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Lecture III

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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CONFORMAL FIELD THEORY

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Please note: These are preliminary notes intended for internal distribution only.

LECTURE III

correction to notes LECTURE II:

$$G_1(x) = (1-x)^{1/2} x^{-1/2}, \quad G_2(x) = (1-x)^{-1/2} x^{1/2}$$

Multi-spinon formulation of $SU(2)_1$ WZW

OPE:
$$\phi^\alpha(z)\phi^\beta(w) = (-1)^q (z-w)^{-1/2} \varepsilon^{\alpha\beta}\left(1 + \tfrac{1}{2}(z-w)^2 T(w) + \dots\right)$$
$$+ (-1)^q (z-w)^{1/2} (t_a)^{\alpha\beta}\left(J^a(w) + \tfrac{1}{2}(z-w)\,\partial J^a(w) \dots\right)$$

Modes:
$$\phi^\alpha(z)\chi_q(0) = \sum_{m\in\mathbb{Z}} z^{m+\frac{q}{2}} \phi^\alpha_{-m-\frac{q}{2}-\frac{1}{4}} \chi_q(0)$$

$$q = 0\,(1) \iff j^3 = \text{integer (halfinteger)}$$

Generalized commutation relation:

$$\sum_{\ell\geq 0} C_\ell^{(-\frac{1}{2})}\left(\phi^\alpha_{-m-\frac{q+1}{2}-\ell+\frac{3}{4}}\,\phi^\beta_{-n-\frac{q}{2}+\ell+\frac{3}{4}} - \begin{pmatrix}\alpha\leftrightarrow\beta\\ m\leftrightarrow n\end{pmatrix}\right)$$

$$= (-1)^q \varepsilon^{\alpha\beta}\, \delta_{m+n+q-1}$$

where $(1-x)^\alpha = \sum_{\ell\geq 0} C_\ell^{(\alpha)} x^\ell$

From this, properties of N-spinon states follow.

Example: $q=0$, $n=m=1$

$$\left(\phi^{+}_{-\frac{3}{4}}\phi^{-}_{-\frac{1}{4}} - \phi^{-}_{-\frac{3}{4}}\phi^{+}_{-\frac{1}{4}}\right)|0\rangle = \phi_{-\frac{3}{4}}\overset{\text{singlet}}{\frown}\phi_{-\frac{1}{4}}|0\rangle = 0\,,$$

(agrees with result from 4-point function)

<u>systematic approach:</u>

look for symmetry structure compatible with multi-spinon structure of the Hilbert space

$\downarrow$

Yangian $Y(sl_2)$

origin: mathematical structure of Haldane-Shastry spin chains with inverse-square exchange.

Definition of $Y(\mathfrak{g})$ ($\mathfrak{g}$ is finite dim. Lie algebra)

$\downarrow$

Hopf algebra generated by Q_0^a, Q_1^a :

RELATIONS

(Y1) $[Q_0^a, Q_0^b] = f^{abc} Q_0^c$

(Y2) $[Q_0^a, Q_1^b] = f^{abc} Q_1^c$

(Y3) $[Q_1^a, [Q_1^b, Q_0^c]] + (\text{cyclic in } a, b, c)$

$$= A^{abc,def} \{Q_0^d, Q_0^e, Q_0^f\}$$

(Y4) $[[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]]$

$$= \left(A^{abp,qrs} f^{cdp} + A^{cdp,qrs} f^{abp}\right) \{Q_0^q, Q_0^r, Q_1^s\}$$

with $f^{adc} f^{dbc} = -2N \delta^{ab}$ $\quad$ ($\mathfrak{g} = su(N)$)

$A^{abp,def} = \frac{h^2}{4} f^{adp} f^{beq} f^{cfr} f^{pqr}$

COMULTIPLICATION

$$\Delta_{\pm}(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a$$

$$\Delta_{\pm}(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a \pm \frac{h}{2} f^{abc} Q_0^b \otimes Q_0^c$$

REMARKS

1) this is non-commutative
non-co-commutative
Hopf algebra, i.e. a quantum group

2) Structure related to Yang's R-matrix

$$R = (u_1 - u_2) + h P^{12}$$

hence the name "Yangian"

3) for $h \to 0$ Yangian goes into half
to loop algebra on g ; $\hat{g}_+$

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back to $su(2)$, WZW:

define
$$\begin{cases} Q_0^a = J_0^a \\ Q_1^a = \frac{1}{2} f^a{}_{bc} \sum_{m>0} J^b_{-m} J^c_m \end{cases}$$

