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# THE WZWM MODEL AT TWO LOOPS

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# THE WZWN model at two loops.

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N.P.B. 408 (1993)

We compute loops in the WZWN model with ordinary quant. field theory and background field formalism

(Witten, Bos: $g \to e^{\pi} g$, $\pi = \pi^a T_a$, $J_+ = \partial_+ g g^{-1}$)

Convent. pert. QFT $\overset{?}{\longleftrightarrow}$ conformal field theory

Dimens. reg. of UV and IR divs ($R^*$), $\epsilon^{\mu\nu}$ tensors, "evanescent counter terms".

We extend the usual background field action to a gauge-inv. action with $\vec{J}_\mu$. Then no UV divs. (chiral truncation yields back action of Witten, Bos).

**The 1PI current corr.fs. are an effective action, equal to WZWN with $k \to k + \tilde{h}$, modulo infrared divs.** ( nonlocality )
**In the connected corr.fs. all loops cancel.**

## General Remarks

1PI diagrams:  $\equiv \Pi^{ab}_{\mu\nu}$ ( but .... )

gauge invariance : $\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$
BPHZ : divs local if lower
(UV and IR) divs subtracted.
} $\Pi^{ab}_{\mu\nu}$ is finite!

1. But  and  not finite $\Rightarrow$  if $\epsilon_{\mu\nu}$ 2-dimens.

2. If $\epsilon^{\mu\nu}$ $n-$ dim: $\epsilon^{\mu\nu}\epsilon_{\rho\sigma} = f(\epsilon)(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$ then no $\Delta\mathcal{L}$
because:
$\Delta\mathcal{L} \sim \mathcal{L}$ (background gauge inv., $n$-dim. rot. inv.),
so only renorm. of quantum fields, which cancels in our case.

3. If $\epsilon^{\mu\nu}$ 2-dimen.: $\epsilon^{\mu\nu}\epsilon_{\rho\sigma} = (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$
Then $\Delta\mathcal{L} = -\frac{\tilde{h}}{32\pi^2\epsilon} \hat{\hat{\delta}}^{\mu\nu} \partial_\mu \pi \partial_\nu \pi$:
evanescent counter terms for  + 

4. No $\Delta\mathcal{L}$ for  or  ,

only for  because $J_\mu$ is 2-dimensional.

Odd number of $\epsilon_{\mu\nu}$? Standard model ?

# The WZWN Model

$$S^{\pm}_W[g] = k(I[g] \pm i\Gamma[g])$$

$$I[g] = \frac{1}{16\pi\chi} \int_{\partial B} d^2x \, Tr \partial_\mu g g^{-1} \partial^\mu g g^{-1} \partial_\lambda g g^{-1}$$

$$\Gamma[g] = \frac{1}{24\pi\chi} \int_B d^3x \epsilon^{\mu\nu\rho} Tr \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g g^{-1}$$

$$f_{ac}{}^d f_{bd}{}^c = -\tilde{h} g_{ab}, \qquad Tr T_a T_b = -\chi g_{ab}$$

$$\int d^3x \epsilon^{\mu\nu\rho} \partial_\rho F = \int d^2x F \; ; \; \epsilon^{12} = +1, \eta^{\mu\nu} = (+1, +1)$$

$$x^{\pm} = \frac{\lambda}{\sqrt{2}}(x^1 \pm ix^2) \; , \; \partial_{\pm} = \frac{1}{\lambda\sqrt{2}}(\partial_1 \mp i\partial_2)$$

$$S^{\pm}_W[hg] = S^{\mu}_W[h] + S^{\pm}_W[g]$$

$$+ \frac{k\lambda^2}{4\pi\chi} \int d^2x Tr \partial_{\pm} g g^{-1} h^{-1} \partial_{\mp} h$$

(Polyakov-Wiegmann 1983)

1. $S^+_W[hg]$ with $h = \exp \pi^a T_a$ as quantum fields and $g$ as background fields, has the Kac-Moody symmetry

$$h \to U(x^+) h U(x^+)^{-1}; \; g \to U(x^+) g$$

(and, of course, conformal symmetry).

2. $S^+_W[\bar{g}^{-1} h] = S^-_W[h^{-1}\bar{g}]$ has same symmetry but with $U(x^-)$.

