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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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CONFORMAL FIELD THEORY

K. SCHOUTENS
Department of Physics
Princeton University
Princeton, NJ
USA

Please note: These are preliminary notes intended for internal distribution only.

LECTURES ON CONFORMAL FIELD THEORY

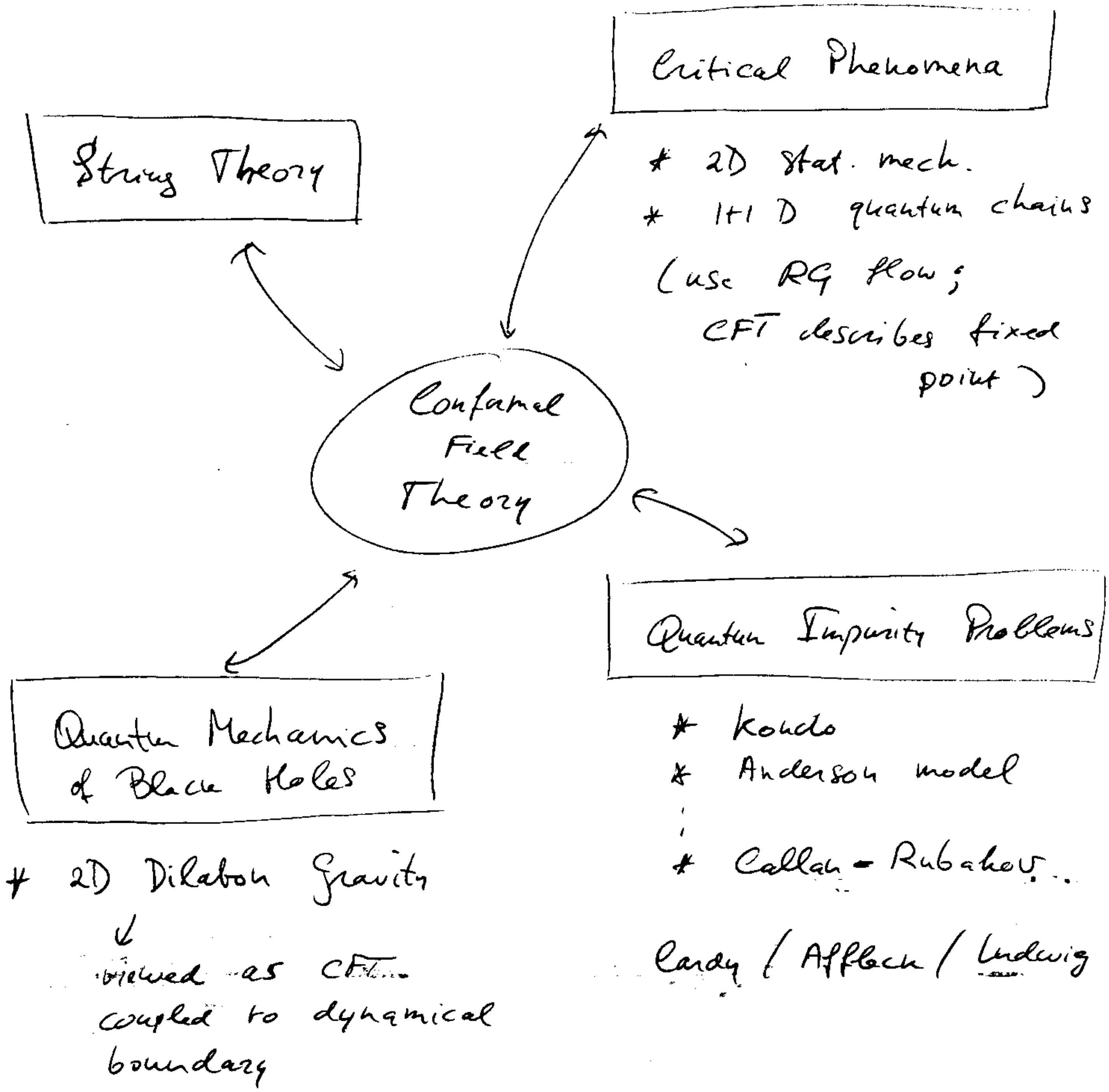
TRIESTE, JULY '94

K. SCHOUTENS, PRINCETON UNIV.

CONTENTS

LECTURE 1

1) General Introduction



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these lectures: CFT pure, no further reference to applications

2. Studying the Spectrum of a CFT

* using conformal symmetry
* " extended symmetry (RCFT)