* these satisfy (Y1) .. (Y4),
hence generate Yangian $Y(sl_2)$

* commuting with the Q's are

$$H_1 = L_0 , \quad H_2 = \sum_{m>0} m J^a_{-m} J^a_m , \quad \text{etc.}$$

$\Rightarrow$ can decompose each L_0 level in irreps of $Y(sl_2)$, each characterized by values of $H_2, H_3, \ldots$

precise content of these multiplets fixed by representation theory of $Y(sl_2)$
(Chari - Pressley)

multiplets can be described in terms of multi-spinon states

<u>EXAMPLE</u> 2-spinon states: $(n_2 \geq n_1 \geq 0)$

$$\Phi^{t,a}_{n_2,n_1} = (t^a)_{\alpha\gamma}\, \phi^{\alpha}_{-n_2-\frac{3}{4}}\, \phi^{\gamma}_{-n_1-\frac{1}{4}} |0\rangle$$

$$\Phi^{s}_{n_2,n_1} = \varepsilon_{\alpha\gamma}\, \phi^{\alpha}_{-n_2-\frac{3}{4}}\, \phi^{\gamma}_{-n_1-\frac{1}{4}} |0\rangle$$

At level $L_0 = 5$, $n_2+n_1 = 4$, have the following multiplets of $Y(sl_2)$

$(4,0)$, $H_2 = 45$: $\Phi^{t,a}_{4,0}$ $\quad \Phi^{s}_{4,0}$

$(3,1)$, $H_2 = 31$: $\Phi^{t,a}_{3,1} - \frac{1}{15}\Phi^{t,a}_{4,0}$ $\quad \Phi^{s}_{3,1} + \frac{3}{15}\Phi^{s}_{4,0}$

$(2,2)$, $H_2 = 25$: $\Phi^{t,a}_{2,2} - \frac{1}{6}\Phi^{t,a}_{3,1} - \frac{11}{120}\Phi^{t,a}_{4,0}$

GENERAL RESULT :

Yangian Highest Weight States are linear combinations of "fully polarized N-spinon states"

$$\phi^{+}_{-\frac{2N-1}{4}-n_N} \cdots \phi^{+}_{-\frac{3}{4}-n_2}\, \phi^{+}_{-\frac{1}{4}-n_1}\, |0\rangle$$

$$n_N \geq \ldots \geq n_2 \geq n_1 \geq 0$$

they can be labelled by $\{n_N, \ldots n_2, n_1\}$

the L_0, H_2 eigenvalues are :

$$\begin{cases} L_0 = \dfrac{N^2}{4} + \displaystyle\sum_{i=1}^{N} n_i \\ H_2 = \displaystyle\sum_{i=1}^{N} 2\left(n_i + \frac{i-1}{2}\right)\left(n_i + \frac{i}{2}\right) \end{cases}$$

The $SU(2)$ content for $\{n_N, \ldots n_2, n_1\}$ is

$$(\tfrac{1}{2}) \otimes (\tfrac{1}{2}) \otimes \ldots \otimes (\tfrac{1}{2})$$

where ${}_{i+1}\otimes_{i}$ symmetrized iff $n_{i+1} = n_i$.

The union of all these multiplets is the full Hilbert space of the theory.

Application: new expressions for the Virasoro characters in this theory

$$\chi^{\mathrm{Vir}}_{j^2} = q^{-\frac{j}{2}} \sum_{m_1, m_2, \ldots}^{k} q^{\frac{1}{2}(m_1^2 + m_2^2 + \ldots - m_1 m_2 - m_2 m_3 \ldots)}$$

$$\times \frac{1}{(q)_{m_1}} \prod_{a \geq 2} \begin{bmatrix} \frac{1}{2}(m_{a-1} + m_{a+1} + \delta_{a,2j+1}) \\ m_a \end{bmatrix}_q$$

where $\left\{ \begin{matrix} m_2, m_4, \ldots m_{2j} \\ m_1, m_3, \ldots m_{2j} \end{matrix} \right\}$ odd, rest even.

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q)_a}{(q)_b (q)_{a-b}} \qquad (q)_a = \prod_{m=1}^{a} (1-q^m)$$

In the sum, m_1 is the number of spinons.

This formula first conjectured by Kedem et al.

About correlation functions: one may write down Ward Identity based on Yangian symmetry. This takes over the role of the KZ equation!

REFERENCES

GENERAL CFT

1) lectures by Cardy / Ginsparg in Les Houches '88
2) reprint volume Itzykson, Saleur, Zuber, 1988

EXTENDED SYMMETRIES

3) Bouwknegt, Schoutens, Phys. Rep 223 (1993) 183

WZW CORRELATORS

4) Knizhnik, Zamolodchikov, NP B 247 (1984) 83

SPINON PICTURE FOR $SU(2)$, $k=1$, WZW

5) Haldane et al., PRL 69 (1992) 2021
6) Bernard, Pasquier, Serban, hep-th / 9404050
7) Bouwknegt, Ludwig, Schoutens, hep-th / 9406020

EXTENSION TO $SU(N)$, $k=1$

8) Schoutens, hep-th / 9401154

CHARACTER IDENTITIES

9) Kedem, Klassen, McCoy, Melzer, PL B307 (1993) 68
10) Melzer, hep-th / 9312043
11) Berkovich, hep-th / 9403073

YANGIAN $Y(g)$

12) Drinfeld, in Proc. Int. Congress of Mathematicians, Berkeley (1986)
13) Chari, Pressley, L'Enseignement Math. 36 (1990) 267; J. reine angew. Math. 417 (1991) 87