3. $S^+_W[h]$ explicitly from $\quad {}_0^1 dt \frac{d}{dt} S^+_W[h(t)]$ with $h(t) = \exp t\pi$, using

$$\frac{d}{dt}[(\partial_\mu h(t)) h^{-1}(t)] = e^{t\pi} \partial_\mu \pi e^{-t\pi}$$

## A gauge-invariant action $S(\vec{\pi}, \vec{J})$

We take as action the GAUGED WZWN model

$$S^{+}_{W}[\bar{g}^{-1}hg] - S^{+}_{W}[\bar{g}^{-1}g] = S[\vec{\pi}, \vec{J}]$$

$$h \to U(x)h\,U(x)^{-1}; g \to U(x)g\ ;\ \bar{g} \to U(x)\bar{g}$$

(For $g = 1$ or $\bar{g} = 1$ the action of Witten, Bos. Noether method extends action with $J_+ = \partial_+ g g^{-1}$ to $J_+$ and $J_- = \partial_- \bar{g}\bar{g}^{-1}$).

$$S[\vec{\pi}, \vec{J}] = S^{+}_{W}[h] - \frac{k\lambda^2}{4\pi\chi}\int d^2x$$

$$Tr(\partial_+ h h^{-1} J_- - J_+ h^{-1}\partial_- h + J_+ h^{-1}[J_-, h])$$

$$= -\frac{k}{8\pi}\int d^2x\big[i\epsilon^{\mu\nu}\vec{\pi}\cdot(\partial_\mu \vec{J}_\nu - \partial_\nu \vec{J} - \vec{J}_\mu \times \vec{J}_\nu)$$

$$+\mathcal{L}^{(2)} + \mathcal{L}^{(3)}(\epsilon^{\mu\nu}) + \mathcal{L}^{(4)} + \dots\big]$$

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu\vec{\pi} - \vec{J}_\mu \times \vec{\pi})^2 =$$

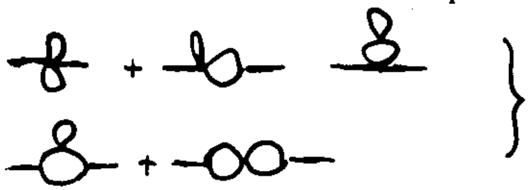
$\mathcal{L}^{(3)} =$

$\mathcal{L}^{(4)} =$

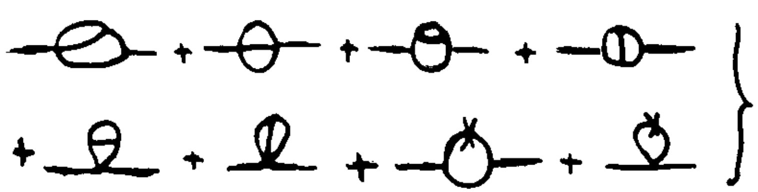
## The two-point function

$$\begin{aligned}\mathcal{L} = &-i\epsilon^{\mu\nu}\vec{\pi}\cdot(\partial_\mu \vec{J}_\nu - \partial_\nu \vec{J}_\mu + \vec{J}_\mu \times \vec{J}_\nu)\\ &-\frac{1}{2}(D_\mu\vec{\pi})^2 - \frac{i}{6}\epsilon^{\mu\nu}\vec{\pi}\cdot(D_\mu\vec{\pi}\times D_\nu\vec{\pi})\\ &+\frac{1}{24}(D_\mu\vec{\pi}\times\vec{\pi})^2 + \mathcal{O}(\pi^5)\end{aligned}$$

$$\pi^{ab}_{\mu\nu}(p) = \frac{\tilde{h}}{4\pi}g^{ab}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})(2 - \ln\frac{p^2}{\mu^2})$$
$$+\frac{\tilde{h}}{4\pi}g^{ab}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})(\frac{1}{12}\frac{\tilde{h}}{k}\ln^2\frac{p^2}{\mu^2})$$



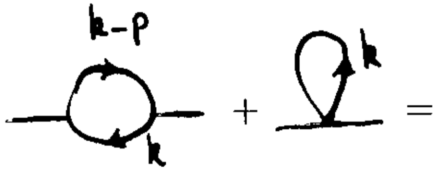
no $\epsilon$-tensors



two $\epsilon$ tensors

$\times$ = evanescent counter term.