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* using multi-particle picture

fields in a CFT

Stress tensor $T(z)$, $\overline{T}(\overline{z})$

OPE: $T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots$

Virasoro primary fields $\phi_{\Delta,\bar{\Delta}}(z,\bar{z})$

OPE: $T(z)\, \phi_\Delta(w) = \frac{\Delta \phi_\Delta(w)}{(z-w)^2} + \frac{\partial \phi_\Delta(w)}{(z-w)} + \dots$

under conformal trafo: (tensor property) $\tilde{\phi}(w) = \left(\frac{dz}{dw}\right)^\Delta \phi(z(w)) \dots$

conformal family:

$\underbrace{\phi_\Delta(z)}_{\text{primary}}$ $\underbrace{\partial\phi_\Delta(z),\ (T\phi_\Delta)(z),\ \dots}_{\text{descendants}}$

field modes: $T(z) = \sum_n L_n z^{-n-2}$

$$\phi_\Delta(z) = \sum_n (\phi_\Delta)_n z^{-n-\Delta} \quad \text{etc.}$$

properties: $[L_m, L_n] = \frac{c}{12}(m^3-m)\delta_{m+n} + (m-n)L_{m+n}$

$$[L_m, (\phi_\Delta)_n] = ((\Delta-1)m-n)(\phi_\Delta)_{m+n}$$

<u>States</u>:

$$\phi_\Delta(0)|0\rangle = \phi_{-\Delta}|0\rangle = |\phi\rangle \qquad \text{\underline{primary} state}$$

satisfies $L_0|\phi\rangle = \Delta|\phi\rangle$

$$L_n|\phi\rangle = 0 \quad , \quad n>0$$

other states obtained as:

$$L_{-N}^{k_N} \cdots L_{-2}^{k_2} L_{-1}^{k_1} |\phi\rangle \qquad \text{\underline{descendant} states}$$

...

together these span a Verma Module.

complication:

some descendant states are orthogonal to <u>all</u> states $\Rightarrow$ null states

example: $L_{-1}|0\rangle$, since

$$\langle 0|L_1 L_{-1}|0\rangle = 0$$

Irreducible rep. of Virasoro algebra:

$$M_{\Delta,c} = \text{Verma module on } |\phi_\Delta\rangle \Big/ \text{null states}$$

Virasoro characters:

$$\chi_{\Delta,c}(q) = \mathrm{tr}_{M_{\Delta,c}}\left(q^{-\frac{c}{24}+L_0} \right)$$

$$= \frac{q^{-\frac{c}{24}+\Delta}}{\prod_{m=1}^{\infty}(1-q^m)} \left(1 - q^{\Delta_1} \ldots \right)$$

subtracts null state at
$L_0 = \Delta + \Delta_1$

results for $c<1$:

* Kac Determinant
* Rocha Caridi character form.
* Friedan Qiu Shenker
  analysis of unitarity

$$\begin{cases} c \in \left(1 - \frac{6}{m(m+1)}\right) & m = 3, 4, \ldots \\ \Delta \in \text{finite set } \Delta^{c}_{r,s} \end{cases}$$

Modular Invariance restricts field content of CFT

$c < 1$: ADE classification

$c \geq 1$: Cardy proved that an infinite # of Virasoro primaries are necessarily present

In so-called Rational CFT (RCFT) finite description possible, based on extension of Virasoro algebra

examples: * affine Kac Moody algebras
* W-algebras

strategy: different highest weight modules (HWM) of Virasoro grouped together to form a single HWM of the extended algebra

examples: * $c=1$ $SU(2)$, $k=1$ WZW (see bel

* $c = \frac{4}{5}$: exceptional invariant

$$Z = |\chi_0 + \chi_3|^2 + |\chi_{\frac{2}{5}} + \chi_{\frac{7}{5}}|^2 + 2|\chi_{\frac{2}{3}}|^2 + 2|\chi_{\frac{1}{15}}|^2$$

understood on the basis of Zamolodchikov's W_3 algebra.

