq \pi$,

$$ds^2 = d\widetilde{\rho}^2 + 4\widetilde{\rho}^2(d\widetilde{\psi} + \cos\widetilde{\theta}d\widetilde{\phi})^2 + \pi^2(d\widetilde{\theta}^2 + \sin^2\widetilde{\theta}d\widetilde{\phi}^2). \tag{22}$$

Let us now express ω given in (16) in this coordinate system near $\rho = \pi$. From Eq.(18) we see that $F \rightarrow F_0$ as $\rho \rightarrow \pi$, so that ω takes the form

$$\omega = F_0\left(\sin\widetilde{\theta}d\widetilde{\phi} \wedge d\widetilde{\theta} - \frac{2}{\pi^2}\widetilde{\rho}d\widetilde{\rho} \wedge (d\widetilde{\psi} + \cos\widetilde{\theta}d\widetilde{\phi})\right). \tag{23}$$

From the form of the metric (22) we see that the one forms $\sin\widetilde{\theta}d\widetilde{\phi}$, $d\widetilde{\theta}$, $d\widetilde{\rho}$, and $\widetilde{\rho}(d\widetilde{\psi} + \cos\widetilde{\theta}d\widetilde{\phi})$ are all well defined near the Bolt. This, in turn, shows that the two form ω is well defined near the Bolt.

Our result establishes the existence of supersymmetric bound states in the two monopole sector, with sixteen fold degeneracy, for each odd value of the total charge of the system.[*] This is in prefect agreement with the predictions of SL(2,Z) symmetry. Generalizing these results to the multi-monopole case will require proving the statement made in the paragraph above Eq.(11).

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[*] Note that since the anti-self-dual harmonic form ω cannot be regarded as a (2,0) or a (0,2) form, it does not correspond to a supersymmetric bound state in the N=2 supersymmetric theories [11].

Note added: The self-dual 2-form ω has been used by Gibbons and Ruback [16] for different purpose earlier, and has been further explored in Ref.[17]. Bound states of monopoles and dyons in non-supersymmetric theories have been discussed by Manton [18].

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