**Example: the 1-loop case.**




$$\frac{\Gamma(1-\epsilon)}{(4\pi)^{\epsilon}}\tilde{h}g^{ab}\int\frac{d^nk}{(2\pi)^n}\left[\frac{k_\mu k_\nu - k_\mu(p-k)_\nu}{k^2(k-p)^2}-\frac{\delta_{\mu\nu}}{k^2}\right](\mu^2)^{-\epsilon}$$

$$=\frac{\tilde{h}g^{ab}}{4\pi}\,\frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)}{\epsilon\Gamma(2-2\epsilon)}\left(\frac{p^2}{\mu^2}\right)^{\epsilon}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)$$ gauge inv.

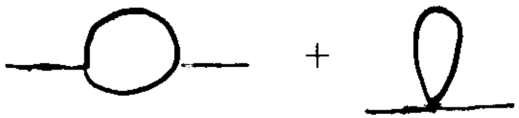
UV finite by gauge inv., so $\epsilon$-pole is IR div.

IR divs. subtracted by

$$\frac{1}{k^2}\to\frac{1}{k^2}+\frac{\pi}{\epsilon}\delta^2(k)$$

Then

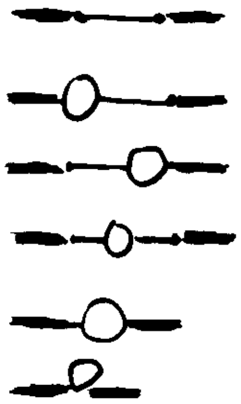
$$\frac{\tilde{h}}{4\pi}g^{ab}\int d^2k\left[\frac{\delta^2(k-p)k_\mu k_\nu}{k^2}-\delta^2(k)\delta_{\mu\nu}\right]=\frac{\tilde{h}g^{ab}}{4\pi\epsilon}\left(\frac{p_\mu p_\nu}{p^2}-\delta_{\mu\nu}\right)$$ gauge inv.



$$=\frac{\tilde{h}}{4\pi}g^{ab}\left(2-\ln\frac{p^2}{\mu^2}\right)\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)$$

(For fermions, no IR divs (primary fields) so no ~~k~~ ln $p^2/\mu^2$)

## Connected graphs.



$\Gamma_\mu(p)\;\Delta_\pi(p^2)\;\Gamma_\nu(p)$

1PI

Sum is zero! at the crit. p.

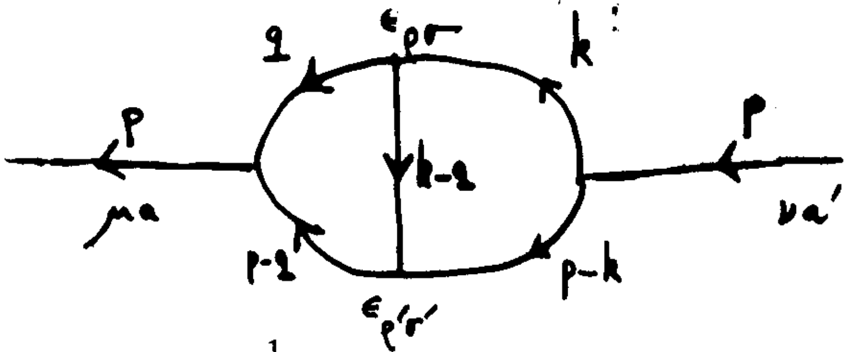
$$\Delta_\pi(p^2)=4g^2\frac{Z_\pi(p^2)}{p^2}\;;\quad\Gamma_\mu(p)=\frac{k}{4\pi}F_\pi(p^2)\epsilon_{\mu\nu}p^\nu.$$

From quantum 2-point and 3-point functions we find: wave-function renorm. and (off-critical point) coupling constant renormalization.

The wave-function renorm. counter terms do not contribute at 2-loop level.

The connected graphs off-critical point are: IR div.
UV finite (small miracle).