Alternative approach to CFT spectra

using many-particle states

$$\prod_i \phi_{-\Delta_i - n_i} |0\rangle$$

Easy example:

free fermions: $\{\phi_n^\alpha, \phi_m^\beta\} = \eta^{\alpha\beta}\delta_{m+n}$

spectrum constructed according to Pauli Principle

Non-trivial example

spinons $\phi^\alpha_{\Delta=\frac{1}{4}}$ in $SU(2)$, $k=1$ WZW

need * generalized commutators
* " Pauli Principle

We will show (lectures 2,3) that these are related to new algebraic structure, the Yangian $Y(sl_2)$

LECTURE II

3. $SU(2)$, $k=1$ WZW model

Virasoro structure

in this theory: Virasoro irreps labeled by $SU(2)$ - spin j

conformal dimensions: $\Delta_j = j^2$

(reason: $\widehat{SU(2)}$ currents $J^{\pm}(z) \leftrightarrow e^{\pm i\sqrt{2}\phi}(z)$

Δ_j state created by $e^{i\sqrt{2}j\phi}(z)$)

Virasoro characters:

$$\chi^{Vir}_{\Delta_j}(q) = \frac{q^{-\frac{1}{24}+j^2}}{\prod_{m=1}^{\infty}(1-q^m)} \left(1 - q^{2j+1}\right) \quad \ldots$$

null state at
$L_0 = j^2 + (2j+1) = (j+1)^2$

example: $j=0$, $\Delta=0$ $\quad |j=0\rangle = |0\rangle$
null state $L_{-1}|0\rangle$ as before

diagram

$L_0 \downarrow$	singlet	doublet	triplet	quadruplet
0	$\|0\rangle$			
$1/4$		$\|j=\frac{1}{2}\rangle$		
1	(null)		$\|j=1\rangle$	
$5/4$		$L_{-1}\|j=\frac{1}{2}\rangle$		
2	$L_{-2}\|0\rangle$		$L_{-1}\|j=1\rangle$	
$9/4$		$\left.\begin{matrix} L_{-2}\|j=\frac{1}{2}\rangle \\ \text{null state}\end{matrix}\right\}$		$\|j=\frac{3}{2}\rangle$
3	$L_{-3}\|0\rangle$		$L_{-1}^2\|j=1\rangle$, $L_{-2}\|j=1\rangle$	
$13/4$				$L_{-1}\|j=\frac{3}{2}\rangle$
4	$L_{-2}^2\|0\rangle$, $L_{-4}\|0\rangle$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ etc.

$$\chi_0^{\text{Vir}} \simeq 1 + q^2 + q^3 + 2q^4 \ldots$$

$$\chi_{\frac{1}{2}}^{\text{Vir}} \simeq q^{1/4}(1 + q + q^2 + \ldots)$$

etc.

$\widehat{su(2)}$, structure

currents $J^a(z)$ $a = +, -, 3$

affine primaries: $\mathbb{1}$, $\phi^\alpha_{\Delta=\frac{1}{4}}$ $\alpha = +, -$

OPE: $J^a(z) J^b(w) = \frac{d^{ab}}{(z-w)^2} + f^{ab}{}_c J^c(w) \frac{1}{z-w} + \ldots$

$$J^a(z)\, \phi^\alpha(w) = (t^a)^\alpha{}_\gamma\, \phi^\gamma(w) \frac{1}{z-w} + \ldots$$

Only two primary sectors w.r.t. $\widehat{su(2)}$,
(but an infinite # w.r.t. Virasoro!)

Affine characters: $\chi_j^{\widehat{su(2)}}$ $j = 0, \frac{1}{2}$

$$\chi_j^{\widehat{su(2)}} = \frac{q^{-1/8}}{\prod_{m=1}^{\infty} (1-q^m)^3} \sum_{n \in \mathbb{Z}} (6n+2j+1)\, q^{\frac{(6n+2j+1)^2}{12}} \quad \ldots$$

$$= \frac{q^{-\frac{1}{24} + \Delta_j}}{\prod_m (1-q^k)^3} (2j+1) \Big(1 - \underbrace{\ldots}_{} \Big)$$

subtract the null states

$$\chi_{j=0} = q^{-1/24}\left(1+3q+9q^2+\dots\right)\left(1-5q^2\dots\right)$$

subtracts $J^{(a}_{-1}J^{b)}_{-1}|0\rangle\Big|_{j=2 \text{ cpt}}$.

$$\chi_{j=\frac{1}{2}} = q^{-1/24+1/4}\left(1+3q+9q^2+\dots\right)\left(2-4q\dots\right)$$

subtracts $J^a_{-1}|j=\tfrac{1}{2}\rangle^\alpha\Big|_{j=\frac{3}{2}}$ cpt.

etc.