2-loop example:




$$J^a_\mu(-p)\frac{1}{2}\tilde{h}^2 J^{a'}_\nu(p)\delta_{aa'}\epsilon_{\rho\sigma}\epsilon_{\rho'\sigma'}$$

$$\int \frac{q_\mu q_\rho (p-q)_{\rho'} k_\sigma (p-k)_{\sigma'} (2k-p)_\nu}{q^2(p-q)^2 k^2 (p-k)^2 (k-q)^2} d^2k d^2q$$

$d^2k$ : UV finite

$d^2q$ : UV finite

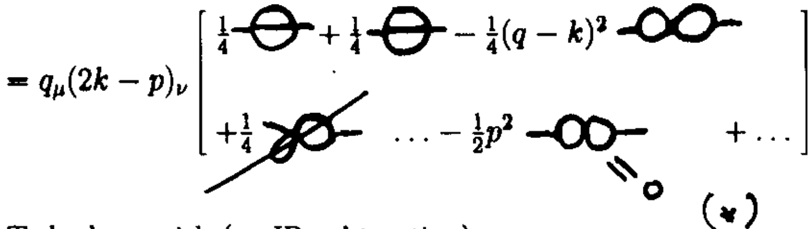
$d^2kd^2q$ : UV finite

IR: none (if replace $q_\rho$ by $(q-k)_\rho$ )

So here we may use

$$\epsilon_{\rho\sigma}\epsilon_{\rho'\sigma'} = \delta_{\rho\rho'}\delta_{\sigma\sigma'} - \delta_{\rho\sigma'}\delta_{\rho'\sigma}$$

$$q_\mu(2k-p)_\nu \times \begin{bmatrix} \frac{1}{4}k^2(q-p)^2 + \frac{1}{4}q^2(k-p)^2 - \frac{1}{4}(q-k)^4 \\ +\frac{1}{4}q^2(k-q)^2 + \frac{1}{4}k^2(k-q)^2 + \frac{1}{4}(k-p)^2(k-q)^2 \\ -\frac{1}{2}p^2(q-k)^2 - \frac{1}{4}k^2(k-p)^2 - \frac{1}{4}q^2(q-p)^2 \\ +\frac{1}{4}(q-p)^2(k-q)^2 \end{bmatrix}$$




Tadpoles vanish (no IR subtraction)

$$(*) \qquad \int \frac{q_\mu(2k-p)_\nu}{q^2k^2(p-q)^2(p-k)^2} = 0$$

5

$$\int \frac{q_\mu(2k-p)_\nu + k_\mu(2q-p)_\nu}{q^2(k-q)^2(p-k)^2}$$

$$\int \frac{q_\mu}{q^2(k-q)^2} = \frac{1}{2}k_\mu \int \frac{1}{q^2(k-q)^2} \qquad (*)$$

$$= \int \frac{2k_\mu k_\nu - \frac{3}{2}k_\mu p_\nu}{(k-p)^2}\Big(\underbrace{\int \frac{1}{q^2(k-q)^2}d^2q}\Big)d^2k$$

$$\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\frac{\pi^{1-\epsilon}\Gamma(1+\epsilon)\Gamma(-\epsilon)^2}{(k^2)^{1+\epsilon}\Gamma(-2\epsilon)} \text{ with } \epsilon = \frac{1}{2}(2-d)$$

Then "G – scheme"

$$\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\int \frac{2k_\mu k_\nu - \frac{3}{2}k_\mu p_\nu}{(k-p)^2(k^2)^{1+\epsilon}}d^2k$$

IR subtraction for other cases:

$$\frac{1}{(k^2)^{1+\tau\epsilon}} + \frac{\pi}{(1+\tau)\epsilon}\delta^2(k)$$

$$\frac{k_\mu k_\nu}{(k^2)^{2+\tau\epsilon}} + \frac{\delta_{\mu\nu}}{2(1+\tau)}\,\frac{\pi}{\epsilon(1-\epsilon)}\,\delta^2(k)$$

Chetyrkin,
Tkachov
Smirnov
Kataev
⋮


$$\int \frac{q_\mu(2k-p)_\nu + k_\mu(2q-p)_\nu}{q^2(k-q)^2(p-k)^2}$$

$$\int \frac{q_\mu}{q^2(k-q)^2} = \frac{1}{2}k_\mu \int \frac{1}{q^2(k-q)^2}$$

$$= \int \frac{2k_\mu k_\nu - \frac{3}{2}k_\mu p_\nu}{(k-p)^2}\Big(\underbrace{\int \frac{1}{q^2(k-q)^2}d^2q}\Big)d^2k$$

$$\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\frac{\pi^{1-\epsilon}\Gamma(1+\epsilon)\Gamma(-\epsilon)^2}{(k^2)^{1+\epsilon}\Gamma(-2\epsilon)} \text{ with } \epsilon = \frac{1}{2}(2-d)$$

Then

$$\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\int \frac{2k_\mu k_\nu - \frac{3}{2}k_\mu p_\nu}{(k-p)^2(k^2)^{1+\epsilon}}d^2k$$

IR subtraction for other cases:

$$\frac{1}{(k^2)^{1+\tau\epsilon}} + \frac{\pi}{(1+\tau)\epsilon}\delta^2(k)$$

$$\frac{k_\mu k_\nu}{(k^2)^{2+\tau\epsilon}} + \frac{\delta_{\mu\nu}}{2(1+\tau)}\,\frac{\pi}{\epsilon(1-\epsilon)}\,\delta^2(k)$$

EXPANSION ABOUT $\epsilon = 0$ TIMES $\epsilon^2$.

| | | |
|---|---|---|
| Gang of 5 | $\frac{1}{2}\epsilon^2$<br>$\frac{1}{2}\epsilon^2$ | $-\frac{1}{2}\epsilon^2$<br>$-\frac{1}{2}\epsilon^2$ |
|  | $\frac{1}{8}+\frac{3}{8}\epsilon \quad +\frac{7}{8}\epsilon^2$<br>$\frac{1}{8}+0 \quad +0$ | $0$<br>$0$ |
|  | $\frac{1}{4}+\frac{5}{4}\epsilon-\frac{3}{2}\ln p^2+\frac{17}{4}\epsilon^2$<br>$\frac{1}{4}+\frac{3}{4}\epsilon-\frac{3}{2}\ln p^2+\frac{9}{4}\epsilon^2$ | $-\epsilon+\frac{3}{2}\ln p^2-5\epsilon^2$<br>$-\epsilon+\frac{3}{2}\ln p^2-3\epsilon^2$ |
|  | $-\frac{1}{8}-\frac{1}{8}\epsilon \quad -\frac{1}{8}\epsilon^2$<br>$-\frac{1}{8}+\frac{1}{8}\epsilon \quad -\frac{1}{8}\epsilon^2$ | $\frac{1}{2}\epsilon^2$<br>$\frac{1}{2}\epsilon^2$ |
|  | $-\frac{1}{8}\epsilon \quad -\frac{1}{16}\epsilon^2$<br>$-\frac{1}{8}\epsilon \quad -\frac{3}{8}\epsilon^2$ | $0$<br>$0$ |
|  | $-\frac{1}{4}-\frac{11}{8}\epsilon+\frac{3}{2}\ln p^2-\frac{85}{16}\epsilon^2$<br>$-\frac{1}{4}-\frac{3}{4}\epsilon+\frac{3}{2}\ln p^2-\frac{7}{4}\epsilon^2$ | $\epsilon-\frac{3}{2}\ln p^2+\frac{21}{4}\epsilon^2$<br>$\epsilon-\frac{3}{2}\ln p^2+3\epsilon^2$ |
|  | $\frac{1}{4}\epsilon^2$<br>$0$ | $-\frac{1}{4}\epsilon^2$<br>$0$ |
| SUM | $0 \quad 0 \quad 0 \quad 0$<br>$0 \quad 0 \quad 0 \quad 0$ | $0 \quad 0 \quad 0$<br>$0 \quad 0 \quad 0$ |

# Final remarks

- Sum IR divs zero? Other genera?
  Proof that background field form = QFT in our case?
- Elitzur's conjecture
- Results do not depend on regul. scheme since gauge inv. does not allow <u>local</u> counter terms
- Odd number of $\epsilon_{\mu\nu}$?
- Our action is a particular action:

1) $S(\bar{g}^{-1}hg)$ at critical point, off critical point $\frac{1}{g^2}(\partial\pi)^2+\ldots+k\;\pi(\partial\pi)^2+\ldots$ gauge invariant.

2) ordinary QFT : $Z_\pi$ (and $Z_g$ off-critical)

3) $S(hg) = \frac{1}{g^2}I(hg)+k\,\Gamma(hg)$
(Witten, Bos, off-critical <u>not</u> gauge inv.

4) $S(\bar{g}^{-1}hg) = \frac{1}{g^2}I(\bar{g}^{-1}hg)+k\,\Gamma(\bar{g}^{-1}hg)$
Two currents, not two gauge inv.
(Leutwyler + Shifman, I.J.M.P. A 1992)
For $g=\bar{g}$ we get our action.