<u>diagram</u>

L_0	0	$\lvert j=0\rangle$	
$\downarrow$	$1/4$		$\lvert j=\tfrac{1}{2}\rangle^\alpha$
	1	$J^a_{-1}\lvert j=0\rangle$	$\dots$
	$5/4$		$J^a_{-1}\lvert j=\tfrac{1}{2}\rangle^\alpha\Big\rvert_{j=\frac{1}{2}}$
	2	$\left\{\begin{matrix} J^a_{-1}J^a_{-1}\lvert j=0\rangle \\ J^a_{-2}\lvert j=0\rangle\end{matrix}\right.$	
	$9/4$		$(j=\tfrac{1}{2})\,(j=\tfrac{3}{2})$
	3	$(j=0)\,(j=1)_2$	$\downarrow$ etc.

Computing the $\langle \phi\phi\phi\phi \rangle$ correlator

(suppressing $\bar{z}$ dependance)

$$G(z_1, z_2, z_3, z_4)_{\alpha_1 \alpha_4}^{\alpha_2 \alpha_3} = \langle \phi_{\alpha_1}(z_1)\, \phi^{\alpha_2}(z_2)\, \phi^{\alpha_3}(z_3)\, \phi_{\alpha_4}(z_4) \rangle$$

Use

$$\langle j^a(z)\, \phi_{\alpha_1}(z_1) \cdots \phi_{\alpha_4}(z_4) \rangle = \sum_{i=1}^{4} \frac{1}{(z-z_i)} (t^a)^{(i)} \langle \phi \cdots \phi \rangle$$

also

$$(j^a \phi^\alpha)(z) = 2 (t^a)^\alpha{}_\beta\, \partial \phi^\beta (z)$$

normal ordered

Combining the two, using $(t_a t^a) = \frac{3}{2} \delta$, sending $z \to z_i$

$$0 = \left(3\, \partial_{z_i} - \sum_{i \neq j} \frac{t_a^{(i)} t^{a(j)}}{(z_i - z_j)} \right) \langle \phi_{\alpha_1}(z_1) \cdots \phi_{\alpha_4}(z_4) \rangle$$

"Knizhnik-Zamolodchikov equation"

Solving the KZ equation:

* from L_{-1}, L_0, L_1 ($sl_2(\mathbb{C})$) Ward identities:

$$G^{\alpha_2\alpha_3}_{\alpha_1\alpha_4}(z_1 \ldots z_4) = \left((z_1-z_4)(z_2-z_3)\right)^{-1/2} G(x)$$

$$x = \frac{z_{12}\, z_{34}}{z_{14}\, z_{32}} \qquad z_{ij} = z_i - z_j$$

* from $SU(2)$ group theory:

$$G^{\alpha_2\alpha_3}_{\alpha_1\alpha_4}(x) = \underbrace{\delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_3}_{\alpha_4}}_{I_1} G_1(x) + \underbrace{\delta^{\alpha_3}_{\alpha_1}\delta^{\alpha_2}_{\alpha_4}}_{I_2} G_2(x)$$

Now work out KZ equation, $i=1$

<u>lemma</u>

$$\left(-t^{(1)}_a\right)^{\beta_1}_{\alpha_1}\left(+t^{a(2)}\right)^{\alpha_2}_{\beta_2}\,\delta^{\beta_2}_{\beta_1}\delta^{\alpha_3}_{\alpha_4} = -\tfrac{3}{2}\,\delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_3}_{\alpha_4} \ldots$$

$$(t^{(1)} t^{(2)})\, I_1 = -\tfrac{3}{2} I_1$$
$$(t^{(1)} t^{(2)})\, I_2 = \tfrac{1}{2} I_2 - I_1$$

$$(t^{(1)} t^{(3)})\, I_1 = \tfrac{1}{2} I_1 - I_2$$
$$(t^{(1)} t^{(3)})\, I_2 = -\tfrac{3}{2} I_2$$

$$(t^{(1)} t^{(4)})\, I_1 = -\tfrac{1}{2} I_1 + I_2$$
$$(t^{(1)} t^{(4)})\, I_2 = I_1 - \tfrac{1}{2} I_2$$

(used that

$$(t_a)^\alpha{}_\gamma (t^a)^\beta{}_\delta = -\frac{1}{2}\delta^\alpha{}_\gamma \delta^\beta{}_\delta + \delta^\alpha{}_\delta \delta^\beta{}_\gamma \quad)$$

find

$$3\partial_{z_1} G = -\frac{3}{2}\frac{1}{z_1-z_4} G + 3\big((z_1-z_4)(z_2-z_3)\big)^{-1/2} \partial_x G(x) \frac{z_{34} z_{24}}{z_{14}^2 z_{32}}$$

$$= \frac{1}{z_{12}}\Big(\big(-\tfrac{3}{2}G_1 - G_2\big) I_1 + \tfrac{1}{2} G_2 I_2 \Big)$$
$$+ \frac{1}{z_{13}}\Big(\tfrac{1}{2} G_1 I_1 + \big(-\tfrac{3}{2} G_2 - G_1\big) I_2 \Big)$$
$$+ \frac{1}{z_{14}}\Big(\big(-\tfrac{1}{2}G_1 + G_2\big) I_1 + \big(-\tfrac{1}{2}G_2 + G_1\big) I_2 \Big)$$

<u>notice</u> $\quad \dfrac{z_{14}^2 z_{32}}{z_{34} z_{24}}\left(\dfrac{1}{z_{14}} - \dfrac{1}{z_{12}}\right) = \dfrac{-1}{x}, \quad \dfrac{z_{14}^2 z_{32}}{z_{34} z_{24}}\left(\dfrac{1}{z_{14}} - \dfrac{1}{z_{12}}\right) = \dfrac{1}{1-x}$

$$3\,\partial_x G(x) = \frac{z_{14}^2 z_{32}}{z_{34} z_{24}} \cdots$$

$$+ \frac{1}{z_{14}}\Big((G_1 + G_2) I_1 + (G_1 + G_2) I_2 \Big)$$
$$+ \frac{1}{z_{12}}\Big(\big(-\tfrac{3}{2}G_1 - G_2\big) I_1 + \tfrac{1}{2} G_2 I_2 \Big)$$
$$+ \frac{1}{z_{13}}\Big(\tfrac{1}{2} G_1 I_1 + \big(-\tfrac{3}{2} G_2 - G_1\big) I_2 \Big)$$

$$3\,\partial_x G(x) = \left(\frac{-1}{x}\left(\frac{3}{2}G_1 + G_2\right) + \frac{1}{x-1}\,\frac{1}{2}\,G_1 \right) I_1$$

$$+ \left(\frac{1}{x}\,\frac{1}{2}\,G_2 - \frac{1}{x-1}\left(\frac{3}{2}\,G_2 + G_1\right) \right) I_2$$

final equation

$$-6\,\partial_x \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \left[\frac{1}{x}\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} + \frac{1}{x-1}\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \right] \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

Solution:

$$G_1(x) = (1-x)^{1/2}\,x^{-1/2}\,, \qquad G_2(x) = (1-x)^{\frac{1}{2}}\,x^{-\frac{1}{2}}$$

Interpretation in terms of 2-spinon states

procedure: map z_i to cylinder: $z_i = e^{\frac{2\pi}{\ell} w_i}$

$w_i = \tau_i + i x_i$, choose $\tau_{i+1} - \tau_i = 0^+$
(time ordered)

remaining function of the x_i:

$$\sim \sum_{n_1 \dots n_4} e^{i \frac{2\pi}{\ell}(n_1 x_1 + \dots n_4 x_4)} \langle 0 | \phi_{n_1} \phi_{n_2} \phi_{n_3} \phi_{n_4} | 0 \rangle$$

Example: triplet channel for α_3, α_4:

$$G_2(z_i) = \left(\frac{z_{12}\, z_{34}}{z_{13} z_{14} z_{23} z_{24}} \right)^{1/2} \underset{(\text{leading})}{\rightsquigarrow} e^{i\frac{2\pi}{\ell}\left(\frac{1}{4}x_1 + \frac{3}{4}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4\right)}$$

$$\langle 0 | \phi_{\frac{1}{4}}\, \phi_{\frac{3}{4}}\, \phi_{-\frac{3}{4}}\, \phi_{-\frac{1}{4}} | 0 \rangle$$

results:

triplet channel of α_3, α_4 indices:

$$\phi_{-\frac{3}{4}}\,\phi_{-\frac{1}{4}}\,|0\rangle$$

$$\phi_{-\frac{7}{4}}\,\phi_{-\frac{1}{4}}\,|0\rangle \qquad \text{etc.}$$

singlet channel

$$\phi_{\frac{1}{4}}\,\phi_{-\frac{1}{4}}\,|0\rangle \sim |0\rangle$$

—

$$\phi_{-\frac{7}{4}}\,\phi_{-\frac{1}{4}}\,|0\rangle \qquad \text{etc.}$$

note 1) mode index of second ϕ shifted by $1/2$

2) two spinon state $\phi^{\alpha}_{-\frac{3}{4}}\,\phi^{\beta}_{-\frac{1}{4}}\,|0\rangle$

vanishes in singlet channel

↙

systematic discussion & derivation in LECTURE III.